Differential Nonabelian Cohomology

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Abstract

Nonabelian cohomology generalizes Čech cohomology with coefficients in sheaves of complexes of abelian groups to cohomology with coefficients in sheaves of ∞-categories. It classifies in particular higher principal bundles and their higher gerbes of sections. There is a differential version which classifies higher bundles with connection. Classes of examples of these arise from possibly twisted lifts of structure groups through shifted central String-like extensions. Using ∞-Lie theory relating smooth ∞-groupoids and L_∞-algebras we construct twisted String 2- and twisted Fivebrane 6-bundles as well as the Chern-Simons 3- and 7-bundles obstructing their untwisting. We interpret the Green-Schwarz mechanism and its magnetic dual version from this point of view.

This exposition is based on [1].

Contents

1 nonabelian cohomology

differential cohomology
in degree n
=
n-dimensional
parallel transport:
local and **smooth**

classical	quantum
assign phases to classical trajectories	assign amplitudes to worldvolumes
$(x \xrightarrow{\gamma} y) \mapsto (E_x \xrightarrow{P\exp(\int_\gamma \nabla)} E_y)$	$(t_1 \longrightarrow t_2) \mapsto (\mathcal{H}_{t_1} \xrightarrow{U(t_2-t_1)=P\exp(\frac{1}{i\hbar}\int_0^1 H\,dt)} \mathcal{H}_{t_2})$
bundle E with connection ∇	spaces of states $\mathcal{H}$ with Hamiltonian H

1.1 Local

locality:
global assignments are fixed by local assignments

$$\mathrm{tra}\left(\begin{tikzcd} {} \arrow[r] \arrow[d] & {} \arrow[d] \arrow[dl, Rightarrow, "\Sigma\ \text{(piece of surface)}"'] \\ {} \arrow[r] & {} \end{tikzcd}\right) = \begin{tikzcd} {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] \arrow[dl, Rightarrow, "\mathrm{tra}(\Sigma_1)"'] & {} \arrow[d] \arrow[dl, Rightarrow, "\mathrm{tra}(\Sigma_3)"'] \\ {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] \arrow[dl, Rightarrow, "\mathrm{tra}(\Sigma_2)"'] & {} \arrow[d] \arrow[dl, Rightarrow, "\mathrm{tra}(\Sigma_4)"'] \\ {} \arrow[r] & {} \arrow[r] & {} \end{tikzcd}$$

formalization:	classical	quantum
∞-functors between ∞-categories	$\mathrm{tra}_\nabla : \mathsf{TargetSpace} \longrightarrow \mathsf{Phases}$	$\mathrm{Z} : \mathsf{Worldvolume} \longrightarrow \mathsf{Amplitudes}$

concrete model used in the following:

$\omega\mathsf{Categories} := \varinjlim\, (n\mathsf{Cat} \hookrightarrow n\mathsf{Cat}{-}\mathsf{Cat})$ [1]

$$\mathrm{tra}_\nabla : \begin{tikzcd}[column sep=small] \gamma_1 \arrow[rr, bend left=60, "x"] \arrow[rr, bend left=25, "\Sigma_1"'] \arrow[rr, dashed, bend right=25, "\Sigma_2"] \arrow[dr, bend right] \arrow[r, phantom, ""] & \overset{\shortparallel}{\Downarrow} V & \gamma_2 \arrow[dl, bend left] \\ & y & \end{tikzcd} \longmapsto \begin{tikzcd}[column sep=small] \mathrm{tra}_\nabla(\gamma_1) \arrow[rr, bend left=60, "E_x"] \arrow[rr, bend left=25, "\mathrm{tra}(\Sigma_1)"'] \arrow[rr, dashed, bend right=25, "\mathrm{tra}(\Sigma_2)"] \arrow[dr, bend right] & \overset{\shortparallel}{\Downarrow} & \mathrm{tra}_\nabla(\gamma_2) \arrow[dl, bend left] \\ & E_y & \end{tikzcd}$$

$\omega\mathsf{Categories}$ is monoidal biclosed [Crans:1995] and carries a model structure [BrownGolasinski:1998, Lack:2002, LafontMétayerWorytkiewicz:2008].

