# Notes from Streetfest

(Sydney, July 2006)
Canberra

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Kapranov

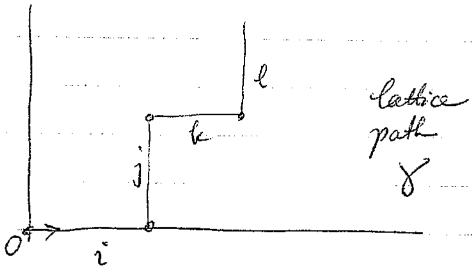
NON-COMM. FOURIER TRANSFORM, CHEN + HOLONOMY

## Motivations

**1.** Non-commutative monomials & paths

$x, y$ do not commute ; monomial $x^i y^j x^k y^l \cdots$

$$\left\{ \sum_{ijkl\ldots} c\, x^i y^j x^k y^l \right\} = \mathbb{C}\langle x, y \rangle$$




Similarly, $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ ; monomials $x^\gamma$ for $\gamma$ a montone lattice path

$$\sum_\gamma c_\gamma x^\gamma \qquad \text{summation over paths}$$

$$\mathbb{C}\langle x_1, \ldots, x_n \rangle \longrightarrow \mathbb{C}[x_1, \ldots, x_n] \qquad \text{homomorphism into commutative r.}$$

$$\sum_\gamma c_\gamma x^\gamma \longmapsto \sum_{\alpha \in \mathbb{Z}^n_+} a_\alpha x^\alpha$$

$$a_\alpha = \sum_{\gamma : 0 \to \alpha} c_\gamma \qquad (\sim \text{Feynman path integration})$$

Want to consider the <u>continuous</u> version of this :

Subdivide the lattice ; allow $x_i^{1/m}$ and look at $m \to \infty$

Assume $\gamma$ continuous and piecewise smooth in $\mathbb{R}^n$, origin $O$.
Can approximate by $\frac{1}{m}$-integer paths, $m \to \infty$.

$$x_i = e^{z_i} \implies x_i^{1/m} = e^{1/m\, z_i} \in \mathbb{C}\langle\langle z_1, \ldots, z_n \rangle\rangle$$

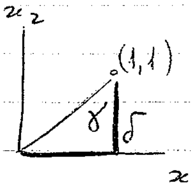
[ Holonomy of connection ] (power series) "$\mathcal{A}$"

Let $\Omega = \sum z_i \, dt_i \in \Omega^1_{\mathbb{R}^n} \otimes \mathcal{A}$. The approximation $\Rightarrow$

$E_\gamma(z)$ = holonomy of $\Omega$ ($\mathcal{A}$ a "connection")

[NC exponential]

eg:




$$E_\gamma(z) = e^{z_1 + z_2} \qquad E_\delta(z) = e^{z_2} \cdot e^{z_1}$$

Idea: NC Fourier transform: representing integrals for paths like this

$$F(z_1, \dots, z_n) \in \mathcal{A} \quad \text{as} \quad \int_\gamma E_\gamma(z) \underbrace{\mathcal{D}\gamma}_{\text{measure}}$$

In terms of <u>Chen's iterated integrals</u>:

Suppose $M$ a $C^\infty$-manifold; $\omega_1, \dots, \omega_p$ 1-forms; $\gamma$ a path in $M$

$\int_\gamma \omega_1 \in \mathbb{C}$ parameterize $\gamma : [0,1] \to M$

$\gamma_{\leq s} : [0,s] \to M$

$\overrightarrow{\int_{(\gamma)}} \omega_1$ a <u>function</u> on $\gamma$

$$s \longmapsto \int_{\gamma_{\leq s}} \omega_1$$

$$\overrightarrow{\int_\gamma} \omega_1 \cdot \omega_2 = \int_\gamma \left( \overrightarrow{\int_{(\gamma)}} \omega_1 \right) \cdot \omega_2 \in \mathbb{C}$$

$\overrightarrow{\int_{(\gamma)}} \omega_1 \cdot \omega_2$ a <u>function</u> on $\gamma$

