Differential Nonabelian Cohomology

with an application to

Characteristic classes and -forms of String(n)-principal 2-bundles

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Abstract

Nonabelian cohomology generalizes Čech cohomology with coefficients in sheaves of complexes of abelian groups to cohomology with coefficients in sheaves of ∞-categories. It classifies higher principal bundles and their higher gerbes of sections. There is a differential refinement which classifies higher bundles with connection.

Interesting examples arise from lifts, and obstructions to lifts, of structure groups through shifted abelian extensions, notably through the ∞-categorical Whitehead tower

$$\mathrm{Fivebrane}(n) \to \mathrm{String}(n) \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to \mathrm{O}(n)$$

of the group $\mathrm{O}(n)$. As an application we discuss String(n)-principal 2-bundles with connection, their characteristic classes and characteristic forms.

This exposition is based on [1].

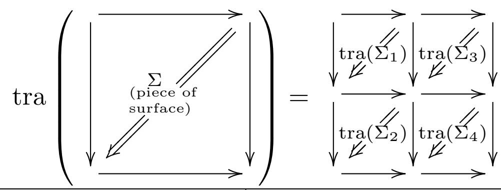
Contents

1 nonabelian cohomology

	classical	quantum
differential cohomology in degree n = n-dimensional *parallel transport*: **local** and **smooth**	assign phases to classical trajectories	assign amplitudes to worldvolumes
	$(\, x \xrightarrow{\gamma} y \,) \mapsto (\, E_x \xrightarrow{P\exp(\int_\gamma \nabla)} E_y \,)$	$(\, t_1 \longrightarrow t_2 \,) \mapsto (\, \mathcal{H}_{t_1} \xrightarrow{U(t_2-t_1)=P\exp(\frac{1}{i\hbar}\int_0^1 H\, dt)} \mathcal{H}_{t_2} \,)$
	bundle E with connection ∇	spaces of states $\mathcal{H}$ with Hamiltonian H

1.1 Local

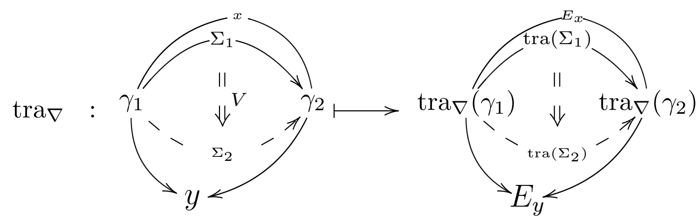
locality:
global assignments are fixed by local assignments



formalization:	classical	quantum
∞-functors between ∞-categories	$\mathrm{tra}_\nabla : \mathsf{TargetSpace} \longrightarrow \mathsf{Phases}$	$Z : \mathsf{Worldvolume} \longrightarrow \mathsf{Amplitudes}$

concrete model used in the following:

$\omega\mathsf{Categories} := \varinjlim (n\mathsf{Cat} \hookrightarrow n\mathsf{Cat}{-}\mathsf{Cat})$ [1]



$\omega\mathsf{Categories}$ is monoidal biclosed [Crans:1995] and carries a model structure [BrownGolasinski:1998, Lack:2002, LafontMétayerWorytkiewicz:2008].

1.2 Smooth

smoothness:
geometry admits probes by

$\mathsf{CartesianSpaces} := \{\, \mathbb{R}^n \xrightarrow{\text{smooth}} \mathbb{R}^m \,\}$

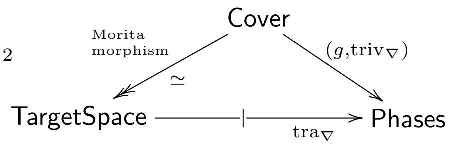
$$\begin{array}{ccccc} \mathsf{Manifolds} & \hookrightarrow & \mathsf{DiffeologicalSpaces} & \hookrightarrow & \mathsf{Spaces} \\ & & \Vert{:=} & & \Vert{:=} \\ & & \mathsf{ConcreteSheaves}(\mathsf{CartesianSpaces}) & \hookrightarrow & \mathsf{Sheaves}(\mathsf{CartesianSpaces}) \end{array}$$