1.2 Smooth

smoothness:
geometry admits probes by
$\mathsf{CartesianSpaces} := \{\ \mathbb{R}^n \xrightarrow{\text{smooth}} \mathbb{R}^m\ \}$

$$\begin{tikzcd} \mathsf{Manifolds} \arrow[r, hook] & \mathsf{DiffeologicalSpaces} \arrow[r, hook] \arrow[d, dash, ":="] & \mathsf{Spaces} \arrow[d, dash, ":="] \\ & \mathsf{ConcreteSheaves(CartesianSpaces)} \arrow[r, hook] & \mathsf{Sheaves(CartesianSpaces)} \end{tikzcd}$$

Definition 1.1 (smooth ω-categories) $\omega\mathsf{Categories}(\mathsf{Spaces}) \simeq \mathsf{Sheaves}(\mathsf{CartesianSpaces}, \omega\mathsf{Categories})$

Proposition 1.2 (homotopy theory of smooth ω-categories) *On* $\omega\mathsf{Groupoids}(\mathsf{Spaces})$ *there is the structure of a* category of fibrant objects *in the sense of [K.-S. Brown:1973] whose fibrations* $\longrightarrow\!\!\!\!\!\twoheadrightarrow$ *are globally and whose weak equivalences* $\xrightarrow{\simeq}$ *and hypercovers* $\overset{\simeq}{\twoheadrightarrow}$ *are stalkwise those of* [BrownGolasinski:1998, LafontMétayerWorytkiewicz:2008].

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[1] These strict ∞-categories are convenient for our purposes due to their relation to *nonabelian homological algebra* and *nonabelian algebraic topology* [BrownHigginsSivera]. They also seem to be sufficient for the purpose of differential cohomology. But all our constructions should generalize to more general kinds of ∞-categories.

Nonabelian cocycles tra_∇ are spans in ωGroupoids(Spaces) [2]

$$\begin{tikzcd} & \mathsf{Cover} \arrow[dl, "\simeq"'] \arrow[dr, "(g,\mathrm{triv}_\nabla)"] & \\ \mathsf{TargetSpace} \arrow[rr, "|" marking, "\mathrm{tra}_\nabla"'] & & \mathsf{Phases} \end{tikzcd}$$

i.e. ω-anafunctors[3]

Examples arise as follows:

1.3 Homotopy and cohomology

We have these fundamental ω-category-valued copresheaves:

$$\Pi : \mathsf{Spaces} \to \omega\mathsf{Categories}(\mathsf{Spaces}) \left\{ \begin{array}{ll} X \mapsto \Pi_\omega(X) & \begin{array}{l} \text{smooth } \underline{\textit{fundamental } \omega\text{-groupoid}} \text{ of } X \\ (k\text{-cells are thin-homotopy classes of maps } D^k \to X \\ \text{well behaved at boundary)} \end{array} \\ \hline X \mapsto \mathcal{P}_n(X) & \begin{array}{l} \text{smooth } \underline{\text{path } n\text{-groupoid}} \\ (\text{truncation of } \Pi_\omega(X) \text{ at } n) \end{array} \end{array} \right.$$

From these we obtain, for $\mathbf{B}G$ a one-object smooth ω-groupoid, the following smooth ω-category valued presheaves:

$$\mathbf{A} : \mathsf{Spaces}^{\mathrm{op}} \to \omega\mathsf{Categories}(\mathsf{Spaces}) \left\{ \begin{array}{ll} X \mapsto \hom(\mathcal{P}_0(X), \mathbf{B}G) & \text{trivial } G\text{-principal } \omega\text{-bundles} \\ \hline X \mapsto \hom(\Pi_\omega(X), \mathbf{B}G) & \begin{array}{l} \text{trivial } G\text{-principal } \omega\text{-bundles} \\ \text{with flat connection} \end{array} \\ \hline X \mapsto \hom(\mathcal{P}_n(X), \mathbf{B}G) & \begin{array}{l} \text{trivial } G\text{-principal } \omega\text{-bundles} \\ \text{with connection} \\ \text{with curvature in degree } n+1 \end{array} \end{array} \right.$$

