Homological BV-BRST methods: from QFT to Poisson reduction

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IMPA

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Than are dreamt of in your philosophy...

Namely: ghosts, anti-ghosts, ghosts for ghosts, ...

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Defining data

$$(\mathfrak{A}_G, S[\chi], \mathfrak{G})$$

- $\mathfrak{A}_G$ space of fields over space-time $\Sigma(=\mathbb{R}^4)$
- $S[\chi]$ classical action functional on fields $\chi \in \mathfrak{A}_G$
- $\mathfrak{G}$ (gauge) symmetry group acting on $\mathfrak{A}_G$ and leaving $S[\chi]$ invariant

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Examples: general Lorentz representation valued fields

- $\mathfrak{A}_G = \{\phi : \Sigma \to V\}$
  $V = \mathbb{R}, \mathbb{C}, \mathbb{R}^4, Dirac's(1/2, 1/2) - Spinors, \ldots$, finite dimentional representation space of Lorentz group.
- $\mathfrak{G}$ finite or infinite dimentional group

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Gauge Theories

- $\mathfrak{A}_G = \{connections\ on\ G - Principal\ bundle\ P \to \Sigma\}$
  If $P \approx \Sigma \times G$ then $\mathfrak{A}_G = \Omega^1(\Sigma, g) \ni A^\mu(x)dx_\mu$
- $\mathfrak{G} \approx Maps\ \{\Sigma \to G\}$ gauge transformations (vertical automorphisms of $P \to \Sigma$)

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Matter fields in gauge theories

$$\mathfrak{A} = \mathfrak{A}_G \times \mathfrak{A}_{Matt}$$

$\psi(x) \in \mathfrak{A}_{Matt} = \{\Sigma \to V[odd]\}$ ($\mathbb{Z}_2$-grading)
entering expressions as anti-commuting (Fermionic) symbols in $\Lambda\mathfrak{A} =$ free graded commutative algebra generated by $\mathfrak{A}$

Scattering matrix elements

$$\langle p_1 p_2 ... p_k | p_A p_B \rangle = \sum_{posible\ \ intermediate\ \ processes} (Feynman\ diagrams)$$

Probability amplitude for the scattering event (quantum amplitudes):

- $|p_A p_B >$ assymptotically free state of 2 "in" particles
- $|p_1 p_2 ... p_k >$ assymptotically free state of k "out" particles

Quantum Field Theory:

rules for obtaining the q-amplitudes from the defining data $(\mathfrak{A}_G, S[\chi], \mathfrak{G})$

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Aspects of Quantum Field Theory

- * Symbolic expressions involving $\int D\phi exp(iS[\phi])$
- perturvative series on Feynman diagrams
- explicit numerical calculations involving integrals

Vacuum-Vacuum q-amplitudes

To get a taste of the symbolic algebra involved...

$$\langle TO[\chi]\rangle = \lim_{t \longrightarrow \infty(1-i\epsilon)} \frac{1}{Z_S} \int_{\mathfrak{A}_G} (\Pi d\chi) \; O[\chi] \, exp\,(iS[\chi])$$

$$Z_S = \int_{\mathfrak{A}_G} (\Pi d\chi) \; exp\,(iS[\chi])$$

$O[\chi]$ is an operator having a polinomial expression in terms of the fields $\chi \in \mathfrak{A}_G$

$S[\chi] = \int_{-t}^{t} \int_{\mathbb{R}^3} \mathcal{L}[\chi]$, being $\mathcal{L}[\chi]$ the lagrangian density over $\Sigma \simeq \mathbb{R}^4$ of the field theory.

Factorization problem

Physical gauge-invariance principle

Physical magnitudes shall depend only on $[\chi] \in \mathfrak{A}_G/\mathfrak{G}$ and not on χ itself.