Definition 1.1 (smooth ω-categories) $\omega\mathsf{Categories}(\mathsf{Spaces}) \simeq \mathsf{Sheaves}(\mathsf{CartesianSpaces}, \omega\mathsf{Categories})$

Proposition 1.2 (homotopy theory of smooth ω-categories) *On* $\omega\mathsf{Groupoids}(\mathsf{Spaces})$ *there is the structure of a* category of fibrant objects *in the sense of [K.-S. Brown:1973] whose* <u>*fibrations*</u> $\longrightarrow\!\!\!\!\!\rightarrow$ *are globally and whose* <u>*weak equivalences*</u> $\xrightarrow{\simeq}$ *and* <u>*hypercovers*</u> $\xrightarrow{\simeq}\!\!\!\!\!\rightarrow$ *are stalkwise those of* [BrownGolasinski:1998, LafontMétayerWorytkiewicz:2008].

[1] These strict ∞-categories are convenient for our purposes due to their relation to *nonabelian homological algebra* and *nonabelian algebraic topology* [BrownHigginsSivera]. They also seem to be sufficient for the purpose of differential cohomology. But all our constructions should generalize to more general kinds of ∞-categories.

Nonabelian cocycles tra_∇ are spans in ωGroupoids(Spaces) [2]



, = morphisms in homotopy category $\mathbf{Ho}$(ωGroupids(Spaces))

Examples arise as follows:

1.3 Homotopy and cohomology

We have these fundamental ω-category-valued copresheaves:

Π : Spaces $\to$ ωCategories(Spaces)

$X \mapsto \Pi_\omega(X)$	smooth <u>*fundamental* ω-groupoid</u> of X (k-cells are thin-homotopy classes of maps $D^k \to X$ well behaved at boundary)
$X \mapsto \mathcal{P}_n(X)$	smooth <u>path n-groupoid</u> (truncation of $\Pi_\omega(X)$ at n)

Write $(C = \mathbf{B}G) \Leftrightarrow (C_0 = \mathrm{pt})$. From the above we get the following smooth ω-category valued presheaves:

$\mathbf{A}$: Spaces$^{\mathrm{op}}$ $\to$ ωCategories(Spaces)

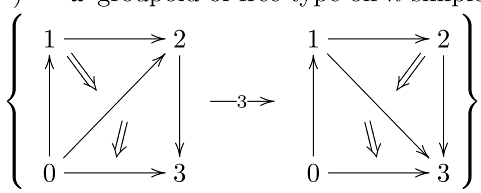
$X \mapsto \hom(\mathcal{P}_0(X), \mathbf{B}G)$	trivial G-principal ω-bundles
$X \mapsto \hom(\Pi_\omega(X), \mathbf{B}G)$	trivial G-principal ω-bundles with flat connection
$X \mapsto \hom(\mathcal{P}_n(X), \mathbf{B}G)$	trivial G-principal ω-bundles with connection with curvature in degree $n+1$

General G-principal bundles arise from *gluing* trivial ones: translate simplices to globes:

G : Δ $\to$ ωCategories

$[n] \mapsto O(\Delta^n)$	nth <u>oriental</u> free ω-category on n-simplex [Street:1987]
$[n] \mapsto U(\Delta^n)$	nth <u>unoriental</u> free weak ω-groupoid on n-simplex
$[n] \mapsto \Pi_\omega(\Delta^n)$	ω-groupoid of free type on n-simplex [BrownSivera:2007]

e.g. the 3-category $O(\Delta^3)$ free on the 3-simplex is $O(\Delta^3) =$

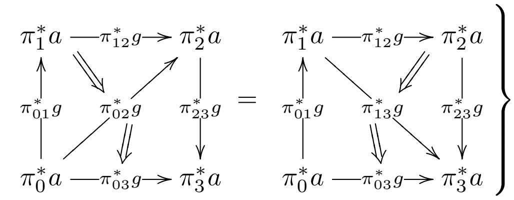


Definition 1.3 (descent and codescent) <u>*codescent ω-category*</u> / <u>*descent ω-category*</u>

$$\mathrm{Codesc}(Y^\bullet, \Pi) := \int^{[n]\in\Delta} \Pi_\omega(\Delta^n) \otimes \Pi(Y^n)$$
$$\mathrm{Desc}(Y^\bullet, \mathbf{A}) := \int_{[n]\in\Delta} \hom(\Pi_\omega(\Delta^n), \mathbf{A}(Y^n))$$

e.g.
$Y^\bullet = (\cdots Y\times_X Y \times_X Y \overset{\pi_{23}}{\underset{\pi_{12}}{\overset{\pi_{13}}{\rightrightarrows}}} Y\times_X Y \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} Y)$
and $\mathbf{A}$ valued in 2-groupoids,