General G-principal bundles arise from *gluing* trivial ones: translate simplices to globes:

$$G : \Delta \to \omega\mathsf{Categories} \left\{ \begin{array}{ll} [n] \mapsto O(\Delta^n) & \begin{array}{l} n\text{th } \underline{\text{oriental}} \\ \text{free } \omega\text{-category on } n\text{-simplex [Street:1987]} \end{array} \\ \hline [n] \mapsto U(\Delta^n) & \begin{array}{l} n\text{th } \underline{\text{unoriental}} \\ \text{free weak } \omega\text{-groupoid on } n\text{-simplex} \end{array} \\ \hline [n] \mapsto \Pi_\omega(\Delta^n) & \omega\text{-groupoid of free type on } n\text{-simplex [BrownSivera:2007]} \end{array} \right.$$

e.g. the 3-category $O(\Delta^3)$ free on the 3-simplex is

$$O(\Delta^3) = \left\{ \begin{tikzcd} 1 \arrow[r] \arrow[dr, phantom, "\Searrow" near start] & 2 \arrow[d] \\ 0 \arrow[u] \arrow[ur] \arrow[r] \arrow[ur, phantom, "\Downarrow" near end] & 3 \end{tikzcd} \xrightarrow{3} \begin{tikzcd} 1 \arrow[r] \arrow[dr] & 2 \arrow[d] \arrow[dl, phantom, "\Swarrow" near start] \\ 0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\Downarrow" near start] & 3 \end{tikzcd} \right\}$$

Definition 1.3 (descent and codescent) $\underline{\textit{codescent } \omega\textit{-category}}$ $\mathrm{Codesc}(Y^\bullet, \Pi) := \int^{[n]\in\Delta} \Pi_\omega(\Delta^n) \otimes \Pi(Y^n)$

$\underline{\textit{descent } \omega\textit{-category}}$ $\mathrm{Desc}(Y^\bullet, \mathbf{A}) := \int_{[n]\in\Delta} \hom(\Pi_\omega(\Delta^n), \mathbf{A}(Y^n))$

e.g.

$$Y^\bullet = (\cdots Y \times_X Y \times_X Y \underset{\pi_{12}}{\overset{\pi_{23}}{\substack{\to \\ \to_{\pi_{13}} \\ \to}}} Y \times_X Y \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} Y)$$

and $\mathbf{A}$ valued in 2-groupoids,

$$\Rightarrow \mathrm{Desc}(Y, \mathbf{A})_0 = \left\{ a \in \mathbf{A}(Y)_0 \,, \begin{tikzcd} \pi_1^* a \arrow[r, "\pi_{12}^* g"] \arrow[dr, phantom, "\Searrow" near start] & \pi_2^* a \arrow[d, "\pi_{23}^* g"] \\ \pi_0^* a \arrow[u, "\pi_{01}^* g"] \arrow[ur, "\pi_{02}^* g" description] \arrow[r, "\pi_{03}^* g"'] \arrow[ur, phantom, "\Downarrow" near end] & \pi_3^* a \end{tikzcd} = \begin{tikzcd} \pi_1^* a \arrow[r, "\pi_{12}^* g"] \arrow[dr, "\pi_{13}^* g" description] & \pi_2^* a \arrow[d, "\pi_{23}^* g"] \arrow[dl, phantom, "\Swarrow" near start] \\ \pi_0^* a \arrow[u, "\pi_{01}^* g"] \arrow[r, "\pi_{03}^* g"'] \arrow[ur, phantom, "\Downarrow" near start] & \pi_3^* a \end{tikzcd} \right\}$$

Definition 1.4 (ω-stack and ω-costack)

$\mathbf{A}$ *is* $\underline{\omega\textit{-stack}}$ $\Leftrightarrow$ $\forall (Y^\bullet \to X) : \ \mathbf{A}(X) \xrightarrow{\simeq} \mathrm{Desc}(Y, \mathbf{A})$

$\mathbf{B}$ *is* $\underline{\omega\textit{-costack}}$ $\Leftrightarrow$ $\forall (Y^\bullet \to X) : \ \mathrm{Codesc}(Y, \mathbf{B}) \xrightarrow{\simeq} \mathbf{B}(X)$

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[2] [K.-S. Brown:1973, Jardine:2006]

[3] 1-Anafunctors in Sets were introduced in [Makkai:1996] and applied in ConcreteSpaces in [Bartels:2004]. The global version of 2-anafunctors is discussed in [9].

