Notes from Streetfest

(Sydney, July 2006)

Canberra

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Getzler

LIE THEORY FOR L_∞-ALGEBRAS (BCH formula)

Higher groups

Work with the nerve of the category [Duskin] '79

Nerves of n-groupoids

Simplicial set $X_\bullet$, $k > 0$, $0 \le i \le k$

horn $\Lambda^i_k \hookrightarrow \Delta_k$ [May's book]

$$X_k \xrightarrow[\xi^i_k]{} \mathrm{Hom}(\Lambda^i_k, X_k)$$

ξ^i_k is surjective, and a bijection if $k > n$. (cf. Joyal - dropping 0 + k^{th} pieces)

* From a Kan complex, realisation of Postnikov tower [Duskin] [Beke]
τ_i, $i \geq 0$

$$\to \tau_2 X_\bullet \to \tau_1 X_\bullet \to \tau_0 X_\bullet$$

$\tau_1 X_\bullet$: Poincaré groupoid; $\tau_0 X_\bullet \cong \pi_0(X_\bullet)$

Definition

A Lie n-groupoid: for $X_\bullet$ a simplicial manifold
ξ^i_k a surjective submersion, bijective for $k > n$

(agrees with usual defn for $n = 1$)

(cf. coverings of a Gr. pretopology " Beke)

AIM

Start with a Lie n-algebra + create a germ about "1" $\Rightarrow$ higher BCH

Definition

A Lie n-group: also have $X_0 = *$ ("reduced")

L_∞-algebras (generalise Lie algebras)

$\mathbb{Z}$-graded vector spaces $\mathfrak{g}^\cdot = (\cdots \mathfrak{g}^{-2}, \mathfrak{g}^{-1}, \mathfrak{g}^0, \mathfrak{g}^1, \cdots)$ finite # of non-zero pieces
($\mathbb{R}$, or $\mathbb{Q}$)

Homogeneous maps $\underbrace{\mathfrak{g}^\cdot \otimes \cdots \otimes \mathfrak{g}^\cdot}_{k \text{ times}} \longrightarrow \mathfrak{g}^\cdot$

$[\,, \cdots, \,]$:

k-bracket homogen. of degree $(2-k)$, graded skew sym

- $k=1$ $\quad [x]$ differential
- $k=2$ $\quad [x,y]$ ordinary Lie bracket

$\vdots$

$\left\{ \begin{array}{l} \text{infinitesimal form of associators for higher} \\ \text{groupoids} \end{array} \right.$

$$\sum_{I \amalg J = \{1,\dots,k\}} \pm [[x_I], x_J] = 0 \quad , \quad I = (i_1 < \cdots < i_{|I|})$$

$$x_I = x_{i_1}, \dots, x_{i_{|I|}}$$

- $k=1$ $\quad [[x]] = 0$ is a differential
- $k=2$ $\quad [[x,y]] = [[x],y] \pm [x,[y]]$ Leibniz
- $k=3$ $\quad [[x,y],z] \pm [[y,z],x] \pm [[z,x],y]$ Jacobi

$$= [[x,y,z]] \pm [[x],y,z]$$

$$\pm [x,[y],z] \pm [x,y,[z]]$$

[Stasheff on A_∞]*
[Kontsevich ~ vanishing differential]

* If k-bracket is zero for $k > 2$; have a <u>dg-Lie algebra</u>

* If differential is zero ; have a "minimal ~~A_∞~~-algebra" (Kontsevich)

* Every L_∞-algebra $\simeq$ one satisfying the 'minimal' condition

Consider

$$0 \longrightarrow \mathfrak{g}^{-1} \xrightarrow{d} \mathfrak{g}^{0} \longrightarrow 0$$

$[,,] : \Lambda^3 \mathfrak{g}^0 \longrightarrow \mathfrak{g}^{-1}$

If $\underset{(d)}{[\,]}$ vanishes then $\mathfrak{g}^0$ is a Lie algebra, $\mathfrak{g}^{-1}$ is a module and $[,,]$ is a cocycle.

Definition A Lie n-algebra : an L_∞-alg. concentrated in degrees $(-n, 0]$

Consider $\Omega^*(\Delta_n) \equiv \Omega_n$ differential forms on n-simplex (analytic, polynomial, ... ?)

Algebraic realisation :

$$t_0 + \cdots + t_n = 1 \qquad \text{hyperplane}$$

$$\Omega_n = K[\underbrace{t_0, \ldots, t_n}_{\text{degree } 0}, \underbrace{dt_0, \ldots, dt_n}_{\substack{\text{anticommuting} \\ \text{degree } 1}}] \Big/ (\Sigma t_i = 1 ; \Sigma dt_i = 0)$$

$\Omega_\bullet$ form a simplicial dg-algebra ("fat" version of K) [Sullivan]

Have a f.d. L_∞-algebra $\mathfrak{g}^\bullet$; associate a Chevalley-Eilenberg complex

$$C^*(\mathfrak{g}^\bullet) = \Lambda^\bullet \mathfrak{g}^\vee = \underbrace{S^\bullet}_{\substack{\text{graded} \\ \text{symmetric}}} \overbrace{\Sigma}^{\text{suspension}} \underbrace{\mathfrak{g}^\vee}_{\text{graded dual}}$$

Differential on this captures the L_∞-algebra structure.