$\vdots$

$$\overrightarrow{\int_\gamma} \omega_1 \dots \omega_p \in \mathbb{C}$$

Can take $\omega_i \in \Omega^1_M \otimes R$ for any associative algebra $R$

If $\omega \in \Omega^1_M \otimes R$ $(\mathrm{End}(E))$ connection form for an ordinary $\nabla$

$$H(\omega) = 1 + \int_\gamma \omega + \vec{\int_\gamma} \omega.\omega + \vec{\int_\gamma} \omega.\omega.\omega + \cdots$$

* holonomy

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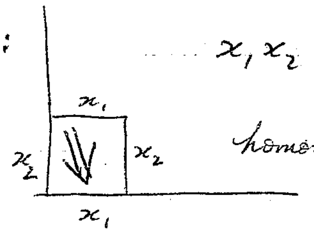
Now for <u>higher dimensional membranes!</u> instead of paths :

For $\mathbb{C}\langle x_1, \ldots, x_n \rangle$, $x_i$ the step in the $i$-th direction

Idea: $x_1 x_2 \Rightarrow x_2 x_1$




Introduce $x_{ij}$, $i<j$, of degree $-1$

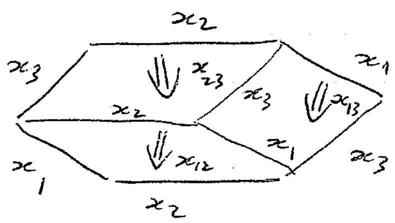
$$\left.\begin{array}{l} \\ d(x_{ij}) = [x_i, x_j] = x_i x_j - x_j x_i \end{array}\right\} \begin{array}{c} B_{\le 2} \\ \text{dg algebra} \end{array}$$

Want to relate to 2-categories :

let $C_{\le 2}$ be the 2-category generated by the 2-skeleton of the cubical lattice in $\mathbb{R}^n$.

Want $C_{\le 2} \longrightarrow B_{\le 2}$

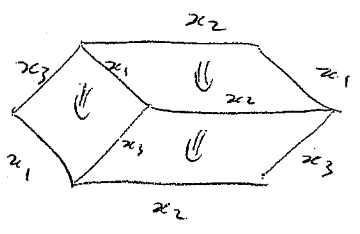
Pasting of cube




giving rule in $B_{\le 2}$

$$x_1 x_{23} + x_{13} x_2 + x_3 x_{12}$$

Alternative choice




$$x_{12} x_3 + x_2 x_{13} + x_{23} x_1$$

$\Longrightarrow$

Extend $B_{\leq 2}$ to $B_{\leq 3}$ by adding $x_{ijk}$ (degree $-2$)

$$d(x_{ijk}) = x_{ij}x_k + x_j x_{ik} + x_{jk}x_i$$

$$- x_i x_{jk} - x_{ik}x_j - x_k x_{ij}$$

$$= [x_{ij}, x_k] + [x_j, x_{ik}] + [x_{jk}, x_i]$$

enveloping algebra of a dg-Lie algebra:

$$B_{\leq 3} = U(L_{\leq 3})$$

<u>Full dg. algebra $B$</u>

generators $x_I$ $\begin{cases} I = \{i_1 < \dots < i_p \mid i_1 \geq 1, \ i_p \leq n\} \\ \text{degree} = -p+1 \end{cases}$

$$d(x_I) = \sum_{\substack{I = J \amalg K \\ \text{disjoint union}}} \underbrace{\varepsilon(J,K)}_{\text{sign of shuffle}} [x_J, x_K]$$

$$d^2 = 0$$

<u>Theorem</u>

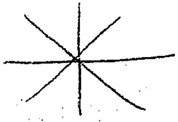
$B$ is a free N.C. resolution of $\mathbb{C}[x_1, \dots, x_n]$. Wrt this $d$,

$$H^j(B) = \begin{cases} \mathbb{C}[\dots] & j = 0 \\ 0 & j \neq 0 \end{cases}$$

<u>Proof</u>

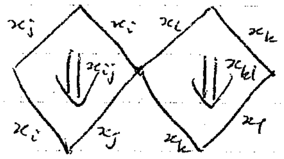
uses $\mathbb{L}$ = Harrison complex of a free graded commutative algebra

$\star$ $C_{\leq n}$ a $n$-category



Want to realize $C_{\leq n}$ inside $B$. <u>Problem</u> for $n=2$:

Consider pasting



What element of $B$ is this?