Definition 1.5 (cohomology and homotopy)		
	cohomology *with coefficients in* $\mathbf{A}$	$H(X,\mathbf{A}) = \lim_{\to_Y} \mathsf{Desc}(Y,\mathbf{A})$
	homotopy *with coefficients in* $\mathbf{B}$	$\pi(X,\mathbf{B}) = \lim_{\leftarrow_Y} \mathsf{Codesc}(Y,\mathbf{B})$

Proposition 1.6 *For all* $n \in \mathbb{N}$. $\mathcal{P}_n$ *is an* ω*-costack, hence so is* Π_ω.

Proof. For $n = 1$ in [8], for $n = 2$ in [10], for $g \geq 3$ conjectural but related by weakening to higher van Kampen theorem [BrownHigginsSivera:2008]. ☐

Proposition 1.7 *Codescent co-represents descent:* $\mathrm{hom}(\mathrm{Codesc}(Y,\Pi),\mathbf{B}G) \simeq \mathsf{Desc}(Y,\mathrm{hom}(\Pi(-),\mathbf{B}G))$.

Definition 1.8 (differential G-cohomology relative Π) *Given a copresheaf* $\Pi : \mathsf{Spaces} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ *we put*

$$H_\Pi(X,\mathbf{B}G) := H(X,\mathrm{hom}(\Pi(-),\mathbf{B}G))\,.$$

Corollary 1.9 *We have*
$$H_\Pi(X,\mathbf{B}G) \simeq \lim_{\to_Y} \left\{ \begin{tikzcd} & \mathrm{Codesc}(Y,\Pi) \arrow[dl, "\simeq"'] \arrow[dr, "(g,\mathrm{triv}_\nabla)"] & \\ \Pi(X) \arrow[rr, "|" marking, "\mathrm{tra}_\nabla"'] & & \mathbf{B}G \end{tikzcd} \right\}$$

1.4 Examples

Proposition 1.10 *For* $\mathbf{A}$ *the image of* $[\mathbf{A}]$ *under the equivalence* [BrownHiggins:1981]

$$\begin{tikzcd} \mathrm{Sheaves}(\mathsf{ChainComplexes}(\mathsf{AbelianGroups})) \arrow[r, hook] & \mathrm{Sheaves}(\mathsf{CrossedComplexes}) \arrow[r, "\simeq"] & \mathrm{Sheaves}(\omega\mathsf{Groupoids}) \\ {[\mathbf{A}]} \arrow[rr, mapsto] & & \mathbf{A} \end{tikzcd}$$

nonabelian cohomology with coefficients in $\mathbf{A}$ *reproduces ordinary Čech cohomology with coefficients in* $[\mathbf{A}]$*:*

$$[H(X,\mathbf{A})] \simeq H(X,[\mathbf{A}])\,.$$

Theorem 1.11 *Let* G_1, G_2 *be a Lie 1- and 2-group, respectively.*

- $H_{\mathcal{P}_0}(X,\mathbf{B}G_1) \simeq \{$*G-principal bundles on* $X\}$
- $H_{\mathcal{P}_0}(X,\mathbf{B}G_2) \simeq \{$*G-principal 2-bundles on* $X\}$ [Bartels:2004, Baković:2008, Wockel:2008]
- $H_{\mathcal{P}_1}(X,\mathbf{B}G_1) \simeq \{$*G-principal bundles with connection on* $X\}$ *[8],*
- $H_{\mathcal{P}_2}(X,\mathbf{B}\mathrm{AUT}(G_1)) \simeq \{$*G-gerbes with connection with curvature in degree 2* $\}$ *[10, 3],*
- $H_{\mathcal{P}_n}(X,\mathbf{B}^nU(1)) \simeq (n+1)$*st Deligne cohomology (for* $n = 1$ *[8], for* $n = 2$ *[10])*

2 the theorem

We construct examples of differential nonabelian cocycles by applying

1. ∞-Lie integration of L_∞-algebras to $\omega\mathsf{Groupoids}(\mathsf{Spaces})$;
2. *lifts* and *extensions* of differential nonabelian cocycles $$\begin{tikzcd} & \mathbf{B}\hat{G} \arrow[d, two heads] & {} \arrow[r, squiggly] & \text{twists} \\ \mathcal{P}_0(X) \arrow[ur, dashed, "\text{lifting problem}", "|" marking] \arrow[r, "g", "|" marking] \arrow[d, hook] & \mathbf{B}G & & \\ \Pi_\omega(X) \arrow[ur, dashed, "\text{extension problem}"', "|" marking] & & {} \arrow[r, squiggly] & \text{curvature and characteristic forms} \end{tikzcd}$$;
3. integration of L_∞-algebraic cocycles [5] to differential nonabelian cocycles.

