Line Bundle Gerbes from 2-Transport

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Abstract

The concept of a line bundle gerbe with connection and curving is a special case of transition data of 2-transport.

Our main aim it to prove theorem 1. Before doing so, we motivate the discussion by some considerations concerning associated 2-bundles.

Let \( \rho : \Sigma(G_2) \to T' \subset \text{Mod}_C \) be a faithful representation of the 2-group \( G_2 \) and

\[
T' \xrightarrow{i} T
\]

some monomorphism. Then we say that \( \text{tra} \) is a associated \( C \)-vector transport with respect to \( (\rho, i) \) if it admits a proper trivialization

\[
\begin{array}{ccc}
\mathcal{P}_U & \xrightarrow{p} & \mathcal{P} \\
\downarrow \text{tra} & & \downarrow \text{tra} \\
G_2 & \xrightarrow{\rho} & T' \\
& \xrightarrow{i} & T
\end{array}
\]

Some familiar associated 2-bundles come from the following class of representations of 2-groups.

**Proposition 1** For any strict 2-group \( G_2 = (H \to G) \), a representation \( \rho : \Sigma(H) \to \text{Vect}_K \) of \( H \) for which the family \( \{ \rho(h) | h \in H \} \) is linearly independent over \( K \) induces a representation

\( \tilde{\rho} : \Sigma(G_2) \to \text{Bim}(\text{Vect}) \xrightarrow{i} \text{Mod}_{\text{Vect}} \)
given by

\[ \hat{\rho} : \Sigma(G_2) \rightarrow \text{Bim}(\text{Vect}) \]

\[ A_\rho \equiv \langle \rho(h) \mid h \in H \rangle \]

Here is the algebra generated by the endomorphisms representing \( H \) and \((A_\rho, g)\) is \( A_\rho \) regarded as a bimodule over itself, with the right action twisted by the automorphism \( g \).

Proof. Let

\[ \rho : \Sigma(H) \rightarrow \text{Vect} \]

and notice that the notation for composition is such that

\[ V \xrightarrow{\rho(h') \circ \rho(h)} V = V \xrightarrow{\rho(h')} V \xrightarrow{\rho(h)} V = V \xrightarrow{\rho(h') \circ \rho(h)} V. \]

Also recall that 2-morphisms in \( \Sigma(G_2) \)

\[ g \]

are labeled by \( g \in G \) and \( h \in H \) with

\[ g'(f) \]

\[ h \]
for arbitrary $f \in H$. What we shall need below is the commutativity of the image of this diagram under $\rho$

\[
\begin{array}{c}
V \\ \Downarrow \rho(g(f)) \end{array} \xrightarrow{\rho(h)} \begin{array}{c} V \\ \Downarrow \rho(g(h)) \end{array}.
\]

In order to construct $\tilde{\rho}$ let now

\[
\text{End}_V \supset A_{\rho} \equiv \langle \rho(h) \mid h \in H \rangle
\]

be the subalgebra of the endomorphism algebra of $V$ which is generated by the linear maps $\rho(h)$ for all $h \in H$. We obtain for each $g \in \text{Aut}(H)$ an automorphism $\rho(g) \in \text{Aut}(A_{\rho})$ of this algebra by setting

$$\rho(g) : \rho(h) \mapsto \rho(g(h))$$

for all $h \in H$, and extended linearly to all of $A_{\rho}$.

Using this, for each $g \in G$ we define an $A_{\rho}$-bimodule

$$(A_{\rho}, g) \equiv A_{\rho} \xrightarrow{\text{Id}} A_{\rho} \xrightarrow{\rho(g)} A_{\rho}$$

which, as an object in $\textbf{Vect}$, is $A_{\rho}$ itself, with both the right and the left $A_{\rho}$ action given by the product in $A_{\rho}$, but with the right action twisted by $\rho(g)$:

$$\begin{align*}
\rho(h) \cdot a & \equiv \rho(h) \circ a \\
 a \cdot \rho(h) & \equiv a \circ \rho(g(h)).
\end{align*}$$

for all $a \in A_{\rho}$.

