Second nonabelian differential cohomology

October 21, 2008

Abstract

Differential nonabelian cohomology characterizes higher dimensional *locally smooth parallel transport* in higher bundles with connection (having higher gerbes of sections). This describes, classically, the gauge action functionals of higher branes and, quantumly, their extended worldvolume quantum field theory. We indicate the structure of the general theory and mention some examples.

This are notes compiled on occasion of a talk at

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The talk itself follows a subset of these notes, in particular omitting all sections labeled "details".
Section 1 is about the general theory;
section 2 about general constructions;
section 3 about examples.

The exposition is based on [1] which builds on the references given in section 4.

Contents

1 Parallel Transport and Differential Cohomology

differential cohomology in degree n = n-dimensional *parallel transport*: **local** and **smooth**	classical	quantum
	assign phases to classical trajectories	assign amplitudes to worldvolumes
	$(\, x \xrightarrow{\gamma} y \,) \mapsto (\, E_x \xrightarrow{P\exp(\int_\gamma \nabla)} E_y \,)$	$(\, t_1 \longrightarrow t_2 \,) \mapsto (\, \mathcal{H}_{t_1} \xrightarrow{U(t_2-t_1)=P\exp(\frac{1}{i\hbar}\int_0^1 H\, dt)} \mathcal{H}_{t_2} \,)$

1.1 Local

locality: global assignments are fixed by local assignments

formalization of parallel transport: ∞-functors between ∞-categories

classical	quantum
$\mathrm{tra} : \mathsf{TargetSpace} \longrightarrow \mathsf{Phases}$	$\mathrm{Z} : \mathsf{Worldvolume} \longrightarrow \mathsf{Amplitudes}$

∞-category: Entity that comprises:

- objects: points in a space;
- morphisms: processes and symmetries/gauge transformations between points;
- k-morphisms: higher order processes, higher order gauge transformations

∞-functor: $f : C \to D$ rule which sends

- each point to a fiber

classical	quantum
space of possible values of wavefunction	space of states

- each process to a process acting on the fibers
- each gauge transformation to a symmetry between fibers

such that composition is respected.

∞-functoriality = locality:	classical	quantum
	locality of action functional	sewing law of path integral

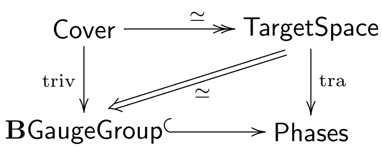
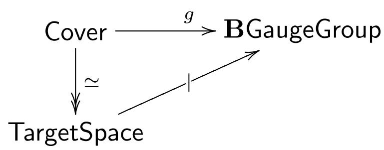
Details. In [1] we model ∞-categories as ω-categories: all compositions are strictly associative and strictly unital. Advantages:

- ω-groupoids (ω-categories where all morphisms are invertible) equivalent to crossed complexes of groups → nonabelian homological algebra [Brown-Higgins:monograph2008], useful in applications in differential geometry and physics;
- simplicial structures are still captured by “freeing” composition operation → free ω-categories, useful for construction of (hyper)covers.

In this context: ω-group G is one-object ∞-groupoid $\mathbf{B}G$.

1.2 Smooth

smoothness: no jumps → locally well-behaved; *locally trivial*

global picture	local picture
∞-functor tra locally equivalent to smooth ∞-functor triv 	smooth ∞-functor out of local resolution  "ω-anafunctor" (generalizing [Makkai:1996, Bartels:2004])

Details. In [1] we use the following concrete model for smooth ∞-categories and smooth ∞-functors between them:

- Spaces := Sheaves(CartesianSpaces);
- smooth ∞-categories = ω-categories internal to Spaces = ωCategories(Spaces) $\simeq$ Sheaves(CartesianSpaces, ωCategories)

In this context we have *homotopy theory*:

Proposition 1.1 (homotopy theory of smooth ω-categories) *On* ωGroupoids(Spaces) *there is the structure of a* category of fibrant objects *in the sense of [K.-S. Brown:1973] whose* *fibrations* $\twoheadrightarrow$ *and* *cofibrations* $\hookrightarrow$ *are globally and whose* *weak equivalences* $\xrightarrow{\simeq}$ *and* *hypercovers* $\overset{\simeq}{\twoheadrightarrow}$ *are stalkwise those of [BrownGolasinski:1998, LafontMétayerWorytkiewicz:2008].*

Write now $\mathbf{X}$ = TargetSpace and $\mathbf{Y}$ = Cover.

Definition 1.2 (nonabelian cohomology)

- *Nonabelian cocycle is ω-anafunctor* $g: \mathbf{X} \longrightarrow\!\!\!\!\!\!\mid\;\; \mathbf{B}G$
- *coboundary is homotopy/transformation* $g \sim g'$.

Formally:

$$H(\mathbf{X}, G) := \mathrm{colim}_{\mathbf{Y}} \ \mathrm{hom}(\mathbf{Y}, \mathbf{B}G)\,,$$

where $\mathrm{hom}(-,-)$ *is internal hom ω-category [BrownHiggins, Crans:1995.]*

Theorem 1.3 *Cohomology classes are the morphisms in the homotopy category* Ho

$$H(\mathbf{X}, \mathbf{B}G)/_\sim = \mathrm{Ho}(\mathbf{X}, \mathbf{B}G)\,.$$

Proof. By using prop. 1.1 in [K.-S. Brown:1973]. □

Recall that $\mathbf{X}$ contains information about all allowed processes = paths between points. There are different possible choices

$$\mathsf{TargetSpace} = \mathbf{X} = \begin{cases} \mathcal{P}_0(X) = X & \text{no paths in } X \to \text{no connection} \to \text{ordinary cohomology;} \\ \Pi_\omega(X) & \text{smooth paths in } X \to \text{flat connection} \to \text{flat } \textit{differential cohomology.} \end{cases}$$

Details. Here k-morphisms in $\Pi_\omega(X)$ are thin-homtopy classes of maps $D^k \to X$ suitably well behaved at the boundary to make composition by gluing well defined.

Given any such choice $\Pi : \mathsf{Spaces} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ and a cover of spaces $Y \twoheadrightarrow X$ we obtain Cover as the *Czech* ∞*-groupoid*

$$\mathsf{Cover} = \int^{[n]\in\Delta} O(\Delta^n) \otimes \Pi(Y^{[n+1]})$$

with $O(\Delta^n)$ a resolution of the point modeled on the n-simplex. Dually, for G the gauge ∞-group and

$$\mathrm{TrivBund}_\Pi(G) := \mathrm{hom}(\Pi(-), \mathbf{B}G) : \mathsf{Spaces}^{\mathrm{op}} \to \omega\mathsf{Categories}(\mathsf{Spaces})$$

the presheaf of trivial G-bundles with Π-connection, we have the ω-category of descent data

$$\mathrm{Desc}(Y, \mathrm{TrivBund}_\Pi(G)) = \int_{[n]\in\Delta} \mathrm{hom}(O(\Delta^n), \mathrm{TrivBund}_\Pi(G)(Y^{[n+1]}))$$

which is *corepresented* by the codescent object Cover:

$$\mathrm{hom}(\mathsf{Cover}, \mathbf{B}G) \simeq \mathrm{Desc}(Y, \mathrm{TrivBund}_\Pi(G))\,.$$

The perspective of the ω-category $\mathrm{Desc}(-,-)$ is useful in computions, for instance for showing the equivalence of $H(X, \mathbf{B}G)$ with Czech cohomology for G an abelian ∞-group.