$\Rightarrow \mathrm{Desc}(Y, \mathbf{A})_0 = \{ a \in \mathbf{A}(Y)_0 ,$



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Definition 1.4 (ω-stack and ω-costack)

$\mathbf{A}$ *is <u>ω-stack</u>* $\Leftrightarrow$ $\forall (Y^\bullet \to X) :\ \mathbf{A}(X) \xrightarrow{\simeq} \mathrm{Desc}(Y, \mathbf{A})$

$\mathbf{B}$ *is <u>ω-costack</u>* $\Leftrightarrow$ $\forall (Y^\bullet \to X) :\ \mathrm{Codesc}(Y, \mathbf{B}) \xrightarrow{\simeq} \mathbf{B}(X)$

[2] [K.-S. Brown:1973, Jardine:2006]

Definition 1.5 (cohomology and homotopy)

cohomology / *with coefficients in* $\mathbf{A}$	$H(X,\mathbf{A}) = \lim_{\to Y} \mathsf{Desc}(Y,\mathbf{A})$
homotopy / *with coefficients in* $\mathbf{B}$	$\pi(X,\mathbf{B}) = \lim_{\leftarrow Y} \mathsf{Codesc}(Y,\mathbf{B})$

Proposition 1.6 *For all* $n \in \mathbb{N}$. $\mathcal{P}_n$ *is an* ω*-costack, hence so is* Π_ω.

Proof. For $n = 1$ in [8], for $n = 2$ in [10], for $g \geq 3$ conjectural but related by weakening to higher van Kampen theorem [BrownHigginsSivera:2008]. □

Proposition 1.7 *Codescent co-represents descent:* $\mathrm{hom}(\mathrm{Codesc}(Y,\Pi), \mathbf{B}G) \simeq \mathsf{Desc}(Y, \mathrm{hom}(\Pi(-), \mathbf{B}G))$.

Definition 1.8 (differential G**-cohomology relative** Π**)** *Given a copresheaf* $\Pi : \mathsf{Spaces} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ *we put*

$$H_\Pi(X, \mathbf{B}G) := H(X, \mathrm{hom}(\Pi(-), \mathbf{B}G)) .$$

Corollary 1.9 *We have*

$$H_\Pi(X,\mathbf{B}G) \simeq \lim_{\to Y} \left\{ \begin{array}{ccc} & \mathrm{Codesc}(Y,\Pi) & \\ \stackrel{\simeq}{\swarrow} & & \searrow^{(g,\mathrm{triv}_\nabla)} \\ \Pi(X) & \xrightarrow[\mathrm{tra}_\nabla]{} & \mathbf{B}G \end{array} \right\}$$

1.4 Examples

Proposition 1.10 *For* $\mathbf{A}$ *the image of* $[\mathbf{A}]$ *under the equivalence* [Whitehead:1956, BrownHiggins:1981]

$$\begin{array}{ccccc} \mathrm{Sheaves}(\mathsf{ChainComplexes}(\mathsf{AbelianGroups})) & \hookrightarrow & \mathrm{Sheaves}(\mathsf{CrossedComplexes}) & \xrightarrow{\simeq} & \mathrm{Sheaves}(\omega\mathsf{Groupoids}) \\ [\mathbf{A}] & \longmapsto & & & \mathbf{A} \end{array}$$

nonabelian cohomology with coefficients in $\mathbf{A}$ *reproduces ordinary Čech cohomology with coefficients in* $[\mathbf{A}]$*:*