The tensor product over $A_{\rho}$ corresponds to the composition of automorphisms

$$(A_{\rho}, g) \otimes_{A_{\rho}} (A_{\rho}, g') = (A_{\rho}, gg'),$$

which shows that $\tilde{\rho}$ is properly functorial on the level of 1-morphisms. For each 2-morphism

$$\begin{array}{c}
\bullet \\ \Downarrow h \\
\bullet
\end{array} \xrightarrow{g} \begin{array}{c}
\bullet \\ \Downarrow g' \\
\bullet
\end{array}$$

define a morphism of bimodules

$$\tilde{\rho}(h) : (A_{\rho}, g) \rightarrow (A_{\rho}, g')$$

$$a \mapsto a \circ \rho(h).$$
This map trivially respects the left $A_\rho$-action. That it also respects the right $A_\rho$ action is a consequence of the commutativity of (1):

\[
\begin{array}{c}
(a, \rho(f)) \xrightarrow{\tilde{\rho}(h) \times \text{Id}} (a \circ \rho(h), \rho(f)) \\
(2) \\
\downarrow \\
(a \circ \rho(g(f))) \xrightarrow{\tilde{\rho}(h)} a \circ \rho(g(f)) \circ \rho(h)
\end{array}
\]

That $\tilde{\rho}$ defined this way is functorial for vertical composition follows from

\[
\begin{array}{c}
\tilde{\rho}
\end{array}
\]

by our remark above. Finally, the 2-functoriality of $\tilde{\rho}$ requires that

\[
\begin{array}{c}
\tilde{\rho}
\end{array}
\]

This can be checked for instance by representing elements of $(A_\rho, g) \otimes_{A_\rho} (A_\rho, \text{Id})$ by $(a, \text{Id}) \in A_\rho \times A_\rho$. Then in particular $\tilde{\rho}(h)((a, \text{Id})) = (a, \text{Id} \circ \rho(h)) \sim (a \circ \rho(g(h)), \text{Id})$.

\[\square\]

**Example 1**

Let $G_2$ be the automorphism 2-group of $\Sigma(U(1))$ $G_2 = (U(1) \to \mathbb{Z}_2)$. Let $\rho : \Sigma(U(1)) \to \text{Vect}_\mathbb{R}$ be the defining 2 real dimensional representation.
In this case we find $A_ρ \simeq \mathbb{C}$, the complex numbers, regarded as an $\mathbb{R}$-algebra. The bimodule $(A_ρ, \text{Id})$ is just $\mathbb{C}$ itself, with the left and right $\mathbb{C}$-action given by multiplication of complex numbers.

$$(A_ρ, \text{Id}) = \mathbb{C}.$$

Denote the nontrivial element of $\mathbb{Z}_2$ by $\sigma$. The bimodule $(A_ρ, \sigma)$ is, as an object, $\mathbb{C}$, with the left $\mathbb{C}$-action given by multiplication of complex numbers and the right $\mathbb{C}$-action given by conjugation followed by multiplication. We write

$$(A_ρ, \sigma) \equiv \mathbb{C}_σ.$$

Concretely, the left and right actions on $\mathbb{C}_σ$ are

$$\begin{aligned}
\mathbb{C} \times \mathbb{C}_σ &\rightarrow \mathbb{C}_σ \\
(c, d) &\mapsto cd
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{C}_σ \times \mathbb{C} &\rightarrow \mathbb{C}_σ \\
(d, c) &\mapsto \bar{c}d.
\end{aligned}$$

Similarly, for any complex vector space $V$, let

$$V_σ \simeq V \otimes \mathbb{C}_σ$$

and

$$\sigma V \simeq \mathbb{C}_σ \otimes V$$

be the $\mathbb{C}$-$\mathbb{C}$-bimodule $V$, as an object, but with the left or right $\mathbb{C}$ action twisted, as indicated.

Notice that we have the canonical isomorphism

$$\sigma V_σ \simeq \bar{V},$$

where $\bar{V}$ is $V$ equipped with the opposite complex structure, and hence in particular the canonical identification

$$\mathbb{C}_σ \otimes \mathbb{C}_σ \simeq \bar{\mathbb{C}} \simeq \mathbb{C}.$$

Denote by $\text{Bim}_\mathbb{C}$ the 2-category of $\mathbb{C}$-$\mathbb{C}$-bimodules, with single object $\mathbb{C}$, bimodules up to canonical isomorphism as 1-morphisms and bimodule intertwiners as 2-morphisms.

