References

[A] M.F. Atiyah, 'Topological quantum field theories', *Publ. Math. Inst. Hautes Etudes Sci., Paris* **68** (1989) p.175–186.

[F] D.S. Freed, 'Higher Algebraic Structures and Quantization', *Preprint* (December 1992).

[FQ] D.S. Freed, F. Quinn, 'Chern-Simons theory with finite gauge group', *Commun. Math. Phys.* (to appear).

[KV] M.M. Kapranov, V.A. Voevodsky, '2-categories and Zamolodchikov's tetrahedra equations', *Preprint* (1992).

[KR] A.N. Kirillov, N.Yu. Reshetikhin, 'Representations of the algebra $U_q(\mathfrak{sl}(2))$, q-orthogonal polynomials and invariants of links', *Infinite dimensional Lie algebras and groups* World Scientific (1988) p.285–342.

[L 1] R.J. Lawrence, 'Extended topological field theories from an algebraic perspective', *In preparation*.

[L 2] R.J. Lawrence, 'Algebras and triangle relations', *Harvard preprint* (1991).

[L 3] R.J. Lawrence, 'On algebras and triangle relations', *Proc. 2nd. Int. Conf. on Topological and Geometric Methods in Field Theory* World Scientific (1992) p.429–447.

[MaSc] Yu.I. Manin, V.V. Schechtman, 'Arrangements of Hyperplanes, Higher Braid Groups and Higher Bruhat Orders', *Adv. Studies in Pure Maths.* **17** (1989) p.289–308.

[M] S.V. Matveev, 'Transformations of special spines and the Zeeman conjecture', *Math. USSR Izvestia* **31** (1988) p.423–434.

[MoSe] G. Moore, N. Seiberg, 'Lectures on RCFT', *Physics, Geometry and Topology* Plenum Press (1990) p.263–361.

[P] U. Pachner, 'Konstruktion methoden und das kombinatorische Homömorphiseproblem für Triangulationen kompakter semilinearen Mannig faltigkeiten', *Abh. Math. Sem. Univ. Hamburg* **57** (1987) p.69–86.

[Q] F. Quinn, 'Lectures on axiomatic topological field theory', *preprint* (August 1992).

[TV] V.G. Turaev, O.Y. Viro, 'State sum invariant of 3-manifolds and quantum 6j-symbols', *Topology* **31** (1992) p.865–902.

[W] E. Witten, 'Quantum field theory and the Jones polynomial', *Commun. Math. Phys.* **121** (1989) p.351–399.

A GEOMETRIC CONSTRUCTION OF THE FIRST PONTRYAGIN CLASS

JEAN-LUC BRYLINSKI
Math. Dept., The Pennsylvania State University
University Park, PA. 16802, USA
and
DENNIS McLAUGHLIN
Math. Dept., Princeton University
Princeton, N. J. 08544, USA

A new obstruction theory for principal bundles is developed, which leads to an integer-valued formula for the first Pontryagin class of a bundle with compact structure group. A geometric representative of this class is given, in terms of a glueing problem for gerbes. The upshot is that there is a natural sheaf of bicategories over the base manifold. Analogous constructions are discussed for finite groups, leading to a proof of the reciprocity theorem of Segal and Witten.

Introduction

In the past few years, there has been a renewal of interest in degree 4 characteristic classes. The inspiration has come mainly from physics.

For a compact simple Lie group G, the inverse transgression in the universal bundle $H^4(BG) \to H^3(G)$ gives the natural correspondence between 3-dimensional Chern-Simons gauge theory and 2-dimensional Wess-Zumino-Witten theories [13].

If G is connected, there are related results concerning the transgression for the free loop space $H^4(BG) \to H^3(LBG)$. The image of this map corresponds to those central extensions of the loop group LG which have the reciprocity property [24].

Perhaps the most relevant development from our point of view has been the discovery that the first Pontryagin class p_1 is the obstruction to defining the spinor bundle on loop space [20] [22] [26]. This means that p_1 plays a role in string theory analogous to that played by the second Stiefel-Whitney class w_2 in point particle physics.

In the case of the tangent bundle of a smooth manifold, there is a well-known formula for an explicit cocycle representing w_2 [9], as well as formulas for p_1 [15] [23]. In this paper, for any principal bundle over any space, we construct geometrically a degree 4 integral Čech cocycle representing the first Pontryagin class. The basic data in this "formula for p_1" are the transition cocycles themselves and tetrahedra in the group which have them as vertices. Our approach to p_1 may be viewed as a generalization of the theory of line bundles with connections, due to A. Weil [25] and Kostant [21].

