Line-2-Bundles and Bundle Gerbes

Urs Schreiber[*]

February 8, 2006

Abstract

A line-2-bundle with 2-connection is defined to be a smooth functor from 2-paths to $\mathbf{Vect}_1$, where $\mathbf{Vect}_1$ is regarded as a 2-category with a single object. Pre-trivializations of a line-2-bundles are defined and shown to be in bijection with abelian bundle gerbes. The 2-category of pre-trivializations of line-2-bundles is shown to be equivalent to the 2-category of abelian bundle gerbes over a fixed fibration.
(Unfinished and unscrutinized private draft.)

[*]E-mail: urs.schreiber at math.uni-hamburg.de

1 Introduction

Parallel transport of points in bundles is most naturally described in terms of parallel transport functors. In fact, bundles with connection can be entirely encoded in a functor from some path category to some transport category.

Here we are interested in an analogous statement for categorifed abelian gauge theory and the parallel transport of abelian strings.

Our main result is that local "pre-trivializations" of smooth 2-functors

$$\mathrm{tra} : \mathcal{P}_2(M) \to \mathbf{Vect}_1$$

from 2-paths (surfaces) in a smooth space to a smooth sub-2-category of the monoidal category of 1-dimensional vector spaces (regarded as a 2-category with a single object) are in bijection with abelian bundle gerbes with connective structure. We expect that this extends to an equivalence of the respective 2-categories.

Our constructions entirely follow those in [13, 15], the only difference being that we are not dealing with *principal* 2-bundles as defined in [5] but with something we call **line-2-bundles**.

The fiber of a principal 2-bundle is defined to be a 2-torsor for its structure 2-group. The 2-torsor condition on the fibers turns out to be too rigid to describe general gerbes *globally*. Our line-2-bundles with 2-connection *locally* look like principal G_2-2-bundles with 2-connection, for G_2 given by the crossed module $G_2 = (U(1) \to 1)$.

A local pre-trivialization of a line-2-bundle should be thought of as local identification of such a line-2-bundle with trivial line-2-bundles, but without trivializing the *transitions* between such local trivial line-2-bundles. The transitions are given by line bundles and constitute a bundle gerbe. Trivializing them, too, leads to a Deligne-cocycle description of the line-2-bundle.

Segal proposed [1] that "string connections" of the kind we are interested in should be 1-functors from 2-dimensional cobordisms to **Vect**. A description of parallel transport of string in abelian gerbes in terms of 1-functors on 2-cobordisms has been discussed in [7]. However, Stolz and Teichner observed [2] that 1-functors are too coarse a tool to capture all aspects of string connections. They instead pass to 2-functors from a sort of 2-paths into some 2-category, such that Segal's picture is obtained as a special case when these 2-paths form cobordisms.

In this sense, following [13, 15], our definition of parallel transport in line-2-bundles given below is more along the lines of Stolz and Teichner's string connections, than Segal's cobordism 1-functors.

In fact, there is a natural motivation for our definition of 2-connection in line-2-bundles obtained from imagining a string as a continuous family of objects in a 1-cobordism category. This is discussed in detail in §2.1.

In §2.2 we then state our definition of a line-2-bundle with 2-connection and discuss how 1-automorphisms of line-2-bundles with 2-connection are related to ordinary line bundles with connection.

In §2.3 this is used to obtain the concept of an abelian bundle gerbe from a line-2-bundle. A "pre-trivialization" of a line 2-bundle is defined to be an operation where the line-2-bundle is locally identified with trivial line-2-bundles which are related by transition 1-morphisms of line-2-bundles. As mentioned above, these 1-morphisms turn out to correspond to ordinary line bundles with connection and are in fact the bundles appearing in the notion of a bundle gerbe.

A full trivialization of a line-2-bundle would then be obtained by further trivializing these transition bundles. Fully trivializing a line-2-bundle yields a locally trivialized $(U(1) \to 1)$-2-bundle along the lines of [13, 15].

The key point of our discussion is that 1- and 2-morphisms of our transport 2-functors (namely their pseudonatural transformations and modifications thereof) automatically encode all the information occuring in the definition of a bundle gerbe. The definition of morphisms of 2-functors is recalled in §A.1.

2 Line-2-Bundles

The study of higher gauge theory decomposes into two tasks:

1. investigating an appropriate arrow-theoretic description for the abstract objects under consideration,
2. realizing this by internalizing the arrow-theory in some suitable ambient category, in our case usually **Diff**, the category of smooth (diffeological) spaces.

The second step is by construction straightforward but tends to be technical. We will present line-2-bundles in the following first working in **Set**, without specifying any extra structure which would determine whether the maps involved are supposed to be continuous, smooth, etc. Most interesting aspects of the theory already appear in this abstract setup.

2.1 Line Bundles with Connection in Terms of Transport Functors

Let K be some field and let $\mathbf{Vect}_1$ be the symmetric monoidal category of 1-dimensional vector spaces over K. In all of the following "vector space" means "vector space over K". In §2.3 we will restrict to $K = \mathbb{C}$.

Fix some set M, the **base space** (to become a topological or a smooth space later on) and fix a groupoid $\mathcal{P}_1(M)$ *over* M. That is, a groupoid such that $\mathrm{Obj}(\mathcal{P}_1(M)) = M$.

Definition 1 *A* **line bundle with connection** *over* M *(with respect to* $\mathcal{P}_1(M)$*) is a functor*

$$\mathrm{tra} : \mathcal{P}_1(M) \to \mathbf{Vect}_1 \, .$$

The vector space $\mathrm{tra}(x)$ represents the fiber over x and the linear map

$$\mathrm{tra}\left(x \xrightarrow{\gamma} y \right) \equiv \mathrm{tra}(x) \xrightarrow{\mathrm{tra}(\gamma)} \mathrm{tra}(y)$$

the parallel transport along γ.

When this notion of line bundle with connection is equipped with a smooth structure, one obtains the ordinary notion of a smooth line bundle with connection.

Definition 2 *Let $E \to M$ be a smooth line bundle. Write* $\mathrm{Trans}(E)$ *for the* **transport groupoid** *of E whose objects are the fibers of E and whose morphisms are the invertible linear maps between these fibers.*

We shall regard $\mathrm{Trans}(E)$ as a subcategory of $\mathbf{Vect}_1$ equipped with a smooth structure, writing $\mathrm{Trans}(E) \subset \mathbf{Vect}_1$. The space of morphisms of $\mathrm{Trans}(E)$ inherits a smooth structure from its representation as $(E \times E)/G$, where G is the structure group.

Proposition 1 *A smooth* **line bundle** $E \to M$ **with connection** ∇ *is a smooth functor*

$$\mathrm{tra}_\nabla : \mathcal{P}_1(M) \to \mathrm{Trans}(E) \subset \mathbf{Vect}_1$$

such that

$$\forall x \in M : \mathrm{tra}_\nabla(x) = E_x \,.$$

Proof. This follows using the result of [3]. See also [4]. □

$\mathbf{Vect}_1$ is a symmetric monoidal category. Using this additional structure the above can be reformulated in terms of cobordisms.

Definition 3 *Given a smooth space M, denote by* $\mathbf{ThCobor}_p(M)$ *the symmetric monoidal catgegory of p-dimensional cobordisms in M.*

Proposition 2 *A smooth* **line bundle** $E \to M$ **with connection** ∇ *is a smooth symmetric monoidal functor*

$$\mathrm{tra}_\nabla : \mathbf{ThCobor}_1(M) \to \mathbf{Vect}_1$$

such that

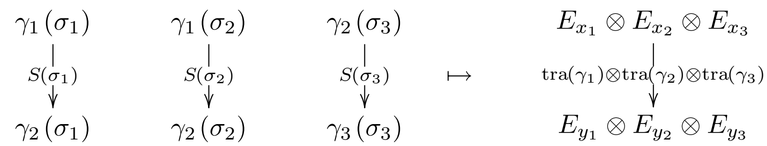
$$\forall x \in M : \mathrm{tra}_\nabla(x) = E_x \,.$$

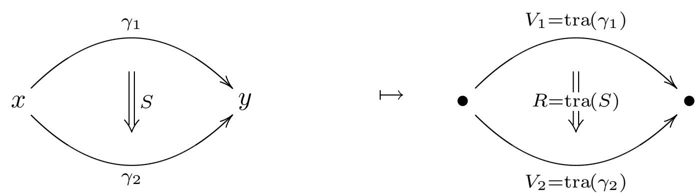
Compare for instance [7].

In the following $\mathbf{Vect}_1$ is notationally treated like its strictification, with all associators, left and right unit and braiding isomorphisms suppressed.

Motivation. In order to motivate the definition of line-2-bundles below, consider the following.

The parallel transport of a "string" γ_1 along a surface S can roughly be thought of as the parallel transport of a continuous family of points. Each line of points would be mapped to the continuous tensor product of the fibers over its points, while the surface would be mapped to the continuous tensor product of the linear operators acting on these fibers. The result would be a 2-functor from 2-paths to $\mathbf{Vect}_1$, where $\mathbf{Vect}_1$ is now regarded as a 2-category with a single object.





This motivates the following definition.

2.2 Definition of Line-2-Bundles

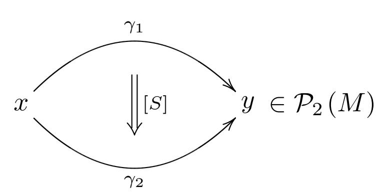
Let $\mathcal{P}_2(M)$ be some 2-groupoid *over* M.