We write
and find in particular

\[
\sigma \rightarrow \rightarrow \sigma \rightarrow \rightarrow
\]

\[
\sigma \rightarrow \rightarrow \sigma \rightarrow \rightarrow
\]

\[
\sigma \rightarrow \rightarrow \sigma \rightarrow \rightarrow
\]

It follows that we get an involutive inner automorphism

\[
\text{Bim}_C \xrightarrow{\text{Ad}_\sigma} \text{Bim}_C
\]

given by conjugation with \( \sigma \) as

\[
s : \quad \sigma \rightarrow \rightarrow \sigma \rightarrow \rightarrow
\]

\[
\sigma \rightarrow \rightarrow \sigma \rightarrow \rightarrow
\]

Given any transport

\[
\text{tra} : \mathcal{P} \rightarrow \text{Bim}_C
\]

we hence obtain what could be called the “opposite” transport

\[
\text{tra}^{\text{op}} \equiv (\text{Ad}_\sigma)^* \text{tra} : \mathcal{P} \rightarrow \text{Bim}_C
\]

We will be interested in transition morphisms in \( \text{Trans}(\mathcal{P}, \text{Bim}_C) \). Consider the case where such a morphism involves \( \sigma \) in its defining tin can equation as follows

\[
p_{12}^L \quad p_{23}^L
\]

\[
p_{13}^L \quad p_{13}^L
\]

\[
\sigma \rightarrow \rightarrow \sigma \rightarrow \rightarrow
\]

\[
\sigma \rightarrow \rightarrow \sigma \rightarrow \rightarrow
\]

The existence of the identity-2-morphisms here says that the transition lines are related by \( L' = \bar{L} \).

This equation can equivalently be rewritten as

\[
p_{12}^L \quad p_{23}^L
\]

\[
p_{13}^L \quad p_{13}^L
\]

\[
\text{tra} \rightarrow \rightarrow \text{tra}
\]

\[
\text{tra} \rightarrow \rightarrow \text{tra}
\]
which says that

\[ f' = \bar{f}. \]

**Example 2**

\[ G_2 = \langle U(n) \rightarrow PU(n) \rangle \]

\[ \rho : U(n) \rightarrow \mathbb{C}^n \]

\[ PU(n) = \{ [g] | g \in U(n) \} \]

\[ A_\rho = \text{End}_{\mathbb{C}^n} \]

\[ (A_\rho, [g]) \times A_\rho \rightarrow \mathbb{C}^n_{[g]} \]

\[ (a, b) \mapsto a \circ \rho(g)^{-1} \circ b \circ \rho(g) \]

**Example 3**

Let \( A \rightarrow X \) be an algebra bundle with typical fiber \( \text{End}_{\mathbb{C}^n} \). Let \( \nabla \) be a connection on \( E \) and let \( \text{tra}_\nabla : P_1(X) \rightarrow \text{Trans}(A) \) be the corresponding parallel transport. The automorphism group of \( \text{End}_{\mathbb{C}^n} \) is \( PU(n) \), hence \( A \) is an associated \( PU(n) \)-bundle.

Define a 2-transport

\[ \text{tra}_{(A, \nabla)} : P_2(X) \rightarrow \text{Bim(\text{Vect})} \]

by

\[
\text{tra}_{(A, \nabla)} : x \quad \begin{array}{c}
\downarrow S \\
\gamma_1
\end{array} \quad y \quad \begin{array}{c}
\downarrow S \\
\gamma_2
\end{array} \quad A_x \quad \begin{array}{c}
\circ f(S) \\
\gamma_1
\end{array} \quad A_y,
\]

where \( f(S) \) is the unique lift of \( \text{tra}_\nabla(\bar{\gamma}_1) \circ \text{tra}_\nabla(\gamma_2) \) to \( A_x \) such that

\[ \exp \left( n \int_S B \right) = \det(f(S)). \]

In order to see that this assignment is indeed 2-functorial, choose a basis \( A_x \xrightarrow{t(x)} \text{End}_{\mathbb{C}^n} \) for all endpoints involved in the computation. Then use the logic of example 2.

