Orientals fail to be weakly equivalent to the point. As a result, the corresponding codescent objects fail to be weakly equivalent to the space they are covering – unless we restrict to cohomology with coefficients in ω-groupoids.

I'd like to fix that. Both because we see that differential cohomology with non-groupoid coefficients is important in QFT, and also because of the feeling that a good definition of descent should work in both cases without modification.

I'll present now what looks like an elegant solution to me. I am particularly fond of this idea because it gives an even more direct connection between cocycles and those connections on vertical paths which I claim (in "∞-Lie integration of L_∞-algebraic cocycles" and then in the examples section) to yield useful classes of examples of cocycles.

I describe an ω-category $P_\omega(S)$ associated to any set S which behaves like an ω-category of paths in the discrete *contractible* space with the elements of the set as its point. I then claim that

- each $P_\omega(S)$ is weakly equivalent to the point;
- for $[n] = \{0, 1, 2, \cdots, n\}$ the $P_\omega([n])$ arrange themselves into a cosimplicial ω-category $P_\omega : \Delta \to \omega\mathrm{Cat}$
- for Gr a 2-groupoid, descent with coefficients in Gr defined using the orientals $O(\Delta^{(-)})$ coincides with that using $P_\omega([-])$.
- Descent over a point using P_ω with coefficients in $\mathbf{B}C$ for C a strict monoidal category is a Frobenius monoid object in C.

But please check. I might be wrong.

0.1 Descent

Definition 0.1 (path category of a set) *For S a set define the* path category $P_1(S)$ *to have S as its set of objects, have finite non-empty sequences $[s_1, s_2, \cdots, s_k]$ of elements $s_i \in S$, $k \in \mathbb{N}$, $k > 0$ as morphisms, with source and target maps picking out the first and last element of a sequence, respectively. The composition of two sequences is obtained by first removing the last element of the first sequence and then concatenating the result with the second.*

Example. With $S = \{0, 1, 2\}$ the path category $P_1(S)$ has morphisms such as $1 \xrightarrow{[1,2]} 2$ and $2 \xrightarrow{[2,1,0]} 0$ whose composite is $[2, 1, 0] \circ [1, 2] = [1, 2, 1, 0]$.

Definition 0.2 (path ω-category of a set) *For S a set, define path n-categories recursively as follows. Suppose $P_n(S)$ has been defined. Then define $P_{n+1}(S)$ by setting*

$$\mathrm{Mor}_{n+1}(P_{n+1}(S)) \subset Mor_1(P_1(Mor_n(P_n(S))))/\sim$$

with the subset and the quotient defined as follows. The set $Mor_1(P_1(Mor_n(P_n(S))))$ naturally inherits the composition along n-cells from the composition in $P_1(Mor_n(P_n(S)))$. We restrict it to those paths of n-cells whose source and target coincide. On that subset we have whiskering operations along $(k < n)$-cells by identity cells using the composition in $P_n(S)$. We define composition along $(k < n)$-cells by following a suitable whiskering with composition along an n-cell. The quotient is the minimal quotient that makes this independent of the choice of whiskering.

[** clearly I need to eventually say this more formally – the following example should make clear what is going on**]

Example. A typical 2-morphism in $P_2(S)$ is of the form

$$\begin{tikzcd}
 & & s_{01} \arrow[dr] \arrow[d, Rightarrow, shift left=-6] & \\
s_1 \arrow[urr] \arrow[r] \arrow[dr] & s_{12} \arrow[r] & s_{13} \arrow[r] \arrow[d, Rightarrow, shift left=-6] & s_4 \\
 & s_{22} \arrow[r] & s_{23} \arrow[ur] &
\end{tikzcd} .$$

Whiskering it with $s_4 \longrightarrow s_5 \longrightarrow s_6$ on the right yields

$$\begin{tikzcd}
 & & s_{01} \arrow[dr] \arrow[d, Rightarrow, shift left=-6] & & & \\
s_1 \arrow[urr] \arrow[r] \arrow[dr] & s_{12} \arrow[r] & s_{13} \arrow[r] \arrow[d, Rightarrow, shift left=-6] & s_4 \arrow[r] & s_5 \arrow[r] & s_6 \\
 & s_{22} \arrow[r] & s_{23} \arrow[ur] & & &
\end{tikzcd}$$

regarded as a path of paths $[s_0, s_{01}, s_4, s_5 s_6] \Rightarrow [s_1, s_{12}, s_{13}, s_4, s_5, s_6] \Rightarrow [s_1, s_{22}, s_{23}, s_4, s_5, s_6]$.

