Line Bundle Gerbes from 2-Transport

September 7, 2006

Abstract

The concept of a line bundle gerbe with connection and curving is a special case of transition data of 2-transport.

Our main aim it to prove theorem 1.

Before doing so, we motivate the discussion by some considerations concerning associated 2-bundles.

Let

$$\rho : \Sigma(G_2) \to T' \subset \mathrm{Mod}_{\mathcal{C}}$$

be a faithful representation of the 2-group G_2 and

$$T' \xrightarrow{i} T$$

some monomorphism. Then we say that tra is a **associated $\mathcal{C}$-vector transport** with respect to (ρ, i) if it admits a proper trivialization

$$\begin{array}{ccc} \mathcal{P}_{\mathcal{U}} & \xrightarrow{\quad p \quad} & \mathcal{P} \\ \Big\downarrow{\scriptstyle \mathrm{tra}^i} & \overset{t}{\swarrow}{\scriptstyle \sim} & \Big\downarrow{\scriptstyle \mathrm{tra}} \\ G_2 \xrightarrow{\rho} T' & \xrightarrow{\quad i \quad} & T \end{array} .$$

Some familiar associated 2-bundles come from the following class of representations of 2-groups.

Proposition 1 *For any strict 2-group $G_2 = (H \to G)$, a representation $\rho : \Sigma(H) \to \mathbf{Vect}_K$ of H for which the family $\{\rho(h) \,|h \in H\}$ is linearly independent over K induces a representation*

$$\tilde{\rho} : \Sigma(G_2) \to \mathrm{Bim}(\mathbf{Vect}) \xrightarrow{i} \mathrm{Mod}_{\mathbf{Vect}}$$

given by

$$\begin{array}{cccc} \tilde{\rho} \; : & \Sigma(G_2) & \to & \mathrm{Bim}(\mathbf{Vect}) \\ & \bullet \underset{g'}{\overset{g}{\Downarrow h}} \bullet & \mapsto & A_\rho \underset{(A_\rho, g')}{\overset{(A_\rho, g)}{\Downarrow (\cdot)\circ\rho(h)}} A_\rho \end{array} .$$

Here

$$A_\rho \equiv \langle \rho(h) \mid h \in H \rangle$$

is the algebra generated by the endomorphisms representing H and (A_ρ, g) is A_ρ regarded as a bimodule over itself, with the right action twisted by the automorphism g.

Proof. Let

$$\begin{array}{ccc} \rho : \Sigma(H) & \to & \mathbf{Vect} \\ \bullet & & V \\ \downarrow h & \mapsto & \downarrow \rho(h) \\ \bullet & & V \end{array}$$

and notice that the notation for compositon is such that

$$V \xrightarrow{\rho(hh')} V = \rho\left(\bullet \xrightarrow{h} \bullet \xrightarrow{h'} \bullet \right) = V \xrightarrow{\rho(h)} V \xrightarrow{\rho(h')} V = V \xrightarrow{\rho(h')\circ\rho(h)} V .$$

Also recall that 2-morphisms in $\Sigma(G_2)$

$$\bullet \underset{g'}{\overset{g}{\Downarrow h}} \bullet$$

are labeled by $g \in G$ and $h \in H$ with

$$\begin{array}{ccc} \bullet & \xrightarrow{h} & \bullet \\ {\scriptstyle g'(f)}\downarrow & & \downarrow{\scriptstyle g(f)} \\ \bullet & \xrightarrow{h} & \bullet \end{array}$$

for arbitrary $f \in H$. What we shall need below is the commutativity of the image of this diagram under ρ

$$\begin{array}{ccc} V & \xrightarrow{\rho(h)} & V \\ {\scriptstyle \rho(g'(f))}\downarrow & & \downarrow{\scriptstyle \rho(g(f))} \\ V & \xrightarrow[\rho(h)]{} & V \end{array} \quad . \tag{1}$$

In order to construct $\tilde{\rho}$ let now

$$\mathrm{End}_V \supset A_\rho \equiv \langle \rho(h) \mid h \in H \rangle$$

be the subalgebra of the endomorphism algebra of V which is generated by the linear maps $\rho(h)$ for all $h \in H$. We obtain for each $g \in \mathrm{Aut}(H)$ an automorphism $\rho(g) \in \mathrm{Aut}(A_\rho)$ of this algebra by setting

$$\rho(g) : \rho(h) \mapsto \rho(g(h))$$

for all $h \in H$, and extended linearly to all of A_ρ.

Using this, for each $g \in G$ we define an A_ρ-bimodule

$$(A_\rho, g) \equiv \; A_\rho \overset{\mathrm{Id}}{\dashrightarrow} A_\rho \overset{\rho(g)}{\dashleftarrow} A_\rho$$

which, as an object in **Vect**, is A_ρ itself, with both the right and the left A_ρ action given by the product in A_ρ, but with the right action twisted by $\rho(g)$:

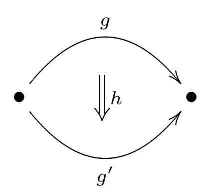
$$\begin{aligned} \rho(h) \cdot a &\equiv \rho(h) \circ a \\ a \cdot \rho(h) &\equiv a \circ \rho(g(h)) \; . \end{aligned} \tag{2}$$

for all $a \in A_\rho$.

