Abstract

Nonabelian cohomology generalizes Čech cohomology with coefficients in sheaves of complexes of abelian groups to cohomology with coefficients in sheaves of $\infty$-categories. It classifies higher principal bundles and their higher gerbes of sections. There is a differential refinement which classifies higher bundles with connection.

Interesting examples arise from lifts, and obstructions to lifts, of structure groups through shifted abelian extensions, notably through the $\infty$-categorical Whitehead tower

$$\text{Fivebrane}(n) \to \text{String}(n) \to \text{Spin}(n) \to \text{SO}(n) \to \text{O}(n)$$

of the group $\text{O}(n)$. As an application we discuss $\text{String}(n)$-principal 2-bundles with connection, their characteristic classes and characteristic forms.

This exposition is based on [1].

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1 nonabelian cohomology

differential cohomology in degree \( n \) = \( n \)-dimensional parallel transport:
local and smooth

<table>
<thead>
<tr>
<th>classical</th>
<th>quantum</th>
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<tr>
<td>assign phases to classical trajectories</td>
<td>assign amplitudes to worldvolumes</td>
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\[
(x \xrightarrow{\gamma} y) \mapsto \left( \begin{array}{l}
E_x \\
\exp(\int \nabla)
\end{array} \right) \quad (t_1 \rightarrow t_2) \mapsto \left( \begin{array}{l}
H_{t_1} \\
\exp\left(\frac{i}{\hbar} \int_0^t H \, dt \right)
\end{array} \right)
\]

bundle \( E \) with connection \( \nabla \)
spaces of states \( \mathcal{H} \) with Hamiltonian \( H \)

1.1 Local

locality:
global assignments are fixed by local assignments

\[
\text{formalization:}
\begin{array}{c}
\infty \text{-functors between } \infty \text{-categories} \\
\text{tr} \varepsilon : \text{TargetSpace} \rightarrow \text{Phases} \\
\text{Z} : \text{Worldvolume} \rightarrow \text{Amplitudes}
\end{array}
\]

concrete model used in the following:

\[
\omega \text{Categories} := \lim_{\longrightarrow} (\text{nCat} \hookrightarrow \text{nCat} - \text{Cat})
\]


1.2 Smooth

smoothness:
groupoids admit probes by

\[
\text{CartesianSpaces} := \{ \mathbb{R}^n \hookrightarrow \mathbb{R}^m \}
\]

\[
\text{ConcreteSheaves(CartesianSpaces)} \rightarrow \text{Sheaves(CartesianSpaces)}
\]

Definition 1.1 (smooth \( \omega \)-categories) \( \omega \text{Categories}(\text{Spaces}) \simeq \text{Sheaves}(\text{CartesianSpaces}, \omega \text{Categories}) \)

Proposition 1.2 (homotopy theory of smooth \( \omega \)-categories) On \( \omega \text{Groupoids}(\text{Spaces}) \) there is the structure of a category of fibrant objects in the sense of [K.-S. Brown:1973] whose fibrations \( \rightarrow \) are globally and whose weak equivalences \( \simeq \) and hypercovers \( \simeq \) are stalkwise those of [BrownGolasinski:1998, LafontMétayerWorytkiewicz:2008].

\( ^1 \)These strict \( \infty \)-categories are convenient for our purposes due to their relation to nonabelian homological algebra and nonabelian algebraic topology [BrownHigginsSivera]. They also seem to be sufficient for the purpose of differential cohomology. But all our constructions should generalize to more general kinds of \( \infty \)-categories.
Nonabelian cocycles \( \tau \gamma \) are spans
in \( \omega \text{-Groupoids}(\text{Spaces}) \)

\[
\begin{array}{ccc}
\text{TargetSpace} & \overset{\gamma}{\longrightarrow} & \text{Phases} \\
\text{Morphisms} & \cong & \text{(g.triv(\gamma))}
\end{array}
\]

\[= \text{morphisms in homotopy category } \text{Ho}(\omega \text{-Groupoids}(\text{Spaces})) \]

Examples arise as follows:

### 1.3 Homotopy and cohomology

We have these fundamental \( \omega \)-category-valued copresheaves:

\[\Pi : \text{Spaces} \to \omega \text{Categories}(\text{Spaces})\]

\[
\begin{cases}
X \mapsto \Pi_\omega(X) & \text{smooth fundamental } \omega \text{-groupoid of } X \\
X \mapsto \mathcal{P}_n(X) & \text{smooth path } n \text{-groupoid}
\end{cases}
\]

\(\text{well behaved at boundary})

Write \((C = BG) \Leftrightarrow (C_0 = \text{pt})\). From the above we get the following smooth \( \omega \)-category valued presheaves:

\[A : \text{Spaces}^{\text{op}} \to \omega \text{Categories}(\text{Spaces})\]

\[
\begin{cases}
X \mapsto \hom(\mathcal{P}_n(X), BG) & \text{trivial } G \text{-principal } \omega \text{-bundles} \\
X \mapsto \hom(\Pi_\omega(X), BG) & \text{trivial } G \text{-principal } \omega \text{-bundles} \\
X \mapsto \mathcal{P}_n(X, BG) & \text{with flat connection}
\end{cases}
\]

with connection

with curvature in degree \( n + 1 \)

General \( G \)-principal bundles arise from \textit{gluing} trivial ones: translate simplices to globes:

\[G : \Delta \to \omega \text{Categories}\]

\[
\begin{cases}
[n] \mapsto O(\Delta^n) & \text{free } \omega \text{-category on } n \text{-simplex} [\text{Street:1987}] \\
[n] \mapsto U(\Delta^n) & \text{nth oriental} \\
[n] \mapsto \Pi_\omega(\Delta^n) & \text{free weak } \omega \text{-groupoid on } n \text{-simplex} \\
[n] \mapsto \mathcal{P}_n(\Delta^n) & \text{nth unoriental}
\end{cases}
\]

\[\omega \text{-groupoid of free type on } n \text{-simplex} [\text{BrownSivera:2007}] \]

\( [n] \mapsto \Pi_\omega(\Delta^n) \)

\[
\begin{cases}
[n] \mapsto O(\Delta^n) & \text{free } \omega \text{-category on } n \text{-simplex} [\text{Street:1987}] \\
[n] \mapsto U(\Delta^n) & \text{free weak } \omega \text{-groupoid on } n \text{-simplex} \\
[n] \mapsto \Pi_\omega(\Delta^n) & \text{free oriental}
\end{cases}
\]

\[\text{e.g. the } 3\text{-category } O(\Delta^3) \text{ free on the } 3\text{-simplex is}
\]

\[
O(\Delta^3) = \left\{ \begin{array}{c}
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\uparrow & & \uparrow \\
0 & \rightarrow & 3 \\
\end{array} \\
\begin{array}{ccc}
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\uparrow & & \uparrow \\
0 & \rightarrow & 3 \\
\end{array}
\end{array}
\end{array} \right\}
\]

\[\text{e.g. } Y^\bullet = (\cdots \times_X Y \times_X Y \times_{\pi_2 Y} \pi_1 Y \times \pi_2 Y \times Y ) \]

\[\Rightarrow \text{Desc}(Y, A)_0 = \left\{ \begin{array}{c}
\begin{array}{ccc}
\pi_1^* a & \rightarrow & \pi_2^* a \\
\pi_0^* g & \rightarrow & \pi_0^* g \\
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\pi_0^* g & \rightarrow & \pi_0^* g \\
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\end{array}
\end{array} \right\}
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\pi_0^* g & \rightarrow & \pi_0^* g \\
\pi_0^* g & \rightarrow & \pi_0^* g \\
\end{array}
\end{array} \right\}
\]

\[\text{and } A \text{ valued in } 2\text{-groupoids,}
\]

\[\text{Definition 1.3 (descent and codescent)}
\]

\[\text{codescent \omega-category} \quad \text{Codesc}(Y^\bullet, \Pi) := \int_{[n] \in \Delta} \Pi_\omega(\Delta^n) \otimes \Pi(Y^n)
\]

\[\text{descent \omega-category} \quad \text{Desc}(Y^\bullet, A) := \int_{[n] \in \Delta} \hom(\Pi_\omega(\Delta^n), \mathcal{A}(Y^n))
\]

\[\text{e.g. } Y^\bullet = (\cdots Y \times_X Y \times_X Y \times_{\pi_2 Y} \pi_1 Y \times \pi_2 Y \times Y ) \]