Definition 4 *A* **line-2-bundle with 2-connection** *is a 2-functor*

$$\mathrm{tra} : \mathcal{P}_2(M) \to \mathbf{Vect}_1$$

with $\mathbf{Vect}_1$ *regarded as a 2-category with a single object.*

Given a 2-path



the transport 2-functor of a line-2-bundle with 2-connection sends it to

$$\mathrm{tra}\left(x \overset{\gamma_1}{\underset{\gamma_2}{\overset{\Downarrow [S]}{\rightrightarrows}}} y \right) \equiv \bullet \overset{V_1 = \mathrm{tra}(\gamma_1)}{\underset{V_2 = \mathrm{tra}(\gamma_2)}{\overset{R = \mathrm{tra}(S)\,\Downarrow}{\rightrightarrows}}} \bullet \in \mathbf{Vect}_1 \,,$$

where V_1 and V_2 are 1-dimensional vector spaces and where $V_1 \xrightarrow{R} V_2$ is a linear operator between them.

Proposition 3 *A line-2-bundle with 2-connection defines a line bundle with connection over the space of 1-morphisms in $\mathcal{P}_2(M)$.*

Definition 5 *The* **2-category of line-2-bundles with 2-connection** *over M (with respect to $\mathcal{P}_2(M)$) is the 2-category*

$$\mathbf{L2B}(M) \equiv \mathbf{Vect}_1^{\mathcal{P}_2(M)}$$

of 2-functors from $\mathcal{P}_2(M)$ to $\mathbf{Vect}_1$.

(See def. 18 in §A.1 for the definition of the 2-category of 2-functors.)

An automorphism of a line bundle corresponds to a *function* (a “0-bundle”). Automorphism of line-2-bundles turn out to correspond to line (1-)bundles.

Proposition 4

1. *1-Automorphisms $\mathrm{tra} \xrightarrow{\phi} \mathrm{tra}$ of line-2-bundles with 2-connection over M are in bijection with flat line bundles with connection on M.*
2. *Composition of 1-automorphisms of line-2-bundles corresponds to taking the tensor product of the corresponding line bundles.*
3. *2-morphisms between 1-automorphisms of line-2-bundles correspond to natural transformation of the transport 1-functors of the corresponding line bundles (and hence to isomorphisms between these line bundles which fix the base space).*

Proof.

1. Write $V = \mathrm{tra}(\gamma)$ for the vector space associated by tra to $x \xrightarrow{\gamma} y$. Then

$$\begin{array}{ccc} \bullet & \xrightarrow{\mathrm{tra}(\gamma)} & \bullet \\ {\scriptstyle \phi(x)}\downarrow & \swarrow\!\!\swarrow {\scriptstyle \phi(\gamma)} & \downarrow{\scriptstyle \phi(y)} \\ \bullet & \xrightarrow{\mathrm{tra}(\gamma)} & \bullet \end{array}$$

is a linear map

$$V \otimes \phi(y) \xrightarrow{\phi(\gamma)} \phi(x) \otimes V\,.$$

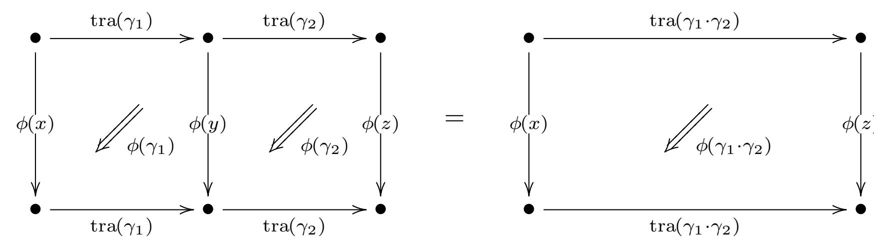
Since V is 1-dimensional this defines a linear map

$$\phi(y) \xrightarrow{\phi(\gamma)} \phi(x)$$

under the isomorphism

$$\begin{aligned}
\mathrm{Hom}(V \otimes \phi(y), \phi(x) \otimes V) &\simeq \mathrm{Hom}(\phi(y), V^* \otimes \phi(x) \otimes V)\\
&\simeq \mathrm{Hom}(\phi(y), \phi(x) \otimes V^* \otimes V)\\
&\simeq \mathrm{Hom}(\phi(y), \phi(x) \otimes K)\\
&\simeq \mathrm{Hom}(\phi(y), \phi(x))\,.
\end{aligned}$$

The functoriality condition on ϕ



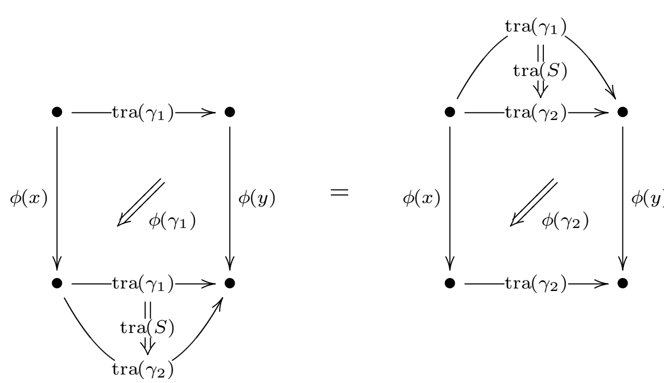
translates similarly into

$$\phi(z) \xrightarrow{\phi(\gamma_2)} \phi(y) \xrightarrow{\phi(\gamma_1)} \phi(x) \;=\; \phi(z) \xrightarrow{\phi(\gamma_1\cdot\gamma_2)} \phi(x)\;\;.$$

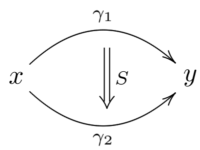
Therefore $\bar{\phi}$ defines a functor

$$\bar{\phi} : \mathcal{P}_1(M) \to \mathbf{Vect}_1$$

and hence a bundle with connection on M. Finally, ϕ has to make the tin can equation hold



.

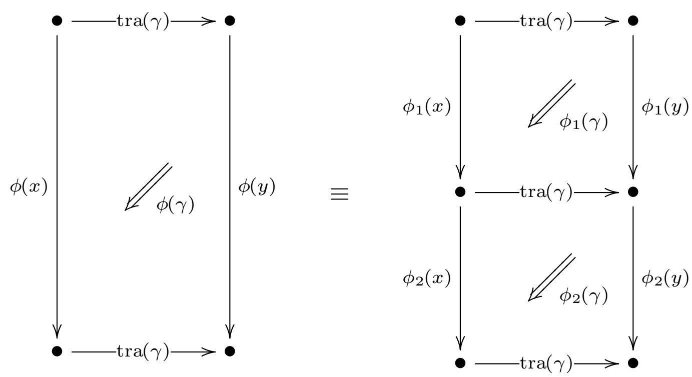
Since we have the same 

on both sides this implies that

$$\phi(\gamma_1) = \phi(\gamma_2)\ .$$

Hence ϕ is *flat*. Running these arguments backwards shows that conversely every flat line bundle on M gives rise to an automorphism $\mathrm{tra} \xrightarrow{\phi} \mathrm{tra}$ ·

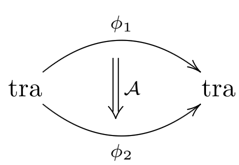
2. The composition



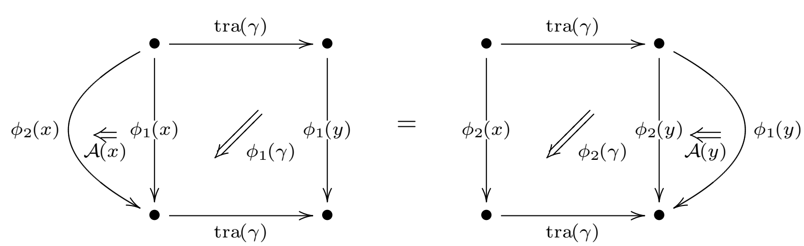
corresponds to

$$\begin{array}{c}\phi(x)\\ \Big\downarrow {\scriptstyle \bar{\phi}(\gamma)}\\ \phi_1(y)\end{array} \quad = \quad \begin{array}{c}\phi_1(x)\otimes\phi_2(x)\\ \Big\downarrow {\scriptstyle \bar{\phi}_1(\gamma)\otimes\bar{\phi}_2(\gamma)}\\ \phi_1(y)\otimes\phi_2(y)\end{array}$$

3. A 2-morphism



satisfies the tin can equation of the following form:



.

Under the above identification of $\phi(\gamma)$ with a linear map $\bar{\phi}(x) \xrightarrow{\bar{\phi}(\gamma)} \bar{\phi}(y)$ this is equivalent to a natural transformation

$$\begin{array}{ccc} \phi_2(x) & \xrightarrow{\bar{\mathcal{A}}(x)} & \phi_1(x) \\ {\scriptstyle \bar{\phi}_1(\gamma)} \downarrow & & \downarrow {\scriptstyle \bar{\phi}_2(\gamma)} \\ \phi_2(y) & \xrightarrow[\bar{\mathcal{A}}(y)]{} & \phi_1(y) \end{array}$$

□

2.3 Bundle Gerbes as Pre-Trivializations of Line-2-Bundles

Definition 6 *A line-2-bundle with 2-connection* tra *is called* **trivial** *precisely if for all* $S \in \mathrm{Mor}_2(\mathcal{P}_2(M))$

$$\mathrm{tra}\left(x \underset{\gamma_2}{\overset{\gamma_1}{\Downarrow [S]}} y \right) \equiv \bullet \underset{K}{\overset{K}{\Downarrow k(S)=\mathrm{tra}(S)}} \bullet \in \mathbf{Vect}_1\,,$$

with $k(S) \in K$ *regarded as a linear map* $k : K \to K$.