2 Constructions

2.1 Universal ∞-bundles

Definition 2.1 (for $n = 2$ in [4]) *For G an ω-group let $\mathbf{E}G$ be the kernel of the source projection of the path fibration of $\mathbf{B}G$, i.e. the pullback*

$$\begin{array}{ccc} \mathbf{E}G & \longrightarrow & (\mathbf{B}G)^I \\ \downarrow & & \downarrow{\scriptstyle \mathrm{dom}} \\ \mathrm{pt} & \longrightarrow & \mathbf{B}G \end{array} .$$

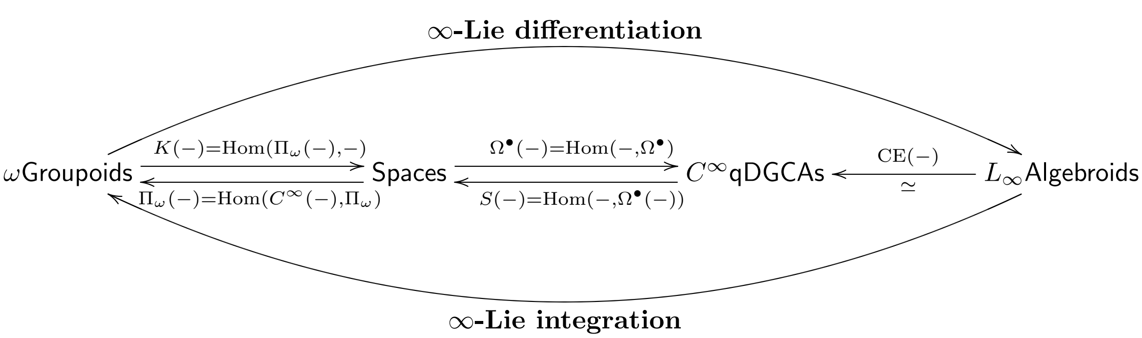
Theorem 2.2 (announced in [4]) *For G a $(n = 1)$- or $(n = 2)$-group: $(G \to \mathbf{E}G \to \mathbf{B}G)$ is the universal G-principal bundle in the sense that every G-principal bundle $P \to X$ [Bartels:2004,Baković:2008,Wockel:2008] with cocycle $g : Y \longrightarrow \mathbf{B}G$ is equivalent to $g^*\mathbf{E}G/_\sim$.*

In components this pullback $g^*\mathbf{E}G$ is given by the formulas given in [Wockel:2008].

For general ω-group G it makes sense to *define* a G-principal bundle $P \to X$ to be one equivalent to $g^*\mathbf{E}G$.

2.2 ∞-Lie theory

A useful method for *constructing* nonabelian differential cocycles is by ∞-Lie integrating morphisms of ∞-Lie algebroids (namely those discussed in [5]):



(compare with [CrainicFernandez, Getzler, Henriques, Ševera, Sullivan, Zhu]).

Here starting with $\mathsf{Quantities} := \mathsf{Copresheaves}(\mathsf{CartesianSpaces})$ we take $C^\infty\mathsf{qDGCA}s$ to be quasi-free differential graded commutative algebras internal to $\mathsf{Quantities}$ as the dual of smooth Lie-∞-algebroids.

- $\Omega^\bullet$, the deRham sheaf, is the object of *infinitesimal paths*;
- Π_ω, the fundamental path co-presheaf, is the object of *finite paths*;

Theorem 2.3 ([1]) *For $n \in \mathbb{N}$ we find the integration of $b^{n-1}\mathfrak{u}(1)$ to be*

$$\Pi_n(S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1)))) = \mathbf{B}\mathbf{B}^{n-1}\mathbb{R}\,.$$

Theorem 2.4 ([1]) *For $\mathfrak{g}$ a semisimple Lie algebra and μ_3 the canonical normalized 3-cocycle, we have the String Lie 2-algebra $\mathfrak{g}_{\mu_3}$ [2, 5] and its Lie integration to a strict 2-group [2]*

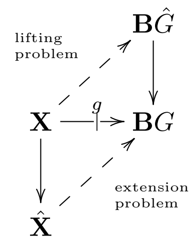
$$\Pi_2(S(\mathrm{CE}(\mathfrak{g}_{\mu_3}))) \simeq \mathbf{B}(\hat{\Omega}G \to PG) =: \mathbf{B}\mathrm{String}(G)\,,$$

where the equivalence is in the homotopy theory of $\omega\mathsf{Categories}(\mathsf{Spaces})$.

