Twisted differential String- and Fivebrane structures

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Abstract

Abelian differential generalized cohomology as developed by Hopkins and Singer has been shown by Freed to formalize the global description of anomaly cancellation problems in String theory, such as notably the Green-Schwarz mechanism. On the other hand, this mechanism, as well as the Freed-Witten anomaly cancellation, are fundamentally governed by the cohomology classes represented by the relevant *nonabelian* $O(n)$- and $U(n)$-principal bundles underlying the tangent and the gauge bundle on target space. In this article we unify the picture by describing *nonabelian* differential cohomology and *twisted* nonabelian differential cohomology and apply it to these situations. We demonstrate that the Freed-Witten mechanism for the B-field, the Green-Schwarz mechanism for the H_3-field, as well as its magnetic dual version for the H_7-field define cocycles in twisted nonabelian differential cohomology that may be addressed, respectively, as twisted $\mathrm{Spin}(n)$-, twisted $\mathrm{String}(n)$- and twisted $\mathrm{Fivebrane}(n)$-structures on target space, where the twist in each case is provided by the obstruction to lifting the gauge bundle through a higher connected cover of $U(n)$. We work out the (nonabelian) L_∞-algebra valued connection data provided by the differential refinements of these twisted cocycles and demonstrate that this reproduces locally the differential form data with the twisted Bianchi identities as known from the string theory literature. The treatment for M-theory leads to models for the C-field and its dual in nonabelian differential cohomology.

This is stuff we are still working on. Handle with care.

Section 2 is based on joint work [58] with
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1 Introduction

String theory and M-theory involve various higher gauge-theoretic entities, locally given by differential form fields of higher degree and globally modeled by higher bundles with connection (higher gerbes with connection, higher differential characters) [23] [58]. Some of these entities arise as obstructions to certain lifts and are required by anomaly cancellation arguments to admit trivializations. A choice of such a trivialization equips the underlying manifold with a certain "structure", such as Spin or Spin^c structure or higher structures such as String and Fivebrane structures [57].

The general situation is that these structures are *twisted* in some way and refined to *differential* structures in the sense of generalized cohomology. A general description of abelian differential generalized cohomology has been given by Hopkins and Singer [31] and was used by Freed [23] to identify some twisted differential structures in String theory with cocycles in twisted abelian differential generalized cohomology. However the twists that appear in the Freed-Witten [26] and in the Green-Schwarz anomaly cancellation mechanism [28], as well as in the magnetic dual Green-Schwarz mechanism [57] themselves originate from and are controlled by nonabelian structures, namely the $O(n)$-principal bundle underlying the tangent bundle of spacetime and the $U(n)$-principal bundle underlying the gauge bundle on spacetime, as well as their lifts to the higher connected structure groups.

In this article we describe a theory of *nonabelian* twisted differential cohomology, that builds on [5] [60] [61] [61] [56] [45] and is discussed in more detail in [58], and we show that the Freed-Witten and the Green-Schwarz mechanism, as well as the magnetic dual Green-Schwarz mechanism, define twisted nonabelian differential cocycles that may be interpreted as twisted differential Spin^c-, String- and Fivebrane-structures, respectively. We thus have a refinement of the treatment in [57] to the twisted and differential case. This means, in particular, that the various abelian background fields appearing in the theory are unified into a natural coherent structure with the nonabelian background fields with which they interact. For instance, the relations between the abelian and the nonabelian differential forms that govern the Green-Schwarz mechanism [28] are hence realized as a (twisted) Bianchi identity of a single nonabelian L_∞-algebra valued connection on a twisted $\mathrm{String}(n)$-principal 2-bundle. A similar structure appears in M-theory (for which the above string theory is essentially a boundary) and, in fact, we get a model for the M-theory degree three C-field in nonabelian cohomology, extending previous models (cf. [19]). Our formalism also provides a model for the dual of the C-field in degree eight.

Aspects of such twisted nonabelian differential cohomology in low degree had been described in [1] in the language of twisted nonabelian bundle gerbes with connection. The formalism that we give in section 2 is meant to provide the fully general picture of such twisted nonabelian differential cohomology in an elegant albeit somewhat abstract language. Its more concrete realization in terms of L_∞-algebra valued connection forms, as introduced in [56], is described in section 4 and the explicit derivation of the twisted Bianchi identites of L_∞-algebra connections corresponding to the Green-Schwarz mechanism and its magnetic dual is in section 5.

Contents of this article.

- Part I: **General abstract nonsense**

  The first part, section 2, of the article considers the general abstract nonsense of twisted nonabelian differential cohomology in the general context of a Grothendieck-Rezk-Lurie $(\infty,1)$-topos. This will sound scary to one half of the readership but is actually an immense clarification: such an $(\infty,1)$-topos is nothing but the formalization of a context in which all the familiar operations of the homotopy theory of topological spaces make sense and work as expected, but where these spaces may be something more general and richer than topological spaces: Specifically, for applications to physics in general and for our applications to differential cohomology in particular they may be *smooth* and wildly *infinite-dimensional* spaces, as described in section 2.2.