The tensor product over A_ρ corresponds to the composition of automorphisms

$$(A_\rho, g) \otimes_{A_\rho} (A_\rho, g') = (A_\rho, gg') \, ,$$

which shows that $\tilde{\rho}$ is properly functorial on the level of 1-morphisms. For each 2-morphism



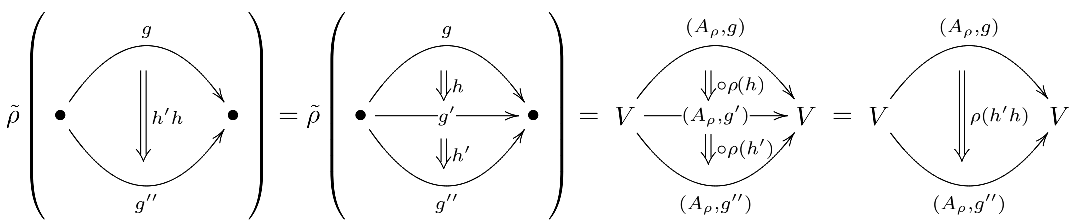
define a morphism of bimodules

$$\begin{aligned} \tilde{\rho}(h) : (A_\rho, g) &\rightarrow A_\rho(g') \\ a &\mapsto a \circ \rho(h) \; . \end{aligned}$$

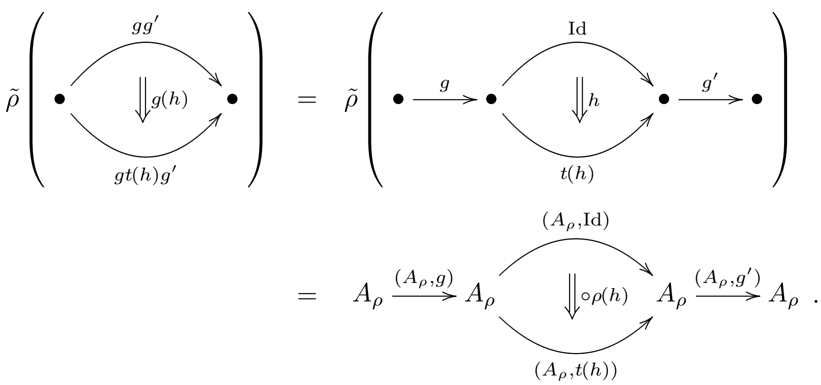
This map trivially respects the left A_ρ-action. That it also respects the right A_ρ action is a consequence of the commutativity of (1):

$$\begin{array}{ccc}
(a,\rho(f)) & \xmapsto{\tilde{\rho}(h)\times\mathrm{Id}} & (a\circ\rho(h)\,,\rho(f)) \\
 & & \big\downarrow (2) \\
\big\downarrow (2) & & a\circ\rho(h)\circ\rho(g'(f)) \\
 & & \big\| (1) \\
a\circ\rho(g(f)) & \xmapsto[\tilde{\rho}(h)]{} & a\circ\rho(g(f))\circ\rho(h)
\end{array} .$$

That $\tilde{\rho}$ defined this way is functorial for vertical composition follows from



by our remark above. Finally, the 2-functoriality of $\tilde{\rho}$ requires that



This can be checked for instance by representing elements of $(A_\rho,g)\otimes_{A_\rho}(A_\rho,\mathrm{Id})$ by $(a,\mathrm{Id})\in A_\rho\times A_\rho$. Then in particular $\tilde{\rho}(h)((a,\mathrm{Id}))=(a,\mathrm{Id}\circ\rho(h))\sim(a\circ\rho(g(h))\,,\mathrm{Id})$. $\square$

Example 1

Let G_2 be the automorphism 2-group of $\Sigma(U(1))$ $G_2=(U(1)\to\mathbb{Z}_2)$. Let $\rho:\Sigma(U(1))\to\mathbf{Vect}_\mathbb{R}$ be the defining 2 real dimensional representation.

In this case we find $A_\rho \simeq \mathbb{C}$, the complex numbers, regarded as an $\mathbb{R}$-algebra. The bimodule (A_ρ, Id) is just $\mathbb{C}$ itself, with the left and right $\mathbb{C}$-action given by multiplication of complex numbers.

$$(A_\rho, \mathrm{Id}) = \mathbb{C}\,.$$

Denote the nontrivial element of $\mathbb{Z}_2$ by σ. The bimodule (A_ρ, σ) is, as an object, $\mathbb{C}$, with the left $\mathbb{C}$-action given by multiplication of complex numbers and the right $\mathbb{C}$-action given by conjugation followed by multiplication. We write

$$(A_\rho, \sigma) \equiv \mathbb{C}_\sigma\,.$$

Concretely, the left and right actions on $\mathbb{C}_\sigma$ are

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C}_\sigma & \xrightarrow{l} & \mathbb{C}_\sigma \\ (c,d) & \mapsto & cd \end{array}$$

and

$$\begin{array}{ccc} \mathbb{C}_\sigma \times \mathbb{C} & \xrightarrow{r} & \mathbb{C}_\sigma \\ (d,c) & \mapsto & \bar{c}d \end{array}\,.$$

Similarly, for any complex vector space V, let

$$V_\sigma \simeq V \otimes \mathbb{C}_\sigma$$

and

$${}_\sigma V \simeq \mathbb{C}_\sigma \otimes V$$

be the $\mathbb{C}$-$\mathbb{C}$-bimodule V, as an object, but with the left or right $\mathbb{C}$ action twisted, as indicated.