2.1 ∞-Lie theory

Definition 2.1 (∞-Lie integration and -differentiation) *By slight variation on* [Sullivan:1977, Ševera:2001, Getzler:2004, Henriques:2006, Ševera:2006] *we set*

$$\begin{tikzcd} \omega\mathsf{Groupoids}(\mathsf{Spaces}) \arrow[r, shift left, "K(-)=\mathrm{Hom}(\Pi_\omega(-),-)"] \arrow[rrr, bend left, "\infty\text{-Lie differentiation}"] & \mathsf{Spaces} \arrow[l, shift left, "\Pi_\omega(-)=\mathrm{Hom}(C^\infty(-),\Pi_\omega)"] \arrow[r, shift left, "\Omega^\bullet(-)=\mathrm{Hom}(-,\Omega^\bullet)"] & C^\infty\mathsf{qDGCAs} \arrow[l, shift left, "S(-)=\mathrm{Hom}(-,\Omega^\bullet(-))"] & L_\infty\mathsf{Algebroids} \arrow[l, "\mathrm{CE}(-)", "\simeq"'] \arrow[lll, bend left, "\infty\text{-Lie integration}"] \end{tikzcd}$$.

Proposition 2.2

- $\Pi_\omega(-)$ *is left adjoint to* $K(-)$ *and* $\Omega^\bullet(-)$ *is left adjoint to* $S(-)$
- *For* $\mathfrak{g}$ *an integrable Lie algebroid, in particular any Lie algebra,* $\Pi_1(S(\mathrm{CE}(\mathfrak{g}))) = \mathbf{B}G$ *for* G *the ordinary simply connected Lie groupoid integrating* $\mathfrak{g}$ *(by comparison with* [CrainincFernandes:2003]*);*
- $\Pi_n(S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1)))) = \mathbf{B}\mathbf{B}^{n-1}\mathbb{R}$*;*
- *for* μ_3 *the normalized canonical 3-cocycle on* $\mathfrak{so}(n)$*, we have* $\Pi_2(S(\mathrm{CE}(\mathfrak{so}_{\mu_3}))) \simeq \mathbf{B}\mathrm{String}_{\mathrm{BCSS}}(n) \simeq \mathbf{B}\mathrm{String}_{\mathrm{Mick}}(n)$*, where the 2-group in the middle is that from [2], and that on the right similarly but coming from Mickelsson's cocycle, and where the equivalences are ω-ana-equivalences;*
- *for* G_2 *the Lie 2-group coming from a strict Lie 2-algebra* $\mathfrak{g}$ *we have* $K(G_2) = S(\mathrm{CE}(\mathfrak{g}))$ *[9].*

2.2 Twisted cohomology

Source of examples of differential nonabelian cocycles: *twisted lifts* and *obstructions.*

For A an abelian group, let $\mathbf{B}^{n-1}A \hookrightarrow \hat{G} \twoheadrightarrow G$ be a *shifted central extension*: we have the weak (homotopy) quotient $\mathbf{B}(\hat{G}//\mathbf{B}^{n-1}A) \overset{\simeq}{\twoheadrightarrow} \mathbf{B}G$.

