Local nets from parallel transport 2-functors

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Abstract

For every 2-functor on the 2-category of paths in a Lorentzian space we can define its endomorphism co-presheaf. We show that this copresheaf is automatically a local net of monoids satisfying the time slice axiom. For suitable codomains of the 2-functor it is a local net of C^*-algebras. It is covariant if the 2-functor is equivariant. One can interpret this as the passage from the Schrödinger to the Heisenberg picture in QM raised to 2-dimensional field theory.

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This exposition is based on [1].

Contents

1 $(0+1)$-dimensional QFT

As warmup and motivation, recall the situation for quantum mechanics.

1.1 Classical

Example: electromagnetic background field for charged particle.

- traditionally: vector bundle with connection $(E \to X, \nabla)$;
- but what is really used (in action functional): the parallel transport

$$\begin{array}{ccc} \text{(paths)} & \to & \text{(morphisms between fibers)} \\ (\, x \xrightarrow{\gamma} y \,) & \mapsto & (\, E_x \xrightarrow{P\exp\int_\gamma \nabla} E_y \,) \end{array} \quad ,;$$

- this assignment has two crucial properties: it is
  - *local=functorial* [** picture goes here, but see also below **]
  - *smooth* (in a sense which can be made precise)

Theorem 1.1 ([9]) *Let a parallel transport functor be a functor from the path groupoid of X to some category of fibers which is smooth in the above sense. We have:*

$$\left\{ \text{parallel transport functors } \mathcal{P}_1(X) \longrightarrow \mathrm{Vect} \right\} \simeq \{\text{smooth vector bundles with connection on } X\}$$

$$\left\{ \text{parallel transport functors } \mathcal{P}_1(X) \longrightarrow G\mathrm{Tor} \right\} \simeq \{\text{smooth } G\text{-principal bundles with connection on } X\}$$

Remark: Precursors. For restriction to closed paths this idea is old [Kobayashi:1954, Milnor:1956, Teleman:1960, Barrett:1991, Lewandowski:1993, Caetano-Picken:1994]. But non-closed paths are crucial for our purpose.

Remark: generalized connections in loop quantum gravity. The ide of encoding connections in terms of their parallel transport functor is the starting point for the quantization of the gravitational field, regarded as a connection on a fiber bundle, in "loop quantum gravity" – but there the smoothness and continuity requirement on the parallel transport functor is dropped: "generalized connections".

Theorem 1.2 ([9]) *A "generalized connection" in this sense comes from a smooth connection on a smooth bundle if and only if it has smooth Wilson lines.*

1.2 Quantum

Now pass to quantum theory of particle charged under (E, ∇).

1.2.1 Schrödinger picture – FQFT

Observation. In the Schrödinger picture the result of quantization is again parallel transport – now on the *worldline*.

$$(\, t_0 \longrightarrow t_1 \,) \mapsto (\, \mathcal{H}_{t_1} \xrightarrow{U(t_1,t_2) = P\exp \frac{1}{i\hbar}\int_{t_0}^{t_1} H(t)\,dt} \mathcal{H}_{t_2} \,)$$

- fibers: spaces $\mathcal{H}$ of states;
- connection: Hamiltonian H;

- parallel transport: time evolution;
- functoriality: sewing axiom of the path integral.

This motivates

Definition 1.3 (functorial QFT [Atiyah, Segal]) *An n-dimensional QFT is a functor*

$$U : n\mathrm{Cob}_S \to \mathsf{VectorSpaces}\,,$$

on the category of n-dimensional cobordisms with S-structure, e.g.

- *S = diffeomorphism classes: topological QFT [Atiyah];*
- *S = conformal: conformal QFT [Segal];*
- *S = Euclidean: euclidean QFT [Stolz-Teichner].*

1.2.2 Heisenberg picture – AQFT

Question. *How does this connect to the Haag-Kastler axioms for QFT (AQFT)?*

Obvious answer in 1d. Pass to Heisenberg picture by forming the

Definition 1.4 (endomorphism co-presheaf)

- *to causal subset* $(t_1, \infty) \subset \mathbb{R}$ *assign algebra* $\mathrm{End}(\mathcal{H}_{t_1})$*;*
- *to inclusiuon of subsets* $(t_2, \infty) \subset (t_1, \infty)$ *assign algebra homommorphism*

$$\mathrm{End}(\mathcal{H}_{t_1}) \xrightarrow{a \mapsto U(t_1,t_2)\, a\, U(t_1,t_2)^{-1}} \mathrm{End}(\mathcal{H}_{t_2}) \;\; .$$

Problem: lack of locality. Same trick won't work for $n \geq 2$, as 1-functors on nCob are *not local enough* to produce local nets of observables.