\[\Rightarrow \text{Desc}(Y, A)_0 = \left\{ \begin{array}{c}
\begin{array}{ccc}
\pi_1^* a & \rightarrow & \pi_2^* a \\
\pi_0^* g & \rightarrow & \pi_0^* g \\
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\pi_1^* a & \rightarrow & \pi_1^* a \\
\pi_0^* g & \rightarrow & \pi_0^* g \\
\pi_0^* g & \rightarrow & \pi_0^* g \\
\pi_0^* g & \rightarrow & \pi_0^* g \\
\pi_0^* g & \rightarrow & \pi_0^* g \\
\pi_0^* g & \rightarrow & \pi_0^* g \\
\end{array}
\end{array} \right\}
\]

\[\text{Definition 1.4 (\omega-stack and \omega-costack)}
\]

\[\text{A is } \omega\text{-stack} \quad \Leftrightarrow \forall(Y^\bullet \to X) : A(X) \xrightarrow{\sim} \text{Desc}(Y, A)
\]

\[\text{B is } \omega\text{-costack} \quad \Leftrightarrow \forall(Y^\bullet \to X) : \text{Codesc}(Y, B) \xrightarrow{\sim} B(X)
\]

\]
Definition 1.5 (cohomology and homotopy)

\[
\begin{align*}
\text{cohomology} & \quad H(X, A) = \lim_{\to Y} \text{Desc}(Y, A) \\
\text{homotopy} & \quad \pi(X, B) = \lim_{\to Y} \text{Codesc}(Y, B)
\end{align*}
\]

Proposition 1.6 For all \( n \in \mathbb{N} \), \( P_n \) is an \( \omega \)-costack, hence so is \( \Pi_\omega \).

Proof. For \( n = 1 \) in [8], for \( n = 2 \) in [10], for \( g \geq 3 \) conjectural but related by weakening to higher van Kampen theorem [BrownHigginsSivera:2008]. \( \square \)

Proposition 1.7 Codescent co-represents descent: \( \hom(\text{Codesc}(Y, \Pi), B G) \simeq \text{Desc}(Y, \hom(\Pi(-), B G)) \).

Definition 1.8 (differential \( G \)-cohomology relative \( \Pi \))

Given a copresheaf \( \Pi : \text{Spaces} \to \omega \text{Categories(Spaces)} \) we put

\[
H_\Pi(X, B G) := H(X, \hom(\Pi(-), B G)).
\]

Corollary 1.9 We have \( H_\Pi(X, B G) \simeq \lim_{\to Y} \left\{ \begin{array}{c}
\text{Codesc}(Y, \Pi) \\
\Pi(X) \end{array} \right\} \]

1.4 Examples

Proposition 1.10 For \( A \) the image of \( [A] \) under the equivalence [Whitehead:1956, BrownHiggins:1981]

\[
\begin{array}{ccc}
\text{Sheaves(ChainComplexes(AbelianGroups))} & \xrightarrow{\simeq} & \text{Sheaves(CrossedComplexes)} \\
[A] & \xrightarrow{\simeq} & \text{Sheaves(\omega Groupoids)} \\
\end{array}
\]

nonabelian cohomology with coefficients in \( A \) reproduces ordinary Čech cohomology with coefficients in \( [A] \):

\[
[H(X, A)] \simeq H(X, [A]).
\]

Theorem 1.11 Let \( G_1, G_2 \) be a Lie 1- and 2-group, respectively.

- \( H_{P_0}(X, B G_1) \simeq \{ G\text{-principal bundles on } X \} \)
- \( H_{P_0}(X, B G_2) \simeq \{ G\text{-principal 2-bundles on } X \} \) [Bartels:2004, Baković:2008, Wockel:2008]
- \( H_{P_1}(X, B G_1) \simeq \{ G\text{-principal bundles with connection on } X \} \) [8],
- \( H_{P_2}(X, B \text{AUT}(G_1)) \simeq \{ G\text{-gerbes with connection with curvature in degree } 3 \} \) [10, 3],
- \( H_{P_n}(X, B^n U(1)) \simeq (n + 1)st \ Deligne \ cohomology \ (\text{for } n \leq 1 \ [8], \ \text{for } n \leq 2 \ [10]) \)
2 the theorem

We will state a fact about characteristic forms on String($n$)-principal 2-bundles.