In anticipation of the situation in the smooth case, we shall write the $k(S)$ in the above formally as $\exp\left(\int_S B\right)$, hence

$$\mathrm{tra}\left(x \underset{\gamma_2}{\overset{\gamma_1}{\Downarrow [S]}} y \right) \equiv \bullet \underset{K}{\overset{K}{\Downarrow \exp\left(\int_S B\right)}} \bullet \in \mathbf{Vect}_1$$

for tra a trivial line-2-bundle. At this point this is just notation, no 2-form has appeared yet.

The next proposition asserts that trivial line-2-bundles have the interesting property that *all* morphisms between them are related to line bundles with connection. For general line-2-bundles this is true in general only for automorphisms, according to prop. 4. Moreover, the line bundle with connection associated to a 1-morphism between trivial line-2-bundles does not have to be flat. Instead, its curvature interpolates between the curving of the two trivial line-2-bundles that it maps between. This is the content of proposition 6 further below.

Proposition 5

1. *1-morphisms of trivial line-2-bundles on M are in bijection with line bundles with connection on M.*

2. *Composition of 1-morphisms of trivial line-2-bundles corresponds to taking the tensor product of the corresponding line bundles.*

3. *2-morphisms between 1-morphisms of trivial line-2-bundles correspond to bundle isomorphisms of the corresponding line bundles.*

Proof. The proof is completely analogous to that of prop. 4. □

In order to establish the connection between line-2-bundles and bundle gerbes, we need to realize the morphisms in $\mathcal{P}_2(M)$ as paths in M. Hence from now on M is assumed to be a smooth space and $\mathcal{P}_2(M)$ is taken to be the **2-groupoid of thin-homotopy classes of 2-paths** in M. Then the following definition makes sense.

Definition 7 *Let* $M' \xrightarrow{f} M$ *be any map. This induces a functor*

$$f^* : \begin{array}{ccc} \mathcal{P}_2(M') & \to & \mathcal{P}_2(M) \\ [S : [0,1]^2 \to M'] & \mapsto & [S \circ f : [0,1]^2 \to M] \end{array}$$

In the present context we find it convenient wo write, for any p-functor $F : \mathcal{P}_p(M) \to T$,

$$\mathcal{P}_p(M') \xrightarrow{f^*F} T \equiv \mathcal{P}_p(M') \xrightarrow{f^*} \mathcal{P}_p(M) \xrightarrow{F} T .$$

We shall use f^*F as a convenient substitute for the **pullback** of F along f (*cf.* §A.2).

Definition 8 *A* **local pre-trivialization** *of a line-2-bundle* tra : $\mathcal{P}_2(M) \to$ **Vect**$_1$ *is*

1. *a surjection (to be thought of as a generalization of a good covering)*

$$\begin{array}{c} \mathcal{U} \\ \downarrow p \\ M \end{array}$$

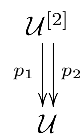
such that there exists a trivial line 2-bundle with connection

$$\mathcal{P}_2(\mathcal{U}) \xrightarrow{\mathrm{tra}_{\mathcal{U}}} \mathbf{Vect}_1$$

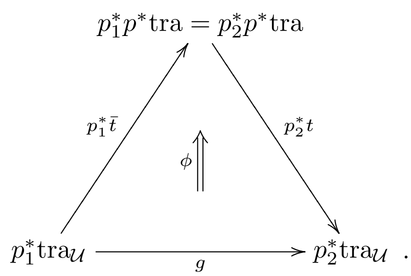
together with a special ambidextrous adjunction *(for instance an adjoint equivalence, see [14] for more details)*

$$p^*\mathrm{tra} \underset{\bar{t}}{\overset{t}{\rightleftarrows}} \mathrm{tra}_{\mathcal{U}} .$$

2. *a choice of 1-morphism* $p_1^*\mathrm{tra}_{\mathcal{U}} \xrightarrow{g} p_2^*\mathrm{tra}_{\mathcal{U}}$ *on*



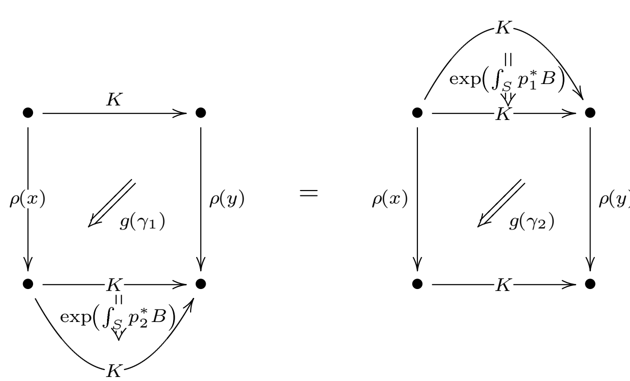
together with a 2-morphism



Proposition 6 *Consider a locally pre-trivialized line-2-bundle for* $K = \mathbb{C}$ *as above. Let* $S \mapsto \exp\left(\int_S B\right) \in \mathbb{C}^\times$ *be the assignment associated with* $\mathrm{tra}_{\mathcal{U}}$ *as in def. 6. Let* (L, ∇) *be the line bundle* $L \to \mathcal{U}^{[2]}$ *with connection associated to the line-2-bundle 1-morphism* $p_1^*\mathrm{tra}_{\mathcal{U}} \xrightarrow{g} p_2^*\mathrm{tra}_{\mathcal{U}}$ *according to prop. 5. Denote by* $\exp\left(\int_S F_\nabla\right) \in \mathbb{C}^\times$ *its holonomy around* ∂S*. Then*

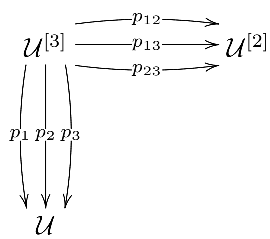
$$\exp\left(\int_{p_{2*}S} B\right) = \exp\left(\int_{p_{1*}S} B\right) \exp\left(\int_S F_\nabla\right) .$$

Proof. The existence of $p_1^*\mathrm{tra}_{\mathcal{U}} \xrightarrow{g} p_2^*\mathrm{tra}_{\mathcal{U}}$ is equivalent to the 2-commutativity of all respective tin cans:

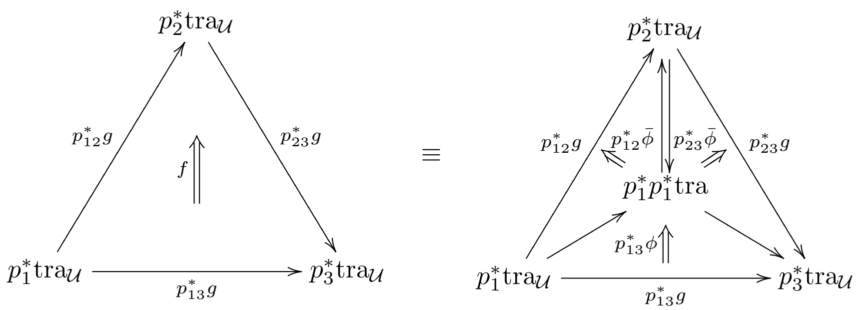


This immediately implies the above statement. □

Definition 9 *Given a locally pre-trivialized line-2-bundle with connection, define the* **local pre-trivialization modification** *on*



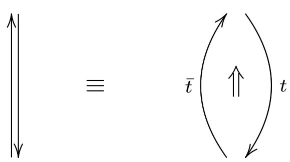
to be



(1)

Here and in the following (in particular in figure 1 on p. 16), two adjacent but oppositely oriented 1-arrows are shorthand for an equivalence

(2)



relating the composition of both 1-morphisms to the identity 1-morphism.

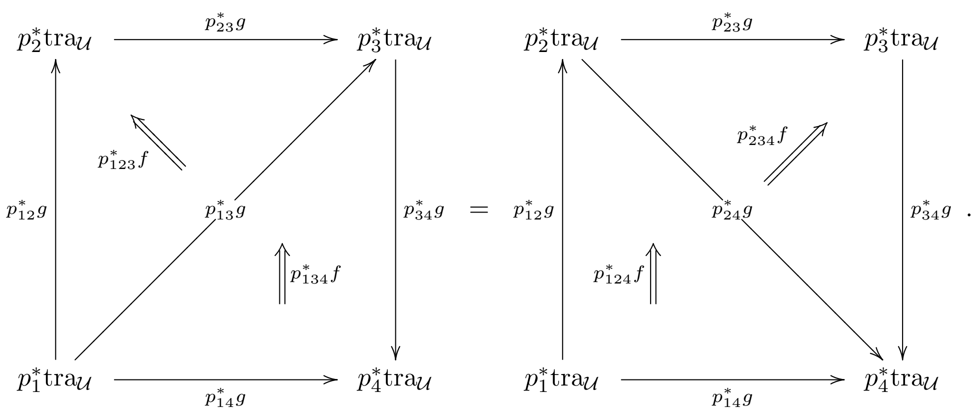
Proposition 7 *Following prop. 5, regard the 1-morphisms p_{ij}^*g in the above definition 9 as flat line bundles with connection. Under this identitification the local pre-trivialization modification f corresponds to a bundle isomorphism*

$$p_{12}^*g \otimes p_{23}^*g \xrightarrow{f} p_{13}^*g$$

fixing the base space.