Theorem 2.5 ([2],[BaezStevenson:2008) *] String(G)-2-bundles have the same classification as topological $|String(G)|$-1-bundles, and $|\mathrm{String}(G)|$, the realization of the nerve of $\mathrm{String}(G)$, is the 3-connected cover of $\mathrm{Spin}(n)$.*

2.3 Obstruction theory

Given a cocycle $g : \mathbf{X} \longrightarrow \mathbf{B}G$ one can try to *lift* it or to *extend* it

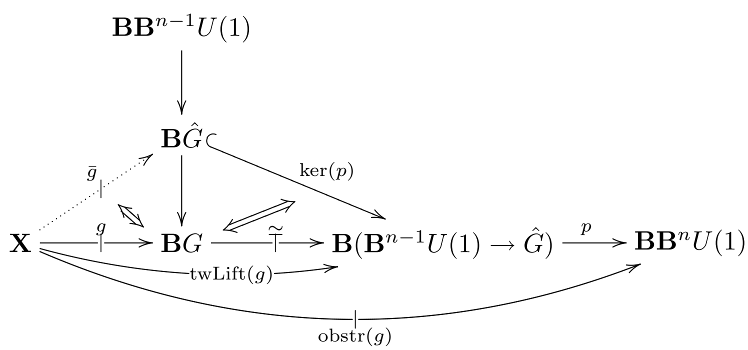


Lifting problem.

Theorem 2.6 ([1]) *The obstruction to lifting a $G = \mathrm{Spin}(n)$-cocycle g through the String-extension*

$$(\mathbf{B}^{n-1}U(1) \to \hat{G} \to G) = (\mathbf{B}U(1) \to \mathrm{String}(n) \to \mathrm{Spin}(n))$$

is the $\mathbf{B}^2U(1)$-cocycle $\mathrm{obstr}(g)$ whose single characteristic class is the first fractional Pontryagin class $\frac{1}{2}p_1(g)$.

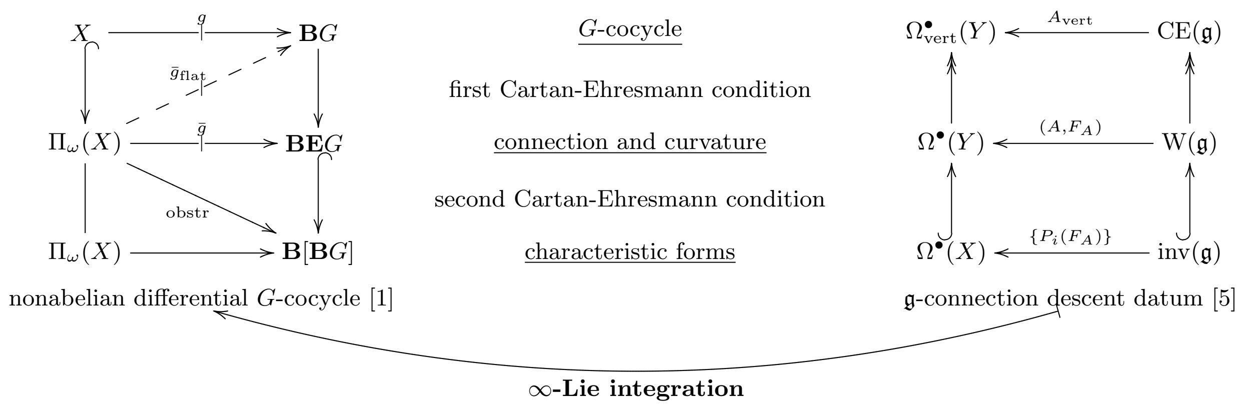


Green-Schwarz mechanism. If the obstruction does not vanish then it is the twist of the *twisted lift* twLift(g). The Green-Schwarz mechanism in heterotic string theory says that the background is a twisted String 2-bundle with connection, generalizing the twisted bundles in twisted K-theory, whose twist is the restriction of the supergravity C-field, a 3-bundle with connection, to the 10-dimensional target space.