$$[H(X,\mathbf{A})] \simeq H(X,[\mathbf{A}]) .$$

Theorem 1.11 *Let* G_1, G_2 *be a Lie 1- and 2-group, respectively.*

- $H_{\mathcal{P}_0}(X,\mathbf{B}G_1) \simeq \{$*G-principal bundles on* $X\}$
- $H_{\mathcal{P}_0}(X,\mathbf{B}G_2) \simeq \{$*G-principal 2-bundles on* $X\}$ [Bartels:2004, Baković:2008, Wockel:2008]
- $H_{\mathcal{P}_1}(X,\mathbf{B}G_1) \simeq \{$*G-principal bundles with connection on* $X\}$ [8],
- $H_{\mathcal{P}_2}(X,\mathbf{B}\mathrm{AUT}(G_1)) \simeq \{$*G-gerbes with connection with curvature in degree 3*$\}$ [10, 3],
- $H_{\mathcal{P}_n}(X,\mathbf{B}^n U(1)) \simeq (n+1)$*st Deligne cohomology (for* $n \leq 1$ [8], *for* $n \leq 2$ [10]*)*

2 the theorem

We will state a fact about characteristic forms on String(n)-principal 2-bundles.

To get there we	first obtain the relevant n-groups from ∞-Lie integration	then consider the notions of	characteristic classes; pseudo-connections; characteristic forms.

2.1 ∞-Lie theory

Definition 2.1 (∞-Lie integration and -differentiation) *By slight variation on* [Sullivan:1977, Ševera:2001, Getzler:2004, Henriques:2006, Ševera:2006] *we set*

∞-Lie differentiation

$$\omega\mathsf{Groupoids}(\mathsf{Spaces}) \underset{\Pi_\omega(-)=\mathrm{Hom}(C^\infty(-),\Pi_\omega)}{\overset{K(-)=\mathrm{Hom}(\Pi_\omega(-),-)}{\rightleftarrows}} \mathsf{Spaces} \underset{S(-)=\mathrm{Hom}(-,\Omega^\bullet(-))}{\overset{\Omega^\bullet(-)=\mathrm{Hom}(-,\Omega^\bullet)}{\rightleftarrows}} C^\infty\mathsf{qDGCAs} \underset{\simeq}{\overset{\mathrm{CE}(-)}{\longleftarrow}} L_\infty\mathsf{Algebroids}\,.$$

∞-Lie integration

Proposition 2.2

- $\Pi_\omega(-)$ *is left adjoint to* $K(-)$ *and* $\Omega^\bullet(-)$ *is left adjoint to* $S(-)$.
- *For* $\mathfrak{g}$ *a Lie algebra,* $\Pi_1(S(\mathrm{CE}(\mathfrak{g}))) = \mathbf{B}G$ *with* G *the ordinary simply connected Lie group integrating* $\mathfrak{g}$ *(by comparison with* [CrainincFernandes:2003]*);*
- $\Pi_n(S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1)))) = \mathbf{B}\mathbf{B}^{n-1}\mathbb{R}$*;*
- *for* G_2 *the Lie 2-group coming from a strict Lie 2-algebra* $\mathfrak{g}$ *we have* $K(G_2) = S(\mathrm{CE}(\mathfrak{g}))$ [9].

Definition 2.3 (string-like extensions) *For* $\mathfrak{g}$ *an* L_∞*-algebra and* $\mu \in \mathrm{CE}(\mathfrak{g})$ *an* L_∞*-algebra* n*-cocycle let* $\mathfrak{g}_\mu$ *be the result of* co-killing μ, *i.e. the pushout*

$$\begin{array}{ccc} \mathrm{CE}(\mathfrak{g}_\mu) & \hookleftarrow & \mathrm{W}(b^{n-1}\mathfrak{u}(1)) \\ \uparrow & & \uparrow \\ \mathrm{CE}(\mathfrak{g}) & \xleftarrow{\mu} & \mathrm{CE}(b^{n-1}\mathfrak{u}(1)) \end{array}$$

Definition 2.4 (String(n) and Fivebrane(n)) *For* $\mu_3, \mu_7 \in \mathrm{CE}(\mathfrak{so}(n))$ *the normalized cocycles, set*

$$\mathbf{B}\mathrm{Spin}(n) = \Pi_1(S(\mathrm{CE}(\mathfrak{so}(n))))$$

$$\mathbf{B}\mathrm{String}(n) := \Pi_2(S(\mathrm{CE}(\mathfrak{so}(n)_{\mu_3})))$$

$$\mathbf{B}\mathrm{Fivebrane}(n) := \Pi_6(S(\mathrm{CE}((\mathfrak{so}(n)_{\mu_3})_{\mu_7})))$$

See [5, 6].