Solution: extended FQFT. Schrödinger picture in n-dimensional QFT must be n-functor $\to$ n-functorial "extended" or "many tiered" QFT [Baez-Dolan, Freed, Stolz-Teichner, Hopkins-Lurie].

2 $(1+1)$-dimensional QFT

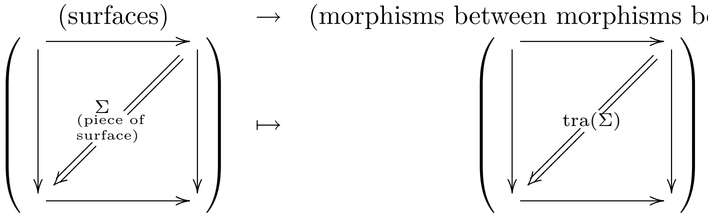
Recall that a 2-category is [** ... **].

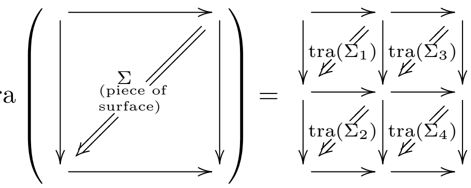
2.1 Classical

Example: B-field background for charged string.

- traditionally: bundle gerbe with connection $(\mathcal{G} \to X, \nabla)$, aka Deligne 3-cocycle;
- but what is really used (in action functional): the parallel transport

(surfaces) $\to$ (morphisms between morphisms between fibers)



- this assignment has two crucial properties: it is
  - *local=2-functorial*: tra 

  - *smooth* (in a sense which can be made precise).

Theorem 2.1 *[10, 11, 4] Let a* <u>*parallel transport 2-functor*</u> *be a 2-functor from the 2-path 2-groupoid of X to some 2-category of fibers which is smooth in the above sense. We have:*

$$\left\{ \textit{parallel transport functors } \mathcal{P}_2(X) \longrightarrow \mathrm{AUT}(G)\mathrm{Tor} \right\} \simeq \{\textit{smooth } G\textit{-gerbes with connection on } X\}$$

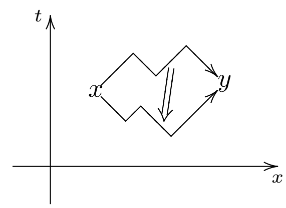
with curvature in degree three[1].

2.2 Quantum

Now quantize.

2.2.1 Schrödinger picture – FQFT

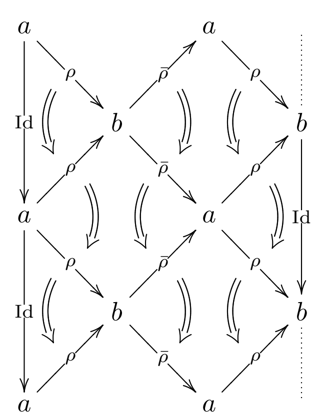
Suppose the result is a

2d extended Minkowskian QFT. On $\mathbb{R}^2$ with its standard Minkowski metric let $P_2(X)$ be the sub-2-category of $\mathcal{P}_2(X)$ containing only piecewise lightlike paths, e.g 

. For C any 2-category, say a 2-functor $Z : P_2(\mathbb{R}^2) \to C$ is an 2d extended Minkowskian QFT if it sends all 2-paths to invertible 2-paths in C. ("unitarity": time evolution is invertible).

Examples .

- Every 2-vector parallel transport [11] yields an example.
- Examples from lattice models:
  - on every edge a Hilbert space of states localized there;
  - on sequences of edges the tensor product of these;
  - on faces the time evolution from the incoming to the outgoing Hilbert spaces.



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[1] Generalization in [2].

2.2.2 Heisenberg picture – AQFT

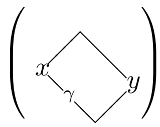
Let $S(\mathbb{R}^2) \subset O(\mathbb{R}^2)$ be the subcategory of the category of open subsets of $\mathbb{R}^2$ given by "causal subsets", i.e. interiors of rectangles all whose sides are lightlike, as usual.