Notice that we have the canonical isomorphism

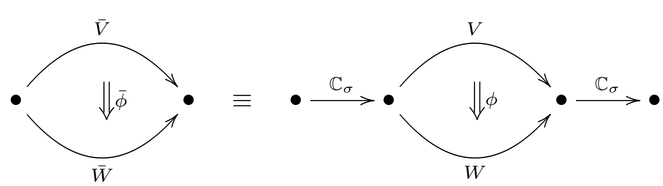
$${}_\sigma V_\sigma \simeq \bar{V}\,,$$

where $\bar{V}$ is V equipped with the opposite complex structure, and hence in particular the canonical identitfication

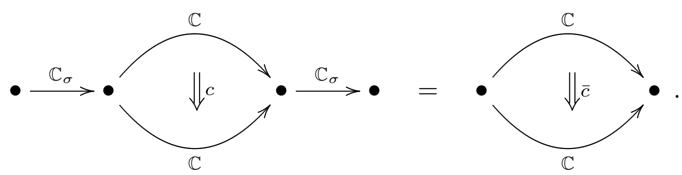
$$\mathbb{C}_\sigma \otimes \mathbb{C}_\sigma \simeq \bar{\mathbb{C}} \simeq \mathbb{C}\,.$$

Denote by $\mathrm{Bim}_{\mathbb{C}}$ the 2-category of $\mathbb{C}$-$\mathbb{C}$-bimodules, with single object $\mathbb{C}$, bimodules up to canonical isomorphism as 1-morphisms and bimodule intertwiners as 2-morphisms.

We write



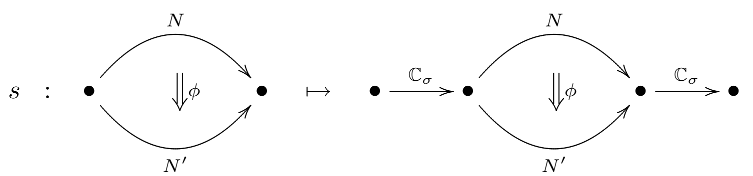
and find in particular



It follows that we get an involutive inner automorphism

$$\mathrm{Bim}_{\mathbb{C}} \xrightarrow{\mathrm{Ad}_{\mathbb{C}_\sigma}} \mathrm{Bim}_{\mathbb{C}}$$

given by conjugation with $\mathbb{C}_\sigma$ as



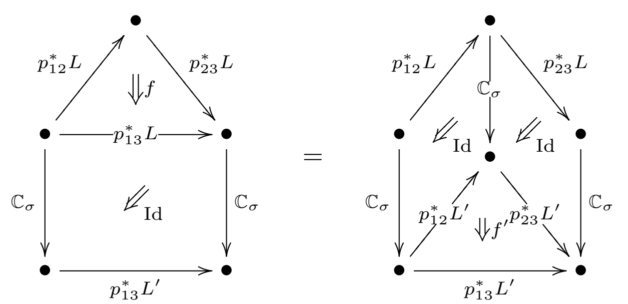
Given any transport

$$\mathrm{tra} : \mathcal{P} \to \mathrm{Bim}_{\mathbb{C}}$$

we hence obtain what could be called the "opposite" transport

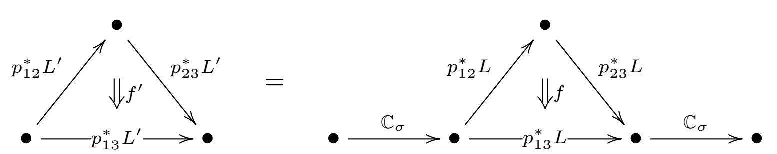
$$\mathrm{tra}^{\mathrm{op}} \equiv (\mathrm{Ad}_{\mathbb{C}_\sigma})^* \mathrm{tra} : \ \mathcal{P} \xrightarrow{\mathrm{tra}} \mathrm{Bim}_{\mathbb{C}} \xrightarrow{\mathrm{Ad}_{\mathbb{C}_\sigma}} \mathrm{Bim}_{\mathbb{C}} \ .$$

We will be interested in transition morphisms in $\mathbf{Trans}(\mathcal{P}, \mathrm{Bim}_{\mathbb{C}})$. Consider the case where such a morphism involves $\mathbb{C}_\sigma$ in its defining tin can equation as follows



The existence of the identity-2-morphisms here says that the transition lines are related by $L' = \bar{L}$.

This equation can equivalently be rewritten as



which says that

$$f' = \bar{f}\,.$$

Example 2

$G_2 = (U(n) \to PU(n))$

$$\rho : U(n) \to \mathbb{C}^n$$

$PU(n) = \{[g] | g \in U(n)\}$
$A_\rho = \mathrm{End}_{\mathbb{C}^n}$

$$\begin{array}{ccc} (A_\rho, [g]) \times A_\rho & \to & \mathbb{C}^n_{[g]} \\ (a,b) & \mapsto & a \circ \rho(g)^{-1} \circ b \circ \rho(g) \end{array}$$

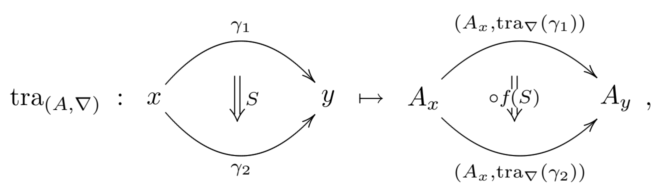
Example 3

Let $A \to X$ be an algebra bundle with typical fiber $End_{\mathbb{C}^n}$. Let ∇ be a connection on E and let $\mathrm{tra}_\nabla : \mathcal{P}_1(X) \to \mathrm{Trans}(A)$ be the corresponding parallel transport. The automorphism group of $\mathrm{End}_{\mathbb{C}^n}$ is $PU(n)$, hence A is an associated $PU(n)$-bundle.