Define the (algebro-geometric) spectrum : for A a dg-commutative algebra [Sullivan]

$$A \longmapsto \underbrace{\langle A \rangle}_{\operatorname{spec} A} = \operatorname{Hom}(A, \Omega_\bullet) \quad \text{simplicial set}$$

[see Quillen]

applying to the Chevalley-Eilenberg complex, get the Maurer-Cartan simplicial set of $\mathfrak{g}$

$$MC_\bullet(\mathfrak{g}) = \mathrm{Spec}\,(C^*(\mathfrak{g}))$$

$$MC_n(\mathfrak{g}) = \left\{ \omega \in \Omega_n \otimes \mathfrak{g} \quad (\text{an } L_\infty\text{-algebra}) \; : \sum_{k=1}^{\infty} \frac{1}{k!}[\omega, \dots, \omega] = 0 \right\}$$

(recall we only consider finite # terms)

Same as the nerve of a dg-Lie algebra. [Hinich]

* If have a Lie 2-algebra $\mathfrak{g}$, can form

$\tau_2(MC_\bullet(\mathfrak{g}))$ 2-groupoid (see String groups - tomorrow!)

* If $\mathfrak{g}$ a Lie algebra,

$$MC_n(\mathfrak{g}) = \{ \text{flat } \mathfrak{g}\text{-connections on } n\text{-simplex} \}$$
$$= C_n(\mathfrak{g})/G \supset \text{nerve of } G$$
$$MC_1(\mathfrak{g}) = P_* G \quad \text{based paths} = \underset{\text{constant 1-forms}}{\Omega^1(\Delta_1, \mathfrak{g})}$$

but we really want G rather than this large space ...

Dupont has an explicit proof of de Rham on a simplicial complex and this provides

$$ds + sd = I - P \quad \text{on } \Omega_n$$

Whitney projection

For $t_0, t_1 ; dt_1$ deg 1

$t_0, t_1, t_2 ;$ $t_0\,dt_1 - t_1\,dt_0$
$t_1\,dt_2 - t_2\,dt_1$ deg 1
$t_2\,dt_0 - t_0\,dt_2$

$dt_1\,dt_2$ deg 2

Theorem

$$MC_n(\mathfrak{g}) \supset \gamma_n(\mathfrak{g}) = \{\omega \in MC_n(\mathfrak{g}) \mid s\omega = 0\}$$ gives what we want

* What about non-nilpotent case? Seems tricky. See stuff on String group, tomorrow.

* dg-stacks : working with a ∞-topos !!

* Applications for $n=3$?? Def$^{\underline{n}}$ theory of modular categories [Kitaev?]

Getzler

OPEN + CLOSED MODULAR OPERADS

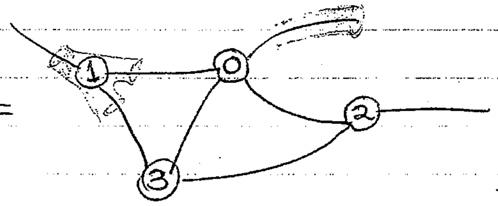
"Physics" - CFT, Strings : operads, but not really categories.

Modular operads

... and their modification to → Fukaya, Floer cohomology etc.

Symm. monoidal cat. $\mathcal{C}$; $P((g,n))$ sequence of objects,
g genus
n # legs of a vertex

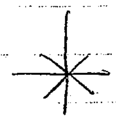
"mod. operad" + compositions along the graph (replacing rooted trees)
(not necessarily planar)

eg: $G =$ 

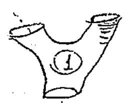
overall genus $= \sum_{\substack{v \\ \text{vertices}}} g(v) + b_1(G)$ first Betti #
$= 8$

giving
legs at 3
$$P((1,3)) \otimes P((0,4)) \otimes P((3,3)) \otimes P((2,3)) \longrightarrow P((8,3))$$

eg: $*$-algebras

eg: $P((g,n)) = \mathcal{M}_{g,n}$ Segal CFT axiomatization 

$= \{$conformal structures & parameterization of boundary$\}$

[think of thickening the lines 

$*$ Want to throw away graphs now, and use surfaces :
Replace the graphs by oriented surfaces with boundaries, together with a configuration of curves in it (disjoint union of closed curves)
These surfaces not necessarily connected.

$\Rightarrow$ a symmetric monoidal category with these as morphisms, $(\Sigma, \Sigma_+, \mathcal{L})$
and oriented surfaces as objects
$\cong$
$\mathcal{G}$ (up to

Once again: morphisms are Σ_-, Σ_+ together with a configuration of curves in Σ_+, together with an isomorphism between Σ_- and the surface obtained from Σ_+ by cutting along the curves.

i.e. surfaces Σ_- are "more" disconnected ; ~~morphisms~~ objects oriented by ~~dividing~~ $\Sigma_+ \setminus \Sigma_+^{fix}$ into 2 subsets exchanged by anti-inv.