BUT we have (for example)

$$Z_S = \int_{\mathfrak{A}_G} (\Pi d\chi) \; exp\,(iS[\chi])$$

if we could $\mathfrak{A}_G \simeq \mathfrak{G} \times \mathfrak{A}_G/\mathfrak{G}$, then problem solved:

$$Z_S \simeq Vol(\mathfrak{G}) \times \int_{\mathfrak{A}_G/G} (\Pi d([\chi])) \; exp\,(iS[\chi])$$

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Factorization problem II: gauge fixing

How to factorize $\mathfrak{A}_G \simeq \mathfrak{G} \times \mathfrak{A}_G/\mathfrak{G}$?!?

Gauge fixing

restrict to a gauge-fixed surface $\{g^a(\chi) = 0\} \subset \mathfrak{A}_G$ transversal to the $\mathfrak{G}$-orbits, $a = 1, ..., dim(G)$

Example: Lorentz gauge fixing in QED

$G = U(1)$, $g^1(A^\mu) = \partial_\mu A^\mu = 0$ "covariant gauge"

Physical gauge-fixing independence principle

Physical magnitudes shall not depend on the choice of gauge fixing g^a's

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By means of Fadeev-Popov's trick, one can get

$$Z_S = Vol(\mathfrak{G}) \times \int_{\mathfrak{A}_G} (\Pi d\chi)\ exp\,(iS[\chi])\ \delta\,(g^a(\chi))\ det\left(\frac{\partial g^a(\chi^\alpha)}{\partial\alpha}\right)$$

Fadeev-Popov-De Witt Theorem

The rhs of the above expression is gauge fixing independent, i.e., it does not depend on the choice of g^a's

Operator q-amplitudes

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Features of the F-P expression:

- it singles out the physical contribution of $\mathfrak{A}_G/\mathfrak{G}$
- it is explicitly Lorentz invariant

BUT

we only know how to (Feynman-diagramatically) handle expressions of the form $\int D\phi exp(iS[\phi])$ ...

Using formal expressions (Fourier transform and Grassmann integration)

$$\delta\left(g^a\right) \rightsquigarrow \int (\Pi db_a)\ exp\left(i\int_\Sigma \frac{\xi}{2} b_a b_a + b_a g^a\right)$$

$$det\left(\frac{\partial g^a(\chi^\alpha)}{\partial \alpha}\right) \rightsquigarrow \int (\Pi dc_a)(\Pi d\bar{c}_b)\, exp\left(-i\int_\Sigma \bar{c}_a \left[\frac{\partial g^a(\chi^\alpha)}{\partial \alpha^b}\right] c_b\right)$$

$a = 1, ..., dim(G)$

- $b_a(x)$ are commuting scalar fields on Σ named *auxiliary fields*
- $c_a : \Sigma \to \mathbb{R}[1]$ ghosts
- $\bar{c}^a : \Sigma \to \mathbb{R}[-1]$ anti-ghosts

The extended Action over the extended field space with ghosts

Then, finally

$$Z_S \propto \int (\Pi d\chi)(\Pi db_a)(\Pi dc_a)(\Pi d\bar{c}_b)\ exp(iS_{FP}[\chi, b_a, c_a, \bar{c}_b])$$

where the extended Fadeev-Popov action functional is

$$S_{FP}[\chi, b_a, c_a, \bar{c}_b] = S[\chi] + \int_\Sigma \frac{\xi}{2} b_a b_a + b_a g^a + \bar{c}_a \left[\frac{\partial g^a(\chi^\alpha)}{\partial \alpha^b}\right] c_b$$

- The above expression has the desired form
- The extended (ghost-graded, vector) field space is $\mathfrak{F}_{FP} = \mathfrak{A}_G \times \langle c_a, \bar{c}^b, b^c \rangle$
- $S_{FP}[\chi, b_a, c_a, \bar{c}_b]$ defines a polynomial (symbolic) expression in $\Lambda\mathfrak{F}_{FP}$

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Why isn't the story over?

- Explicit gauge-invariance of the original action $S[\chi]$ was a fundamental tool for proving renormalizability
- Now, in the F-P expressions, S_{FP} is not gauge symmetric (not $\mathfrak{G}$-ivariant)... how to prove renormalizability then?

A generalized symmetry involving ghosts!

S_{FP} has another symmetry: BRST symmetry.