One proof of the formula involves the lifting of the $\mathbb{Z}$-valued Čech cocycle to a cocycle with values in a smooth version of the Deligne complex of sheaves, using a connection on the principal bundle. However, this is not the way the formula was found. We were thinking about the geometrical meaning of p_1 in connection with the theory of gerbes and gauge theory, and we found a canonical geometric object corresponding to p_1, which recasts classical obstruction theory in the language of categories. We then realized that, using the holonomy of a gerbe around the boundary of a tetrahedron, we could write down an explicit formula for a representative of p_1. The geometric construction appears in fact to have a deeper meaning, as the origin of the Chern-Simons topological quantum field theory of Witten. For compact Lie groups, this is still a conjecture, presented in [8], but for a finite group G and a class in $H^3(G, \mathbb{C}^*)$, an analog of the conjecture can be established, as is explained in §3.

1. Statement of the main result

Let G be a connected compact almost simple Lie group, with $\pi_1(G) = \mathbb{Z}/N \cdot \mathbb{Z}$, a finite number. Then $H_2(G, \mathbb{Z}) = 0$ and $H_3(G, \mathbb{Z}) = \mathbb{Z}$. Suppose that $P \to M$ is a principal G-bundle over M. We define $p_1 \in H^4(M, \mathbb{Z})$ to be the class obtained by transgression of N times the generator of $H^3(G, \mathbb{Z})$.

Choose an open covering of M by contractible open sets U_i, indexed by the set I, such that all non-empty intersections of these open sets are contractible. Choose sections $s_i : U_i \to P$, and let g_{ij} be the associated transition cocycles. So g_{ij} is the continuous function from $U_i \cap U_j$ to G, characterized by $s_j = s_i \cdot g_{ij}$.

For $y \in U_i \cap U_j$, choose a path $\gamma_{ij}(y,t)$ in G, from the identity to $g_{ij}(y)$. We require that γ_{ij} be a continuous function of $(y,t) \in (U_i \cap U_j) \times [0,1]$, and a smooth function of t. For $y \in U_i \cap U_j \cap U_k$, denote by $\gamma_{ijk}(y)$ the loop given by the composition $\gamma_{ij} * g_{ij}\gamma_{jk} * \gamma_{ik}^{-1}$ (Figure 1).

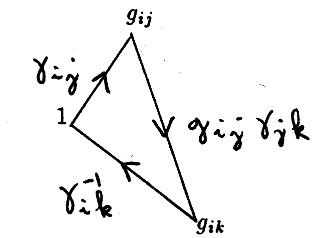

Figure 1

The loop γ_{ijk} may not be a boundary, but $N \cdot \gamma_{ijk}$ will bound a 2-simplex σ_{ijk}. Again one chooses $\sigma_{ijk}(y,x)$ to be a continuous function $(U_i \cap U_j \cap U_k) \times \Delta_2 \to G$, and a smooth function on the 2-simplex Δ_2. For $y \in U_i \cap U_j \cap U_k \cap U_l$, the linear combination $g_{ij}\sigma_{jkl} - \sigma_{ikl} + \sigma_{ijl} - \sigma_{ijk}$ is a singular cycle. We think of this as a singular 2-cycle on $Map(U_i \cap U_j \cap U_k \cap U_l, G)$; it is then the boundary of a 3-chain $T_{ijkl}(y)$, which we symbolically draw as a tetrahedron in Figure 2 (note that if $N = 1$, we may choose T_{ijkl} to consist of just one 3-simplex).

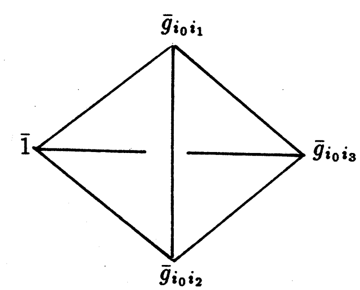

Figure 2

Finally, for $y \in U_i \cap U_j \cap U_k \cap U_l$, let V_{ijklm} be the 3-cycle

$$T_{ijkl} - T_{ijkm} + T_{ijlm} - T_{iklm} + g_{ij}T_{jklm}.$$

Theorem 1. *Let ν be the closed, bi-invariant integral 3-form on G whose cohomology class generates $H^3(G, \mathbb{Z})$. Define*

$$\beta_{ijklm}(y) = \int_{V_{ijklm}(y)} \nu.$$

Then β is a $\mathbb{Z}$-valued Čech cocycle of degree 4 which represents $-p_1$.

Remarks.