Proof. This is an immediate consequence of prop. 5. □

Proposition 8 *The local pre-trivialization modification of a locally pre-trivialized line-2-bundle satisfies the following tetrahedron equation:*



Proof. Write out the pre-trivialization modification f in terms of the pre-trivialization 2-morphism ϕ according to its definition. The proposition follows by rearranging terms as shown in figure 1 on p. 16. $\square$

Definition 10 (Murray[8]) *A* **bundle gerbe** *over a manifold M is*

- *a surjective submersion*

$$\begin{array}{c} Y \\ \downarrow \\ M \end{array}$$

- *a $\mathbb{C}^\times$-bundle*

$$\begin{array}{c} L \\ \downarrow \\ Y^{[2]} \end{array}$$

- *over* $Y^{[3]} \overset{p_{12},\,p_{13},\,p_{23}}{\rightrightarrows} Y^{[2]}$ *a bundle isomorphism*

$$p_{12}^*L \otimes p_{23}^*L \xrightarrow{f} p_{13}^*L$$

which is associative in the sense that on $Y^{[4]} \overset{p_{123},\,p_{124},\,p_{134},\,p_{234}}{\rightrightarrows} Y^{[3]}$ *the dia-*

gram

$$\begin{array}{ccc} p_{12}^*L \otimes p_{23}^*L \otimes p_{34}^*L & \xrightarrow{p_{123}^* f \otimes \mathrm{Id}} & p_{13}^*L \otimes p_{34}^*L \\ \downarrow{\scriptstyle \mathrm{Id} \otimes p_{234}^* f} & & \downarrow{\scriptstyle p_{134}^* f} \\ p_{12}^*L \otimes p_{24}^*L & \xrightarrow[p_{124}^* f]{} & p_{14}^*L \end{array}$$

commutes.

A **connective structure** *on a bundle gerbe (also known as* **connection and curving** *on a bundle gerbe) is*

- *a connection* ∇ *on* L
- *a 2-form* $\omega \in \Omega^2(Y)$ *on* Y

such that on $Y^{[2]} \overset{p_2}{\underset{p_1}{\rightrightarrows}} Y$ *the equation*

$$p_2^*\omega - p_1^*\omega = F_\nabla$$

holds.

Here everything in sight is supposed to be smooth. In order to relate this to line-2-bundles we now introduce smooth line-2-bundles. In fact, for comparison we need smooth structure only on *trivial* line-2-bundles.

Definition 11

1. *A* **smooth trivial line-2-bundle** *is a trivial line-2-bundle* $\mathrm{tra} : \mathcal{P}_2(M) \to \mathbf{Vect}_1$ *such that* $\mathrm{tra}(S) = \exp\left(\int_S B\right)$ *does come from a 2-form* B *(compare def. 6).*

2. *Similarly a* **smooth 1-morphism** *of trivial smooth line 2-bundles is a 1-morphism coming from a smooth line bundle with connection in the sense of prop. 5.*

3. *Accordingly, a* **smooth 2-morphism** *of such smooth 1-morphisms is a 2-morphism coming from a smooth isomorphism of such line bundles with connection in the sense of prop. 5.*

This finally allows to state the following proposition.

Proposition 9 *Locally pre-trivialized smooth line-2-bundles for* $K = \mathbb{C}$ *with 2-connection are in bijection with abelian bundle gerbes with connective structure.*

Proof. Using the above notation, identify Y with $\mathcal{U}$. By prop. 5 the trivialization transition g defines a line bundle with connection on $\mathcal{U}^{[2]}$ and vice versa. Hence identify

$$g \leftrightarrow (L, \nabla)\,.$$

The picture obtained is

$$\begin{array}{ccccccc} g & & & & L & & \\ \downarrow & & & & \downarrow & & \\ \mathcal{U}^{[2]} & \rightrightarrows & \mathcal{U} & \leftrightarrow & Y^{[2]} & \rightrightarrows & Y \\ & & \downarrow & & & & \downarrow \\ & & M & & & & M \end{array}$$

Identify the gerbe product with the inverse of the modification f using the third item of prop. 5. By prop. 8 this does satisfy the required associativity condition.

In order to match the connection data, observe that the line-2-bundle $\mathrm{tra}_{\mathcal{U}}$ is trivial by assumption and hence defines, according to def. 11, a global 2-form B on $\mathcal{U}$. Identify this 2-form with the curving ω of the bundle gerbe. Prop. 6 says that $\mathrm{tra}_{\mathcal{U}}$ and (L, ∇) satisfy the condition of a gerbe connection

$$p_2^* B - p_1^* B = F_\nabla\,.$$

□

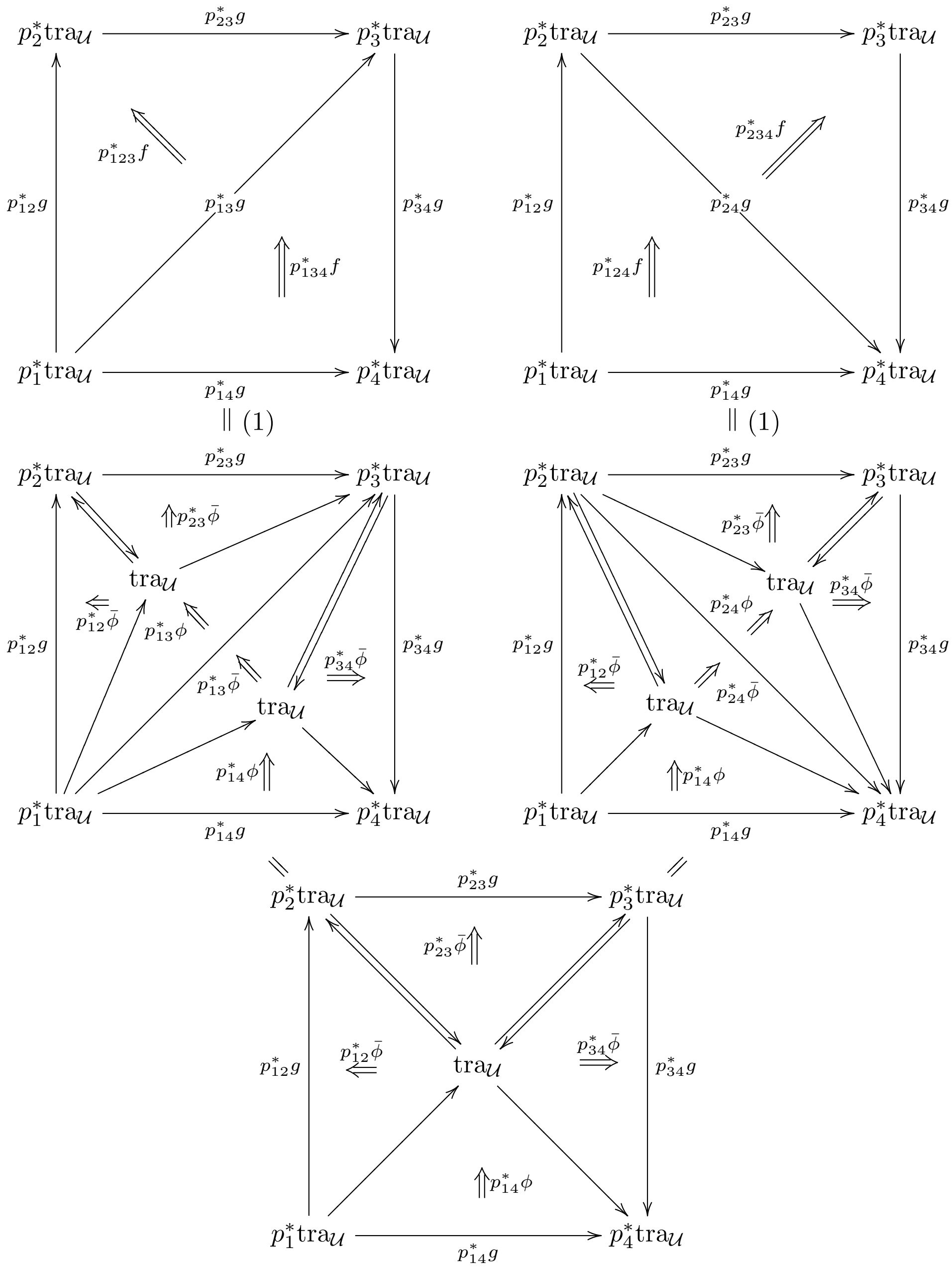


Figure 1: **Proof of the tetrahedron law** stated in prop. 8 on p. 12. The proof makes essential use of equation (1) on p. 12. Antiparallel arrows are shorthand for an equivalence, as declared in equation 2 on p. 12.

2.4 Stable Isomorphisms and the 2-Category of Local Pre-Trivializations

2.4.1 The 2-Category of Local Pre-Trivializations

A pre-trivialization of a line-2-bundle is essentially a 2-functor from Čech-simplices to the 2-category of trivial line-2-bundles. This naturally motivates a notion of 1- and 2-morphisms between pre-trivializations.