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Ghost grading and differential

- $\Lambda\mathfrak{F}_{FP} = \langle \chi, c, \bar{c}, b, \partial_I \chi, \partial_I c, ... \rangle$ Free commutative graded algebra
- Total ghost number *tgn* grading:
  $tgn(\chi, b) = 0, \; tgn(c) = 1 \; tgn(\bar{c}) = -1$
- $S_{FP}[\chi, b_a, c_a, \bar{c}_b]$ is a polynomial expression $\Rightarrow$ defines a $tgn = 0$ element in $\Lambda\mathfrak{F}_{FP}$
- $\exists\; s : \Lambda\mathfrak{F}_{FP} \to \Lambda\mathfrak{F}_{FP}$ of $tgn(s) = +1$ such that $(\Lambda\mathfrak{F}_{FP}, s)$ is a commutative differential graded algebra.

$$s^2 = 0$$

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Ghost grading and differential II

$$\begin{aligned} s(\chi_\mu^a) &= \partial_\mu c^a + f_{bc}^a \chi_\mu^b c^c \\ s(c^a) &= -\frac{1}{2} f_{bc}^a c^b c^c \\ s(\bar{c}_a) &= -b_a \\ s(b_a) &= 0 \end{aligned}$$

f_{bc}^a denote the structure constants of $\mathfrak{g} = Lie(G)$

s is extended as a super derivation and is called the BRST operator.

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Properties of BRST operator

Props:

- Relation to gauge transformation expression

$$\delta_\epsilon \chi^a_\mu = \theta s(\chi^a_\mu)$$

  θ parameter anti-commuting with ghosts ($\mathbb{Z}_2$-module structure) with $\epsilon^a(x) = \theta c^a(x)$ infinitesimal gauge parameter

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Classical BRST cohomology

- Gauge invariance of $S[\chi] \Leftrightarrow\ s-$ zero cocycle

$$s(S[\chi]) = s(S_{FP}[\chi, b_a, c_a, \bar{c}_b]) = 0$$

- Gauge fixing choices (ghost terms in S_{FP}) $\Leftrightarrow\ s-$ zero coboundaries

$$S_{FP}[\chi, b_a, c_a, \bar{c}_b] = S[\chi] + s(\Psi[\chi, b_a, \bar{c}_b])$$

$$\Psi[b_a, c_a, \bar{c}_b] = \int_\Sigma \bar{c}_b g^b[\chi] + \frac{\xi}{2}\bar{c}_b b_a$$

$tgn(\Psi) = -1$ known as "gauge fixing fermion".

Classical observables = 0th BRST cohomology

$H^0_s(\Lambda\mathfrak{F}_{FP}) \simeq Funct(\mathfrak{A}_G/\mathfrak{G})$ are observables that can be quantized through gauge-fixing and yield the same result for any gauge-fixing choice.

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Higher $H_s^{i>0}(\Lambda\mathfrak{F}_{FP})$

without physical meaning... but (finite dim examples) geometrical meaning.

Quantization I

BRST quantization:

- Start with classical data $(\Lambda\mathfrak{F}_{FP}, s, S[\chi])$
- choose gauge-fixing fermion $\Psi[\chi, b_a, c_a, \bar{c}_b]$ of $tgn = -1$
- define q-vacuum-amplitudes $\langle TO[\chi]\rangle$ through F-P expression

F-P-dW Theorem revisited

These $\langle TO[\chi]\rangle$ are well defined regardeless the choice of Ψ
q-vacuum-amplitudes depend on $[\chi] \in \mathfrak{A}_G/\mathfrak{G}$ as desired

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Scattering matrix elements and Quantum BRST cohomology

S-Matrix elements:

$$\langle p_1 p_2 ... p_k | p_A p_B \rangle = \sum_{posible\ intermediate\ processes} (Feynman\ diagrams)$$

involve k-particles states $|p_1 p_2 ... p_k >$ in a Hilbert space $\mathfrak{H}$ on which the quantized fields $\check{\chi}$ act.