1. The formula is combinatorial in nature, since if M is the underlying topological space of a simplicial complex, we may take a covering by the stars of the vertices. A Čech cocycle for this cover will then give a simplicial cocycle. It would be interesting to relate our approach with the combinatorial formula for p_1 of the tangent bundle of a smooth manifold, due to Gabrielov, I. Gel'fand, Losik, and to MacPherson [15] [23]. The formula for p_1 may be generalized to all Chern classes and Pontryagin classes. We also point out other formulas for characteristic classes, due to Gel'fand and MacPherson [17] and to Goncharov [19].

2. Volumes of tetrahedra also appear in the work of Cheeger and Simons [10] on differential characters. Their tetrahedra are geodesic tetrahedra in the fibers of a sphere bundle associated to a flat vector bundle.

2. Holonomy of gerbes and the proof of the formula

There is available a direct proof of the formula [6], which however does not shed light on its geometric significance. We will focus here on obstruction theory for a principal bundle. First we recall the geometrical meaning of low degree cohomology with $\mathbb{Z}$ coefficients.

For X a paracompact space, the boundary map in the exponential exact sequence induces an isomorphism: $H^p(X,\underline{\mathbb{T}})\tilde{\to}H^{p+1}(X,\mathbb{Z})$. It is well-known that, for X a smooth manifold, the sheaf cohomology group $H^1(X,\underline{\mathbb{T}})$ classifies line bundles over X. There is a similar interpretation of $H^2(X,\underline{\mathbb{T}})$ as equivalence classes of gerbes bound by the sheaf $\underline{\mathbb{T}}$ [18] [4] [5]. Such a gerbe $\mathcal{C}$ is a sheaf of categories over X, in the sense that for U open in X, there is given a category $\mathcal{C}_U$, and for $V \subset U$, there is a "restriction functor" $\mathcal{C}_U \to \mathcal{C}_V$. This functor will be denoted by $P \mapsto P_{/V}$. Each category $\mathcal{C}_U$ is to be a groupoid, which means that every morphism is invertible. It is assumed that two objects P_1 and P_2 of $\mathcal{C}_U$ are locally isomorphic, in the sense that any $x \in U$ has an open neighborhood V such that the restrictions of P_1 and P_2 to V are isomorphic. Also X is covered by open sets over which the category is not empty. However, it may (and will often) happen that the global category $\mathcal{C}_X$ is empty.

A glueing axiom must be satisfied, which allows one to obtain an object of $\mathcal{C}_X$ from an open cover (U_i), objects P_i of $\mathcal{C}_{U_i}$, and isomorphisms $(P_i)_{/U_i\cap U_j}\tilde{\to}(P_j)_{/U_i\cap U_j}$

which satisfy the natural cocycle condition. With all these conditions, one has a gerbe over X. We say that the gerbe $\mathcal{C}$ is bound by $\underline{\mathbb{T}}$ if the sheaf of automorphisms of any local object is isomorphic to $\underline{\mathbb{T}}$. A typical example is: $\mathcal{C}_U$ is the category of hermitian line bundles over U, a morphism $L_1 \to L_2$ is an isomorphism of line bundles, the restriction functors are the obvious ones. This example is indeed typical in the sense that any gerbe bound by $\underline{\mathbb{T}}$ is locally equivalent to this one.

We now turn to the question of differentiable structures. For line bundles, there is the Weil-Kostant theory of line bundles with connection and their curvature [25] [21], and the relation with so-called "smooth Deligne cohomology", due to Deligne (see [12] for the holomorphic case). We recall this briefly. The smooth Deligne complex of sheaves $\mathbb{Z}(p)^{\infty}_{\mathcal{D}}$ may be described as the complex of sheaves

$$\underline{\mathbb{T}} \xrightarrow{d\log} \sqrt{-1}\cdot \underline{A}^1_X \to \cdots \to \sqrt{-1}\cdot \underline{A}^{p-1}_X,$$

where $\underline{\mathbb{T}}$ is placed in degree 1. Given a hermitian line bundle with hermitian connection (L,∇), one derives a class in the hypercohomology group $H^2(X,\mathbb{Z}(2)^{\infty}_{\mathcal{D}})$, which is the total cohomology of the double complex of Čech cochains with coefficients in $\mathbb{Z}(2)^{\infty}_{\mathcal{D}}$. Let (U_i) be a nice open covering of X, and let s_i be a non-vanishing section of L over U_i. Set $g_{ij} = \frac{s_i}{s_j}$ and $\alpha_i = \frac{\nabla(s_i)}{s_i}$. Then $(g_{ij}, -\alpha_i)$ is a Čech cocycle with coefficients in $\mathbb{Z}(2)^{\infty}_{\mathcal{D}}$. In this way, Deligne identifies the group of isomorphism classes of pairs (L,∇) with $H^2(X,\mathbb{Z}(2)^{\infty}_{\mathcal{D}})$.