The composition along the 0-cell b of

$$\begin{tikzcd}
a \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, bend right=60, ""{name=D}] & b & b \arrow[r, bend left=60, ""{name=U2, below}] \arrow[r, bend right=60, ""{name=D2}] & c
\arrow[Rightarrow, from=U, to=D] \arrow[Rightarrow, from=U2, to=D2]
\end{tikzcd}$$

is defined to be composition along 1-cells of

$$\begin{tikzcd}
a \arrow[r, bend left=60, ""{name=A1, below}] \arrow[r, bend right=60, ""{name=A2}] & b \arrow[r, bend left=60] & c & & a \arrow[r, bend left=60] & b \arrow[r, bend left=60, ""{name=B1, below}] \arrow[r, bend right=60, ""{name=B2}] & c \\
 & & & = & & & \\
a \arrow[r, bend right=60] & b \arrow[r, bend left=60, ""{name=C1, below}] \arrow[r, bend right=60, ""{name=C2}] & c & & a \arrow[r, bend left=60, ""{name=D1, below}] \arrow[r, bend right=60, ""{name=D2}] & b \arrow[r, bend right=60] & c
\arrow[Rightarrow, from=A1, to=A2] \arrow[Rightarrow, from=B1, to=B2] \arrow[Rightarrow, from=C1, to=C2] \arrow[Rightarrow, from=D1, to=D2]
\end{tikzcd} ,$$

where the equality is enforced by the equivalence relation.

Definition 0.3 (path ω-category of a set) *As usual, set*

$$P_\omega(S) := \mathrm{colim}_n \, P_n(S) \, .$$

Proposition 0.4 *For all sets S the terminal morphism from $P_\omega(S)$ to the point is a weak equivalence*

$$P_\omega(S) \xrightarrow{\simeq_s} \mathrm{pt}$$

Proof. By construction, there is an k-morphism in $P_\omega(S)$ going between every two parallel $(k-1)$-morphisms. Therefore the morphism to the point is essentially k-surjective for all $k \in \mathbb{N}$. $\Box$

Notice the following way to look at the simplicial category Δ, which is particularly suggestive in the present context:

Definition 0.5 (simplicial category) *The category* Δ *is the full subcategory of* Cat *on categories* $[n]$ *freely generated from linear graphs of length* n*.*

$$[0] = \{0\}$$

$$\begin{tikzcd} {[1] = \{}\ 0 \arrow[r] & 1\ \} \end{tikzcd}$$

$$\begin{tikzcd} {[2] = \{}\ 0 \arrow[r] \arrow[rr, bend right] & 1 \arrow[r] & 2\ \} \end{tikzcd}$$

$$\vdots$$

Let $P_\omega^{\geq}([n])$ be the full sub-ω-category of $P_\omega([n])$ on those 1-morphisms along which the sequences of objects are non-decreasing.

Definition 0.6 (ω-anafunctors) *We say that a span*

$$\begin{tikzcd} & C \arrow[dl, "\simeq_w"'] \arrow[dr] & \\ A & & B \end{tikzcd}$$

*of ω-categories is an ω-*anafunctor $\begin{tikzcd} A \arrow[r, "|" marking] & B \end{tikzcd}$.

Proposition 0.7 *For all* $n \in \mathbb{N}$ *there is an ω-anafunctor*

$$\begin{tikzcd} {[n]} \arrow[r, "|" marking] & P_\omega([n]) \end{tikzcd}$$

given by

$$\begin{tikzcd} & P_\omega^{\geq}([n]) \arrow[dl, "\simeq_w"'] \arrow[dr, hook] & \\ {[n]} \arrow[rr, "|" marking] & & P_\omega([n]) \end{tikzcd} \quad ,$$

where the weak equivalence $P_\omega([n]) \xrightarrow{\simeq_w} [n]$ *sends all 1-morphisms to the unique 1-morphisms in* $[n]$ *between their source and target objects.*

Proposition 0.8 *This ana-embedding of* $[n]$ *into* $P_\omega([n])$ *uniquely induces the structure of a cosimplicial ω-category on* $P_\omega([n])$*:*

$$P_\omega : \Delta \to \omega\mathrm{Cat}\,.$$

Definition 0.9 (descent) *Given*

- *a subcategory* $C \subset \mathsf{Spaces}$*;*
- *an ω-category valued presheaf*
  $$\mathbf{A} : \mathsf{Spaces}^{\mathrm{op}} \to \omega\mathrm{Cat}(\mathsf{Spaces})\,;$$
- *a regular epimorphism* $\pi : Y \twoheadrightarrow X$ *in* C*, so that the simplicial space* $Y^\bullet : \Delta^{\mathrm{op}} \to C$ *built from all pullbacks of* π *along itself exists*

then

$$\mathrm{Desc}(Y^\bullet, C) := [\Delta, \omega\mathrm{Cat}(\mathsf{Spaces})](P_\omega, \mathbf{A}(Y^\bullet))$$

is the descent ω-category *within* C *of* X *relative to* Y *with coefficients in* $\mathbf{A}$*. Here*

$$\mathbf{A}(Y^\bullet) : \Delta \xrightarrow{(Y^\bullet)^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\mathbf{A}} \omega\mathrm{Cat}(\mathsf{Spaces})\ .$$

