Module Categories and internal Bimodules

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Abstract

A theorem by Ostrik says that under some conditions every module category of a monoidal category $\mathcal{C}$ is equivalent to a category of modules internal to $\mathcal{C}$. I note that the 2-category $\mathbf{BiMod}(\mathcal{C})$ of bimodules internal to $\mathcal{C}$ sits inside the 2-category ${}_{\mathcal{C}}\mathbf{Mod}$ of module categories over $\mathcal{C}$:

$$\mathbf{BiMod}(\mathcal{C}) \subset {}_{\mathcal{C}}\mathbf{Mod}$$

in a certain sense. Ostrik's theorem suggests the conjecture that, when it applies, we actually have an equivalence of 2-categories.

$$\mathbf{BiMod}(\mathcal{C}) \simeq {}_{\mathcal{C}}\mathbf{Mod}\,.$$

Definition 1

1. *A* **2-monoid** *or* **monoidal category** $\mathcal{C}$ *is a coherent monoid*

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$

$$\begin{array}{ccc} \mathcal{C}\times\mathcal{C}\times\mathcal{C} & \xrightarrow{\mathcal{C}\times\otimes} & \mathcal{C}\times\mathcal{C} \\ {\scriptstyle \otimes\times\mathcal{C}}\downarrow & \swArrow\simeq & \downarrow{\scriptstyle\otimes} \\ \mathcal{C}\times\mathcal{C} & \xrightarrow[\otimes]{} & \mathcal{C} \end{array}$$

in **Cat**.

2. *A left* **2-module** *or left* **module category** ${}_{\mathcal{C}}\mathcal{M}$ *is a coherent left module*

$$\mathcal{C} \times {}_{\mathcal{C}}\mathcal{M} \xrightarrow{\ell} {}_{\mathcal{C}}\mathcal{M}$$

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$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} \times {}_{\mathcal{C}}\mathcal{M} & \xrightarrow{\mathcal{C} \times \ell} & \mathcal{C} \times {}_{\mathcal{C}}\mathcal{M} \\ {\scriptstyle \otimes \times \mathcal{C}} \downarrow & \swArrow \simeq & \downarrow {\scriptstyle \ell} \\ \mathcal{C} \times {}_{\mathcal{C}}\mathcal{M} & \xrightarrow[\ell]{} & {}_{\mathcal{C}}\mathcal{M} \end{array}$$

in **Cat**.

3. *A* **morphism of left $\mathcal{C}$-modules** *is a coherent morphism of left modules*

$$ {}_{\mathcal{C}}\mathcal{M} \xrightarrow{\phi} {}_{\mathcal{C}}\mathcal{M}' $$

$$\begin{array}{ccc} \mathcal{C} \times {}_{\mathcal{C}}\mathcal{M} & \xrightarrow{\ell} & {}_{\mathcal{C}}\mathcal{M} \\ {\scriptstyle \mathcal{C} \times \phi} \downarrow & \swArrow \simeq & \downarrow {\scriptstyle \phi} \\ \mathcal{C} \times {}_{\mathcal{C}}\mathcal{M}' & \xrightarrow[\ell']{} & {}_{\mathcal{C}}\mathcal{M}' \end{array}$$

in **Cat**, *hence a functor.*

4. *A* **2-morphism of left $\mathcal{C}$-modules** *is a natural transformation*

$$ {}_{\mathcal{C}}\mathcal{M} \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} {}_{\mathcal{C}}\mathcal{M}' \quad (R : \phi_1 \Rightarrow \phi_2) \; . $$

5. *The* **2-category of left $\mathcal{C}$ modules** *is the sub-2-category* ${}_{\mathcal{C}}$**Mod** *of* **Cat** *whose*

   - *objects are left $\mathcal{C}$-modules*
   - *morphisms are morphisms of left $\mathcal{C}$-modules*
   - *2-morphisms are 2-morphisms of left $\mathcal{C}$-modules.*