An analogous story exists for lifts through $\mathbf{B}^5U(1) \to \mathsf{Fivebrane}(n) \to \mathsf{String}(n)$ [6]. Here the obstructing $\mathbf{B}^5U(1)$-bundle has characteristic class $\frac{1}{6}p_2$ and describes the magnetic dual Green-Schwarz mechanism.

Extension problem. The obstruction to extend an ordinary cocycle $g : X \longrightarrow \mathbf{B}G$ to a flat differential cocycle through $X \hookrightarrow \Pi_\omega(X)$ are the characteristic forms of any non-flat lift. In particular, non-flat (and non-"fake-flat") G-principal parallel transport is flat $\mathbf{BE}G$-parallel transport $\bar{g} : \Pi_\omega(X) \longrightarrow \mathbf{BE}G$ with constraints. Flatness of $\bar{g}$ is the *Bianchi identity.*

Details.



3 Examples and Applications

3.1 Classical

Definition 3.1 *Write $\mathcal{P}_n(X)$ for the path n-groupoid of X, the truncation of $\Pi_\omega(X)$ at n-morphisms.*

Theorem 3.2 ([10]) *For G a 2-group we have*

$$\bar{H}_{\mathrm{ff}}(X, \mathbf{B}G) := H(\mathcal{P}_2(X), \mathbf{B}G) \simeq \{\textit{locally trivializable 2-functors } \mathrm{tra} : \mathcal{P}_2(X) \to G\mathrm{Tor}\} ;$$

and for $G = \mathrm{AUT}(H)$

$$\cdots \simeq \{\textit{nonabelian } H\textit{-gerbes with fake flat connection}\} ;$$

and for $G = \mathbf{B}U(1)$

$$\cdots \simeq \{U(1)\textit{-gerbes with general connection}\} .$$

For $\phi : \Sigma \to X$ a surface and $\Sigma \in \Pi_2(\Sigma)$ a fundamental chain, $\mathrm{tra}(\phi(\Sigma))$ is the surface holonomy, reproducing for $G = \mathbf{B}U(1)$ the familiar surface holonomy of abelian gerbes, which is the gauge part of the action functional for the string.

3.2 Quantum

One formulation of local quantum physics is in terms of *local nets* of algebras [HaagKastler]. These are obtained as the *endomorphisms co-opresheaves* of parallel transport.

Definition 3.3 (endomorphism co-presheaf of 2-transport, [7]) *For $Z : \mathcal{P}_2(\mathbb{R}^2) \to T$ a 2-transport with values in the 2-C^*-category T, let*

$$A_Z : \mathsf{CausalSubsets}(\mathbb{R}^2) \to \mathsf{Algebras}$$

be the assignment which sends

- *O with past boundary $\gamma(O)$ to $\mathrm{End}_T(Z(\gamma(O)))$;*
- *$(O' \hookrightarrow O)$ to*

$$\mathrm{End}_T(Z(w_O(\gamma(O')))) \xhookrightarrow{\mathrm{Ad}_Z} \mathrm{End}_T(Z(\gamma(O))) \ ,$$

  where $w_O(\cdot)$ is rewhiskering with lightlike paths.

Theorem 3.4 ([7]) *A_Z is a local net of C^*-algebras satisfying the time-slice axiom. If Z is G-equivariant then A_Z is G-covariant.*

4 References

References

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