

Differential Nonabelian Cohomology

Urs Schreiber

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Abstract

Nonabelian cohomology generalizes Čech cohomology with coefficients in sheaves of complexes of abelian groups to cohomology with coefficients in sheaves of ∞ -categories. It classifies in particular higher principal bundles and their higher gerbes of sections. There is a differential version which classifies higher bundles with connection. Classes of examples of these arise from possibly twisted lifts of structure groups through shifted central String-like extensions. Using ∞ -Lie theory relating smooth ∞ -groupoids and L_∞ -algebras we construct twisted String 2- and twisted Fivebrane 6-bundles as well as the Chern-Simons 3- and 7-bundles obstructing their untwisting. We interpret the Green-Schwarz mechanism and its magnetic dual version from this point of view.

This exposition is based on [1].

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1 nonabelian cohomology

	classical	quantum
<i>differential cohomology</i> in degree n	assign phases to classical trajectories	assign amplitudes to worldvolumes
= n -dimensional <i>parallel transport</i> : local and smooth	$(x \xrightarrow{\gamma} y) \mapsto (E_x \xrightarrow{P \exp(\int_{\gamma} \nabla)} E_y)$	$(t_1 \longrightarrow t_2) \mapsto (\mathcal{H}_{t_1} \xrightarrow{U(t_2-t_1)=P \exp(\frac{1}{i\hbar} \int_0^1 H dt)} \mathcal{H}_{t_2})$
	bundle E with connection ∇	spaces of states \mathcal{H} with Hamiltonian H

1.1 Local

locality:

global assignments are fixed by local assignments

$$\text{tra} \left(\begin{array}{ccc} \longrightarrow & & \longrightarrow \\ \downarrow & \nearrow & \downarrow \\ & \Sigma & \\ \downarrow & \nwarrow & \downarrow \\ \longrightarrow & & \longrightarrow \end{array} \right) = \begin{array}{ccc} \text{tra}(\Sigma_1) & \text{tra}(\Sigma_3) & \\ \downarrow & \downarrow & \\ \text{tra}(\Sigma_2) & \text{tra}(\Sigma_4) & \\ \downarrow & \downarrow & \\ \longrightarrow & & \longrightarrow \end{array}$$

formalization:	classical	quantum
∞ -functors between ∞ -categories	$\text{tra}_{\nabla} : \text{TargetSpace} \longrightarrow \text{Phases}$	$Z : \text{Worldvolume} \longrightarrow \text{Amplitudes}$

concrete model used in the following:

$$\omega\text{Categories} := \varinjlim (n\text{Cat} \hookrightarrow n\text{Cat} - \text{Cat})$$

$$1 \quad \text{tra}_{\nabla} : \begin{array}{ccc} & x & \\ \curvearrowright \Sigma_1 & & \curvearrowright \\ \parallel & & \parallel \\ \downarrow V & & \downarrow \\ & \Sigma_2 & \\ \curvearrowleft & & \curvearrowleft \\ & y & \end{array} \longmapsto \begin{array}{ccc} & E_x & \\ \curvearrowright \text{tra}(\Sigma_1) & & \curvearrowright \\ \parallel & & \parallel \\ \downarrow \text{tra}_{\nabla} & & \downarrow \text{tra}_{\nabla} \\ & \text{tra}(\Sigma_2) & \\ \curvearrowleft & & \curvearrowleft \\ & E_y & \end{array}$$

$\omega\text{Categories}$ is monoidal biclosed [Crans:1995] and carries a model structure [BrownGolasinski:1998, Lack:2002, LafontMétayerWorytkiewicz:2008].

1.2 Smooth

$$\begin{array}{ccc} \text{Manifolds}^{\subset} & \longrightarrow & \text{DiffeologicalSpaces}^{\subset} & \longrightarrow & \text{Spaces} \\ \text{smoothness:} & & & & \\ \text{geometry admits probes by} & & & & \\ \text{CartesianSpaces} := \{ \mathbb{R}^n \xrightarrow{\text{smooth}} \mathbb{R}^m \} & & \text{ConcreteSheaves}(\text{CartesianSpaces})^{\subset} & \longrightarrow & \text{Sheaves}(\text{CartesianSpaces}) \end{array}$$

Definition 1.1 (smooth ω -categories) $\omega\text{Categories}(\text{Spaces}) \simeq \text{Sheaves}(\text{CartesianSpaces}, \omega\text{Categories})$

Proposition 1.2 (homotopy theory of smooth ω -categories) *On $\omega\text{Groupoids}(\text{Spaces})$ there is the structure of a category of fibrant objects in the sense of [K.-S. Brown:1973] whose fibrations \longrightarrow are globally and whose weak equivalences $\xrightarrow{\cong}$ and hypercovers $\xrightarrow{\cong} \twoheadrightarrow$ are stalkwise those of [BrownGolasinski:1998, LafontMétayerWorytkiewicz:2008].*

¹These strict ∞ -categories are convenient for our purposes due to their relation to *nonabelian homological algebra* and *nonabelian algebraic topology* [BrownHigginsSivera]. They also seem to be sufficient for the purpose of differential cohomology. But all our constructions should generalize to more general kinds of ∞ -categories.