To get there we first obtain the relevant $n$-groups from $\infty$-Lie integration then consider the notions of characteristic classes; pseudo-connections; characteristic forms.

2.1 $\infty$-Lie theory


\[ K(-) = \text{Hom}(\Pi_\omega(-),-) \]
\[ S(-) = \text{Hom}(-,\Pi_\omega(-)) \]
\[ \Omega^\bullet(-) = \text{Hom}(-,\Omega^\bullet(-)) \]

\[ \ast \to \infty \text{-Lie differentiation} \]
\[ \ast \to \infty \text{-Lie integration} \]

Proposition 2.2

- $\Pi_n(-)$ is left adjoint to $K(-)$ and $\Omega^\bullet(-)$ is left adjoint to $S(-)$.
- For $\mathfrak{g}$ a Lie algebra, $\Pi_1(S(\text{CE}(\mathfrak{g}))) = B G$ with $G$ the ordinary simply connected Lie group integrating $\mathfrak{g}$ (by comparison with [CrainicFernandes:2003]);
- $\Pi_n(S(\text{CE}(b^{n-1}u(1)))) = B B^{n-1}\mathbb{R}$;
- for $G_2$ the Lie 2-group coming from a strict Lie 2-algebra $\mathfrak{g}$ we have $K(G_2) = S(\text{CE}(\mathfrak{g}))$ [9].

Definition 2.3 (string-like extensions) For $\mathfrak{g}$ an $L_\infty$-algebra and $\mu \in \text{CE}(\mathfrak{g})$ an $L_\infty$-algebra $n$-cocycle let $\mathfrak{g}_\mu$ be the result of co-killing $\mu$, i.e. the pushout

\[ \begin{array}{c}
\text{CE}(\mathfrak{g}_\mu) & \leftarrow & \mathfrak{g} \\
\text{w}(b^{n-1}u(1)) & \leftarrow & \text{CE}(\mathfrak{g})\end{array} \]

Definition 2.4 (String($n$) and Fivebrane($n$)) For $\mu_3, \mu_7 \in \text{CE}(\mathfrak{so}(n))$ the normalized cocycles, set

\[ \text{BString}(n) := \Pi_2(S(\text{CE}(\mathfrak{so}(n)_{\mu_3}))) \]
\[ \text{BFivebrane}(n) := \Pi_6(S(\text{CE}((\mathfrak{so}(n)_{\mu_3})_{\mu_7}))) \]

See [5, 6].

Theorem 2.5 In $\text{Ho}(\omega \text{Groupids}(\text{Spaces}))$ we have $\text{BString}(n) \simeq \text{BString}_{\text{CSS}}(n)$, with right hand from [2].
2.2 Principal ω-bundles

Definition 2.6 (universal G-principal ω-bundles) Let \( p : EG \rightarrow BG \) be the pullback \[^4\] \[
\begin{array}{ccc}
EG & \longrightarrow & (BG)^1 \longrightarrow BG \\
\downarrow & & \downarrow_{d_1} \\
pt & \longrightarrow & BG
\end{array}
\]

Proposition 2.7 (generalizing [4]) This yields an exact sequence \[
\begin{array}{ccc}
\ker(p) & \xrightarrow{1} & EG \\
G & \longrightarrow & BG
\end{array}
\]

Definition 2.8 (G-principal ω-bundle) A G-principal bundle over \( X \) is \( P \longrightarrow X \) such that there is \( g \in \text{Ho}(X, BG) \) with \[
\begin{array}{ccc}
Y_0 \times G & \longrightarrow & G \\
\downarrow & & \downarrow \\
P & \xleftarrow{\simeq} & g^*EG \\
\downarrow & & \downarrow \\
X & \xleftarrow{\simeq} & Y \\
\downarrow & & \downarrow \\
g & \longrightarrow & BG
\end{array}
\]

Theorem 2.9 ([4]) For \( G \) an \( n \)-group with \( n \leq 2 \) this reproduces the existing notion of G-principal \( n \)-bundle \([\text{Bartels}:2004, \text{Baković}:2008, \text{Wockel}:2008]\).