Vacuum state

$|\Omega > \in \mathfrak{H}$ is one of these states, the one corresponding to no particles at all... i.e. vacuum state.

How is gauge-fixing operation represented in particle state space $\mathfrak{H}$?

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Scattering matrix elements and Quantum BRST cohomology II

Quantum representation of BRST differential algebra $(\mathfrak{H}, \check{Q})$

$$[\check{Q}, \check{\Phi}]_{\pm} = i(s\Phi)^{\vee}$$

$$[\check{Q}, \check{Q}]_{\pm} = 2\check{Q}^2 = 0$$

$\mathfrak{H}$ is also ghost graded (ghost particles)

Quantum BRST cohomology

$H^0_Q(\mathfrak{H})$ physical quantum particle states (with no ghosts, gauge-fixing independent)

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Well behaved QFT states

When is it "well behaved"

- No-ghost theorem for $H^0_Q(\mathfrak{H})$
- compatibility with inner product in $\mathfrak{H}$ I: restricted S-matrix unitary
- compatibility with inner product in $\mathfrak{H}$ II: physical states with positive definite norm

Gauge theories

There exists a well behaved quantum representation $(\mathfrak{H}, \check{Q})$ of the classical BRST cohomology for gauge theories $(\mathfrak{A}_G, S[\chi], \mathfrak{G})$

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- compatibility with inner product in $\mathfrak{H}$ II: physical states with positive definite norm

Gauge theories

There exists a well behaved quantum representation $(\mathfrak{H}, \check{Q})$ of the classical BRST cohomology for gauge theories $(\mathfrak{A}_G, S[\chi], \mathfrak{G})$

Final remarks on BRST quantization

- BRST symmetry is a tool for proving Renormalizability
- $[\check{Q}, -]$ suggests looking for classical inner representation $s = \{Q, -\}$
- How to get $(\Lambda\mathfrak{F}_{FP}, s, S_{FP}[\chi])$ for a general $(\mathfrak{A}_G, S[\chi], \mathfrak{G})$ without F-P trick?
- How to handle reducible symmetries? (p-form field theories)
- How to handle open symmetries? (supergravity, TFT)

Solution: BV formalism

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Solution: BV formalism

BV ingredients

- Enlargement $\mathfrak{F}_{BV} = \mathfrak{F}_{FP} \times \mathfrak{F}^{\sharp}_{FP}$, by adding an anti-field $\phi^{\sharp}_{\alpha} \in \mathfrak{F}^{\sharp}_{FP}$ for each field $\phi_{\alpha} \in \mathfrak{F}_{FP}$
for gauge theories ϕ_{β} runs over $\chi, c, \bar{c}, b$,
- ghost gradings $tgn(\phi^{\sharp}) = -tgn(\phi) - 1$.
- commutative graded algebra $\Lambda\mathfrak{F}_{BV}$ has an odd Poisson bracket (of tgn $+1$) defined on generators $\phi_{\beta}, \phi^{\sharp}_{\alpha} \in \mathfrak{F}_{BV}$ by

$$\begin{aligned} \{\phi_{\beta}, \phi^{\sharp}_{\alpha}\} &= \delta_{\beta\alpha} \\ \{\phi_{\beta}, \phi_{\alpha}\} &= \{\phi^{\sharp}_{\beta}, \phi^{\sharp}_{\alpha}\} = 0 \end{aligned}$$

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BV action

- $S_{BV}[\phi_\beta, \phi^\sharp_\alpha]$ with $tgn = 0$, satisfying the classical Master equation

$$\{S_{BV}, S_{BV}\} = 0$$

- $(\Lambda\mathfrak{F}_{BV}, D = \{S_{BV}, -\})$ is a C-DGA $(tgn(D) = +1)$

0th BV cohomology

"cotangent classical observables" $\approx Fun(T^*[1](\mathfrak{A}_G/\mathfrak{G}))$