Definition 12 *The* **2-category of local pre-trivializations** *with respect to* $\mathcal{U} \xrightarrow{p} M$ *is the 2-category defined as follows:*

1. *objects are local pre-trivializations* $\mathcal{G} = (\mathrm{tra}_{\mathcal{U}}, t, \phi)$ *of line-2-bundles with respect to* $\mathcal{U}$

2. *a morphism* $\mathcal{G} \xrightarrow{\epsilon} \mathcal{G}'$ *is a morphism*

$$\mathrm{tra} \xrightarrow{f} \mathrm{tra}'$$

together with a map

$$\epsilon : \{t, \bar{t}, g\} \longrightarrow \mathrm{Mor}_2(\mathbf{L2B}(\mathcal{U}))$$

given by

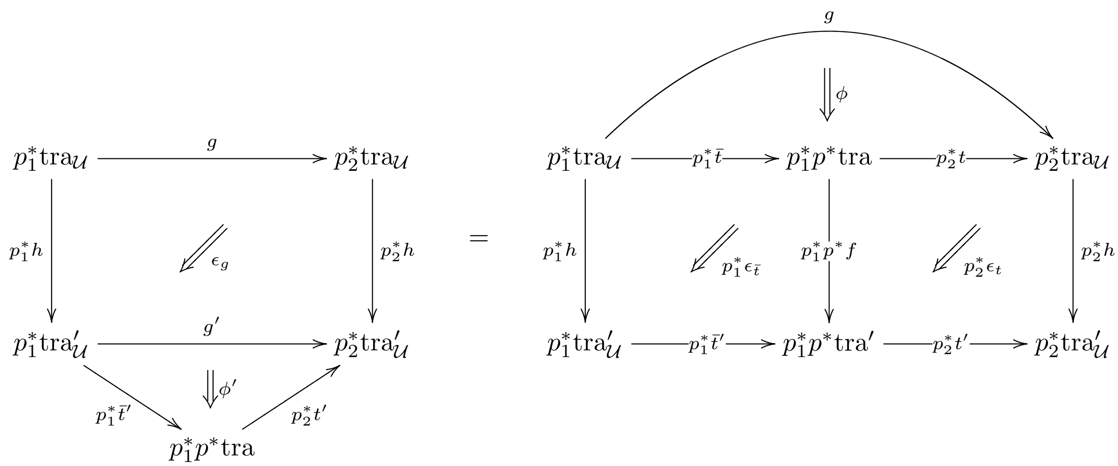
$$t \mapsto \begin{array}{ccc} p^*\mathrm{tra} & \xrightarrow{t} & \mathrm{tra}_{\mathcal{U}} \\ {\scriptstyle p^*f}\downarrow & \swarrow\!\!\!\!\swarrow \epsilon_t & \downarrow{\scriptstyle h} \\ p^*\mathrm{tra}' & \xrightarrow[t']{} & \mathrm{tra}'_{\mathcal{U}} \end{array}$$

$$\bar{t} \mapsto \begin{array}{ccc} \mathrm{tra}_{\mathcal{U}} & \xrightarrow{\bar{t}} & p^*\mathrm{tra} \\ {\scriptstyle h}\downarrow & \swarrow\!\!\!\!\swarrow \epsilon_{\bar{t}} & \downarrow{\scriptstyle p^*f} \\ \mathrm{tra}'_{\mathcal{U}} & \xrightarrow[\bar{t}']{} & p^*\mathrm{tra}' \end{array}$$

$$g \mapsto \begin{array}{ccc} p_1^*\mathrm{tra}_{\mathcal{U}} & \xrightarrow{g} & p_2^*\mathrm{tra}_{\mathcal{U}} \\ {\scriptstyle p_1^*h}\downarrow & \swarrow\!\!\!\!\swarrow \epsilon_g & \downarrow{\scriptstyle p_2^*h} \\ p_1^*\mathrm{tra}'_{\mathcal{U}} & \xrightarrow[g']{} & p_2^*\mathrm{tra}'_{\mathcal{U}} \end{array} \qquad (3)$$

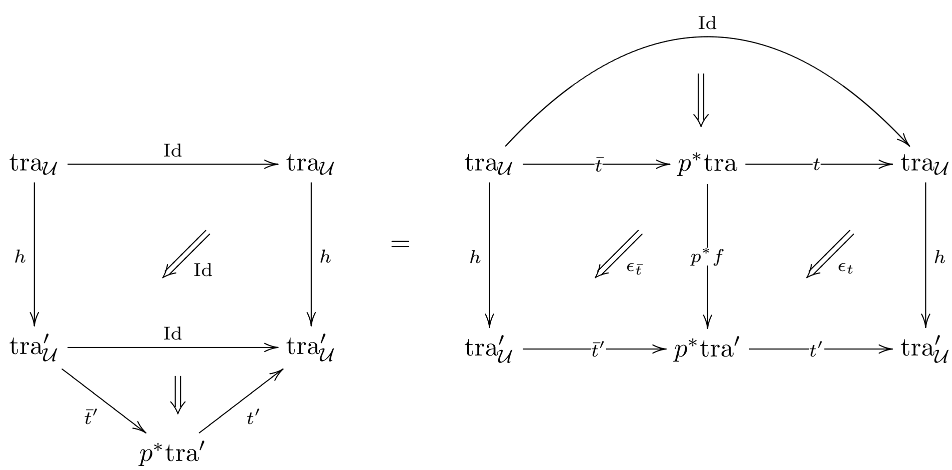
such that all relevant tin can equations hold:

(a) tin can based on the transition modification



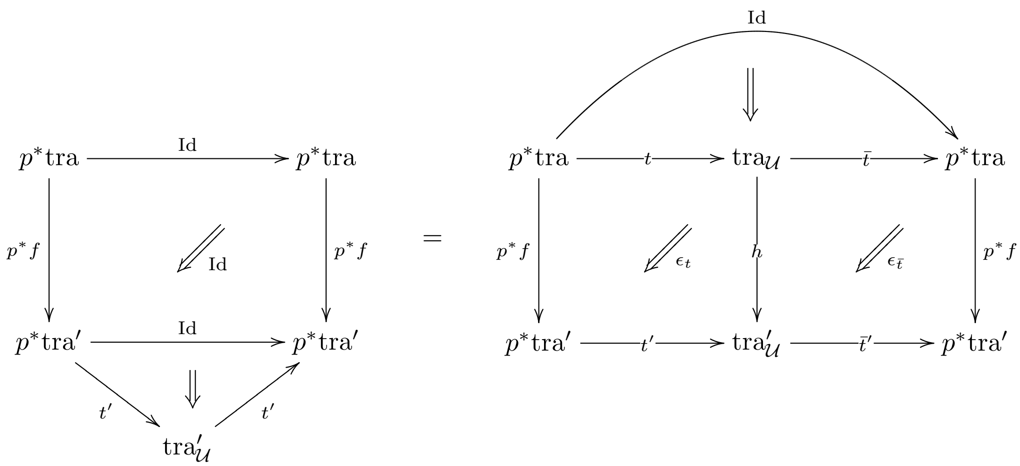
(4)

(b) tin can based on the unit on $t \circ \bar{t}$



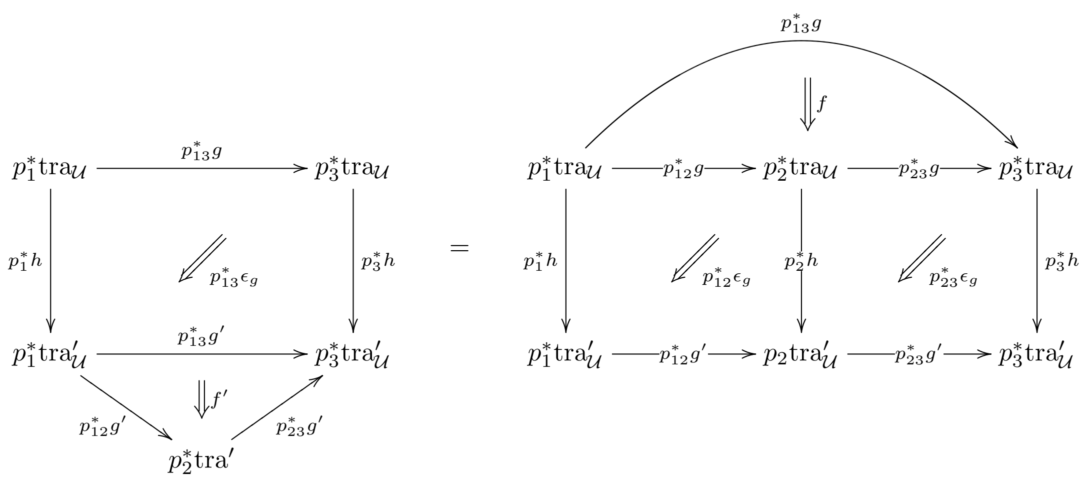
(5)

tin can based on the unit on $\bar{t} \circ t$



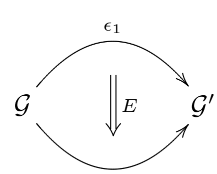
. (6)

Note that this implies in particular the following tin can equation:



(7)

3. *a 2-morphism between 1-morphisms between local pre-trivializations*



is a "modification of the above pseudonatural transformations" in the sense that it is a map

$$E : \{h, f\} \longrightarrow \mathrm{Mor}_2(\mathbf{L2B}(\mathcal{U}))$$

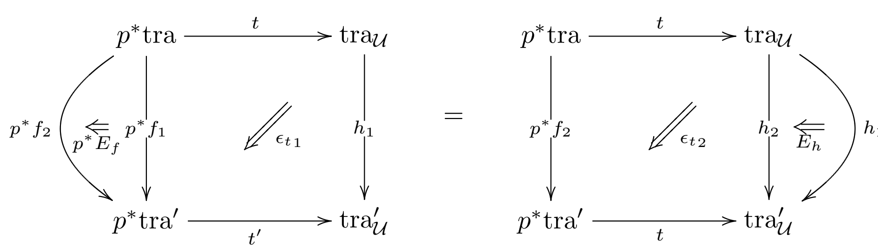
given by

$$h \mapsto \quad \mathrm{tra}_{\mathcal{U}} \underset{h_2}{\overset{h_1}{\Downarrow E_h}} \mathrm{tra}'_{\mathcal{U}}$$

and

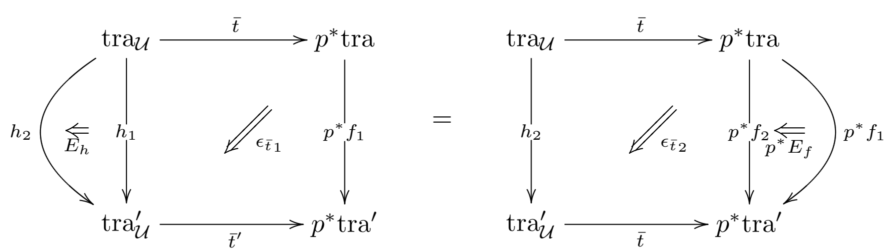
$$f \mapsto \quad \mathrm{tra} \underset{f_2}{\overset{f_1}{\Downarrow E_f}} \mathrm{tra}'$$

such that the modification tin can equations



(8)

and



(9)

hold.