2.3 Characteristic classes and forms

Approximate nonabelian cocycles by \{ \begin{enumerate}
\item \text{abelian cocycles} \\
\item \text{abelian differential cocycles}
\end{enumerate} \}
nonabelian \text{differential} cocycles by \{ \begin{enumerate}
\item \text{characteristic classes} \\
\item + \text{characteristic forms}
\end{enumerate} \}

Definition 2.10 (characteristic classes)
\[
\begin{array}{c}
\text{universal characteristic classes of \( \omega \)-group} G: \quad \{ c \in \text{Ho}(BG, B^nU(1)) \} \\
\text{characteristic classes of \( \omega \)-bundle} P : \quad \{ g^*c \in \text{Ho}(X, B^nU(1)) \}
\end{array}
\]

Corollary 2.11 \( \text{Ho}(X, B^nU(1))/_{\simeq} \simeq H^{n+1}(X, \mathbb{Z}) \) hence the class \([ g^*c ] \in H^{n+1}(X, \mathbb{Z}) \).

Proof. From prop. 1.10. \( \square \)

Remark. Similar to \([\text{BaezStevenson}:2008, \text{GinotStiénon}:2008]\) but staying within \( \text{Ho}(\text{Groupids}(\text{Spaces})) \), i.e. without passing to topological realizations.

Theorem 2.12 Let \( \mu_3, \mu_7 \in \text{CE}(\mathfrak{so}(n)) \) be the normalized Lie algebra 3- and 7-cocycles.

1. there are universal characteristic classes \( c_{3,7} := \int \mu_{3,7}/\mathbb{Z} \in \text{Ho}(B\text{Spin}(n), B^{3,7}U(1)) \);
2. \( c_3 \) vanishes when pulled back along \( p : B\text{String}(n) \rightarrow B\text{Spin}(n) \) to \( p^*c_3 \in \text{Ho}(B\text{String}(\text{Spin}), B^3U(1)) \);
3. \( c_7 \) vanishes when pulled back further along \( q : B\text{Fivebrane}(n) \rightarrow B\text{String}(n) \) to \( (q \circ p)^*c \in \text{Ho}(B\text{Fivebrane}(n), B^7U(1)) \);
4. for \( P \simeq g^*E\text{Spin}(n) \) a Spin\((n)\)-principal bundle on \( X \), we have \( g^*c_3 = \frac{1}{2}p_1[P] \), the first fractional Pontryagin class of \( P \);
5. for \( \hat{P} \simeq \hat{g}^*E\text{String}(n) \) a String\((n)\)-principal 2-bundle on \( X \) lifting \( P \), we have \( \hat{g}^*c_7 = \frac{1}{6}p_2[P] \), the second fractional Pontryagin class of \( P \).

Proof. After choosing a suitable surjectively equivalent resolution of \( BG \) whose \( k \)-morphisms are generated from smooth \( k \)-simplices in \( \text{Spin}(n) \), this becomes a corollary of \([\text{BrylinskiMcLaughlin}:1993,1996]\). \( \square \)

Remark. After passing to topological realizations this reproduces statements in \([\text{BaezStevenson}:2008],[\text{DouglasHillHenriques}:2008]\), and [6].
2.4 Characteristic forms

From now on:

for \( g \) a Lie \( n \)-algebra

with Chevalley-Eilenberg algebra \( CE(g) \)

and Weil algebra \( W(g) \)

Notice the shift in the truncation degree on the left: \( n + 1 \) instead of \( n \).

Proposition 2.13 We obtain surjectively equivalent models

\[
\Pi_2(S(CE(g))) \xrightarrow{\simeq} BSpin(n)
\]

\[
\Pi_3(S(CE(so_3))) \xrightarrow{\simeq} BString(n)
\]

\[
\Pi_7(S(CE((so_{13})_{13}))) \xrightarrow{\simeq} BFivebrane(n)
\]

Proof. The \((n + 1)\)st homotopy groups vanish in each case. \( \square \)

Definition 2.14 (pseudo-connections) Pseudo-differential \( G \)-cohomology (classifying pseudo-connections on \( G \)-principal bundles), \( \hat{H}_{pseudo}(-, BG) \), is cohomology with coefficients in the \( \omega \)-category valued presheaf

\[X \mapsto \text{hom}(\left( \begin{array}{cc} \mathcal{P}_0(X) & BG \\ \Pi_c(X) & BEG \end{array} \right) \), so that a cocycle is a commuting diagram\]

\[X \xrightarrow{\gamma} BG\]

The analogous notion for the inclusion \( BB^0U(1) \xrightarrow{\longrightarrow} B(B^{n-1}R \to B^{n-1}U(1)) \) yields \( \hat{H}_{pseudo}(-, BU(1)) \).