Define a 2-transport

$$\mathrm{tra}_{(A,\nabla)} : \mathcal{P}_2(X) \to \mathrm{Bim}(\mathbf{Vect})$$

by

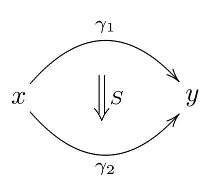


where $f(S)$ is the unique lift of $\mathrm{tra}_\nabla(\bar{\gamma}_1) \circ \mathrm{tra}_\nabla(\gamma_2)$ to A_x such that

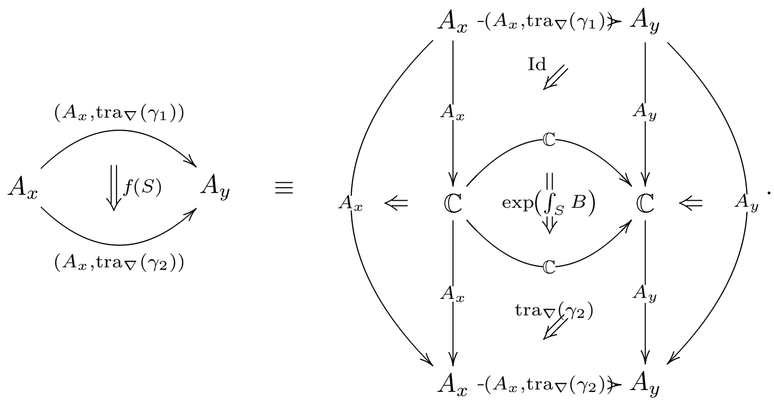
$$\exp\left(n \int_S B\right) = \det(f(S))\ .$$

In order to see that this assignment is indeed 2-functorial, choose a basis $A_x \xrightarrow{t(x)} \mathrm{End}_{C^n}$ for all endpoints involved in the computation. Then use the logic of example 2.

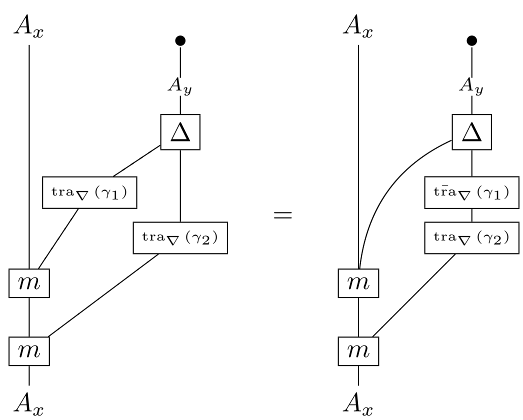
Diagrammatically, we would naturally associate to each surface



the 2-cell



In terms of string diagrams the right hand side reads



This can be seen to be almost the same bimodule homomorphism as above, up to a scalar multiple. Where before we had the determinant, this involves a trace. As a result, this second assignment is not 2-functorial in general.

Proposition 2 *The 2-transport from example 3 is an associated* **Vect***-vector transport with respect to*

$$(U(1) \to \mathbb{Z}_2) \xrightarrow{\tilde{\rho}} \mathrm{Bim}_{\mathbb{C}} \xrightarrow{i} \mathrm{Bim}(\mathbf{Vect}) \;\;,$$

with $\tilde{\rho}$ the representation from example 1.

Proof.

Locally we may always identify, $A|_U \xrightarrow[\sim]{\tau} \mathrm{End}_V$, the bundle A with the endomorphism bundle of some vector bundle V with connection tra_V. This can be expressed in terms of bimodules as

$$\begin{array}{ccc} A_x & \xrightarrow{(A_x,\mathrm{tra}_\nabla(\gamma))} & A_y \\ {\scriptstyle (A_x,\tau(x))}\Big\downarrow & \Swarrow_{\mathrm{Id}} & \Big\downarrow{\scriptstyle (A_y,\tau(y))} \\ \mathrm{End}_{V_x} & \xrightarrow[(\mathrm{End}_{V_x},\mathrm{Ad}_{\mathrm{tra}_V(\gamma)})]{} & \mathrm{End}_{V_y} \end{array} \quad .$$