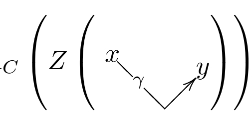
Definition 2.2 (endomorphism co-presheaf of 2-functor) *Given any extended 2-dimensional FQFT, i.e. a 2-functor*

$$Z : P_2(\mathbb{R}^2) \to C$$

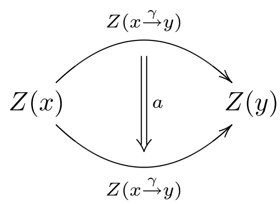
we define a functor

$$\mathcal{A}_Z : S(\mathbb{R}^2) \to \text{Monoids}\,.$$

On objects it acts as $\mathcal{A}_Z$: 

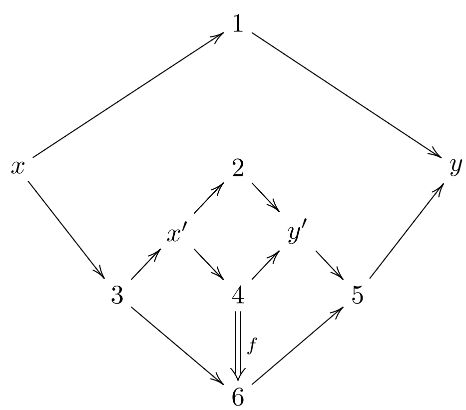
$\mapsto$ End_C 

, *where on the right we form the monoid of 2-endomorphism a in C on the 1-morphism $Z(x \xrightarrow{\gamma} y)$ in C that is the past boundary of $O_{x,y}$,*



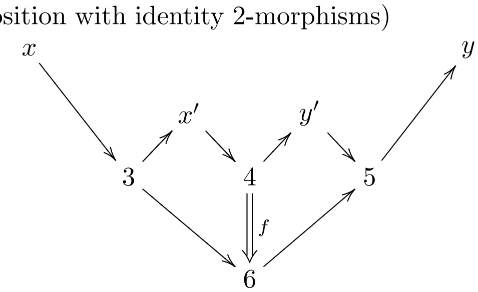
. *On morphisms $\mathcal{A}_Z$ is defined to act as follows.*

For any inclusion $O_{x',y'} \subset O_{x,y} \in S(\mathbb{R}^2)$ consider



in $P_2(\mathbb{R}^2)$. Here the obvious projections along light-like directions (for instance from x' onto $x \to 6$ yielding 3) is used. It is at this point that the light-cone structure crucially enters the construction.

Let f' be the 2-morphism obtained by whiskering (= horizontal composition with identity 2-morphisms) the indicated 2-morphism f with the 1-morphisms $x \to 3$ and $5 \to y$. $f' :=$



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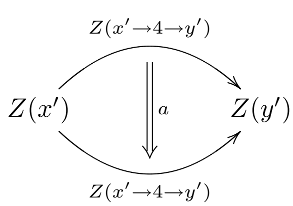
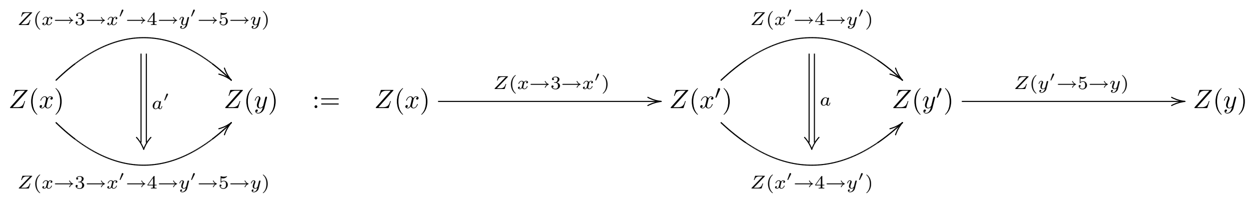
For any $a \in \text{End}_C Z(x', 4, y')$,



, let a' be the corresponding re-whiskering by

$Z(x, 3, x')$ from the left and by $Z(y', 5, y)$ from the right:



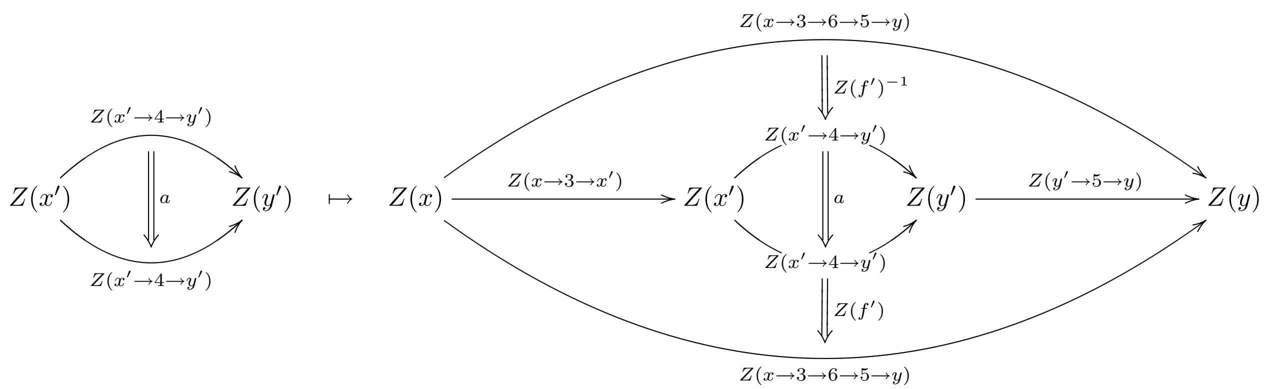
Then we obtain a co-restriction map

$$\mathrm{End}_C(Z(x', 4, y')) \hookrightarrow \mathrm{End}_C(Z(x, 3, 6, 5, y))$$

by setting

$$a \mapsto Z(f') \circ a' \circ Z(f')^{-1} \,,$$

i.e.



Theorem 2.3 ([1]) *$\mathcal{A}_Z$ is a*

- *copresheaf (corestriction maps are functorial);*
- *which is a net (corestriction maps are injections) of monoids;*
- *which are local (monoids on spacelike separated regions commute with each other);*
- *satisfying the time slice axiom (assignment to causal subset fixed by assignment to any Cauchy surface).*

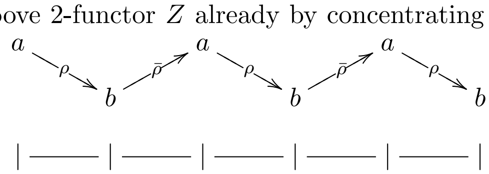
Proof. Basic mechanism: 2-functoriality induces respect for composition and for exchange law:

FQFT		**AQFT**
n-functoriality	$\leftrightarrow$	locality

Details in section A. ☐

Example. Recall lattice model from previous example. The algebras assigned by the corresponding net $\mathcal{A}_Z$ to the elementary causal bigon $O_{\rho,\bar\rho}$ and $O_{\bar\rho,\rho}$ are $\mathcal{A}_Z(O_{\rho,\bar\rho}) = \mathrm{End}_{\mathcal{C}}(\bar\rho \circ \rho)$ and $A_Z(O_{\bar\rho,\rho}) = \mathrm{End}_{\mathcal{C}}(\rho \circ \bar\rho)$.

If $\mathcal{C}$ is a 2-C^*-category and ρ is an "irreducible 1-morphism generating a 2-C^*-category of depth two" as in section 4 of [Zito], then these are C^*-Hopf algebras H and $\hat{H}$ which are duals of each other [Mueger, Zito]. Due to the fact that the 2-morphisms in the above diagrams do not mix ρ and $\bar\rho$, we can understand the nature of the net A_Z obtained from the above 2-functor Z already by concentrating on the endomorphism algebras assigned to a horizontal zig-zag