Remark. This \( \hat{H}_{pseudo}(-, BU(1)) \) reproduces the notion of pseudoconnections from [BehrendXu:2006].

Proposition 2.15 Every element in \( \hat{H}_{pseudo}(X, BU(1)) \) is cohomologous to one that extends to a diagram

\[
\begin{array}{ccc}
\Pi_c(X) & \xrightarrow{\gamma} & B^nU(1) \\
\downarrow & & \downarrow \\
\Pi_\omega(X) & \xrightarrow{Fa} & B^{n+1}R
\end{array}
\]

cocycle

\[
\begin{array}{ccc}
\Pi_c(X) & \xrightarrow{} & B^nU(1) \\
\downarrow & & \downarrow \\
\Pi_\omega(X) & \xrightarrow{} & B^{n-1}R
\end{array}
\]

connection

\[
\begin{array}{ccc}
\Pi_c(X) & \xrightarrow{} & B^nU(1) \\
\downarrow & & \downarrow \\
\Pi_\omega(X) & \xrightarrow{} & B^{n+1}R
\end{array}
\]

curvature

Remark. This is the \( \infty \)-Lie integration of the \( g \)-connection diagrams in [5].

Proposition 2.16 For \( c \in H_0(BG, B^nU(1)) \) a universal class of the form \( \int \mu/Z \) as in theorem 2.12, and for \( g \in \hat{H}_{pseudo}(X, BG) \) a pseudo-differential cocycle, there is canonically a pseudo-differential \( B^nU(1) \)-cocycle \( g^c \in H_{pseudo}(X, B^nU(1)) \) given by a diagram

\[
\begin{array}{ccc}
\Pi_c(X) & \xrightarrow{\gamma} & B^nU(1) \\
\downarrow & & \downarrow \\
\Pi_\omega(X) & \xrightarrow{} & BEG \\
\downarrow & & \downarrow \\
\Pi_\omega(X) & \xrightarrow{} & B^{n+1}R
\end{array}
\]

characteristic class

\[
\begin{array}{ccc}
\Pi_c(X) & \xrightarrow{\gamma} & B^nU(1) \\
\downarrow & & \downarrow \\
\Pi_\omega(X) & \xrightarrow{} & B^{n-1}R \\
\downarrow & & \downarrow \\
\Pi_\omega(X) & \xrightarrow{} & B^{n+1}R
\end{array}
\]

secondary characteristic form

\[
\begin{array}{ccc}
\Pi_c(X) & \xrightarrow{\gamma} & B^nU(1) \\
\downarrow & & \downarrow \\
\Pi_\omega(X) & \xrightarrow{} & B^{n+1}R
\end{array}
\]

characteristic form

Proof. By \( \infty \)-Lie integrating the diagrammatics in [5] and using the equivalence \( \text{Hom}(\Pi_\omega(X), B^{n+1}R) \simeq \Omega_{\text{closed}}^{n+1}(X) \) from [9]. \( \square \)
Corollary 2.17 Let $P \xrightarrow{\sim} g^*E\text{Spin}(n)$ be a Spin($n$)-principal bundle with connection $\nabla$ with String($n$)-principal lift $\hat{P} \xrightarrow{\sim} \hat{g}^*E\text{String}(n)$ as in theorem 2.12. Then

1. the characteristic form refining the class $g^*c_3$ is $\frac{1}{2}P_4(F_\nabla)$;
2. the characteristic form refining the class $\hat{g}^*c_7$ is $\frac{1}{6}P_8(F_\nabla)$,

where $P_4, P_8 \in W(\mathfrak{so}_n)_{\text{basic}}$ are the invariant polynomials related by transgression to $\mu_3$ and $\mu_7$.

References