Pasting $\Rightarrow$ 2 different results.
However, the difference is $d[x_{ij}, x_{kl}]$

So could consider quotient algebra by this ... define

$$D_{\leq 2} = B \Big/ d\left(\begin{matrix}\text{all commutators involving} \\ x_I \,,\; |I| \geq 2\end{matrix}\right)$$

$$= B_{\leq 2} \Big/ d([x_{ij}, x_{kl}])$$

$C_{\leq 2}$ quotiented by translations $\hookrightarrow D_{\leq 2}$

<u>Continuous version</u>

Extend $\mathbb{C}\langle\langle z_1, \dots, z_n \rangle\rangle$ by $z_{ij}, z_{ijk}, \dots$ as above.

Get $\hat{B} = B \underset{\substack{\mathbb{C}\langle z_1, \dots \rangle \\ \rotatebox{90}{$\subseteq$} \\ B_0}}{\otimes} \dots \quad ; \quad (t_1, \dots t_n) \in \mathbb{R}^n$

$$\Omega = \sum z_i \, dt_i + \sum_{i<j} z_{ij} \, dt_i \, dt_j + \sum_{i<j<k} z_{ijk} \, dt_i \, dt_j \, dt_k + \dots$$

total degree 1

<u>Claim</u>: $d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$      <u>FLAT CONNECTION</u>

<u>Relation of this to connections on gerbes</u>:

Crossed module

$$G^{-1} \xrightarrow{d} G^0 \qquad G^0 \text{ acts on } G^{-1}$$

i.e. 2-groupoids with 1 object

If $G^i$ are Lie groups ; $\mathfrak{g}^i$ the Lie algebras

$$\left(\mathfrak{g}^\bullet\right) \quad \mathfrak{g}^{-1} \xrightarrow{d} \mathfrak{g}^0 \qquad = \text{dg - Lie algebra in degrees } -1, 0$$

Let $M$ a manifold.

$$\Omega \in \left(\Omega_M^\bullet \otimes \mathfrak{g}^\bullet\right)^1 \quad ; \quad F = d\Omega + \tfrac{1}{2}[\Omega, \Omega] \in \left(\Omega_M^\bullet \otimes \mathfrak{g}^\bullet\right)^2$$

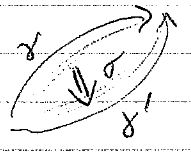
$$= \Omega_1 + \Omega_2 \qquad\qquad = \tilde{\Omega}_2 \in \Omega_M^2 \otimes \mathfrak{g}^0$$

usual connection

$$+ \tilde{\Omega}_3 \in \Omega_M^3 \otimes \mathfrak{g}^{-1}$$

If $F = 0$ then the claim is that $\forall$ membranes $\sigma$

$$\exists \quad H(\sigma) \in U(\mathfrak{g}^\bullet)^{-1} \quad \text{with} \quad dH(\sigma) = H(\gamma) - H(\gamma')$$




Depends on $\sigma$ up to reparameterisation.

Chen's iterated integrals written as functions / forms on space of paths, $\mathcal{P}(M)$.

$$\gamma \longmapsto \int_\gamma^{\rightarrow} \omega_1 \cdots \omega_p \qquad \omega_i \in \Omega^1$$

If $\omega_i \in \Omega_M^{l_i}$ then have

$$\int_\gamma^{\rightarrow} \omega_1 \cdots \omega_p \in \Omega_{\mathcal{P}(M)}^{\Sigma l_i - 1}$$

$$H = \sum \int^{\rightarrow} \Omega \cdots \Omega \in \left(\Omega_{\mathcal{P}(M)}^\bullet \otimes U(\mathfrak{g}^\bullet)\right)^0$$

"Chen's Holonomy"

If $F=0$, then $dH=0$ on each $\mathcal{P}(\overset{\text{start}}{a}, \overset{\text{end}}{b})$.