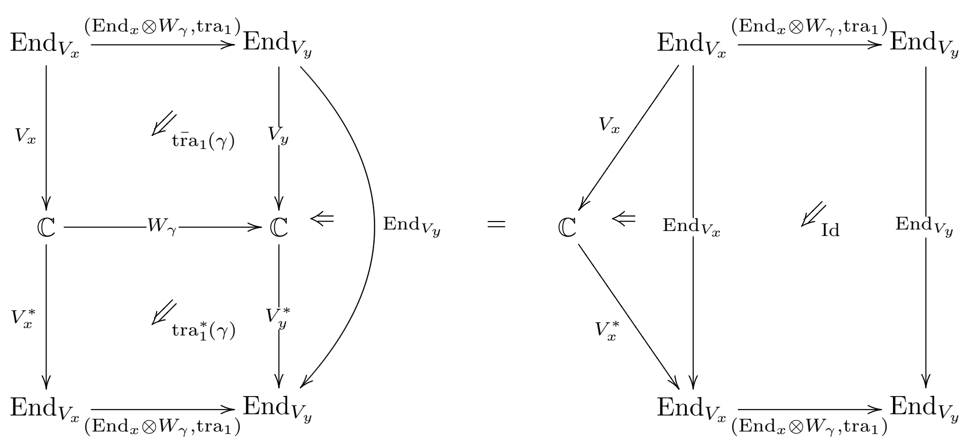
Consider the bimodule homomorphisms

$$\begin{array}{ccc} \mathrm{End}_{V_x} & \xrightarrow{(\mathrm{End}_x,\mathrm{Ad}_{\mathrm{tra}_V(\gamma)})} & \mathrm{End}_{V_y} \\ {\scriptstyle V_x}\Big\downarrow & \Swarrow_{\bar{\mathrm{tra}}_V(\gamma)} & \Big\downarrow{\scriptstyle V_y} \\ \mathbb{C} & \xrightarrow[\mathbb{C}]{} & \mathbb{C} \end{array}$$

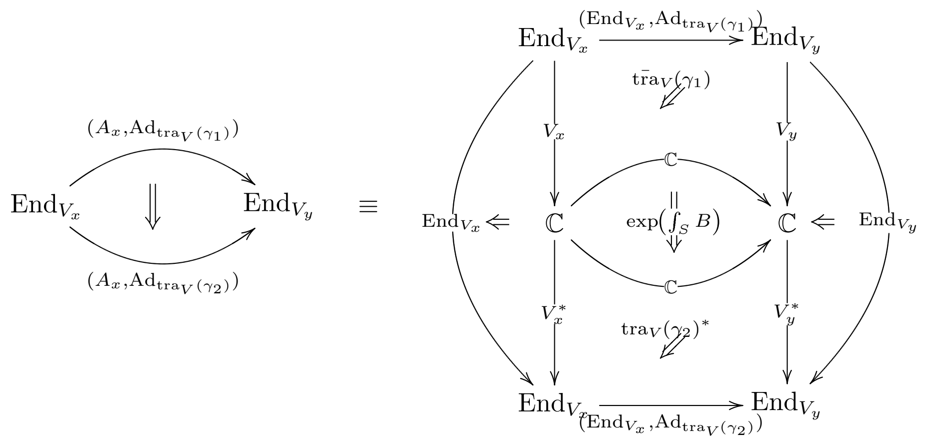
and

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\mathbb{C}} & \mathbb{C} \\ {\scriptstyle V_x^*}\Big\downarrow & \Swarrow_{\mathrm{tra}_V^*(\gamma)} & \Big\downarrow{\scriptstyle V_y^*} \\ \mathrm{End}_{V_x} & \xrightarrow[(\mathrm{End}_x,\mathrm{Ad}_{\mathrm{tra}_V(\gamma)})]{} & \mathrm{End}_{V_y} \end{array} \quad .$$

These fit into an adjoint equivalence due to



Therefore, given a 2-form $B \in \Omega^2(U)$, we can form the 2-transport

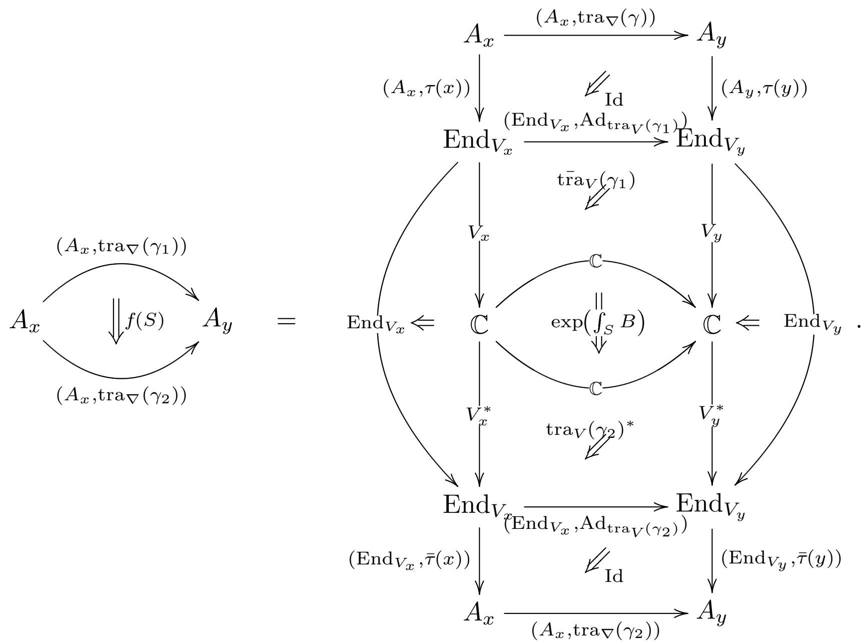


The composition of 2-cells on the right corresponds to the bimodule homomorphism which sends $a \in \mathrm{End}_{V_x}$ to

$$
\begin{aligned}
& \left(V_x \xrightarrow{a} V_x \right) \\
\mapsto\ & \left(\sum_i V_y \xrightarrow{e^i} \mathbb{C} \xrightarrow{e_i} V_y \xrightarrow{\bar{\mathrm{tra}}_V(\gamma_1)} V_x \xrightarrow{a} V_x \xrightarrow{\mathrm{tra}_V(\gamma_1)} V_y \right) \\
\mapsto\ & \exp\left(\int_S B\right) \left(\sum_i V_x \xrightarrow{\mathrm{tra}_V(\gamma_2)} V_y \xrightarrow{e^i} \mathbb{C} \xrightarrow{e_i} V_y \xrightarrow{\bar{\mathrm{tra}}_V(\gamma_1)} V_x \xrightarrow{a} V_x \xrightarrow{\mathrm{tra}_V(\gamma_1)} V_y \xrightarrow{\bar{\mathrm{tra}}_V(\gamma_1)} V_x \right) \\
\mapsto\ & \exp\left(\int_S B\right) \left(\sum_i V_x \xrightarrow{\mathrm{tra}_V(\gamma_2)} V_y \xrightarrow{\bar{\mathrm{tra}}_V(\gamma_1)} V_x \xrightarrow{a} V_x \right),
\end{aligned}
$$