Diagrammatically, we would naturally associate to each surface

\[
\begin{array}{c}
x \quad \begin{array}{c}
\downarrow S \\
\gamma_1
\end{array} \quad y \quad \begin{array}{c}
\downarrow S \\
\gamma_2
\end{array}
\end{array}
\]
the 2-cell

\[
\begin{array}{c}
\xymatrix{ 
  A_x \ar[r]_{f(S)} & A_y \\
  (A_x, \text{tra}_V(\gamma_1)) \ar@{-}[u] & (A_x, \text{tra}_V(\gamma_2)) \ar@{-}[u]
}
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{ 
  A_x \ar[r]^{	ext{Id}} & A_y \\
  \exp(\int_B B) \ar@{-}[u] & \ar@{-}[u] \\
  \text{tra}_V(\gamma_2) \ar@{-}[u] & \ar@{-}[u]
}
\end{array}
\]

In terms of string diagrams the right hand side reads

This can be seen to be almost the same bimodule homomorphism as above, up to a scalar multiple. Where before we had the determinant, this involves a trace. As a result, this second assignment is not 2-functorial in general.

**Proposition 2** The 2-transport from example 3 is an associated **Vect**-vector transport with respect to

\[
(U(1) \to \mathbb{Z}_2) \xrightarrow{\hat{\rho}} \text{Bim}_C \xrightarrow{i} \text{Bim}(\text{Vect}) ,
\]

with \(\hat{\rho}\) the representation from example 1.

**Proof.**
Locally we may always identify, \(A|_U \xrightarrow{\tau} \text{End}_V\), the bundle \(A\) with the endomorphism bundle of some vector bundle \(V\) with connection \(\text{tra}_V\). This can
be expressed in terms of bimodules as

\[
A_x \xrightarrow{(A_x, \text{tr}_\gamma(\gamma))} A_y.
\]

\[
\text{End}_{V_x} \xleftarrow{(\text{End}_{V_x}, \text{Ad}_{\text{tr}_\gamma(\gamma)})} \text{End}_{V_y}.
\]

Consider the bimodule homomorphisms

\[
\begin{array}{ccc}
\text{End}_{V_x} & \xrightarrow{\text{Id}} & \text{End}_{V_y} \\
V_x & \xrightarrow{\text{tr}_\gamma(\gamma)} & V_y \\
C & \xrightarrow{\text{Id}} & C
\end{array}
\]

and

\[
\begin{array}{ccc}
C & \xrightarrow{\text{Id}} & C \\
V_x^* & \xrightarrow{\text{tr}_\gamma^*(\gamma)} & V_y^* \\
\text{End}_{V_x} & \xleftarrow{(\text{End}_{V_x}, \text{Ad}_{\text{tr}_\gamma(\gamma)})} & \text{End}_{V_y}
\end{array}
\]

These fit into an adjoint equivalence due to

\[
\begin{array}{ccc}
\text{End}_{V_x} & \xrightarrow{(\text{End}_{V_x} \otimes W_\gamma, \text{tr}_\gamma)} & \text{End}_{V_y} \\
V_x & \xrightarrow{\text{tr}_\gamma(\gamma)} & V_y \\
C & \xrightarrow{\text{Id}} & C \\
V_x^* & \xrightarrow{\text{tr}_\gamma^*(\gamma)} & V_y^* \\
\text{End}_{V_x} & \xleftarrow{(\text{End}_{V_x} \otimes W_\gamma, \text{tr}_\gamma)} & \text{End}_{V_y}
\end{array}
\]
Therefore, given a 2-form $B \in \Omega^2(U)$, we can form the 2-transport

\[
\begin{array}{c}
\text{End}_{V_x} \xrightarrow{(A_x, \text{Ad}_{\text{tra}_V(\gamma_1)})} \text{End}_{V_y} \\
\downarrow \\
\text{End}_{V_x} \xrightarrow{(A_x, \text{Ad}_{\text{tra}_V(\gamma_2)})} \text{End}_{V_y}
\end{array}
\]

\[
\equiv \begin{array}{c}
\text{End}_{V_x} \xrightarrow{\exp(\int_{\gamma_1} B)} \text{End}_{V_y} \\
\downarrow \\
\text{End}_{V_x} \xrightarrow{\exp(\int_{\gamma_2} B)} \text{End}_{V_y}
\end{array}
\]