Proposition 2.3 *For μ_3 and μ_7 the normalized canonical 3- and 7-cocycles on $\mathfrak{so}(n)$, respectively, and setting $\mathbf{B}\mathrm{String}(n) := \Pi_2(S(\mathrm{CE}(\mathfrak{so}(n)_{\mu_3})))$ and $\mathbf{B}\mathrm{Fivebrane}(n) := \Pi_6(S(\mathrm{CE}((\mathfrak{so}(n)_{\mu_3})_{\mu_7})))$ we have shifted central extensions*

- $\mathbf{B}U(1) \to \mathrm{String}(n) \to \mathrm{Spin}(n)$ *[2];*
- $\mathbf{B}^5U(1) \to \mathrm{Fivebrane}(n) \to \mathrm{String}(n)$*.*

Proposition 2.4 *There is an ω-ana-inverse* $\begin{tikzcd}\mathbf{B}G \arrow[r, "|" marking] & \mathbf{B}(\hat{G}//\mathbf{B}^{n-1}A)\end{tikzcd}$ *to* $\begin{tikzcd}\mathbf{B}(\hat{G}//\mathbf{B}^{n-1}A) \arrow[r, two heads, "\simeq"] & G\end{tikzcd}$ *post-composition by which yields morphisms*

$$\begin{tikzcd}
H_\Pi(-,\mathbf{B}G) \arrow[r, "\mathrm{twistedLift}"] \arrow[rr, bend right, "\mathrm{obstr}"'] & H_\Pi(-,\mathbf{B}(\hat{G}//\mathbf{B}^{n-1}A)) \arrow[r, "\mathrm{twist}"] & H_\Pi(-,\mathbf{B}\mathbf{B}^nA)
\end{tikzcd}$$

such that $\mathrm{obstr}(g)$ *is the obstruction to lifting a G-cocycle g to a $\hat{G}$-cocycle.*

Theorem 2.5

shifted cental extension	**twisted lifts and obstructions**
$\mathbb{Z}_2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n)$	SO(n)-*bundle* *twisted* SO(n)-*bundle* *Spin lifting gerbe* [MurraySinger:2003] *obstruction class in* $H^2(-,\mathbb{Z}_2)$ $\begin{tikzcd} g \arrow[r, mapsto] \arrow[rrr, bend right, "w_2(-)"'] & \mathrm{twistedLift}(g) \arrow[r, mapsto] & \mathrm{obstr}(g) \arrow[r, mapsto] & {[\mathrm{obstr}(g)]} \end{tikzcd}$
$\mathbf{B}U(1) \to \mathrm{String}(n) \to \mathrm{Spin}(n)$	Spin(n)-*bundle* *twisted* String(n)-*2-bundle* *Chern-Simons lifting 3-bundle* *obstruction class in* $H^4(-,\mathbb{Z})$ $\begin{tikzcd} g \arrow[r, mapsto] \arrow[rrr, bend right, "\frac{1}{2}p_1(-)"'] & \mathrm{twistedLift}(g) \arrow[r, mapsto] & \mathrm{obstr}(g) \arrow[r, mapsto] & {[\mathrm{obstr}(g)]} \end{tikzcd}$
$\mathbf{B}^5U(1) \to \mathrm{Fivebrane}(n) \to \mathrm{String}(n)$	String(n)-*2-bundle* *twisted* Fivebrane(n)-*6-bundle* *Chern-Simons lifting 7-bundle* *obstruction class in* $H^8(-,\mathbb{Z})$ $\begin{tikzcd} g \arrow[r, mapsto] \arrow[rrr, bend right, "\frac{1}{6}p_2(-)"'] & \mathrm{twistedLift}(g) \arrow[r, mapsto] & \mathrm{obstr}(g) \arrow[r, mapsto] & {[\mathrm{obstr}(g)]} \end{tikzcd}$

Proof. Refine to differential cochomology (next section) and read off characteristic classes from characteristic forms. Observe that this realizes the construction and theorem of [BrylinskiMcLaughlin:1993,1996] in the top abelian component. ☐

The vanishing of these obstructions is known, respectively, as *Spin-structure*, *String-structure*, *Fivebrane-structure* [6], [DouglasHillHenriques:2008].

Green-Schwarz mechanism. In terms of the differential form data obtained from the above [7] and comparing with [Freed:2000], one sees the relation to the Green-Schwarz mechanism:

In heterotic string theory (for the case of vanishing E_8 background field, for simplicity) the B-field of the heterotic background theory is part of the connection on a twisted $\mathrm{String}(n)$-2-bundle whose twist is the Chern-Simons 3-bundle of the above theorem.