As regards gerbes, two levels of differentiable structures on gerbes bound by $\underline{\mathbb{T}}$ were introduced in [4], where they were baptized "connective structure" and "curving". A "connective structure" associates to each object P of $\mathcal{C}_U$ a sheaf $Co\ (P)$ of "connections" on P, which is a principal homogeneous space under $\sqrt{-1}\cdot\underline{A}^1_X$. A "curving" associates to each connection $\nabla \in Co\ (P)$ its curvature $K(\nabla)$, which is an honest purely imaginary 2-form on X. The conditions satisfied by these notions are explained in [4]. Note in particular that $K(\nabla+\alpha) = K(\nabla)+d\alpha$, for $\alpha \in \sqrt{-1}\cdot\underline{A}^1_X$ a 1-form on X.

The equivalence classes of gerbes on X with connective structure and curving are classified by the smooth Deligne cohomology group $H^2(X,\underline{\mathbb{T}} \xrightarrow{d\log} \sqrt{-1}\cdot\underline{A}^1_X \to \sqrt{-1}\cdot\underline{A}^2_X)$, i.e. the group $H^3(X,\mathbb{Z}(3)^{\infty}_{\mathcal{D}})$. The 3-curvature Ω of such an object is the 3-form which is equal to $dK(\nabla)$, for ∇ a connection on an object of $\mathcal{C}$ defined locally. It is deduced from a morphism of complexes of sheaves from $\mathbb{Z}(3)^{\infty}_{\mathcal{D}}$ to $\sqrt{-1}\cdot\underline{A}^3_X$, put in degree 3.

Now we turn to a compact Lie group G, which we assume to be simply-connected. A concrete example of a gerbe on G can be found from the path-loop fibration $PG \to G$ (see also [5] [7]). This is the universal ΩG-bundle. Locally on G, it is possible to lift the structure group to $\widetilde{\Omega G}$, an extension of ΩG by the circle $\mathbb{T}$. These local liftings are the objects of a category, in which the morphisms are constrained to induce the identity on PG. This sheaf of categories is a gerbe bound by the sheaf $\underline{\mathbb{T}}$. The corresponding obstruction in $H^2(G,\underline{\mathbb{T}}) \tilde{\to} H^3(G,\mathbb{Z})$ is the obstruction to finding the global lift. For G simply-connected, this obstruction class is a generator of $H^3(G,\mathbb{Z})$.

As G is 2-connected, one easily checks that $H^3(G,\mathbb{Z}(3)^\infty_\mathcal{D})$ is isomorphic to the group $\Lambda^3(G)$ of closed 3-forms on G with integral periods. Now consider a G-bundle $P \to M$ as in §2. To prove Theorem 1, we take the gerbe Q on G with 3-curvature $\Omega := 2\pi \cdot \sqrt{-1} \cdot \nu$, and we try to glue it all over M. To analyze the glueing problem, one may look at gerbes on P, bound by $\underline{\mathbb{T}}$, which are equipped with a connective structure "along the fibers of π", and a relative curving which assigns a relative 2-form to each connection. Such gerbes are classified by the cohomology group $H^2(P, \underline{\mathbb{T}} \stackrel{d\,log}{\to} \sqrt{-1} \cdot \underline{A}^1_{P/M} \to \sqrt{-1} \cdot \underline{A}^2_{P/M})$, where $\underline{A}^j_{P/M}$ is the sheaf of germs of real relative j-forms.

One has an exact sequence of complexes of sheaves on P:

$$0 \to \pi^{-1}\underline{\mathbb{T}}_M \to \left(\underline{\mathbb{T}} \to \sqrt{-1} \cdot \underline{A}^1_{P/M} \to \sqrt{-1} \cdot \underline{A}^2_{P/M}\right) \to \sqrt{-1} \cdot \underline{A}^3_{P/M,cl}[-2] \to 0,$$

where $\underline{A}^3_{P/M,cl}$ denotes the sheaf of germs of closed relative 3-forms, and for $K^\bullet$ a complex, $K^\bullet[j]$ denotes the complex obtained by translating $K^\bullet$ by j steps to the left. The obstruction κ to constructing a class in $H^2(P, \underline{\mathbb{T}} \to \sqrt{-1} \cdot \underline{A}^1_{P/M} \to \sqrt{-1} \cdot \underline{A}^2_{P/M})$ mapping to $2\pi\sqrt{-1} \cdot \nu$ belongs to $H^3(M,\underline{\mathbb{T}}) \subset H^3(P, \pi^{-1}\,\underline{\mathbb{T}}_M)$, and it maps to p_1 under the exponential isomorphism $H^3(M,\underline{\mathbb{T}}) \tilde{\to} H^4(M,\mathbb{Z})$.