Example 1

Let $A \in \mathcal{C}$ be a monoid internal to the 2-monoid $\mathcal{C}$

$$ A \otimes A \xrightarrow{m} A \; . $$

Let $\mathbf{Mod}_A$ be the category of right A-modules internal to $\mathcal{C}$. For any morphism

$$\begin{array}{c} N_A \\ \big\downarrow f \\ N'_A \end{array} \quad \in \quad \mathrm{Mor}\,(\mathbf{Mod}_A) \subset \mathrm{Mor}\,(\mathcal{C})$$

and any morphism $U \xrightarrow{g} V \in \mathrm{Mor}\,(\mathcal{C})$ we get a new morphism

$$\begin{array}{c} U \otimes N_A \\ \big\downarrow g \otimes f \\ V \otimes N'_A \end{array} \quad \in \quad \mathrm{Mor}\,(\mathbf{Mod}_A)$$

in a way that is clearly functorial. This makes $\mathbf{Mod}_A$ into a left $\mathcal{C}$-module

$$\begin{array}{ccccc} \ell & : & \mathcal{C} \times \mathbf{Mod}_A & \to & \mathbf{Mod}_A \\ & & \left(\begin{array}{c} U \\ \big\downarrow g \\ V \end{array} \times \begin{array}{c} N_A \\ \big\downarrow f \\ N'_A \end{array} \right) & \mapsto & \begin{array}{c} U \otimes N_A \\ \big\downarrow g \otimes f \\ V \otimes N'_A \end{array} \end{array} .$$

Coherence of this left action is inherited from the coherence of the associator in $\mathcal{C}$.

Theorem 1 (Ostrik [1]) *Let $\mathcal{C}$ be a category which is*

- *monoidal*
- *semisimple*
- *rigid*
- *has finitely many irreducible objects*
- *has an irreducible unit object.*

Let ${}_{\mathcal{C}}\mathcal{M}$ be a module category over $\mathcal{C}$ which is

- *semisimple*
- *indecomposable.*

Then there exists an algebra object $A \in \mathcal{C}$ which is

- *semisimple*
- *indecomposable*

such that ${}_{\mathcal{C}}\mathcal{M}$ is equivalent to the category $\mathbf{Mod}_A$ of internal right A-modules:

$$\exists\ A \in \mathcal{C} \quad : \quad {}_{\mathcal{C}}\mathcal{M} \simeq \mathbf{Mod}_A \,.$$

Remark. Every monoidal category contains the trivial algebra object $\mathbb{1}$, the tensor unit, equipped with the trivial product $\mathbb{1} \otimes \mathbb{1} \longrightarrow \mathbb{1}$. Every object of $\mathcal{C}$ may be regarded as a $\mathbb{1}$-$\mathbb{1}$ bimodule, and $\otimes$ may be regarded as the tensor product over $\mathbb{1}$: $\otimes = \otimes_{\mathbb{1}}$. In the same vein, every right A-module N_A in $\mathcal{C}$ may be regarded as a $\mathbb{1}$-A-bimodule ${}_{\mathbb{1}}N_A$ internal to $\mathcal{C}$.

Definition 2 *Given a 2-monoid $\mathcal{C}$, the (weak)* **2-category of bimodules in** $\mathcal{C}$, $\mathbf{BiMod}(\mathcal{C})$, *is the (weak) 2-category whose*

1. *objects are algebra objects A in $\mathcal{C}$*
2. *morphisms $A \xrightarrow{{}_A N_B} B$ are A-B-bimodules in $\mathcal{C}$*
3. *2-morphisms* $A \underset{{}_A N'_B}{\overset{{}_A N_B}{\Downarrow \rho}} B$ *are bimodule homomorphisms ("intertwiners") in $\mathcal{C}$*

and where horizontal composition is given by the tensor product $\otimes_B$ of bimodules, while vertical composition is the composition of homomorphisms of bimodules.

Remark. We write

$$
{}_A\mathbf{Mod}_B \equiv \mathrm{Hom}_{\mathbf{BiMod}(\mathcal{C})}(A, B) \; .
$$

In particular[1]

$$
\begin{aligned}
\mathcal{C} &= {}_{\mathbb{1}}\mathbf{Mod}_{\mathbb{1}} \\
\mathbf{Mod}_A &= {}_{\mathbb{1}}\mathbf{Mod}_A \\
{}_A\mathbf{Mod} &= {}_A\mathbf{Mod}_{\mathbb{1}} \; .
\end{aligned}
$$

Horizontal composition in $\mathbf{BiMod}(\mathcal{C})$ gives functors

$$
{}_A\mathbf{Mod}_B \times {}_B\mathbf{Mod}_C \xrightarrow{\otimes_B} {}_A\mathbf{Mod}_C \; .
$$

The coherently weak associativity of these functors makes all ${}_A\mathbf{Mod}_A$ into 2-monoids, all categories ${}_A\mathbf{Mod}_B$ into left ${}_A\mathbf{Mod}_A$-modules and all categories ${}_B\mathbf{Mod}_A$ into right ${}_A\mathbf{Mod}_A$-modules, for all monoids A, B internal to $\mathcal{C}$.