$H_0 \in \Omega^0_{\mathcal{P}(M)} \otimes U(\mathfrak{g}^\bullet)^0$

$H_1 \in \Omega^1_{\mathcal{P}(M)} \otimes U(\mathfrak{g}^\bullet)^{-1}$

$$H(\gamma) = \int_\gamma H_1 \quad \text{over paths in } \mathcal{P}(M)$$

We had $\Omega \in (\Omega^\bullet_{\mathbb{R}^n} \otimes \hat{\mathcal{B}}^\bullet)^1$ for $\hat{\mathcal{B}}^\bullet = U(\hat{\mathcal{L}}^\bullet)$

$\mathcal{L}^\bullet$ sit in degrees $\{-n+1, \ldots, 0\}$

$L^{\geq -1} / d(L^{-2}) \leftarrow$ (i.e. $d[z_{ij}, z_{kl}]$) Lie crossed module *

Smooth membranes in $\mathbb{R}^n$ (modulo reparameterization)

$U(\hat{\mathcal{L}}^{\geq -1} / d\hat{\mathcal{L}}^{-2})$

Berger

ITERATED WREATH PRODUCT

Will talk about
1. $\Delta$ + 1-fold loop spaces
2. New categories out of $\Delta$ + $n$-fold loop spaces
3. $n$-operads vs $E_n$-operads

Ref: G. Segal ('74) Theorem Special $\Gamma$-spaces as models for $\infty$-loop spaces

Special $\Delta$-spaces model 1-fold loop spaces

Motivation: comparison of Segal and May approaches to loop spaces

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Objects of $\Delta$ $[n] = \{0 < 1 < \cdots < n\}$ (geometric)

Arrows of $\Delta$ functors $[m] \to [n]$

$$\Delta \underset{\text{dense}}{\subset} \underline{Cat} \xrightarrow[\mathcal{N}]{} \underline{Set}^{\Delta^{op}} \equiv \widehat{\Delta}$$

nerve

$\widehat{\Delta}$ is a classifying topos for linear orders $\Rightarrow$ nice geometric realization:

$$\begin{tikzcd}
\widehat{\Delta} \arrow[r, "\text{left exact}"'] & \underline{Top} \\
 & \Delta^n \\
\Delta \arrow[uu, "y"] \arrow[uur, bend right] & \\
{[n]} \arrow[uur, mapsto, bend right] &
\end{tikzcd}$$

determined by linear order in $\underline{Top}$

$\Delta_0 \subset \Delta$ morphisms are the free morphisms in $\Delta$
$= \{\phi \mid \phi(i+1) = \phi(i) + 1 \ \forall i\}$

Let $G_1$ be the category $\cdot \rightrightarrows \cdot$

$$\begin{tikzcd}
{[n]} & \underline{Cat} \arrow[d, bend left, "\text{forgetful}"] \arrow[d, phantom, "\dashv"] \\
\underset{n_*}{\cdot \to \cdot \to \cdot \to \cdot} \arrow[u, mapsto] & \widehat{G_1} \arrow[u, bend left, "\text{free}"]
\end{tikzcd}$$

$\Delta_0$ is a site for topology = epimorphic family of morphisms.