We can explicitly write down a cocycle representing κ. Over the open set U_i, we have: $\pi^{-1}(U_i) \tilde{\to} U_i \times G$, so from Q one obtains by pull-back a gerbe $\mathcal{C}_i$ on $U_i \times G$, with relative connective structure and curving. Over $U_i \cap U_j$, we may find an equivalence between the restrictions of the gerbes $\mathcal{C}_i$ and $\mathcal{C}_j$. In fact, such an equivalence is naturally obtained from the choice of a path $\gamma_{ij}(y)$ from 1 to $g_{ij}(y)$, which depends smoothly on $y \in U_i \cap U_j$, as this path gives a path in the group of diffeomorphisms of $(U_i \cap U_j) \times G$, from Id to the diffeomorphism $(y,g) \mapsto (y, g_{ij}(y) \cdot g)$, and one has a notion of "parallel transport" for a gerbe Q along such a path of diffeomorphisms (see [5, Chapter 6]).

Over $U_i \cap U_j \cap U_k$, the composition of three equivalences gives rise to an equivalence of $\mathcal{C}_i$ with itself, hence to a hermitian line bundle $\mathcal{L}_{ijk}$. The choice of the two-simplex σ_{ijk} with boundary γ_{ijk} produces a section $u_{ijk}(y)$ of norm 1 of this line bundle. One verifies from general principles that over $U_i \cap U_j \cap U_k \cap U_l$, the line bundle $\mathcal{L}_{jkl} \otimes \mathcal{L}^{\otimes -1}_{ikl} \otimes \mathcal{L}_{ijl} \otimes \mathcal{L}^{\otimes -1}_{ijk}$ is canonically trivialized. Hence $\kappa_{ijkl} := \frac{u_{jkl} \cdot u_{ijl}}{u_{ikl} \cdot u_{ijk}}$ is a smooth function from $U_i \cap U_j \cap U_k \cap U_l$ to $\mathbb{T}$, which gives a $\underline{\mathbb{T}}$-valued Čech cocycle.

To compute this function κ_{ijkl}, one interprets it as the inverse of the <u>holonomy</u> of the gerbe Q on G around the boundary of the tetrahedron T_{ijkl}. This holonomy $H(\Sigma)$ is defined for any closed oriented surface Σ mapping to G; it is computed from an object P of the restriction of Q to Σ, with connection ∇ and curvature K. Then we have $H(\Sigma) = exp(-\iint_\Sigma K) \in \mathbb{T}$; this is easily seen to be independent of the choice of (P, ∇).

In case $\Sigma = \partial T$, an application of Stokes' theorem shows that $H(\Sigma)$ equals $exp(-\int_T \Omega)$. Hence $\kappa_{ijkl}(y) = exp(2\pi\sqrt{-1} \cdot \int_{T_{ijkl}(y)} \nu)$, which proves Theorem 1.

3. Conclusion

Recently, Breen [3] has constructed the geometric objects classified by $H^3(X,\underline{\mathbb{T}})$. He calls them 2-gerbes bound by the sheaf $\underline{\mathbb{T}}$. Such an object is a sheaf of bicategories [2], where the 1-arrows between two objects form a gerbe, and the 2-arrows between given 1-arrows form a $\underline{\mathbb{T}}$-torsor. We have actually encountered this structure in the above discussion of obstruction theory for gerbes, and we record it in the following

Theorem 3. *Assume $\pi_1(G) = 0$. Let ν be a closed left-invariant 3-form on G, with integral periods, and $\pi : P \to M$ a principal G-bundle. The bicategory over U, whose objects are gerbes bound by $\underline{\mathbb{T}}$ over $\pi^{-1}(U)$, with relative connective structure and curving such that the 3-curvature is $\Omega = 2\pi\sqrt{-1} \cdot \nu$, gives a 2-gerbe on M, bound by $\underline{\mathbb{T}}$. The cohomology class in $H^3(M,\underline{\mathbb{T}})$ defined by this 2-gerbe is the transgression of Ω in the fibration $P \to M$.*

This 2-gerbe gives a geometric interpretation of the corresponding "holonomy" gerbe on LM and of the reciprocity law it satisfies (see the introduction), which involves constructing an object of a gerbe over the space of mappings $\Sigma \to M$, for Σ a surface with boundary. Details will be forthcoming.