Definition 1.5 (cohomology and homotopy)

$\begin{array}{l} \text{cohomology} \\ \text{with coefficients in } \mathbf{A} \end{array}$	$H(X, \mathbf{A}) = \lim_{\rightarrow Y} \text{Desc}(Y, \mathbf{A})$
$\begin{array}{l} \text{homotopy} \\ \text{with coefficients in } \mathbf{B} \end{array}$	$\pi(X, \mathbf{B}) = \lim_{\leftarrow Y} \text{Codesc}(Y, \mathbf{B})$

Proposition 1.6 For all $n \in \mathbb{N}$. \mathcal{P}_n is an ω -costack, hence so is Π_ω .

Proof. For $n = 1$ in [8], for $n = 2$ in [10], for $g \geq 3$ conjectural but related by weakening to higher van Kampen theorem [BrownHigginsSivera:2008]. \square

Proposition 1.7 Codescent co-represents descent: $\text{hom}(\text{Codesc}(Y, \Pi), \mathbf{BG}) \simeq \text{Desc}(Y, \text{hom}(\Pi(-), \mathbf{BG}))$.

Definition 1.8 (differential G -cohomology relative Π) Given a copresheaf $\Pi : \text{Spaces} \rightarrow \omega\text{Categories}(\text{Spaces})$ we put

$$H_\Pi(X, \mathbf{BG}) := H(X, \text{hom}(\Pi(-), \mathbf{BG})).$$

Corollary 1.9 We have $H_\Pi(X, \mathbf{BG}) \simeq \lim_{\rightarrow Y} \left\{ \begin{array}{c} \text{Codesc}(Y, \Pi) \\ \swarrow \simeq \quad \searrow (g, \text{triv}_\nabla) \\ \Pi(X) \xrightarrow{\quad | \quad} \mathbf{BG} \\ \text{tra}_\nabla \end{array} \right\}$

1.4 Examples

Proposition 1.10 For \mathbf{A} the image of $[\mathbf{A}]$ under the equivalence [BrownHiggins:1981]

$$\text{Sheaves}(\text{ChainComplexes}(\text{AbelianGroups})) \hookrightarrow \text{Sheaves}(\text{CrossedComplexes}) \xrightarrow{\simeq} \text{Sheaves}(\omega\text{Groupoids})$$

$$[\mathbf{A}] \longmapsto \mathbf{A}$$

nonabelian cohomology with coefficients in \mathbf{A} reproduces ordinary Čech cohomology with coefficients in $[\mathbf{A}]$:

$$[H(X, \mathbf{A})] \simeq H(X, [\mathbf{A}]).$$

Theorem 1.11 Let G_1, G_2 be a Lie 1- and 2-group, respectively.

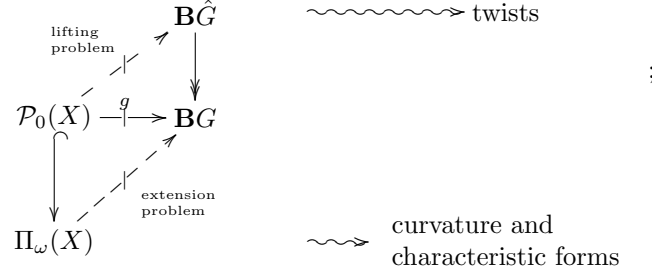
- $H_{\mathcal{P}_0}(X, \mathbf{BG}_1) \simeq \{G\text{-principal bundles on } X\}$
- $H_{\mathcal{P}_0}(X, \mathbf{BG}_2) \simeq \{G\text{-principal 2-bundles on } X\}$ [Bartels:2004, Baković:2008, Wockel:2008]
- $H_{\mathcal{P}_1}(X, \mathbf{BG}_1) \simeq \{G\text{-principal bundles with connection on } X\}$ [8],
- $H_{\mathcal{P}_2}(X, \mathbf{BAUT}(G_1)) \simeq \{G\text{-gerbes with connection with curvature in degree 2}\}$ [10, 3],
- $H_{\mathcal{P}_n}(X, \mathbf{B}^n U(1)) \simeq (n+1)\text{st Deligne cohomology (for } n=1 \text{ [8], for } n=2 \text{ [10])}$