Proposition 10 *Let* tra *and* tra$'$ *be transport 2-functors with local pre-trivializations* $\mathcal{G}$ *and* $\mathcal{G}'$*, respectively. For every morphism*

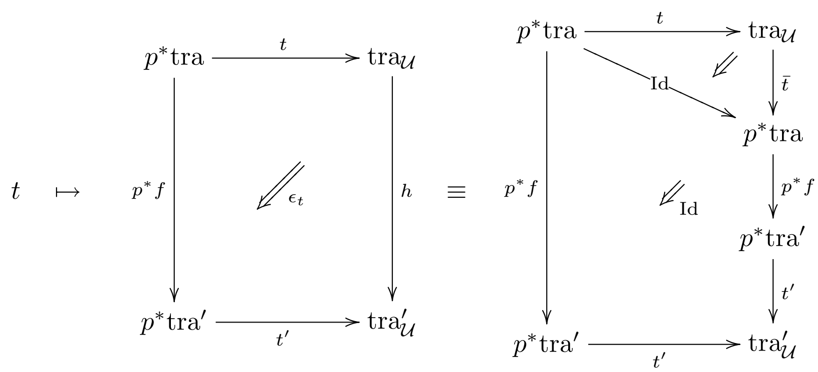
$$\mathrm{tra} \xrightarrow{f} \mathrm{tra}'$$

there is (at least) one morphism

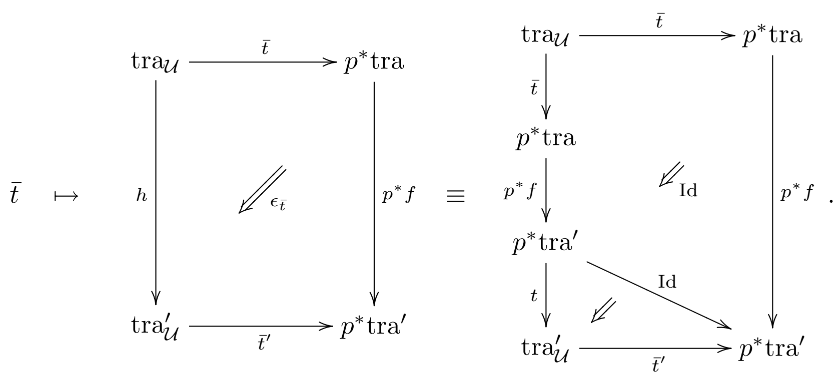
$$\mathcal{G} \xrightarrow{\epsilon(f)} \mathcal{G}'$$

in the 2-category of pre-trivializations.

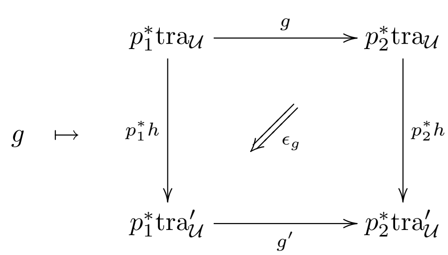
Proof. We explicitly construct the morphism $\mathcal{G} \xrightarrow{\epsilon(f)} \mathcal{G}'$ in the obvious way by setting



and



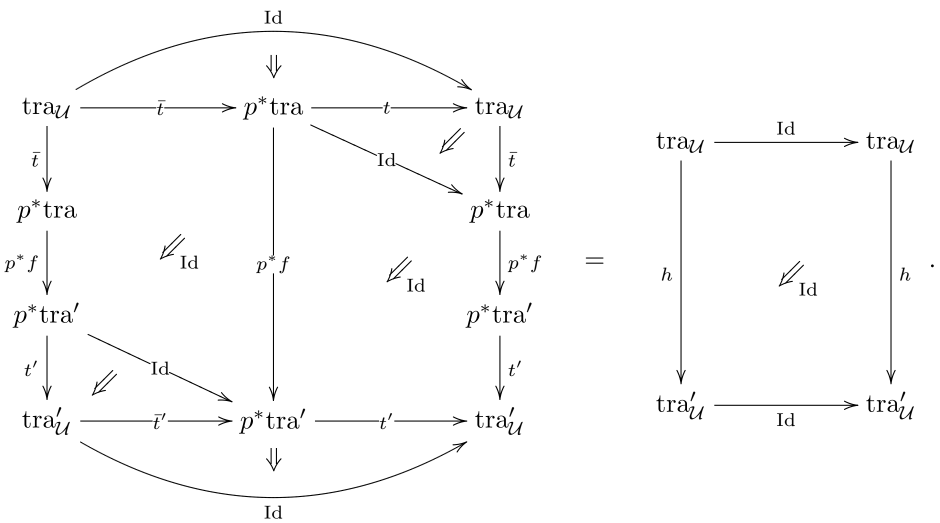
Then we define



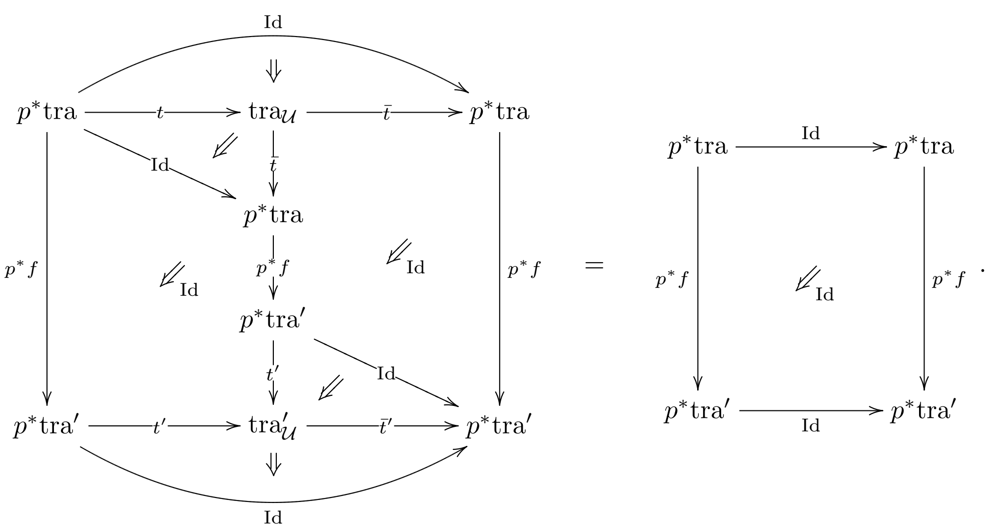
to be the unique solution of the tin can equation (4).

We then need to check that the remaining tin can equations (5) and (6) are satisfied. This turns out to be a consequence of the triangle identities and the speciality condition satisfied by the special ambidextrous adjunction between t

and $\bar{t}$. The zig-zag identity of the adjunction implies that



This is equivalent to (5). The speciality property implies that



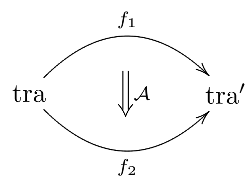
This is equivalent to (6). $\square$

Corollary 1 *Let* tra *be a transport 2-functor with two local pre-trivializations* $\mathcal{G}$ *and* $\mathcal{G}'$*. There is (at least) one morphism*

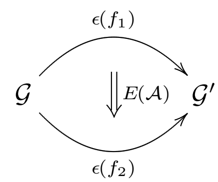
$$\mathcal{G} \xrightarrow{\epsilon(\mathcal{G},\mathcal{G}')} \mathcal{G}' \ .$$

Proof. Set $f = \mathrm{Id}$ in the above proposition. $\square$

Proposition 11 *Let* tra *and* tra' *be transport 2-functors with local pre-trivializations* $\mathcal{G}$ *and* $\mathcal{G}'$*, respectively. For every 2-morphisms of transport 2-functors*

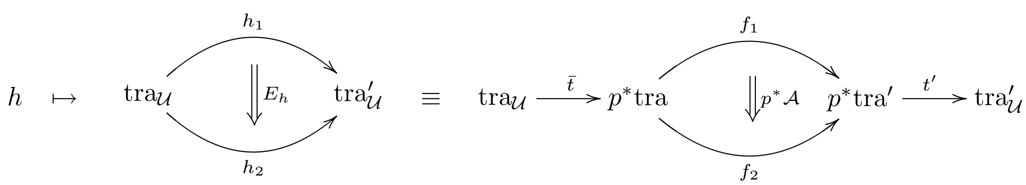


there is (at least) one 2-morphism

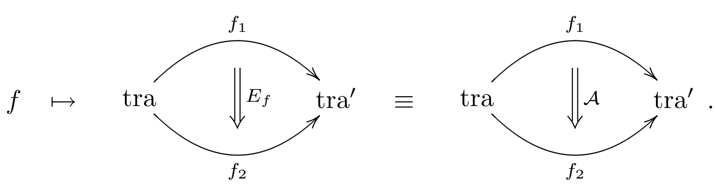


of local pre-trivializations.