hence to $a \circ \bar{\mathrm{tra}}_V(\gamma_1) \circ \mathrm{tra}_V(\gamma_2)$, up to a scalar factor. This is indeed, locally, the assignment of the 2-transport from example 3. We re-obtain the global

2-transport by pulling this back along τ



$\square$

Proposition 3 *The transitions of the local trivialization from prop. 2 are i-transitions, for the obvious embedding* $i : \Sigma(\Sigma(\mathbb{C})) \longrightarrow \Sigma(1\mathrm{DVect}_{\mathbb{C}})$.

This motivates the study of $\mathrm{Tra}(i)$, the 2-category of i-transitions. Let $p : Y \to X$ be a surjective submersion.

Theorem 1 *The 2-category of ($\mathcal{P}_2(Y) \xrightarrow{p} \mathcal{P}_2(X)$)-local ($\Sigma(\mathbb{C}) \xrightarrow{i} 1\mathrm{DVect}_{\mathbb{C}}$)-transitions is equivalent (isomorphic, even) to the 2-category of line bundle gerbes for fixed Y*

$$\mathrm{Tra}(p, i) \simeq \mathrm{BunGer}(Y) \; .$$

We prove this using a couple of lemmas.

Lemma 1 *Let* $\mathrm{tra} : \mathcal{P}_2 \to \Sigma(1\mathrm{DVect})$ *be a 2-transport which assigns 1-dimensional vector spaces to paths and linear maps to surfaces.*

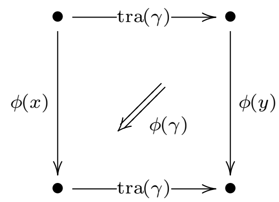
1. *1-Automorphisms* $\mathrm{tra} \xrightarrow{\sim} \mathrm{tra}$ *are in bijection with flat transport* $\mathcal{P}_1 \to 1\mathrm{DVect}$*, i.e. with flat line bundle with connection.*

2. *Composition of these 1-automorphisms corresponds to taking the tensor product of the corresponding line bundles.*

3. *2-morphisms between these 1-automorphisms correspond to natural transformations of the transport 1-functors of the corresponding line bundles and hence to isomorphisms between these line bundles which fix the base space.*

Proof.

1. Write $V = \mathrm{tra}(\gamma)$ for the vector space associated by tra to $x \xrightarrow{\gamma} y$. Then



is a linear map

$$V \otimes \phi(y) \xrightarrow{\phi(\gamma)} \phi(x) \otimes V .$$

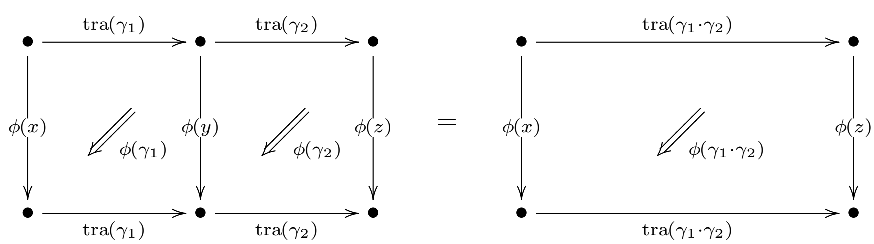
Since V is 1-dimensional this defines a linear map

$$\phi(y) \xrightarrow{\phi(\gamma)} \phi(x)$$

under the isomorphism

$$\begin{aligned} \mathrm{Hom}(V \otimes \phi(y), \phi(x) \otimes V) &\simeq \mathrm{Hom}(\phi(y), V^* \otimes \phi(x) \otimes V) \\ &\simeq \mathrm{Hom}(\phi(y), \phi(x) \otimes V^* \otimes V) \\ &\simeq \mathrm{Hom}(\phi(y), \phi(x) \otimes K) \\ &\simeq \mathrm{Hom}(\phi(y), \phi(x)) . \end{aligned}$$

The functoriality condition on ϕ



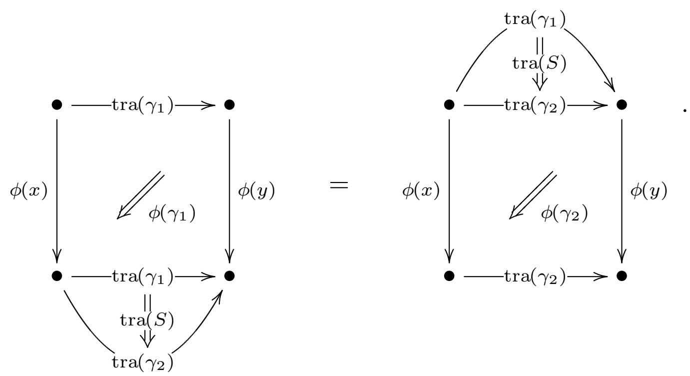
translates similarly into

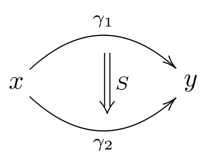
$$\phi(z) \xrightarrow{\phi(\gamma_2)} \phi(y) \xrightarrow{\phi(\gamma_1)} \phi(x) \;=\; \phi(z) \xrightarrow{\phi(\gamma_1 \cdot \gamma_2)} \phi(x) \;.$$