2 the theorem

We construct examples of differential nonabelian cocycles by applying

1. ∞ -Lie integration of L_∞ -algebras to $\omega\text{Groupoids}(\text{Spaces})$;

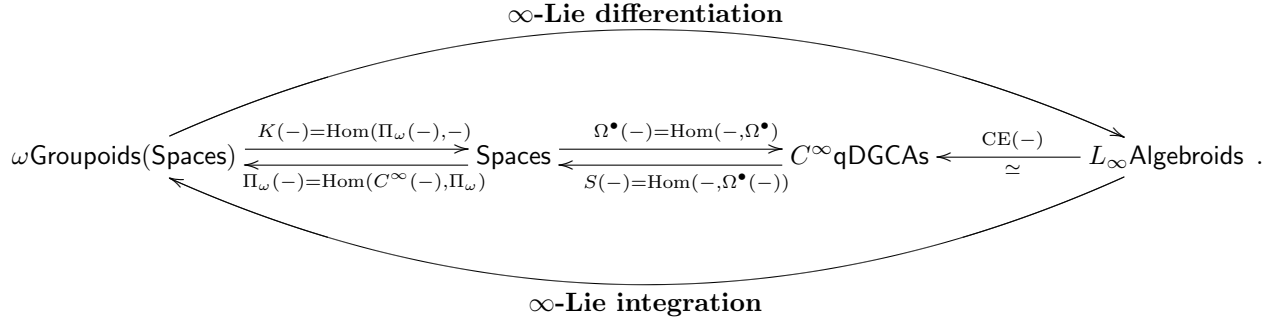
2. *lifts* and *extensions* of differential nonabelian cocycles



3. integration of L_∞ -algebraic cocycles [5] to differential nonabelian cocycles.

2.1 ∞ -Lie theory

Definition 2.1 (∞ -Lie integration and -differentiation) *By slight variation on [Sullivan:1977, Ševera:2001, Getzler:2004, Henriques:2006, Ševera:2006] we set*



Proposition 2.2

- $\Pi_\omega(-)$ is left adjoint to $K(-)$ and $\Omega^\bullet(-)$ is left adjoint to $S(-)$
- For \mathfrak{g} an integrable Lie algebroid, in particular any Lie algebra, $\Pi_1(S(\text{CE}(\mathfrak{g}))) = \mathbf{B}G$ for G the ordinary simply connected Lie groupoid integrating \mathfrak{g} (by comparison with [CrainicFernandes:2003]);
- $\Pi_n(S(\text{CE}(b^{n-1}\mathfrak{u}(1)))) = \mathbf{B}\mathbb{B}^{n-1}\mathbb{R}$;
- for μ_3 the normalized canonical 3-cocycle on $\mathfrak{so}(n)$, we have $\Pi_2(S(\text{CE}(\mathfrak{so}_{\mu_3}))) \simeq \mathbf{B}\text{String}_{\text{BCSS}}(n) \simeq \mathbf{B}\text{String}_{\text{Mick}}(n)$, where the 2-group in the middle is that from [2], and that on the right similarly but coming from Mickelsson's cocycle, and where the equivalences are ω -ana-equivalences;
- for G_2 the Lie 2-group coming from a strict Lie 2-algebra \mathfrak{g} we have $K(G_2) = S(\text{CE}(\mathfrak{g}))$ [9].

2.2 Twisted cohomology

Source of examples of differential nonabelian cocycles: *twisted lifts* and *obstructions*.

For A an abelian group, let $\mathbf{B}^{n-1}A \hookrightarrow \hat{G} \twoheadrightarrow G$ be a *shifted central extension*: we have the weak (homotopy) quotient $\mathbf{B}(\hat{G}/\mathbf{B}^{n-1}A) \xrightarrow{\simeq} \mathbf{B}G$.

Proposition 2.3 For μ_3 and μ_7 the normalized canonical 3- and 7-cocycles on $\mathfrak{so}(n)$, respectively, and setting $\mathbf{B}\text{String}(n) := \Pi_2(S(\text{CE}(\mathfrak{so}(n)_{\mu_3}))$ and $\mathbf{B}\text{Fivebrane}(n) := \Pi_6(S(\text{CE}((\mathfrak{so}(n)_{\mu_3})_{\mu_7}))$ we have shifted central extensions

- $\mathbf{B}U(1) \rightarrow \text{String}(n) \rightarrow \text{Spin}(n)$ [2];
- $\mathbf{B}^5U(1) \rightarrow \text{Fivebrane}(n) \rightarrow \text{String}(n)$.