. If we to restrict to

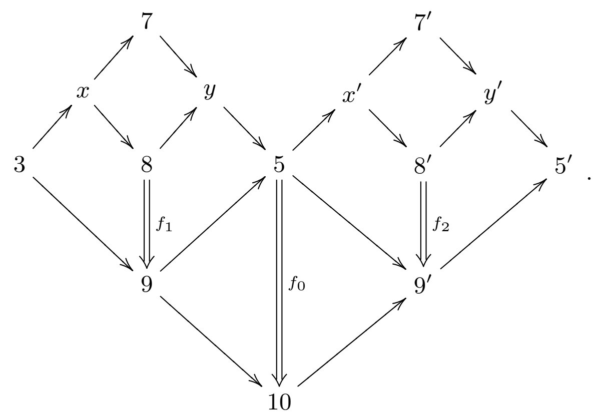
evaluating the net A_Z on zig-zags of even length, this gives rise to a net on the latticized real axis with the property that algebras $A_Z(I_1)$ and $A_Z(I_2)$ commute if the intervals I_1 and I_2 are not just disjoint but differ by at least one lattice spacing. Precisely these kind of 1-dimensional nets are considered in [NillSzlachányi], where they are addressed as *Hopf spin chain models*.

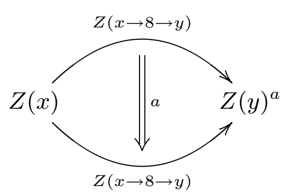
Open questions:

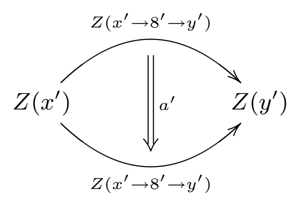
- which 2-functors give nets of type III von Neumann algebra factors? (contunuum limit of lattice models?);
- my main motivation: we interpret [12] the construction in [FuchsRunkelSchweigert] as saying that full rational 2D CFT is, topologically, a cocycle for parallel 2-transport with coefficients in $(\mathbf{B}\mathrm{Bimod}(\mathcal{C}))^I$ for $\mathcal{C}$ a modular tensor category. Aim: refine to full differential cocycle which locally describes conformal nets as above.

A Some diagram proofs

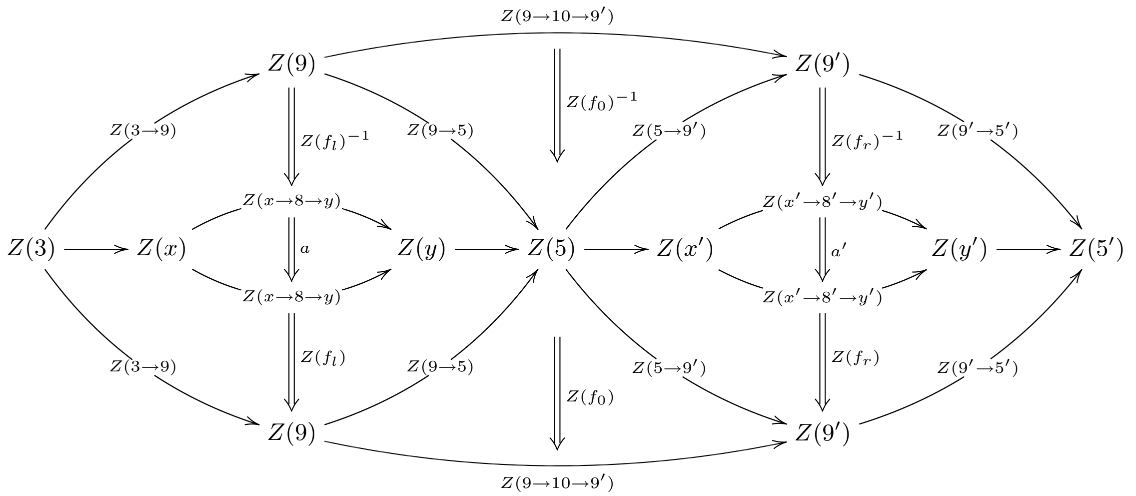
Parts of the proof of theorem 2.3. To see locality, let $O_{x,y}$ and $O_{x',y'}$ be two spacelike separated causal subsets inside $O_{(3,5')}$. The relevant pasting diagram in $P_2(\mathbb{R}^2)$ is of the form



Now given any two endomorphisms 

and 

we can either first include a in $\mathrm{End}_C(Z(3 \to 9 \to 10 \to 9' \to 5'))$ and then a', or the other way around. Either way, the total endomorphism in $\mathrm{End}_C(Z(3 \to 9 \to 10 \to 9' \to 5'))$ is



This means that the inclusions of a and a' in $\mathrm{End}_C(Z(3 \to 9 \to 10 \to 9' \to 5'))$ commute. □

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