STREETFEST

WORKSHOP

A.N.U (Canberra)

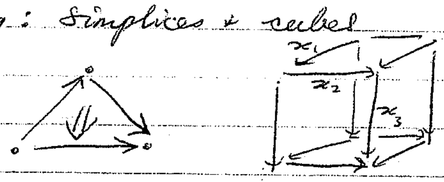
July '05

97 pages

*Kapranov*

# NC FOURIER TRANSFORM II (Prequel)

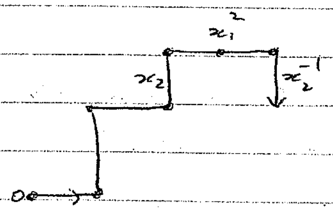
Street - free $n$ categories eg: simplices + cubes




monomials in $x_1, x_2, x_3$

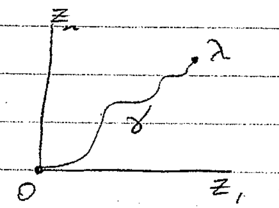
Chen (1950's): papers about paths

Let $F(n)$ free group on $x_1, \dots, x_n$ = lattice paths in $\mathbb{R}^n$ modulo translations = (begin paths at 0)




modulo cancellations

Associate to this path $\gamma$,




$$E(\gamma)(z_1, z_2, \dots, z_n) \xrightarrow{c} \exp\left(\sum \lambda_i z_i\right)$$

See previous talk

$E_{(\gamma)}(z_1, \dots, z_n) \in A$ Hopf algebra, $z_i$ primitive, $A = \mathbb{C}\langle\langle z_1, \dots z_n\rangle\rangle$
grouplike element

Consider $\mathcal{E} \subset A^*$ grouplike elements in $A$; $\mathcal{E}$ the $\varprojlim \mathcal{E}_d$ for $\mathcal{E}_d$ the free degree $d$ unipotent group. (Lie)

Let $\Pi = \{\gamma$ modulo reparameterization and cancellation$\}$

$\Pi \hookrightarrow \mathcal{E}$ embedding (Chen), dense ($\to \mathcal{E}_d$ surjective)

## Idea of NC Fourier Transform

$\{$NC functions of $z_1, \dots, z_n\} \longleftrightarrow$ with measures on $\Pi$

$$\mu \longmapsto F(z) = \int_{\sigma \in \Pi} E_\sigma(z)\, d\mu$$

Example : $e^{-z_1^2 \cdots -z_n^2}$

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Let $M$ a $C^\infty$-manifold ; $P_M$ groupoid of paths in $M$ (piecewise) modulo reparam. and cancellation

want to view as a Lie algebroid :

Definition A Lie algebroid is

A vector bundle $E \xrightarrow{q} TM$ with Lie algebra structure in sections for $q$ a morphism of Lie algebras, and for sections $s$ and $t$

$$[fs, t] - f\cdot[s,t] = -(\mathcal{L}_{q(t)} f)\, s$$

★ $\tilde{P}M \xrightarrow[q]{} TM$ the Lie algebroid of $P_M$,

with $(\tilde{P}_M)_x \equiv T_{id(x)} P_M(x, \cdot)$

$=$ free Lie algebras on $T_x M$ $=$ $T_x M \oplus \Lambda^2 T_x M \oplus \cdots$

Theorem

{free Lie algebras on $TM$} has the structure of a Lie algebroid with $q$ = projection to $TM$.

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Let $\xi$ a vector field on $M$, a section of $\tilde{P}_M$ :

$$[\xi, \eta]_{\tilde{P}} = [\xi, \eta]_{\text{Lie}} + \underbrace{[\xi, \eta]_{\substack{\text{Free Lie} \\ \text{on } TM}}}_{= \xi \wedge \eta}$$

Consider $\underline{\text{Bun}}_\nabla(M)$ = category of vector bundles with connections

$\exists$ fiber functors $\underline{\text{Bun}}_\nabla(M) \xrightarrow[\phi_x]{} \underline{\text{Vect}}$

$\boxed{D : E \to E}$

with

$\gamma$, path $x$ to $y$ $\longmapsto$ $M_\gamma : \phi_x \Rightarrow \phi_y$ preserving $\otimes$ for bundles

$\tilde{P}_M$ = NC derivations ($D$) s.t.

$$D_{E\otimes F} = D_E \otimes 1_F + 1_E \otimes D_F$$

eg: $\xi \in \mathrm{Vect}(M)$
$(E, \nabla) \longmapsto \nabla_\xi$

eg: $\omega \in \Gamma(\Lambda^2 TM)$
$(E, \nabla) \longmapsto (F_\nabla, \omega)$

* Concept of connections and curvature encoded in structure of $\tilde{P}_M$

* Extend idea to differential operators ....