Proposition 2.4 There is an ω -ana-inverse $\mathbf{B}G \dashrightarrow \mathbf{B}(\hat{G}/\mathbf{B}^{n-1}A)$ to $\mathbf{B}(\hat{G}/\mathbf{B}^{n-1}A) \xrightarrow{\cong} G$ post-composition by which yields morphisms

$$\begin{array}{ccc}
 H_{\Pi}(-, \mathbf{B}G) & \xrightarrow{\text{twistedLift}} & H_{\Pi}(-, \mathbf{B}(\hat{G}/\mathbf{B}^{n-1}A)) & \xrightarrow{\text{twist}} & H_{\Pi}(-, \mathbf{B}\mathbf{B}^n A) \\
 & \searrow & & \nearrow & \\
 & & \text{obstr} & &
 \end{array}$$

such that $\text{obstr}(g)$ is the obstruction to lifting a G -cocycle g to a \hat{G} -cocycle.

Theorem 2.5

shifted central extension	twisted lifts and obstructions			
$\mathbb{Z}_2 \rightarrow \text{Spin}(n) \rightarrow \text{SO}(n)$	$\text{SO}(n)$ -bundle	twisted $\text{SO}(n)$ -bundle	<i>Spin lifting gerbe</i> [MurraySinger:2003]	obstruction class in $H^2(-, \mathbb{Z}_2)$
	g	$\xrightarrow{\text{twistedLift}(g)}$	$\xrightarrow{\text{obstr}(g)}$	$\xrightarrow{[\text{obstr}(g)]}$
		$\xrightarrow{w_2(-)}$		
$\mathbf{B}U(1) \rightarrow \text{String}(n) \rightarrow \text{Spin}(n)$	$\text{Spin}(n)$ -bundle	twisted $\text{String}(n)$ -2-bundle	<i>Chern-Simons lifting 3-bundle</i>	obstruction class in $H^4(-, \mathbb{Z})$
	g	$\xrightarrow{\text{twistedLift}(g)}$	$\xrightarrow{\text{obstr}(g)}$	$\xrightarrow{[\text{obstr}(g)]}$
		$\xrightarrow{\frac{1}{2}p_1(-)}$		
$\mathbf{B}^5U(1) \rightarrow \text{Fivebrane}(n) \rightarrow \text{String}(n)$	$\text{String}(n)$ -2-bundle	twisted $\text{Fivebrane}(n)$ -6-bundle	<i>Chern-Simons lifting 7-bundle</i>	obstruction class in $H^8(-, \mathbb{Z})$
	g	$\xrightarrow{\text{twistedLift}(g)}$	$\xrightarrow{\text{obstr}(g)}$	$\xrightarrow{[\text{obstr}(g)]}$
		$\xrightarrow{\frac{1}{6}p_2(-)}$		

Proof. Refine to differential cohomology (next section) and read off characteristic classes from characteristic forms. Observe that this realizes the construction and theorem of [BrylinskiMcLaughlin:1993,1996] in the top abelian component. \square

The vanishing of these obstructions is known, respectively, as *Spin-structure*, *String-structure*, *Fivebrane-structure* [6], [DouglasHillHenriques:2008].

Green-Schwarz mechanism. In terms of the differential form data obtained from the above [7] and comparing with [Freed:2000], one sees the relation to the Green-Schwarz mechanism:

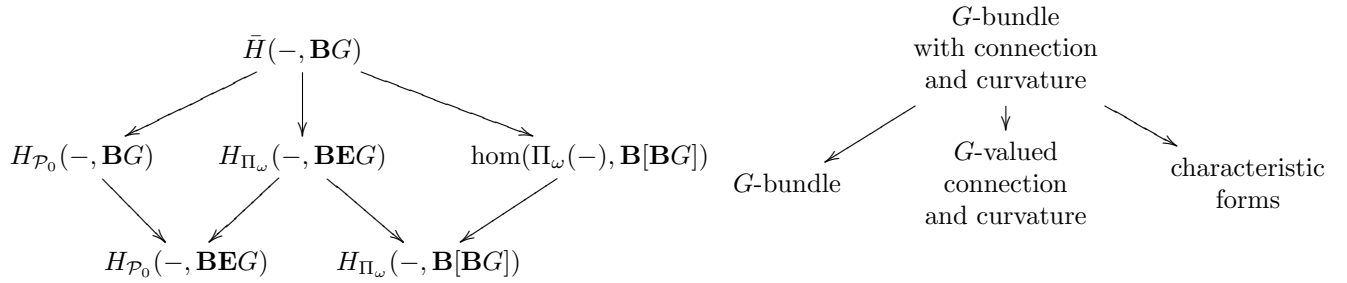
In heterotic string theory (for the case of vanishing E_8 background field, for simplicity) the B -field of the heterotic background theory is part of the connection on a twisted $\text{String}(n)$ -2-bundle whose twist is the Chern-Simons 3-bundle of the above theorem.