$\otimes$ in $\mathcal{G}_0$ is disjoint union

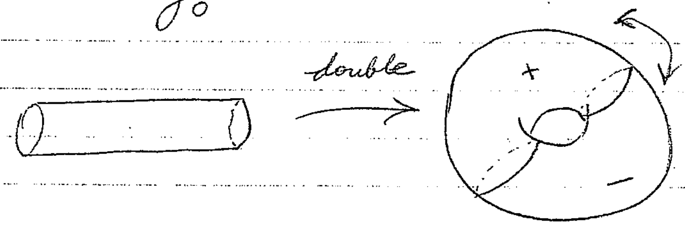
A modular operad is a symmetric monoidal functor from $\mathcal{G}_0$ to $\mathcal{C}$

Open CFT

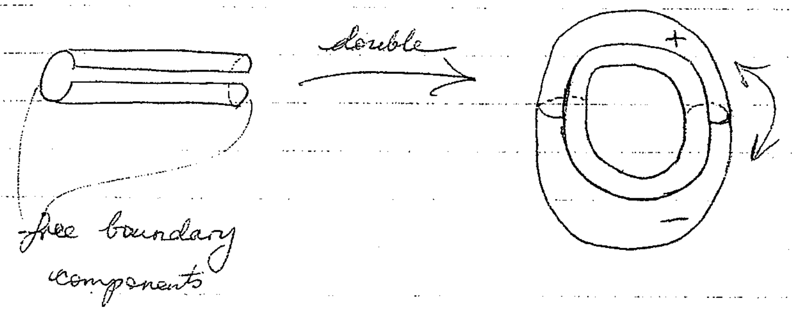
Will modify $\mathcal{G}$ by allowing surfaces with anti-involution [Moore-Segal]
An anti-involution on Σ is an orientation reversing involution.
These preserve the curve configuration (but not piece by piece).
The boundary "labels" are then sets with involution.

Call this category $\mathcal{G}_u$ (u is for "unoriented")

* Focus on $\mathcal{G}_0$



[Liu]



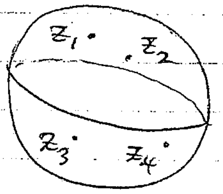
fixed pts of anti-involution give free boundary

Examples : Deligne-Mumford compactifications

$\overline{\mathcal{M}}_{g,n}$ n labelled points on genus g Riemann surfaces → Moduli space of conformal structures compactified

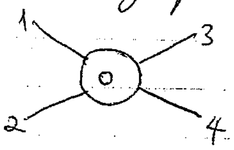
$\overline{\mathcal{M}}_{g,n}$ gives an operad

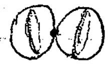
eg: $g=0,\ n=4$



$\xrightarrow{\text{cross ratio}} \mathbb{C}\setminus\{0,1\}$

graph



meridean circle gives  pinched into circle

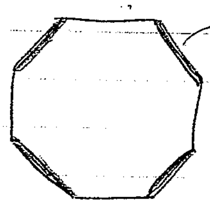
$\overline{\mathcal{M}}_{0,4} \cong \mathbb{CP}^1 \qquad \mathcal{M}_{0,4} \cong \mathbb{C}\setminus\{0,1\}$

In general, $\dim \overline{\mathcal{M}}_{g,n} = 3g-3+n$ [Dijkgraaf]

<u>Classification</u> - what surfaces can arise? For objects in $\mathcal{G}$

Let h = # holes (boundary meeting fixed pt set), n = # boundary components (circles), m = " " " (arcs)

eg:



glue two of these along here (4 boundary)

$g=0,\ h=1,\ n=0,\ m=4$

$\mathcal{M}_{g,n;h,m}$ has dimension $6g-6+3h+2n+m$

($\sim$ classification of open/closed Frobenius algebras)

* Can compactify using manifolds with corners, or Deligne-Mumford

$\overline{\mathcal{M}}_{g,n;h,m}$ compact orbifolds with corners

eg: $g=0,\ n=0,\ h=1,\ m\geq 3$; $\dim = m-3$ (see example above)

$\Longrightarrow$ $\boxed{\text{associahedron}}$

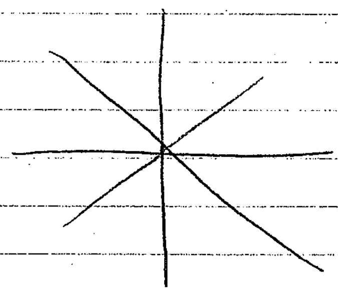
eg: $g=0$, $n=1$, $h=1$, $m \geq 1$; $\dim = m-1$

$\Longrightarrow$ $\boxed{\text{cyclohedron}}$

Theorem

The 2-skeleton of each component of $\overline{\mathcal{M}}_{g,n;h,m}$ is connected.

$\approx$ TCFT classification (open/closed)



* D-branes are objects of the additive category = the open sector

[Costello]

... $g=0$ truncation, an operad,