Proposition 0.10 (equivalence to Street's definition) *For* $\mathbf{A} : \mathsf{Spaces}^{\mathrm{op}} \to 2\mathrm{Grpd}(\mathsf{Spaces})$ *this definition of descent coincides with that by Street.*

Proof. Using the presence of inverses and the Frobenius property on Street's descent data implied by them, Street's descent objects map into the descent objects just defined.

Conversely, we notice that by the lack of nontrivial 3-morphisms in the coefficient object, all paths of paths in $P_\omega([n])$ have to be labeled by *equal* 2-morphism. Since all paths of paths are generated from those that add or delete a vertex, all of them are already specified by the triangles appearing in Street's definition. That all pasting composites built from these triangles are in fact equal follows as in [**?**] from the tetrahedron law in the presence of inverses. ☐

See figure 1.

If the coefficient object does not take values in ωgroupoids, then the notion of descent in definition 0.9 is strictly more strict than that by Street.

$$\begin{tikzcd}
0 \arrow[r, bend left=60, "\mathrm{Id}", ""'{name=U}] \arrow[r, ""{name=D}] & 1
\arrow[Rightarrow, from=U, to=D]
\end{tikzcd}$$

unit

in $P_\omega([1])$

$$\begin{tikzcd}
0 \arrow[r, bend left=60, "\mathrm{Id}", ""'{name=U}] \arrow[r, ""{name=D}] & 1
\arrow[Rightarrow, from=D, to=U]
\end{tikzcd}$$

co-unit

$$\begin{tikzcd}
& 1 \arrow[dr] \arrow[d, Rightarrow] & \\
0 \arrow[ur] \arrow[rr] & {} & 2
\end{tikzcd}$$

product

in $P_\omega([2])$

$$\begin{tikzcd}
& 1 \arrow[dr] & \\
0 \arrow[ur] \arrow[rr] & {} \arrow[u, Rightarrow] & 2
\end{tikzcd}$$

co-product

$$\begin{tikzcd}
& 0 \arrow[dr] \arrow[d, Rightarrow] & & & & \\
0 \arrow[ur, ""{name=L}] \arrow[rr] \arrow[ur, bend left=90, "\mathrm{Id}", ""'{name=C}] & {} & 1 \arrow[r, phantom, "\longrightarrow"] & {} & 0 \arrow[r] & 1
\arrow[Rightarrow, from=C, to=L]
\end{tikzcd}$$

left unit law

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr, phantom, "\Swarrow" near start, "\Swarrow" near end] & 2 \arrow[d] & {} & 1 \arrow[r] \arrow[dr] & 2 \arrow[d] \\
0 \arrow[u] \arrow[ur] \arrow[r] & 3 \arrow[r, phantom, "\longrightarrow"] & {} & 0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\Searrow" near start, "\Searrow" near end] & 3
\end{tikzcd}$$

associativity

in $P_\omega([3])$

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr, phantom, "\Nearrow" near start, "\Nearrow" near end] & 2 \arrow[d] & {} & 1 \arrow[r] \arrow[dr] & 2 \arrow[d] \\
0 \arrow[u] \arrow[ur] \arrow[r] & 3 \arrow[r, phantom, "\longrightarrow"] & {} & 0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\Nwarrow" near start, "\Nwarrow" near end] & 3
\end{tikzcd}$$

co-associativity

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr, phantom, "\Swarrow" near start, "\Swarrow" near end] & 2 & {} & 1 \arrow[r] \arrow[dr] & 2 \\
0 \arrow[u] \arrow[ur] \arrow[r] & 3 \arrow[u] \arrow[r, phantom, "\longrightarrow"] & {} & 0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\Searrow" near start, "\Searrow" near end] & 3 \arrow[u]
\end{tikzcd}$$

Frobenius property

Figure 1: **Algebraic structure in** $P_\omega([n])$. When the coefficient object has no nontrivial morphisms above 2, paths of paths in $P_\omega([n])$ map to Frobenius algebroids (monoidoids, in general).