Bicategories are also implicit in the Chern-Simons field theory of Witten [27]. This is a 2+1-dimensional topological quantum field theory associated to a characteristic class $\alpha \in H^4(BG;\mathbb{Z})$, for G a compact Lie group. It has several layers of geometric structure which are completely unexpected from the viewpoint of classical topology. Our observation is that all this structure falls naturally into place if one represents α by a 2-gerbe $\mathcal{B}$. We will illustrate this in the case where G is a finite group [13]. To a finite group G and a class in $H^3(G,\mathbb{C}^*)$, Dijkgraf and Witten associate a TQFT in 2+1 dimensions; this has been studied also by mathematicians [14].

Our first task is to construct the vector space $V(\Sigma)$ associated to a surface Σ without boundary. For a space X, let $\mathcal{M}_X = Map(X;BG)$ denote the "moduli space" of (necessarily flat) G-bundles on X. Beilinson has given a purely cohomological construction of a flat line bundle $\mathcal{L}$ on $\mathcal{M}_\Sigma$. The procedure is to pull back $\alpha \in H^3(BG;\mathbb{C}^*) \cong H^4(BG;\mathbb{Z})$ by the evaluation $ev : \mathcal{M}_\Sigma \times \Sigma \to BG$ and then integrate over the fiber Σ. This produces a class $\int_\Sigma ev^*\alpha \in H^1(\mathcal{M}_\Sigma;\mathbb{C}^*)$ and $\mathcal{L}$ is the corresponding flat line bundle. The vector space $V(\Sigma)$ is defined to be the space of horizontal global sections of $\mathcal{L}$. The transgression $\alpha \mapsto \int_\Sigma ev^*\alpha$ is realized geometrically by mapping $\mathcal{B}$ to the $\mathbb{C}^*$-torsor $p_*ev^*\mathcal{B}$, where $p : \mathcal{M}_\Sigma \times \Sigma \to \mathcal{M}_\Sigma$ is the projection. This $\mathbb{C}^*$-torsor is described in the following statement.

Theorem 4. *Let S be the set of triples (ϕ, A, z), where ϕ is a point of $\mathcal{M}_\Sigma$, A is an object of the restriction of $ev^*\mathcal{B}$ to $p^{-1}(\phi)$ and $z \in \mathbb{C}^*$. Define an equivalence relation on S by setting*

$$(\phi, A_1, z_1) = (\phi, A_2, z_2)$$

if $A_2 \cong A_1 \otimes Q$ for Q a $\mathbb{C}^*$-gerbe on Σ and $z_2 = (\int_\Sigma [Q])z_1$, where $[Q]$ denotes the cohomology class of Q.

Then the quotient of S by this relation is a principal homogeneous space for the action of $\mathbb{C}^*$ defined by $w \cdot (\phi, A, z) = (\phi, A, wz)$. The cohomology class of this $\mathbb{C}^*$-bundle in $H^1(\mathcal{M}_\Sigma;\mathbb{C}^*)$ is exactly $\int_\Sigma ev^*\alpha$.

The assignment $\phi \mapsto [(\phi, A, z)]$ defines a typical section of $\mathcal{L}$. From this point of view it is clear that reversing the orientation of Σ changes $V(\Sigma)$ to its dual.

We now show how a 3-manifold M with boundary Σ determines a vector in $V(\Sigma)$. Set $v_M(\psi) = [(\psi, A, z)]$, where now ψ is a point of $\mathcal{M}_M$ and A is an object of $ev^*\mathcal{B}$ on $\{\phi\} \times M$. Then v_M defines a section of the pullback of $\mathcal{L}$ to $\mathcal{M}_M$. For another choice of object $A' = A \otimes Q$, where Q is a $\mathbb{C}^*$-gerbe on M, we have $[(\psi, A', z)] = [(\psi, A, (\int_\Sigma [Q])z)]$. But the restriction map $H^2(M;\mathbb{C}^*) \to H^2(\Sigma;\mathbb{C}^*)$ is trivial, so that $\int_\Sigma [Q] = 1$. It follows that v_M is independent of the choice of object A and therefore defined globally on $\mathcal{M}_M$. Note also that it is constant (i.e. horizontal) on each component. Now for any space X with basepoint x_0, a point η of $\mathcal{M}_X$ can be represented as a homomorphism $\tilde{\eta} \in Hom(\pi_1(X, x_0), G)$. We will take the basepoint to lie in Σ and set $v_M(\tilde{\psi}) = v_M(\psi)$, for $\psi \in \mathcal{M}_M$. We now define a section w_M of $\mathcal{L}$. Let ϕ be a point of $\mathcal{M}_\Sigma$, corresponding to a homomorphism $\tilde{\phi} : \pi_1(\Sigma) \to G$. Then we put:

$$w_M(\phi) = \sum_{f \in \{Hom(\pi_1(M), G) : f|_{\pi_1(\Sigma)} = \tilde{\phi}\}} v_M(f)$$

then gives the required global section of $\mathcal{L}$ on $\mathcal{M}_M$.