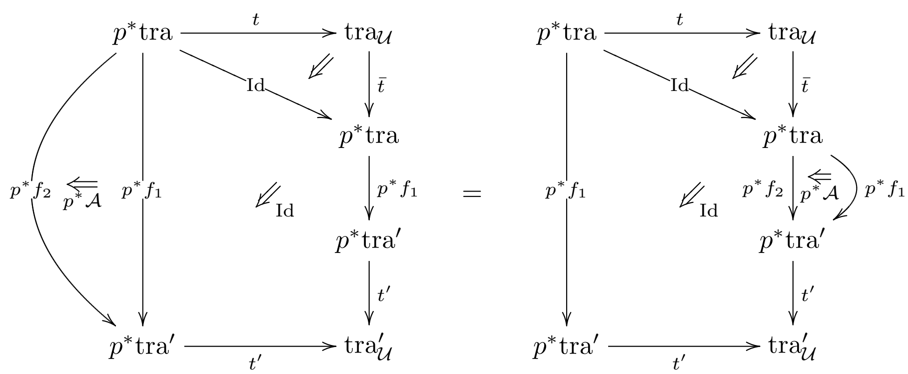
Proof. We construct such a 2-morphism in an obvious way and check its properties. Set



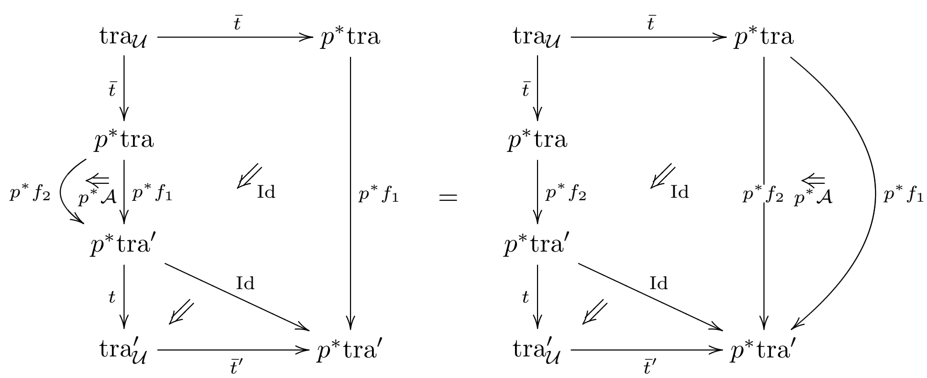
and



This trivially satisfies the equation



equivalent to (8) and the equation



equivalent to (9). □

We have hence shown that the 2-category of transport 2-functors can be injected into the 2-category of their local pre-tricializations. This should actually be an equivalence of 2-categories, but checking this in detail requires more work.

Conjecture 1 *The 2-category of transport 2-functors (on M) is equivalent to that of local pre-trivializations of transport 2-functors on M.*

2.4.2 Stable Isomorphisms

We want to compare this 2-category of pre-trivializations of line 2-bundles with some notion of 2-category (or bicategory) of bundle gerbes. 2-categories of bundle gerbes have been discussed in section 3.4 of [9].

There is a "naive" notion of 1-morphism between bundle gerbes obtained by simply looking at isomorphisms of the bundles involved. This however turns out to be ill-suited for a good notion of 2-catgegory of bundle gerbes. The "right" notion is slightly more sophisticated and called a "stable isomorphism", as defined below. It is this stable isomorphism which is automatically produced by the obvious 1-morphisms of pre-trivializations of line-2-bundles. This is the content of prop. 12 below.

Definition 13 (Murray,Stevenson [10]) [1] *Given two bundle gerbes with connective structure (L,Y) and (L',Y) a* **stable isomorphism**

$$(L,Y) \xrightarrow{(H,\mathcal{E})} (L',Y)$$

is a line bundle with connection $H \longrightarrow Y$ *together with an isomorphism*

$$p_1^*H \otimes L \xrightarrow{\mathcal{E}} L' \otimes p_2^*H$$

of line bundles with connection on $Y^{[2]}$.

Remarks.

1. There is an obvious compatibility condition on the bundle isomorphism coming from the products f and f' in (L,Y) and (L',Y) and the bundle isomorphism $\mathcal{E}$ above. Here we shall demand explicitly that for a stable isomorphism

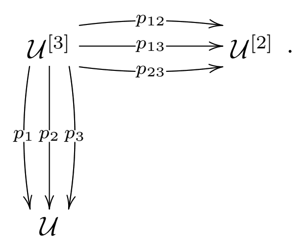
   $$(L,Y) \xrightarrow{(H,\mathcal{E})} (L',Y)$$

   to exist the diagram

$$\begin{array}{ccccc} p_1^*H \otimes p_{12}^*L \otimes p_{23}^*L & \xrightarrow{p_{12}^*\mathcal{E}\otimes \mathrm{Id}_{p_{23}^*L}} & p_{12}^*L' \otimes p_2^*H \otimes p_{23}^*L & & \\ & & & \searrow^{\mathrm{Id}_{p_{12}^*L'}\otimes p_{23}^*\mathcal{E}} & \\ \downarrow{\scriptstyle \mathrm{Id}_{p_1^*H}\otimes f} & & & & p_{12}^*L' \otimes p_{23}^*L' \otimes p_3^*H \\ & & & \swarrow_{f'\otimes \mathrm{Id}_{p_3^*H}} & \\ p_1^*H \otimes p_{13}^*L & \xrightarrow[p_{13}^*\mathcal{E}]{} & p_{13}^*L' \otimes p_3^*H & & \end{array} \tag{10}$$

[1] What I state here is only a "truncated" version of the original definition. See remark 2 below.

commutes, where all bundles appearing have been pulled back to



2. Usually the definition for stable isomorphisms of bundle gerbes is stated slightly differently and more generally for two bundle gerbes that are not necessarily defined over the same fibration Y. This however makes the composition of stable isomorphisms a little involved (see pp. 31 in [9]). For the time being I restrict attention to the simplified version stated above. The generalization of 1-morphisms of pre-trivializations of line-2-bundles to the more general case is obvious and straightforward.

Proposition 12 *Every 1-morphism of pre-trivializations of line-2-bundles gives rise to a stable isomorphism of the associated bundle gerbes.*

Proof. According to def.12 a 1-morphism of pre-trivializations comes with a 2-morphism (3) of trivial line-2-bundles. According to prop. 5 this line-2-bundle 2-morphism defines an isomorphism of line bundles with connection

$$p_1^* h \otimes g' \xrightarrow{\bar{\epsilon}_g} g \otimes p_2^* h$$

The tin can equation (7) is then equivalent to the compatibility condition 10. □

Remark. If we suppress the information given by t and $\bar{t}$ in the definition of a local pre-trivialization the converse of prop. 12 also becomes true and we get a bijection between 1-morphisms of pre-trivializations of line-2-bundles and stable isomorphisms of the respective bundle gerbes.

[...]

2.5 Gerbe Modules

We have been working inside $\mathbf{Vect}_1$ so far, the category of 1-dimensional vector spaces. There is however no need to make that restriction. Quite naturally, our line-2-bundles with 2-connection can be regarded as 2-functors into all of **Vect**. Because the groupoid of paths will have to be mapped to the Picard group of **Vect**, the image of these 2-functors will still lie in $\mathbf{Vect}_1 \subset \mathbf{Vect}$. But the difference is that now *morphisms* between line-2-bundles may involve higher dimensional vector spaces.

In particular, it may happen that a line-2-bundle with 2-connection tra is not trivializable in $\mathbf{Vect}_1$, but is trivializable in $\mathbf{Vect}$. If this is the case, we have a pseudonatural transformations

$$\mathrm{tra} \xrightarrow{t} \mathrm{tra}_0$$

to a trivial 2-transport tra_0 which involves naturality tin cans that do live in $\mathbf{Vect}$ but not in $\mathbf{Vect}_1$. By prop. 10 every such morphism gives rise to a morphism of pre-trivializations of tra and tra_0. One finds that such a morphism describes precisely a **bundle gerbe module** [11].

[...]

2.6 Full Local Trivializations of Line-2-Bundles

The invertible 2-morphisms of the form $\bullet \underset{K}{\overset{K}{\Downarrow k \in K}} \bullet$ form a (strict) sub-2-category of (the strictified version of) $\mathbf{Vect}_1$. For $K = \mathbb{C}$ this sub-2-category is equivalent to the strict 2-group coming from the crossed module $(\mathbb{C}^\times \to 1)$ (regarded as a 2-group with a single object). This again contains the sub-2-group coming from the crossed module $(U(1) \to 1)$.

A full local trivialization of a line 2-bundle with 2-connection is a choice of local equivalences to trivial line-2-bundles such that the transitions come to lie in $(U(1) \to 1) \subset \mathbf{Vect}_1$.

Proposition 13 *Full local trivializations of line-2-bundles with 2-connection are classified by Deligne cohomology in degree 2.*

[...]

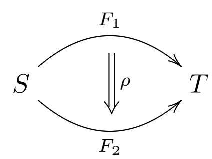
Acknowledgements. The diagrams were created using the LaTeXpackage XY-pic written by Kristoffer Rose and Ross Moore.

A Appendix

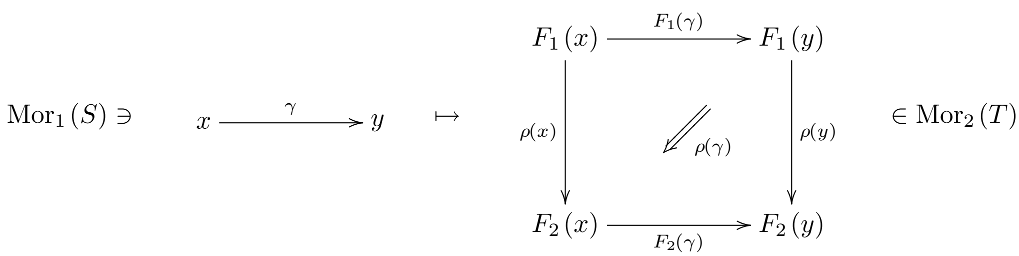
A.1 2-Categories

1-morphisms and 2-morphisms between 2-functors are called pseudonatural transformations and modifications, respectively. These are defined as follows (*cf.* [12]).