This can be interpreted as saying that the Chern-Simons 3-bundle with connection is *magnetic 5-brane charge* which twists the ordinary Bianchi-identity $dH_3 = 0$ of the 3-form curvature H_3 of the electric B -field to $dH_3 \propto \langle F_{\nabla_{\mathfrak{so}(n)}} \wedge F_{\nabla_{\mathfrak{so}(n)}} \rangle$. The Green-Schwarz mechanism is the assertion that adding this magnetic charge introduces an anomaly in the higher Yang-Mills action functional that cancels the anomaly from the chiral fermions of the theory.

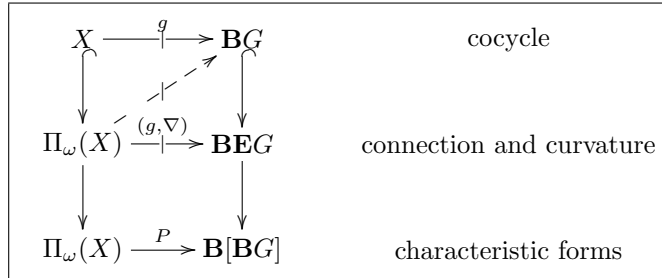
A similar statement applies to the electric-magnetic dual picture, in which strings and 5-branes interchange role [6].

2.3 Differential cohomology

To obtain the characteristic forms for the cocycles appearing in theorem 2.5, define non-flat differential cohomology $\bar{H}(-, \mathbf{BG})$ as a measure for the obstruction to the extension from ordinary, $H_{\mathcal{P}_0}(-, \mathbf{BG})$, to flat differential cohomology, $H_{\Pi_\omega}(-, \mathbf{BG})$, namely as the pullback

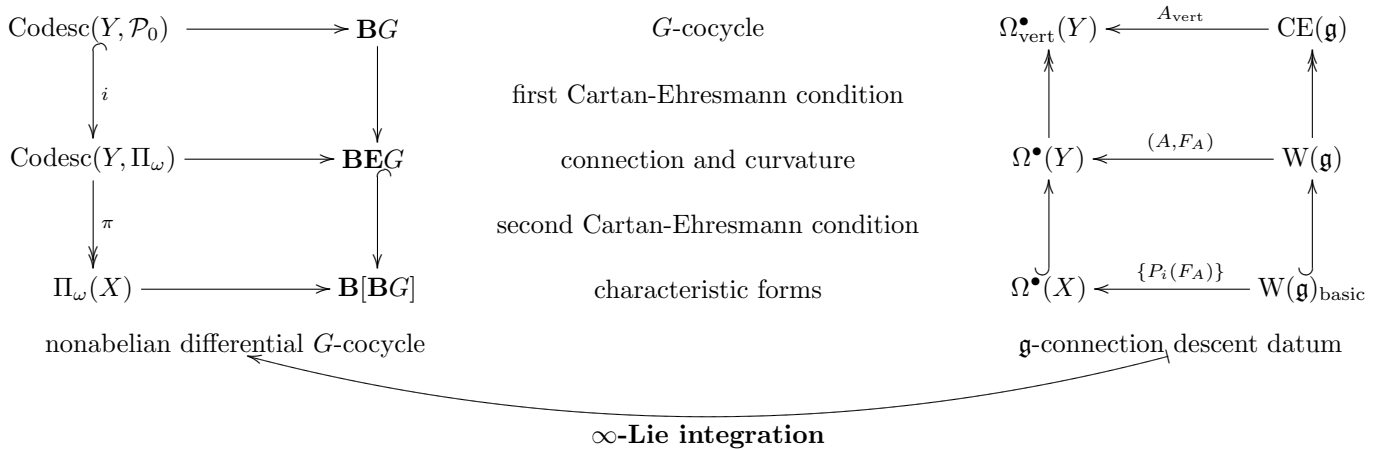


meaning that a cocycle in $\bar{H}(X, \mathbf{BG})$ is a commuting diagram



where the characteristic forms are a measure for the obstruction to the existence of the dashed morphism, which would be a flat differential cocycle.

One obtains such differential cocycles by ∞ -Lie-integrating the corresponding diagrams of L_∞ Algebroids in [5]:



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