The next step is to consider two oriented manifolds M_1, M_2 with $\partial M_1 = \Sigma$ and $\partial M_2 = -\Sigma$. Let M be the manifold obtained by glueing M_1 and M_2 along their common boundary. It follows easily from Van Kampen's Theorem that

$$(w_{M_1}, w_{M_2}) = \frac{1}{|G|} \sum_{\tilde{\psi} \in Hom(\pi_1(M), G)} \psi^*\alpha[M],$$

where $(\, , \,)$ is the pairing between $V(\Sigma)$ and $V(\Sigma)^*$ and $[M]$ is the fundamental class of M. This recovers the invariant $Z(M)$ of Dijkgraaf-Witten [13].

There is another, deeper level of structure associated to α. This reflects the connection between Chern-Simons theory and conformal field theory discovered by Witten [27]. We can see this abstractly using the transgression procedure applied to 1-manifolds rather than surfaces. Consider the case of the circle S^1. As above, let $ev : \mathcal{M}_{S^1} \times S^1 \to BG$ be the evaluation map and $p : \mathcal{M}_{S^1} \times S^1 \to \mathcal{M}_{S^1}$ the projection. Pulling back α and integrating over S^1, we obtain a class $\int_{S^1} ev^*\alpha \in H^2(\mathcal{M}_{S^1};\mathbb{C}^*)$. Geometrically, this corresponds to a $\mathbb{C}^*$-gerbe $p_*ev^*\mathcal{B}$ on $\mathcal{M}_{S^1}$. Denote this gerbe by $\mathcal{C}$. The objects of $\mathcal{C}$ over the point $\phi \in \mathcal{M}_{S^1}$ are the global objects of the restriction of $ev^*\mathcal{B}$ to $p^{-1}(\phi)$. Given any two such objects A_1, A_2, there is an equivalence $A_2 = A_1 \otimes Q$ for some well-defined $\mathbb{C}^*$-gerbe Q on S^1. Then in $\mathcal{C}$ we set $Hom(A_1, A_2) = p_*ev^*Q$; here p_*ev^*Q is a $\mathbb{C}^*$-torsor. This describes the fiber of $\mathcal{C}$ at ϕ.

The gerbe $\mathcal{C}$ has several remarkable properties.

Theorem 5. *Let Σ be an oriented surface whose boundary is a disjoint union of r parametrized circles $C_1, ..., C_r$ and let $b_j : \mathcal{M}_\Sigma \to \mathcal{M}_{C_j}$ be the natural restriction.*

(1) There is a canonical global object A_Σ *of the pull back gerbe* $\mathcal{C}_\Sigma = \bigotimes_{j=1}^{r} b_j^*\mathcal{C}$.

(2) A_Σ *is invariant under* $Diff^+(\Sigma)$.

(3) The object A_Σ *is natural with respect to the operation of glueing surfaces along boundary circles.*

This theorem is one version of the reciprocity law of Segal and Witten [24] [13]. The object A_Σ may be described pointwise as follows. For each map $\phi : \Sigma \to BG$, choose a global object A of $\phi^*\mathcal{B}$. For any other choice A', we have $A' = A \otimes Q$ where Q is a gerbe on Σ. But considered as objects of $\mathcal{C}_\Sigma$, A and A' differ by tensoring with the $\mathbb{C}^*$-torsor $\bigotimes_{j=1}^{r} b_j^*\mathcal{H}(Q)$, where $\mathcal{H}(Q) := p_* ev^* Q$ is the torsor over $\mathcal{M}_{S^1}$ described before Theorem 5. Now this $\mathbb{C}^*$-torsor is canonically trivial as $\sum_{j=1}^{r} C_j$ is homologous to zero in Σ. Therefore, we obtain a canonical global object of $\mathcal{C}_\Sigma$, which proves (1). The rest of the Theorem follows easily.

There is a compatibility between this layer of structure and the vector space V constructed above; Suppose that two surfaces Σ_1, Σ_2 are glued together along their boundary circles giving a new surface Σ without boundary. For $i = 1, 2$, let $r_i : \mathcal{M}_\Sigma \to \mathcal{M}_{\Sigma_i}$ be the natural restriction maps. Then the gerbes $r_1^*\mathcal{C}_{\Sigma_1} \otimes r_2^*\mathcal{C}_{\Sigma_2}$ and $\mathcal{C}_\Sigma$ on $\mathcal{M}_\Sigma$ are canonically equivalent. The canonical global objects $r_1^* A_{\Sigma_1} \otimes r_2^* A_{\Sigma_2}$ and A_Σ correspond to each other in this equivalence.