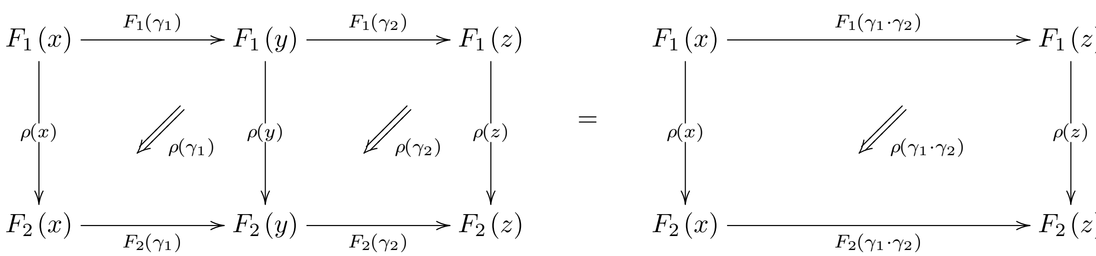
Definition 14 *Let* $S \xrightarrow{F_1} T$ *and* $S \xrightarrow{F_2} T$ *be two 2-functors. A* **pseudonatural transformation**



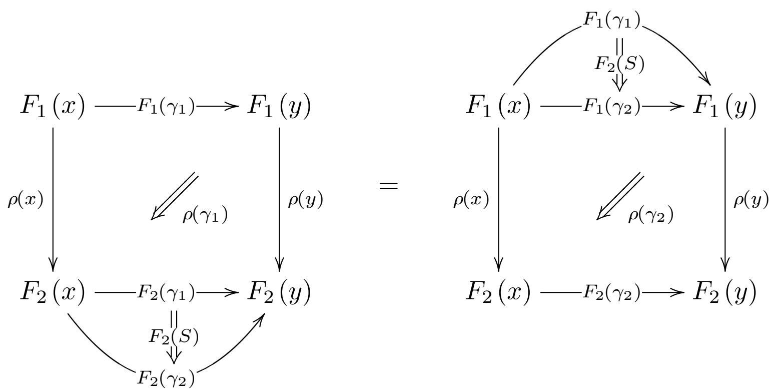
is a map



which is functorial in the sense that

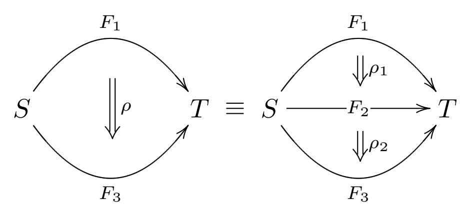


and which makes the pseudonaturality tin can 2-commute

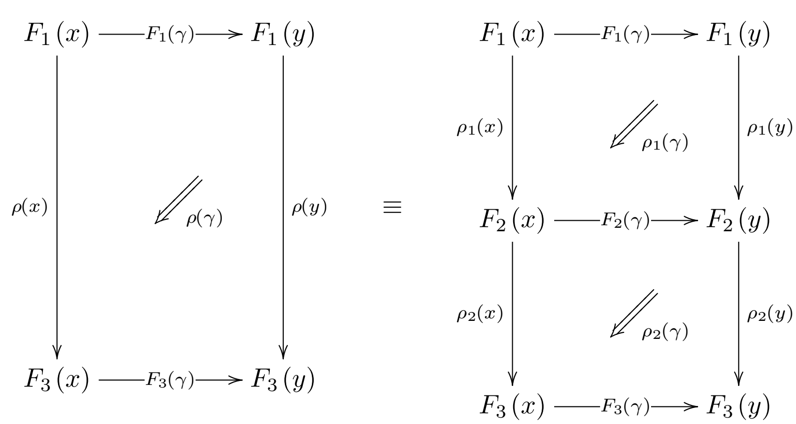


for all $x \underset{\gamma_2}{\overset{\gamma_1}{\Downarrow S}} y \in \mathrm{Mor}_2(S)$.

Definition 15 *The vertical* **composition of pseudonatural transformations**

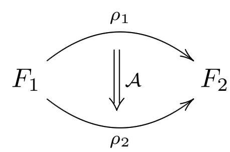


is given by

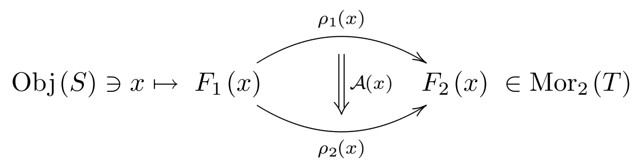


Definition 16 *Let* $F_1 \xrightarrow{\rho_1} F_2 \quad F_1 \xrightarrow{\rho_2} F_2$ *be two pseudonat-*

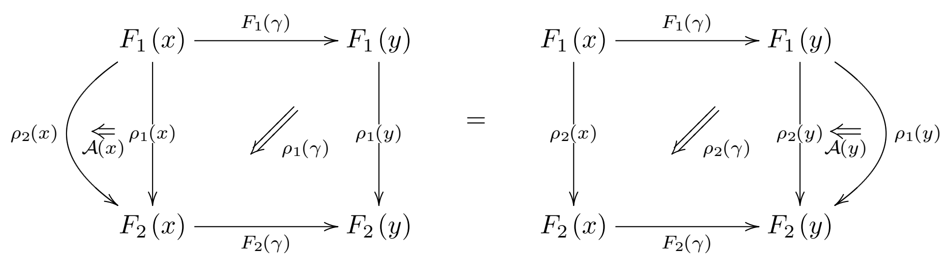
ural transformations. A **modification** *(of pseudonatural transformations)*



is a map



such that

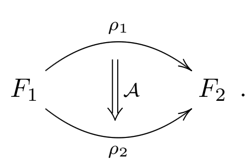


for all $x \xrightarrow{\gamma} y \in \mathrm{Mor}_1(S)$.

Definition 17 *The horizontal and vertical composite of modifications is, respectively, given by the horizontal and vertical composites of the maps to 2-morphisms in* $\mathrm{Mor}_2(T)$.

Definition 18 *Let S and T be two 2-categories. The* **2-functor 2-category** T^S *is the 2-category*

1. *whose objects are functors* $F : S \to T$
2. *whose 1-morphisms are pseudonatural transformations* $F_1 \xrightarrow{\rho} F_2$
3. *whose 2-morphisms are modifications*



A.2 Pullback of Line-2-Bundles

Proposition 14 *Let* $\mathrm{tra} : \mathcal{P}_1(M) \to \mathrm{Trans}(E) \subset \mathbf{Vect}_1$ *be a line bundle with connection on* M*. Let* $f : M' \to M$ *be a smooth map. Then the transport functor on the pullback bundle* f^*E *is* $f^*\mathrm{tra}$ *satisfying*

$$\begin{array}{ccc} \mathcal{P}_1(M') & \xrightarrow{f^*} & \mathcal{P}_1(M) \\ {\scriptstyle f^*\mathrm{tra}}\Big\downarrow & \swarrow\!\!\swarrow\, \eta & \Big\downarrow{\scriptstyle \mathrm{tra}} \\ \mathrm{Trans}(f^*E) & \longrightarrow & \mathrm{Trans}(E) \end{array} \quad .$$

[...]

References

[1] G. Segal, Elliptic Cohomology. *Séminaire Bourbaki* **695** (1988) 187-201.

[2] S. Stolz, P. Teichner, What is an elliptic object?, available at `http://math.ucsd.edu/~teichner/Papers/Oxford.pdf`

[3] J. Barret, Holonomy and path structures in General Relativity, *Int. J. Theor. Phys.* **30** (1991), no. 9, 1171-1215

[4] M. Mackaay, Holonomy and parallel transport for abelian gerbes, to appear in *Adv. Math.* also available as `math.DG/0007053`

[5] T. Bartels, Categorified gauge theory: 2-bundles, preprint available as `math.CT/0410328`.

[6] H. Pfeiffer, F. Girelli , Higher gauge theory — differential versus integral formulation, *J. Math. Phys. 45* **10** 7 (2004), 3949-3971, also available as `hep-th/0309173`

[7] U. Bunke, P. Turner, S. Willerton, Gerbes and homotopy quantum field theories, *Alg. & Geom. Top.* **4** (2004), 407-437

[8] M. Murray, Bundle gerbes, *J. Lond. Math. Soc* **54** (1996), 403-416

[9] D. Stevenson, The Geometry of Bundle Gerbes, PhD thesis, available as `math.DG/0004117`

[10] M. Murray, D. Stevenson, Bundle gerbes: stable isomorphism and local theory, *J. Lond. Math. Soc* **62** (2000), 925-937, also available as `math.DG/9908135`

[11] P. Bouwknegt, A. Carey, V. Mathai, M. Murray, D. Stevenson, Twisted K-theory and K-theory of Bundle Gerbes, *Commun.Math.Phys.* **228** (2002) 17-49, `hep-th/0106194`

[12] H. Pfeiffer, Higher gauge theory and a non-Abelian generalization of 2-form electrodynamics, *Annals. Phys.* **308** (2003), 447-477, also available as `hep-th/0304074`

[13] J. Baez and U. Schreiber, Higher gauge theory, available as `math.DG/0511710`

[14] U. Schreiber, FRS Formalism from 2-Transport, private notes, available at `http://www.math.uni-hamburg.de/home/schreiber/FRSfrom2Transport.pdf`

[15] U. Schreiber, From Loop Space Mechanics to Nonabelian Strings, PhD thesis, available as hep-th/0509163