The composition of 2-cells on the right corresponds to the bimodule homomorphism which sends $a \in \text{End}_{V_x}$ to

\[
\begin{align*}
&\left( V_x \xrightarrow{a} V_x \right) \\
\mapsto &\left( \sum_i V_y \xrightarrow{e^i} \mathbb{C} \xrightarrow{e_i} V_y \xrightarrow{\text{tra}_V(\gamma_1)} V_x \xrightarrow{a} V_x \xrightarrow{\text{tra}_V(\gamma_1)} V_y \right) \\
\mapsto &\exp\left( \int_{\gamma_2} B \right) \left( \sum_i V_x \xrightarrow{\text{tra}_V(\gamma_2)} V_y \xrightarrow{e^i} \mathbb{C} \xrightarrow{e_i} V_y \xrightarrow{\text{tra}_V(\gamma_1)} V_x \xrightarrow{a} V_x \xrightarrow{\text{tra}_V(\gamma_1)} V_y \right) \\
\mapsto &\exp\left( \int_{\gamma_2} B \right) \left( \sum_i V_x \xrightarrow{\text{tra}_V(\gamma_2) \text{tra}_V(\gamma_1)} V_y \xrightarrow{\text{tra}_V(\gamma_1)} V_x \xrightarrow{a} V_x \right),
\end{align*}
\]

hence to $a \circ \text{tra}_V(\gamma_1) \circ \text{tra}_V(\gamma_2)$, up to a scalar factor. This is indeed, locally, the assignment of the 2-transport from example 3. We re-obtain the global
Proposition 3 The transitions of the local trivialization from prop. 2 are $i$-transitions, for the obvious embedding $i : \Sigma(\Sigma(C)) \longrightarrow \Sigma(1DVect_C)$.

This motivates the study of $\text{Tra}(i)$, the 2-category of $i$-transitions. Let $p : Y \to X$ be a surjective submersion.

Theorem 1 The 2-category of $\text{Tra}(p, i)$ is equivalent (isomorphic, even) to the 2-category of line bundle gerbes for fixed $Y$

$$\text{Tra}(p, i) \simeq \text{BunGer}(Y).$$

We prove this using a couple of lemmas.

Lemma 1 Let $\text{tra} : \mathcal{P}_2 \to \Sigma(1DVect)$ be a 2-transport which assigns 1-dimensional vector spaces to paths and linear maps to surfaces.

1. 1-Automorphisms $\text{tra} \sim \text{tra}$ are in bijection with flat transport $\mathcal{P}_1 \to 1DVect$, i.e. with flat line bundle with connection.

2. Composition of these 1-automorphisms corresponds to taking the tensor product of the corresponding line bundles.
3. 2-morphisms between these 1-automorphisms correspond to natural transformations of the transport 1-functors of the corresponding line bundles and hence to isomorphisms between these line bundles which fix the base space.

Proof.

1. Write \( V = \text{tra}(\gamma) \) for the vector space associated by \( \text{tra} \) to \( x \xrightarrow{\gamma} y \). Then

\[
\begin{array}{c}
\bullet \quad \text{tra}(\gamma) \quad \bullet \\
\phi(x) \\
\phi(y)
\end{array}
\]

is a linear map

\[
V \otimes \phi(y) \xrightarrow{\phi(\gamma)} \phi(x) \otimes V.
\]

Since \( V \) is 1-dimensional this defines a linear map

\[
\phi(y) \xrightarrow{\phi(\gamma)} \phi(x)
\]

under the isomorphism

\[
\text{Hom}(V \otimes \phi(y), \phi(x) \otimes V) \simeq \text{Hom}(\phi(y), \phi(x) \otimes V)
\]

\[
\simeq \text{Hom}(\phi(y), \phi(x) \otimes V^*)
\]

\[
\simeq \text{Hom}(\phi(y), \phi(x) \otimes K)
\]

\[
\simeq \text{Hom}(\phi(y), \phi(x)).
\]