Theorem 2.5 *In* $\mathbf{Ho}(\omega\mathsf{Groupids}(\mathsf{Spaces}))$ *we have* $\mathbf{B}\mathrm{String}(n) \simeq \mathbf{B}\mathrm{String}_{\mathrm{BCSS}}(n)$*, with right hand from* [2].

2.2 Principal ω-bundles

Definition 2.6 (universal G-principal ω-bundles) *Let* $p: \mathbf{E}G \twoheadrightarrow \mathbf{B}G$ *be the pullback* [4]

$$\begin{array}{ccccc} \mathbf{E}G & \longrightarrow & (\mathbf{B}G)^I & \xrightarrow{d_2} & \mathbf{B}G \\ \downarrow & & \downarrow d_1 & & \\ \mathrm{pt} & \longrightarrow & \mathbf{B}G & & \end{array}$$

Proposition 2.7 (generalizing [4]) *This yields an exact sequence*

$$\begin{array}{ccccc} \ker(p) & & & & \\ \| := & & & & \\ G & \hookrightarrow & \mathbf{E}G & \twoheadrightarrow & \mathbf{B}G \end{array}$$

Definition 2.8 (G-principal ω-bundle) *A G-principal bundle over X is* $P \twoheadrightarrow X$ *such that there is* $g \in \mathbf{Ho}(X, \mathbf{B}G)$ *with*

$$\begin{array}{ccccc} & & \mathbf{Y}_0 \times G & \longrightarrow & G \\ & & \downarrow & & \downarrow \\ P & \overset{\simeq}{\twoheadleftarrow} & g^*\mathbf{E}G & \longrightarrow & \mathbf{E}G \\ \downarrow & & \downarrow & & \downarrow \\ X & \overset{\simeq}{\twoheadleftarrow} & \mathbf{Y} & \xrightarrow{g} & \mathbf{B}G \end{array}$$

Theorem 2.9 ([4]) *For G an n-group with $n \leq 2$ this reproduces the existing notion of G-principal n-bundle* [Bartels:2004, Baković:2008, Wockel:2008].

2.3 Characteristic classes and forms

Approximate $\begin{array}{l}\text{nonabelian cocycles}\\ \text{nonabelian } \textit{differential} \text{ cocycles}\end{array}$ by families of $\left.\begin{array}{l}\text{abelian cocycles}\\ \text{abelian } \textit{differential} \text{ cocycles}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{characteristic classes}\\ + \text{ characteristic forms}\end{array}\right.$

Definition 2.10 (characteristic classes)

universal characteristic classes of ω-group G: $\quad c \in \mathbf{Ho}(\mathbf{B}G, \mathbf{B}^n U(1))$

characteristic classes of ω-bundle $P \overset{\simeq}{\twoheadleftarrow} g^*\mathbf{E}G$ *:* $\quad g^*c \in \mathbf{Ho}(X, \mathbf{B}^{\mathbf{n}} U(1))$

Corollary 2.11 $\mathbf{Ho}(X, \mathbf{B}^n U(1))/_\sim \simeq H^{n+1}(X, \mathbb{Z})$ *hence the class* $[g^*c] \in H^{n+1}(X, \mathbb{Z})$.

Proof. From prop. 1.10. ☐

Remark. Similar to [BaezStevenson:2008, GinotStiénon:2008] but staying within $\mathbf{Ho}(\omega\mathsf{Groupids}(\mathsf{Spaces}))$, i.e. without passing to topological realizations.