Let $\mathcal{D}_M$ = NC-diff$^{\underline{l}}$ operators ; $P \longmapsto P_{E,\nabla} : E \to E$

eg: a vector field on $M$

$\tilde{}$ $\boxed{\mathcal{D}\text{-modules for "curved" vector bundles}}$

eg: $(M, g_{ij})$ a Riemannian manifold

$(E, \nabla) \longrightarrow d^* \circ d$ Laplace operator $E \to E$

For $M = \mathbb{R}^n$, coords $t_1, \dots, t_n$ ; $\Delta = \nabla_{t_1}^2 + \dots + \nabla_{t_n}^2$

Laplace

$\Theta = e^{-\Delta}$ Heat operator, kernel $K(x,y) : E_x \to E_y$

For $(E, \nabla)$ : $K_E(x,y) = \int\limits_{\substack{\gamma:[0,1]\to M \\ \gamma(0)=x \\ \gamma(1)=y}} H_\gamma \, \mathcal{D}w$

Wiener measure

with $H_\gamma : E_x \to E_y$ holonomy along $\gamma$

Usually, $\gamma \longmapsto H_\gamma$ defined on a measure $0$ set $\Longrightarrow$

cf. $\boxed{\text{Connes NCG version ??}}$

Stochastic integration, extension of this map up to inessential ambiguity

Here, consider $D_{\mathbb{R}^n} = C^\infty(\mathbb{R}^n)$ or, free algebra "action" on $\mathbb{C}<\nabla_{t_1}, \dots, \nabla_{t_n}>$

$$z_i \rightsquigarrow \nabla_{t_i}$$

The NC F. transform of $e^{-\sum z_i^2}$ is given by the measure on $\Pi$, which is the pushdown of the Wiener measure on $\text{Map}[0,1] \to \mathbb{R}^n$, $0 \mapsto 0$

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$$\mathcal{E} \longrightarrow \cdots \longrightarrow \mathcal{E}_2 \xrightarrow{p_{21}} \mathcal{E}_1 = \mathbb{R}^n$$

$\mathcal{E}_2$: nilpotent order 2 = Heisenberg

$\ker(p_{21}) = \Lambda^2 \mathbb{R}^n$

$$\Longrightarrow \quad \cdots \longrightarrow \mu_2 \longrightarrow \mu_1$$

$\mu_2$: Heat measure for nilpotent groups

$\mu_1$: Gaussian measure

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Recall lecture I - think of extending to membranes

Replacement of $\mathcal{E}$? A "membrane 2-group"

Recall a dg-Lie algebra $\mathfrak{g}^{-1} \longrightarrow \mathfrak{g}^0$ for $\mathfrak{g}^0$ free Lie on $z_1, \dots, z_n$

Group version: crossed module $G^{-1} \xrightarrow{d} G^0$ for $G^0$ free grp on $n$ generators

for image of $d$ the commutator

Want a central extension of $[G^0_n, G^0_n]$ by an Abelian group $K$

( element of $H^2([\,,\,], K)$ )

$B[\,,\,] = $ 1-skeleton $Sk_1(\mathbb{R}^n_\square)$ decomposition into cubes

$[\,,\,]^{Ab} = Z_1(\mathbb{R}^n_\square)$ 1-cycles

$$\begin{tikzcd}
{[\,,\,]} \arrow[d] & G^{-1} \arrow[l] \arrow[d] \\
{[\,,\,]^{Ab}} & C_2(\mathbb{R}^n_\square) \arrow[l, "\partial"]
\end{tikzcd}$$ 2-chains

giving the crossed module.

$\pi_1 = \operatorname{coker}(d) = \mathbb{Z}^n$

$\pi_2 = \ker(d) = Z_2(\mathbb{R}^n_\square)$

"Membrane 2-groupoid"