The functoriality condition on \( \phi \)

\[
\begin{array}{c}
\bullet \quad \text{tra}(\gamma_1) \quad \bullet \\
\phi(x) \\
\phi(y) \\
\phi(z)
\end{array}
\]

\[
\begin{array}{c}
\bullet \quad \text{tra}(\gamma_2) \quad \bullet \\
\phi(\gamma_1) \\
\phi(\gamma_2) \\
\phi(\gamma_1 \cdot \gamma_2)
\end{array}
\]

translates similarly into

\[
\phi(z) \xrightarrow{\phi(\gamma_2)} \phi(y) \xrightarrow{\phi(\gamma_1)} \phi(x) = \phi(z) \xrightarrow{\phi(\gamma_1 \cdot \gamma_2)} \phi(x).
\]
Therefore $\tilde{\phi}$ defines a functor

$$\tilde{\phi} : \mathcal{P}_1(M) \to \text{Vect}_1$$

and hence a bundle with connection on $M$. Finally, $\phi$ has to make the tin can equation hold

Since we have the same $x \to y$ on both sides this implies that

$$\phi(\gamma_1) = \phi(\gamma_2) .$$

Hence $\phi$ is flat. Running these arguments backwards shows that conversely every flat line bundle on $M$ gives rise to an automorphism $\text{tra} \to \phi \to \text{tra}$. 

2. The composition

$$\phi(x) \quad \phi(y) \quad \equiv \quad \phi_1(x) \quad \phi_1(y)$$

$$\phi(\gamma) \quad \phi_1(\gamma) \quad \phi_2(\gamma)$$

corresponds to

$$\phi(\gamma) = \phi_1(\gamma) \otimes \phi_2(\gamma)$$
3. A 2-morphism

\[ \phi_1 \xrightarrow{\text{tra}} A \xrightarrow{\text{tra}} \phi_2 \]

satisfies the tin can equation of the following form:

\[ \phi_1(x) \xrightarrow{A(x)} \phi_1(y) = \phi_2(x) \xrightarrow{A(y)} \phi_2(y) \]

Under the above identification of \( \phi(\gamma) \) with a linear map

\[ \hat{\phi}(x) \xrightarrow{\hat{A}(x)} \hat{\phi}(y) \]

this is equivalent to a natural transformation

\[ \phi_2(x) \xrightarrow{\hat{A}(x)} \phi_1(x) \]

\[ \hat{\phi}_1(\gamma) \xrightarrow{\hat{A}(y)} \hat{\phi}_2(\gamma) \]

\[ \phi_2(y) \xrightarrow{\hat{A}(y)} \phi_1(y) \]

\[ \square \]

In the same way one proves

**Lemma 2**

1. 1-morphisms of i-trivial 2-transport are in bijection with line bundles with connection.

2. Composition of such 1-morphisms corresponds to taking the tensor product of the corresponding line bundles.

3. 2-morphisms between such 1-morphisms of trivial line-2-bundles correspond to bundle isomorphisms of the corresponding line bundles.

**Lemma 3** Let \( \text{tra}_B, \text{tra}_{B'} : \mathcal{P}_2 \to \Sigma \text{1DVect} \) be i-trivial 2-transport coming from the 2-forms \( B, B' \in \Omega^2(\text{Lie}(U(1))) \). Let \( \text{tra}_B \xrightarrow{\text{tra}_{B'}} \text{tra}_{B'} \) be the morphism given by the line bundle with connection \( \nabla \) by lemma 2. Then

\[ B' = B + F_\nabla. \]
Proof. The existence of \( \text{tra}_B \xrightarrow{\text{tra}} \text{tra}_B' \) is equivalent to the 2-commutativity of all respective tin cans:

\[
\begin{array}{c}
K \\
\rho(x) \downarrow \downarrow \\
\exp(f \rho(p_B^1)) \\
\end{array}
\begin{array}{c}
K' \\
\rho(y) \downarrow \downarrow \\
\exp(f \rho(p_B^2)) \\
\end{array}
\]

This immediately implies the above statement. \( \square \)

**Definition 1 (Murray)** A line bundle gerbe over a manifold \( M \) is

- a surjective submersion

\[
\begin{array}{c}
Y \\
\downarrow \\
M \\
\end{array}
\]

- a \( \mathbb{C}^\times \)-bundle

\[
\begin{array}{c}
L \\
\downarrow \\
Y^{[2]} \\
\end{array}
\]