Theorem 2.12 *Let* $\mu_3, \mu_7 \in \mathrm{CE}(\mathfrak{so}(n))$ *be the normalized Lie algebra 3- and 7-cocycles.*

1. *there are universal characteristic classes* $c_{3,7} := \int \mu_{3,7}/\mathbb{Z} \in \mathbf{Ho}(\mathbf{B}\mathrm{Spin}(n), \mathbb{B}^{3,7}U(1))$*;*
2. c_3 *vanishes when pulled back along* $p : \mathbf{B}\mathrm{String}(n) \to \mathbf{B}\mathrm{Spin}(n)$ *to* $p^*c_3 \in \mathbf{Ho}(\mathbf{B}\mathrm{String}(\mathrm{Spin}), \mathbf{B}^3 U(1))$*;*
3. c_7 *vanishes when pulled back further along* $q : \mathbf{B}\mathrm{Fivebrane}(n) \to \mathbf{B}\mathrm{String}(n)$ *to* $(q \circ p)^*c \in \mathbf{Ho}(\mathbf{B}\mathrm{Fivebrane}(n), \mathbf{B}^7 U(1))$;
4. *for* $P \simeq g^*\mathbf{E}\mathrm{Spin}(n)$ *a* $\mathrm{Spin}(n)$*-principal bundle on* X*, we have* $g^*c_3 = \frac{1}{2}p_1[P]$*, the first fractional Pontryagin class of* P*;*
5. *for* $\hat{P} \simeq \hat{g}^*\mathbf{E}\mathrm{String}(n)$ *a* $\mathrm{String}(n)$*-principal 2-bundle on* X *lifting* P*, we have* $\hat{g}^*c_7 = \frac{1}{6}p_2[P]$*, the second fractional Pontryagin class of* P*.*

Proof. After choosing a suitable surjectively equivalent resolution of $\mathbf{B}G$ whose k-morphisms are generated from smooth k-simplices in $\mathrm{Spin}(n)$, this becomes a corollary of [BrylinskiMcLaughlin:1993,1996]. ☐

Remark. After passing to topological realizations this reproduces statements in [BaezStevenson:2008], [DouglasHillHenriques:2008], and [6].

2.4 Characteristic forms

From now on:
for $\mathfrak{g}$ a Lie n-algebra
with Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$
and Weil algebra $\mathrm{W}(\mathfrak{g})$
set

$$\begin{array}{ccc} \mathbf{B}G & \hookrightarrow & \mathbf{B}\mathcal{E}G \\ \downarrow{:=} & & \downarrow{:=} \\ \Pi_{n+1}(S(\mathrm{CE}(\mathfrak{g}))) & \hookrightarrow & \Pi_{n+1}(S(\mathrm{W}(\mathfrak{g}))) \end{array} .$$

Notice the shift in the truncation degree on the left: $n+1$ instead of n.

Proposition 2.13 *We obtain surjectively equivalent models*

$$\begin{array}{l} \Pi_2(S(\mathrm{CE}(\mathfrak{so}))) \xrightarrow{\simeq} \mathbf{B}\mathrm{Spin}(n) \\ \Pi_3(S(\mathrm{CE}(\mathfrak{so}_{\mu_3}))) \xrightarrow{\simeq} \mathbf{B}\mathrm{String}(n) \\ \Pi_7(S(\mathrm{CE}((\mathfrak{so}_{\mu_3})_{\mu_7}))) \xrightarrow{\simeq} \mathbf{B}\mathrm{Fivebrane}(n) \end{array} .$$

Proof. The $(n+1)$st homotopy groups vanish in each case. □

Definition 2.14 (pseudo-connections) *Pseudo-differential G-cohomology (classifying pseudo-connections on G-principal bundles), $\bar{H}_{\mathrm{pseud}}(-,\mathbf{B}G)$, is cohomology with coefficients in the ω-category valued presheaf*

$$X \mapsto \hom\left(\begin{array}{c} \mathcal{P}_0(X) \\ \downarrow \\ \Pi_\omega(X) \end{array} , \begin{array}{c} \mathbf{B}G \\ \downarrow \\ \mathbf{B}\mathcal{E}G \end{array} \right),$$

so that a cocycle is a commuting diagram

$$\begin{array}{ccc} X & \stackrel{g}{\longmapsto} & \mathbf{B}G \\ \downarrow & & \downarrow \\ \Pi_\omega(X) & \longmapsto & \mathbf{B}\mathcal{E}G \end{array} .$$

The analogous notion for the inclusion $\mathbf{B}\mathbf{B}^{n-1}U(1) \hookrightarrow \mathbf{B}(\mathbf{B}^{n-1}\mathbb{R} \to \mathbf{B}^{n-1}U(1))$ yields $\bar{H}_{\mathrm{pseud}}(-,\mathbf{B}U(1))$.