The next (and final) step is to apply the transgression procedure to manifolds of dimension 0 and interpret the resulting cohomology class geometrically. But this is exactly the problem of representing α by a 2-gerbe $\mathcal{B}$. Abstractly, we know that $\mathcal{B}$ can be found and that the Dijkgraaf-Witten theory can be recovered as above. A concrete description of $\mathcal{B}$ will be presented in [8].

We conclude from this discussion that 2-gerbes are the fundamental geometric objects in Chern-Simons theory. A similar observation has been made by D. Kazhdan.

Acknowledgements

Each author thanks the N.S.F for its support though research grants. The second author thanks MSRI for support during part of this research.

References

1. A. Beilinson, *Higher regulators and values of L-functions*, J. Sov. Math. **30** (1985), 2036–2070.
2. J. Bénabou, *Introduction to bicategories*, Midwest Category Seminar, Lecture Notes in Math. vol. 47, Springer-Verlag, 1967, pp. 1–77.
3. L. Breen, *Théorie de Schreier supérieure*, Ann. Sci. Ec. Norm. Sup. **25** (1992), 465–514.
4. J.-L. Brylinski, *The Kaehler geometry of the space of knots on a smooth threefold*, preprint 1990.
5. J.-L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization*, Progress in Math. vol. 107, Birkhäuser.
6. J.-L. Brylinski and D. McLaughlin, *Čech cocycles for characteristic classes*, preprint 1992.
7. J.-L. Brylinski and D. McLaughlin, *The geometry of degree four characteristic classes*, preprint 1992.
8. J.-L. Brylinski and D. McLaughlin, *The geometry of degree four characteristic classes II*, (in preparation).
9. J. Cheeger, *A combinatorial formula for Stiefel-Whitney classes*, Topology of Manifolds, Markham, 1970, pp. 470–471.
10. J. Cheeger and J. Simons, *Differential characters and geometric invariants*, Lecture Notes in Math. vol. 1167, Springer-Verlag, 1985, pp. 50–80.
11. S. S. Chern and J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. **99** (1974), 48–69.
12. P. Deligne, *Le symbole modéré*, Publ. Math. IHES **73** (1991), 147–181.
13. R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990), 393–429.
14. D. Freed and F. Quinn, *Chern-Simons theory with finite gauge group*, Comm. Math. Phys. (to appear).
15. A. Gabrielov, I. M. Gel'fand and M. V. Losik, *Combinatorial calculation of characteristic classes*, Funct. Anal. Appl. **9** (1975), 48–50, 103–115, 186–202.
16. K. Gawedzki, *Topological actions in two-dimensional quantum field theories*, Nonperturbative Quantum Field Theories, ed. G't Hooft, A. Jaffe, G. Mack, P. K. Mitter, R. Stora, NATO ASI Series vol. 185, Plenum Press, 1988, pp. 101–142.
17. I. Gel'fand and R. MacPherson, *A combinatorial formula for the Pontrjagin classes*, Bull. Amer. Soc. **26, n0. 2** (1992), 304–309.
18. J. Giraud, *Cohomologie Non-Abélienne*, Ergeb. der Math. vol. 64, Springer-Verlag, 1971.
19. A. Goncharov, *Explicit construction of characteristic classes*, Adv. in Soviet Math. (in press).
20. T. P. Killingback, *World sheet anomalies and loop geometry*, Nucl. Phys. B **288** (1987), 578–588.
21. B. Kostant, *Quantization and unitary representations. Part I: Prequantization*, Lecture Notes in Math. vol. 170, Springer-Verlag, 1970, pp. 87–208.
22. D. A. McLaughlin, *Orientation and string structures on loop space*, Pac. J. Math. **155 no. 1** (1992), 143–156.
23. R. MacPherson, *The combinatorial formula of Gabrielov, Gel'fand and Losik for the first Pontryagin class*, Séminaire Bourbaki. Exposés 498-506, Lecture Notes in math. vol. 677, Springer-Verlag, 1977, pp. 105–124.

24. G. Segal, *The definition of conformal field theory*, to appear.

25. A. Weil, *Variétés Kähleriennes*, Hermann, 1958.

26. E. Witten, *The index of the Dirac operator on loop space*, Elliptic Curves and Modular Forms in Algebraic Topology, Lecture Notes in Math. vol. 1326, Springer-Verlag, 1988, pp. 161–181.

27. E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351–389.