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$$S_{BV}[\phi_\beta, \phi^\sharp_\alpha] = S_{min}[\chi, c, \chi^\sharp, c^\sharp] - b^A \bar{c}^\sharp_A$$

$$S_{min} = S[\chi] + c^A f^r_A[\chi]\chi^\sharp_r + \frac{1}{2}c^A c^B f^C_{AB}[\chi]c^\sharp_C + \frac{1}{2}c^A c^B f^{rs}_{AB}[\chi]\chi^\sharp_r\chi^\sharp_s + higher\ te$$

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Master equation for S_{BV} imply compatibility conditions for structure functions $f^r_A[\chi], f^C_{AB}[\chi], f^{rs}_{AB}[\chi], \ldots$

gauge fixing within BV formalism

- set anti-fields

$$\phi^\sharp_\alpha = \frac{\partial \Psi[\phi]}{\partial \phi^\alpha}$$

Cannonical transformation

$(\phi^\alpha, \phi^\sharp_\alpha)$ to $(\phi^\alpha, \tilde{\phi}^\sharp_\alpha = \phi^\sharp_\alpha - \frac{\partial \Psi[\phi]}{\partial \phi^\alpha})$ s.t. $\tilde{\phi}^\sharp_\alpha = 0$

- (generalized) BRST operator on $\Lambda \mathfrak{F}_{FP} = \langle \phi^\alpha \rangle \subset \Lambda \mathfrak{F}_{BV}$

$$s(\phi^\alpha) = -\{S_{BV}[\phi^\alpha, \phi^\sharp_\alpha], \phi^\alpha\}|_{\phi^\sharp_\alpha = \frac{\partial \Psi[\phi]}{\partial \phi^\alpha}}$$

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gauge fixing within BV formalism II

- def the Gauge-fixed action

$$S^{\Psi}_{FP}[\phi^\alpha] = S_{BV}\left[\phi^\alpha, \phi^\sharp_\alpha = \frac{\partial \Psi[\phi]}{\partial \phi^\alpha}\right]$$

It is easy to check that $s^2 = 0$ and $s(S^{\Psi}_{FP}) = 0$

- for $\mathfrak{F}_{BV} = \mathfrak{F}_{FP} \times \mathfrak{F}^\sharp_{FP}$ coming from gauge theory $(\mathfrak{A}_G, S[\chi], \mathfrak{G})$, setting $S_{BV}\left[\phi^\alpha, \phi^\sharp_\alpha\right] = S[\chi] + s(\phi^\alpha)\phi^\sharp_\alpha$, then

$$S^{\Psi}_{FP}[\phi^\alpha] = S[\chi] + s(\Psi[\phi])$$

moreover, for closed transformation algebras, s coincides with the BRST operator, yielding the early BRST $(\mathfrak{F}_{FP}, s, S^{\Psi}_{FP})$ construction. $H^0_s(\Lambda\mathfrak{F}_{FP})$ gives the classical observables.

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gauge-fixing independence and Quantum master equation

- vacuum-vacuum amplitud $Z_{S^{\Psi}_{FP}}$ is gauge-fixing (Ψ) independent, if quantum master equation is full-filled

$$\{S_{BV}, S_{BV}\} - 2i\hbar\Delta S_{BV} = 0 \;\; at \;\; \phi^{\sharp}_{\alpha} = \frac{\partial\Psi[\phi]}{\partial\phi^{\alpha}}$$

where

$$\Delta S_{BV} = \frac{\partial_R}{\partial\phi^{\sharp}_{\alpha}}\frac{\partial_L}{\partial\phi^{\alpha}}S_{BV}$$

general quantum amplitudes for operators

$\langle O[\phi^{\alpha}]\rangle$ is gauge-fixing $\Psi-$independent, when S_{BV} satisfyies the QME and $O[\phi^{\alpha}]$ is $s-$invariant:

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Final remarks on BV formalism

- general framework for (open, reducible) symmetries $\mathfrak{G} = \mathit{Diff}(\Sigma), \mathit{End}(A \to \Sigma), \ldots$
- More powerfull tool for renormalizability of gauge theories (Zinn-Justin) (for sums of diagrams)
- treatment of anomalies (symmetry loss after quantization)

We have learned...

the moral

That ghosts exist! and are usefull...

Thank you, see you next monday.