Therefore $\bar{\phi}$ defines a functor

$$\bar{\phi} : \mathcal{P}_1(M) \to \mathbf{Vect}_1$$

and hence a bundle with connection on M. Finally, ϕ has to make the tin can equation hold



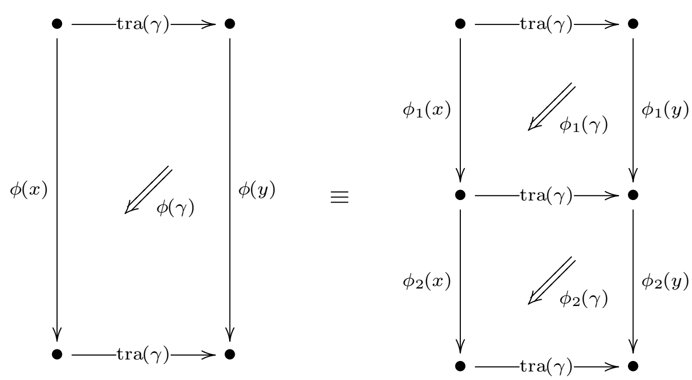
Since we have the same 

on both sides this implies that

$$\phi(\gamma_1) = \phi(\gamma_2) \ .$$

Hence ϕ is *flat*. Running these arguments backwards shows that conversely every flat line bundle on M gives rise to an automorphism $\mathrm{tra} \xrightarrow{\phi} \mathrm{tra}$.

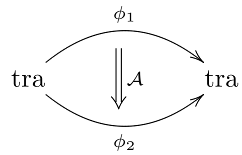
2. The composition



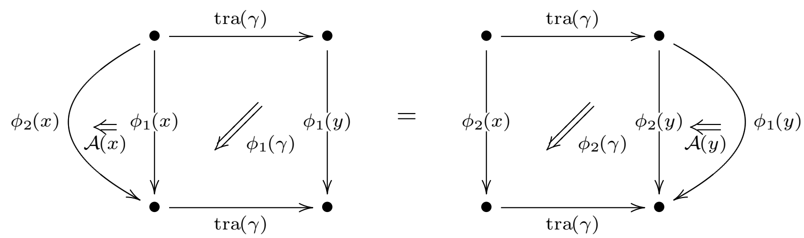
corresponds to

$$\begin{array}{ccc} \phi(x) & & \phi_1(x) \otimes \phi_2(x) \\ \downarrow \bar{\phi}(\gamma) & = & \downarrow \bar{\phi}_1(\gamma) \otimes \bar{\phi}_2(\gamma) \\ \phi_1(y) & & \phi_1(y) \otimes \phi_2(y) \end{array}$$

3. A 2-morphism

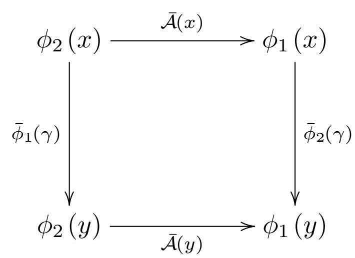


satisfies the tin can equation of the following form:



.

Under the above identification of $\phi(\gamma)$ with a linear map $\bar{\phi}(x) \xrightarrow{\bar{\phi}(\gamma)} \bar{\phi}(y)$ this is equivalent to a natural transformation



□

In the same way one proves

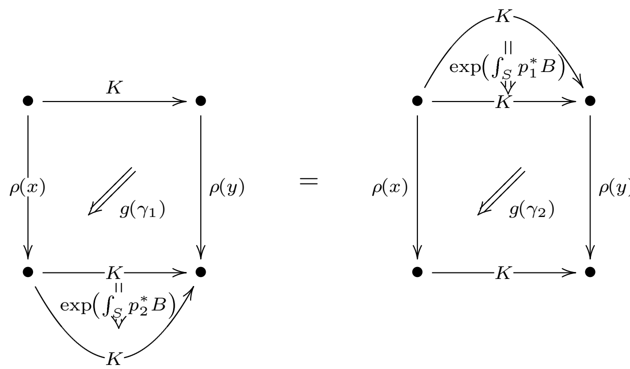
Lemma 2

1. *1-morphisms of i-trivial 2-transport are in bijection with line bundles with connection.*
2. *Composition of such 1-morphisms corresponds to taking the tensor product of the corresponding line bundles.*
3. *2-morphisms between such 1-morphisms of trivial line-2-bundles correspond to bundle isomorphisms of the corresponding line bundles.*

Lemma 3 *Let* $\mathrm{tra}_B, \mathrm{tra}_{B'} : \mathcal{P}_2 \to \Sigma 1\mathrm{DVect}$ *be i-trivial 2-transport coming from the 2-forms* $B, B' \in \Omega^2(\mathrm{Lie}(\mathrm{U}(1)))$. *Let* $\mathrm{tra}_B \xrightarrow{\mathrm{tra}_\nabla} \mathrm{tra}_{B'}$ *be the morphism given by the line bundle with connection* ∇ *by lemma 2. Then*

$$B' = B + F_\nabla \,.$$

Proof. The existence of $\mathrm{tra}_B \xrightarrow{\mathrm{tra}_\nabla} \mathrm{tra}_{B'}$ is equivalent to the 2-commutativity of all respective tin cans:



This immediately implies the above statement. □

Definition 1 (Murray) *A* **line bundle gerbe** *over a manifold M is*

- *a surjective submersion*

$$\begin{array}{c} Y \\ \downarrow \\ M \end{array}$$

- *a $\mathbb{C}^\times$-bundle*

$$\begin{array}{c} L \\ \downarrow \\ Y^{[2]} \end{array}$$

- *over* $Y^{[3]} \overset{p_{12}}{\underset{p_{23}}{\overset{p_{13}}{\rightrightarrows}}} Y^{[2]}$ *a bundle isomorphism*

$$p_{12}^* L \otimes p_{23}^* L \xrightarrow{f} p_{13}^* L$$

which is associative in the sense that on $Y^{[4]} \xrightarrow[p_{234}]{p_{123},\, p_{124},\, p_{134}} Y^{[3]}$ *the diagram*

$$\begin{array}{ccc} p_{12}^* L \otimes p_{23}^* L \otimes p_{34}^* L & \xrightarrow{p_{123}^* f \otimes \mathrm{Id}} & p_{13}^* L \otimes p_{34}^* L \\ {\scriptstyle \mathrm{Id} \otimes p_{234}^* f} \downarrow & & \downarrow {\scriptstyle p_{134}^* f} \\ p_{12}^* L \otimes p_{24}^* L & \xrightarrow[p_{124}^* f]{} & p_{14}^* L \end{array}$$

commutes.

A **connective structure** *on a bundle gerbe (also known as* **connection and curving** *on a bundle gerbe) is*

- *a connection* ∇ *on* L
- *a 2-form* $\omega \in \Omega^2(Y)$ *on* Y

such that on $Y^{[2]} \underset{p_1}{\overset{p_2}{\rightrightarrows}} Y$ *the equation*

$$p_2^*\omega - p_1^*\omega = F_\nabla$$

holds.

Lemma 4 *(p,i)-transition tetrahedra are in bijection with line bundle gerbes with connection and curving.*

Proof. Using the above notation, identify Y with $\mathcal{U}$. By prop. **??** the trivialization transition g defines a line bundle with connection on $\mathcal{U}^{[2]}$ and vice versa. Hence identify

$$g \leftrightarrow (L, \nabla)\,.$$

The picture obtained is

$$\begin{array}{ccccccc} g & & & & L & & \\ \downarrow & & & & \downarrow & & \\ \mathcal{U}^{[2]} & \rightrightarrows & \mathcal{U} & \leftrightarrow & Y^{[2]} & \rightrightarrows & Y \\ & & \downarrow & & & & \downarrow \\ & & M & & & & M \end{array}$$

Identify the gerbe product with the inverse of the modification f using the third item of prop. **??**. By prop. **??** this does satisfy the required associativity condition.

In order to match the connection data, observe that the line-2-bundle $\mathrm{tra}_{\mathcal{U}}$ is trivial by assumption and hence defines, according to def. **??**, a global 2-form B on $\mathcal{U}$. Identify this 2-form with the curving ω of the bundle gerbe. Prop. **??** says that $\mathrm{tra}_{\mathcal{U}}$ and (L, ∇) satisfy the condition of a gerbe connection

$$p_2^*B - p_1^*B = F_\nabla\,.$$

□

Definition 2 (Murray,Stevenson) *Given two bundle gerbes with connective structure* (L, Y) *and* (L', Y) *a* **stable isomorphism**

$$(L, Y) \xrightarrow{(H,\mathcal{E})} (L', Y)$$

is a line bundle with connection $H \longrightarrow Y$ *together with an isomorphism*

$$p_1^*H \otimes L \xrightarrow{\mathcal{E}} L' \otimes p_2^*H$$

of line bundles with connection on $Y^{[2]}$ *satisfying*

$$\begin{array}{ccccc}
p_1^*H \otimes p_{12}^*L \otimes p_{23}^*L & \xrightarrow{p_{12}^*\mathcal{E}\otimes \mathrm{Id}_{p_{23}^*L}} & p_{12}^*L' \otimes p_2^*H \otimes p_{23}^*L & & \\
 & & & \searrow^{\mathrm{Id}_{p_{12}^*L'}\otimes p_{23}^*\mathcal{E}} & \\
\Big\downarrow{\scriptstyle \mathrm{Id}_{p_1^*H}\otimes f} & & & & p_{12}^*L' \otimes p_{23}^*L' \otimes p_3^*H \\
 & & & \swarrow_{f'\otimes \mathrm{Id}_{p_3^*H}} & \\
p_1^*H \otimes p_{13}^*L & \xrightarrow[p_{13}^*\mathcal{E}]{} & p_{13}^*L' \otimes p_3^*H & &
\end{array} \tag{3}$$

Lemma 5 *1-morphisms in* $\mathrm{Tra}(p, i)$ *are in bijection with stable isomorphisms of bundle gerbes.*

Proof. According to def.**??** a 1-morphism of pre-trivializations comes with a 2-morphism (**??**) of trivial line-2-bundles. According to prop. **??** this line-2-bundle 2-morphism defines an isomorphism of line bundles with connection

$$p_1^*h \otimes g' \xrightarrow{\bar{\epsilon}_g} g \otimes p_2^*h$$

The tin can equation (**??**) is then equivalent to the compatibility condition 3. □