This can be interpreted as saying that the Chern-Simons 3-bundle with connection is *magnetic 5-brane charge* which twists the ordinary Bianchi-identity $dH_3 = 0$ of the 3-form curvature H_3 of the electric B-field to $dH_3 \propto \langle F_{\nabla_{\mathfrak{so}(n)}} \wedge F_{\nabla_{\mathfrak{so}(n)}} \rangle$. The Green-Schwarz mechanism is the assertion that adding this magnetic charge introduces an anomaly in the higher Yang-Mills action functional that cancels the anomaly from the chiral fermions of the theory.

A similar statement applies to the electric-magnetic dual picture, in which strings and 5-branes interchange role [6].

2.3 Differential cohomology

To obtain the characteristic forms for the cocycles appearing in theorem 2.5, define non-flat differential cohomology $\bar{H}(-, \mathbf{B}G)$ as a measure for the obstruction to the extension from ordinary, $H_{\mathcal{P}_0}(-, \mathbf{B}G)$, to flat differential cohomology, $H_{\Pi_\omega}(-, \mathbf{B}G)$, namely as the pullback

$$\begin{tikzcd}
 & & \bar{H}(-, \mathbf{B}G) \arrow[dll] \arrow[d] \arrow[drr] & & \\
H_{\mathcal{P}_0}(-, \mathbf{B}G) \arrow[dr] & & H_{\Pi_\omega}(-, \mathbf{BE}G) \arrow[dl] \arrow[dr] & & \mathrm{hom}(\Pi_\omega(-), \mathbf{B}[\mathbf{B}G]) \arrow[dl] \\
 & H_{\mathcal{P}_0}(-, \mathbf{BE}G) & & H_{\Pi_\omega}(-, \mathbf{B}[\mathbf{B}G]) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \begin{array}{c} G\text{-bundle} \\ \text{with connection} \\ \text{and curvature} \end{array} \arrow[dl] \arrow[d] \arrow[dr] & \\
G\text{-bundle} & \begin{array}{c} G\text{-valued} \\ \text{connection} \\ \text{and curvature} \end{array} & \begin{array}{c} \text{characteristic} \\ \text{forms} \end{array}
\end{tikzcd}
,$$

meaning that a cocycle in $\bar{H}(X, \mathbf{B}G)$ is a commuting diagram

$$\begin{tikzcd}
X \arrow[r, "g", "|" marking] \arrow[d, hook] & \mathbf{B}G \arrow[d, hook] & \text{cocycle} \\
\Pi_\omega(X) \arrow[r, "{(g,\nabla)}", "|" marking] \arrow[ur, dashed, "|" marking] \arrow[d] & \mathbf{BE}G \arrow[d] & \text{connection and curvature} \\
\Pi_\omega(X) \arrow[r, "P"] & \mathbf{B}[\mathbf{B}G] & \text{characteristic forms}
\end{tikzcd}$$

where the characteristic forms are a measure for the obstruction to the existence of the dashed morphism, which would be a flat differential cocycle.

One obtains such differential cocycles by ∞-Lie-integrating the corresponding diagrams of L_∞Algebroids in [5]:

$$\begin{tikzcd}
\mathrm{Codesc}(Y, \mathcal{P}_0) \arrow[r] \arrow[dd, hook, "i"] & \mathbf{B}G \arrow[dd] & G\text{-cocycle} & \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\
 & & \text{first Cartan-Ehresmann condition} & & \\
\mathrm{Codesc}(Y, \Pi_\omega) \arrow[r] \arrow[dd, two heads, "\pi"] & \mathbf{BE}G \arrow[dd, hook] & \text{connection and curvature} & \Omega^\bullet(Y) \arrow[uu, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] \arrow[uu, two heads] \\
 & & \text{second Cartan-Ehresmann condition} & & \\
\Pi_\omega(X) \arrow[r] & \mathbf{B}[\mathbf{B}G] & \text{characteristic forms} & \Omega^\bullet(X) \arrow[uu, hook] & \mathrm{W}(\mathfrak{g})_{\mathrm{basic}} \arrow[l, "{\{P_i(F_A)\}}"'] \arrow[uu, hook] \\
\text{nonabelian differential } G\text{-cocycle} & & & & \mathfrak{g}\text{-connection descent datum} \arrow[llll, bend left, mapsto, "\textbf{∞-Lie integration}"]
\end{tikzcd}$$

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