- over \( Y^{[3]} \) a bundle isomorphism

\[
p_{12}^*L \otimes p_{23}^*L \xrightarrow{f} p_{13}^*L
\]

which is associative in the sense that on \( Y^{[4]} \) the diagram

\[
\begin{array}{c}
p_{12}^*L \otimes p_{23}^*L \otimes p_{34}^*L \\
\downarrow \downarrow \downarrow \\
Id \otimes p_{23}^*f \\
\end{array}
\begin{array}{c}
p_{13}^*L \otimes p_{34}^*L \\
\downarrow \\
p_{14}^*L \\
\end{array}
\begin{array}{c}
p_{12}^*L \otimes p_{24}^*L \\
\downarrow \downarrow \\
p_{14}^*L \\
\end{array}
\]

commutes.
A connective structure on a bundle gerbe (also known as connection and curving on a bundle gerbe) is

- a connection $\nabla$ on $L$
- a 2-form $\omega \in \Omega^2(Y)$ on $Y$

such that on $Y^{[2]} \xrightarrow{p_2} Y$ the equation

$$p_2^*\omega - p_1^*\omega = F_{\nabla}$$

holds.

**Lemma 4** $(p,i)$-transition tetrahedra are in bijection with line bundle gerbes with connection and curving.

**Proof.** Using the above notation, identify $Y$ with $U$. By prop. ?? the trivialization transition $g$ defines a line bundle with connection on $U^{[2]}$ and vice versa. Hence identify

$$g \leftrightarrow (L, \nabla).$$

The picture obtained is

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
U^{(2)} & \xrightarrow{g} & U \\
\downarrow & & \downarrow \\
M & \xrightarrow{} & M \\
\end{array}
\begin{array}{ccc}
\downarrow & & \downarrow \\
L & \xrightarrow{} & Y^{[2]} \xrightarrow{p_2} Y \\
\downarrow & & \downarrow \\
M & \xrightarrow{} & M \\
\end{array}
$$

Identify the gerbe product with the inverse of the modification $f$ using the third item of prop. ?? By prop. ?? this does satisfy the required associativity condition.

In order to match the connection data, observe that the line-2-bundle $\text{tra}_U$ is trivial by assumption and hence defines, according to def. ??, a global 2-form $B$ on $U$. Identify this 2-form with the curving $\omega$ of the bundle gerbe. Prop. ?? says that $\text{tra}_U$ and $(L, \nabla)$ satisfy the condition of a gerbe connection

$$p_2^*B - p_1^*B = F_{\nabla}.$$

□

**Definition 2 (Murray, Stevenson)** Given two bundle gerbes with connective structure $(L, Y)$ and $(L', Y)$ a **stable isomorphism**

$$
(L, Y) \xrightarrow{(H, \xi)} (L', Y)
$$
is a line bundle with connection $H \rightarrow Y$ together with an isomorphism

$$p_1^*H \otimes L \xrightarrow{\varepsilon} L' \otimes p_2^*H$$

of line bundles with connection on $Y^{[2]}$ satisfying

$$p_1^*H \otimes p_1^*L \otimes p_2^*L \xrightarrow{p_1^*\varepsilon \otimes \text{Id}_{p_2^*L}} p_1^*L' \otimes p_2^*H \otimes p_2^*L$$

(3)

$$\xrightarrow{\text{Id}_{p_1^*H} \otimes f} p_1^*L \otimes p_1^*L \otimes p_3^*H$$

$$\xrightarrow{p_1^*\varepsilon \otimes \text{Id}_{p_3^*H}} p_1^*L' \otimes p_2^*L' \otimes p_3^*H$$

Lemma 5 1-morphisms in $\text{Tra}(p,i)$ are in bijection with stable isomorphisms of bundle gerbes.

Proof. According to def. a 1-morphism of pre-trivializations comes with a 2-morphism of trivial line-2-bundles. According to prop. this line-2-bundle 2-morphism defines an isomorphism of line bundles with connection

$$p_1^*h \otimes g' \xrightarrow{\varepsilon} g \otimes p_2^*h$$

The tin can equation is then equivalent to the compatibility condition 3. $\square$