Remark. This $\bar{H}_{\mathrm{pseud}}(-,\mathbf{B}U(1))$ reproduces the notion of pseudoconnections from [BehrendXu:2006].

Proposition 2.15 *Every element in $\bar{H}_{\mathrm{pseud}}(X,\mathbf{B}U(1))$ is cohomologous to one that extends to a diagram*

$$\begin{array}{cccl} X & \stackrel{g}{\longmapsto} & \mathbf{B}^n U(1) & \textit{cocycle} \\ \downarrow & & \downarrow & \\ \Pi_\omega(X) & \longmapsto & \mathbf{B}(\mathbf{B}^{n-1}\mathbb{R} \to \mathbf{B}^{n-1}U(1)) & \textit{connection} \\ \downarrow & & \downarrow & \\ \Pi_\omega(X) & \xrightarrow{F_A} & \mathbf{B}^{n+1}\mathbb{R} & \textit{curvature} \end{array}$$

Remark. This is the ∞-Lie integration of the $\mathfrak{g}$-connection diagrams in [5].

Proposition 2.16 *For $c \in \mathbf{Ho}(\mathbf{B}G, \mathbf{B}^n U(1))$ a universal class of the form $\int \mu/\mathbb{Z}$ as in theorem 2.12, and for $\bar{g} \in \bar{H}_{\mathrm{pseud}}(X,\mathbf{B}G)$ a pseudo-differential cocycle, there is canonically a pseudo-differential $\mathbf{B}^n U(1)$-cocycle $\bar{g^*c} \in \bar{H}_{\mathrm{pseud}}(X,\mathbf{B}^n U(1))$ given by a diagram*

$$\begin{array}{cccccl} X & \longmapsto & \mathbf{B}G & \stackrel{\int \mu/\mathbb{Z}}{\longmapsto} & \mathbf{B}^n U(1) & \textit{characteristic class} \\ \downarrow & & \downarrow & & \downarrow & \\ \Pi_\omega(X) & \longmapsto & \mathbf{B}\mathcal{E}G & \longmapsto & \mathbf{B}(\mathbf{B}^{n-1}U(1) \to \mathbf{B}^{n-1}\mathbb{R}) & \textit{secondary characteristic form} \\ \downarrow & & & & \downarrow & \\ \Pi_\omega(X) & & \longrightarrow & & \mathbf{B}^{n+1}\mathbb{R} & \textit{characteristic form} \end{array}$$

Proof. By ∞-Lie integrating the diagrammatics in [5] and using the equivalence $\mathrm{Hom}(\Pi_\omega(X),\mathbf{B}^{n+1}\mathbb{R}) \simeq \Omega^{n+1}_{\mathrm{closed}}(X)$ from [9]. □

Corollary 2.17 *Let* $P \overset{\simeq}{\twoheadleftarrow} g^*\mathbf{E}\mathrm{Spin}(n)$ *be a* $\mathrm{Spin}(n)$*-principal bundle with connection* ∇ *with* $\mathrm{String}(n)$*-principal lift* $\hat{P} \overset{\simeq}{\twoheadleftarrow} \hat{g}^*\mathbf{E}\mathrm{String}(n)$ *as in theorem 2.12. Then*

1. *the characteristic form refining the class* g^*c_3 *is* $\frac{1}{2}P_4(F_\nabla)$*;*
2. *the characteristic form refining the class* $\hat{g}^*c_7$ *is* $\frac{1}{6}P_8(F_\nabla)$*,*

where $P_4, P_8 \in \mathrm{W}(\mathfrak{so}_n)_{\mathrm{basic}}$ *are the invariant polynomials related by transgression to* μ_3 *and* μ_7*.*

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