[1] More precisely, we should write ${}_A\mathbf{Mod}_B(\mathcal{C})$ in order to indicate the ambient 2-monoid $\mathcal{C}$. For our purposes however we can fix once and for all some 2-monoid $\mathcal{C}$ and hence notationally suppress the depence of everything on this choice.

Example 2

The left 2-action from example 1 can now equivalently be written as

$$\ell = \otimes_{\mathbb{1}} : {}_{\mathbb{1}}\mathbf{Mod}_{\mathbb{1}} \times {}_{\mathbb{1}}\mathbf{Mod}_A \to {}_{\mathbb{1}}\mathbf{Mod}_A \,.$$

Definition 3 *Define the following map*

$$E \;:\; \mathbf{BiMod}(\mathcal{C}) \;\to\; {}_{\mathcal{C}}\mathbf{Mod}$$

$$A \;\underset{{}_AN'_B}{\overset{{}_AN_B}{\Downarrow \rho}}\; B \quad\mapsto\quad {}_{\mathbb{1}}\mathbf{Mod}_A \;\underset{\otimes_A(--,{}_AN'_B)}{\overset{\otimes_A(--,{}_AN_B)}{\Downarrow \otimes_A(\mathrm{Id},\rho)}}\; {}_{\mathbb{1}}\mathbf{Mod}_B$$

Here the notation on the right is supposed to mean the following. The functor $\otimes(--, {}_AN_B)$ acts as

$$\otimes(--, {}_AN_B) \;:\; {}_{\mathbb{1}}\mathbf{Mod}_A \;\to\; {}_{\mathbb{1}}\mathbf{Mod}_B$$

$$\mathbb{1} \;\underset{{}_{\mathbb{1}}M'_A}{\overset{{}_{\mathbb{1}}M_A}{\Downarrow \phi}}\; A \quad\mapsto\quad \mathbb{1} \;\underset{{}_{\mathbb{1}}M'_A}{\overset{{}_{\mathbb{1}}M_A}{\Downarrow \phi}}\; A \xrightarrow{{}_AN_B} B \,,$$

and the natural transformation $\otimes_A(\mathrm{Id}, \rho)$ is given by the map

$$\mathrm{Obj}({}_{\mathbb{1}}\mathbf{Mod}_A) \ni {}_{\mathbb{1}}M_A \mapsto \left(\mathbb{1} \xrightarrow{{}_{\mathbb{1}}M_A} A \;\underset{{}_AN'_B}{\overset{{}_AN_B}{\Downarrow \rho}}\; B \right) \in \mathrm{Mor}({}_{\mathbb{1}}\mathbf{Mod}_B)$$

which makes the naturality squares

$$\begin{array}{ccc}
{}_{\mathbb{1}}M_A \otimes_A {}_AN_B & \xrightarrow{\mathrm{Id}\otimes_A\rho} & {}_{\mathbb{1}}M_A \otimes_A {}_AN'_B \\
\Big\downarrow{\scriptstyle \phi\otimes_A\mathrm{Id}} & & \Big\downarrow{\scriptstyle \phi\otimes_A\mathrm{Id}} \\
{}_{\mathbb{1}}M'_A \otimes_A {}_AN_B & \xrightarrow[\mathrm{Id}\otimes_A\rho]{} & {}_{\mathbb{1}}M'_A \otimes_A {}_AN'_B
\end{array}$$

commute.

Proposition 1 *E is a 2-functor.*

Proof. Follows from the exchange law in $\mathbf{BiMod}(\mathcal{C})$. □

Remark. The 2-functor E is clearly injective on objects as well as on 1- and 2-morphisms. Hence it "embeds" $\mathbf{BiMod}(\mathcal{C})$ into ${}_{\mathcal{C}}\mathbf{Mod}$. So in any case we have

$$\mathbf{BiMod}(\mathcal{C}) \subset {}_{\mathcal{C}}\mathbf{Mod}$$

in some suitable sense of inclusion of 2-categories.

But Ostrik's theorem (theorem 1) says that if $\mathcal{C}$ is semisimple, rigid, has finitely many irreducible objects and an irreducible unit object, then E is also surjective on objects, up to equivalence. This motivates the following

Conjecture 1 *If $\mathcal{C}$ has all the properties listed in theorem 1, then E is an equivalence of 2-categories.*

I don't yet have a proof for this. But I think one would have to follow Ostrik's proof of theorem 1 on p. 10 of [1] and use functoriality of the internal Hom.

References

[1] V. Ostrik, Module Categories, weak Hopf Algebras and Modular Invariants, available as `math.QA/0111139`