$$
\begin{tikzcd}
\mathbf{B}\mathrm{String} \arrow[d] & X \arrow[l, dashed, "\hat{g}_{TX}"'] \arrow[r, dashed, "\hat{g}_E"] \arrow[dl, "g_{TX}"] \arrow[dr, "g_E"] \arrow[d, bend right, "\frac{1}{6}c_2(E)"'] \arrow[d, bend left, "\frac{1}{2}p_1(TX)"] \arrow[d, phantom, "\overset{H_3}{\Longleftrightarrow}"] & \mathbf{B}U\langle 8\rangle \arrow[d] \\
\mathbf{B}\mathrm{Spin} \arrow[r, "\frac{1}{2}p_1"] & \mathbf{B}^3U(1) & \mathbf{B}U\langle 6\rangle \arrow[l, "\frac{1}{6}c_2"']
\end{tikzcd}
$$

abelian cohomology

nonabelian cohomology

Figure 1: **Abelian versus nonabelian cohomology.** The relation between abelian and nonabelian cohomology illustrated in a simple example in degree four: The Green-Schwarz mechanism is the statement that two classes in abelian cohomology, namely in 4th differential integral cohomology, coincide. But these classes are particularly obstruction classes to String-lifts in nonabelian cohomology. The middle part of the diagram labelled "abelian cohomology" identifies the cocycles in $H^4(X, \mathbb{Z})$ and the coboundary between them, but does not specify where these cocycles come from. The outer part of the diagram, labelled "nonabelian cohomology" does specify the object whose class is the one identified by the middle part. This is discussed in section 3. The situation becomes more pronounced when this setup is refined to *differential* nonabelian cohomology, discussed in section 5.

The purpose of this first part is to demonstrate where the structures considered in the second part are conceptually rooted. The reader inclined to take for granted that there is good reason to consider these notions is advised to skip section 2. Throughout the article we provide pointers to those parts of section 2 that underlie the issue under discussion.

- Part II: **Topological and differential twisted structures**

  The second part of the article

  - first discusses how the anomaly cancellation mechanisms in String-theory can be understood topologically as twisted higher structures given by twisted nonabelian topological cocycles (section 3);
  - then describes the nonabelian L_∞-algebraic constructions that encode differential refinements of these nonabelian cocycle (section 4);
  - and demonstrates that the twisted Bianchi identites of nonabelian L_∞-algebraic connections on twisted String(n)-principal 2- and Fivebrane(n)-principal 6-bundles coincide with the familiar abelian and nonabelian differential form data appearing in the Green-Schwarz mechanism and its magnetic dual version.
  - Finally, a directly analogous treatment for M-theory yields a model for the C-field and its dual in nonabelian differential cohomology.

  This will sound scary to the other half of the readership, but is actually an immense clarification: a plethora of geometric and algebraic entities considered in String theory is organized this way into a precise and fully natural general abstract structure.

Background and rough idea. The geometric description of the above twisted structures involves twisted cocycles, where cocycles are (generalized) maps $g : X \to A$ between (generalized) spaces X and A which classify (generalized higher) bundles $P \to X$. This is described in section 2. Since the cocycle $g : X \to A$ is usually realized on a local resolution of X, in the form of a Čech cocycle, the cocycle description is usually called a *local* or patchwise description, while the corresponding bundle $P \to X$ represents a *global* description.

	Global structures	**Local structures**	
	Bundles	Cocycles	
		Topological	**Differential**
ordinary	fibers are *sets*	cocycles with coefficients in sheaves with values in *sets*	Cartan-Ehresmann connections
homotopy / categorified	fibers are ∞-*groupoids*	cocycles with coefficients in sheaves with values in ∞-*groupoids*	L_∞-connections

A simple example of this is the Hopf bundle. In the global description, this is just the Hopf fibration $S^1 \to S^3 \to S^2$. The local description consists of choices of patches on S^2 and corresponding transition functions from their intersections to $U(1)$. A standard choice of patches is for instance the upper hemisphere D^+ and the lower hemisphere D^-. The corresponding covering space $Y = \{D^+ \cup D^-\}$ defines the Kan simplicial space $Y_\bullet = \{\underbrace{Y \times_{S^2} Y \times_{S^2} \cdots \times_{S^2} Y}_{n+1}\}_{n\in\mathbb{N}}$ which, as a generalized space, is equivalent to X in a suitable sense. The cocycle $g : S^2 \to U(1)$ classifying the Hopf fibration is then the composite $X \xrightarrow{\simeq} Y_\bullet \longrightarrow U(1)$ with the map $Y_\bullet \to U(1)$ determined by the transition function. Tools for formalizing these statements about maps between generalized spaces, in particular for the case where all generalized spaces involved are taken to be *smooth* are recalled in section 2.2.

We conceive differential refinements of the above cocycles on X essentially as cocycles on the generalized space $\Pi(X)$, the smooth homotopy ∞-groupoid of X, which can be thought of as the conflation of all homotopy groups of X for all basepoints of X. Then Čech cocycles on $\Pi(X)$ are flat Deligne cocycles on X (higher bundles or gerbes with connection). It turns out that such a differential cocycle is locally given by (higher) differential form data [5] [60] [61] [62] [45] as one would expect for a connection on a (higher) bundle. The differential form data one finds in fact organizes itself into the L_∞-connections described in [56]. This is discussed in more detail in section 2.1.6 and section 4.

Next consider the notion of twisted cocycles. The word "twist" is, unfortunately, commonly used for various different phenomena and is thus rather undescriptive by itself. Here we deal with two different kinds of phenomena, both of which often called "twists" – which happen to be really two aspects of a single phenomenon (that we make precise in section 2.1.3): we consider certain morphisms

$$\underbrace{\alpha}_{\substack{\text{some (obstruction)}\\ (n+1)\text{-cocycle}}} \xrightarrow{\qquad\qquad\qquad\qquad \eta \qquad\qquad\qquad\qquad} \underbrace{p}_{\substack{\text{another (obstruction)}\\ (n+1)\text{-cocycle}}}$$

$$\underbrace{\text{trivialization of } p \text{ relative to } \alpha \;\simeq\; \alpha\text{-twisted } p\text{-structure} \;\simeq\; (p,\alpha)\text{-bi-twisted } n\text{-cocycle } \eta}$$

between degree $(n+1)$-cocycles in cohomology theories that characterize higher bundles with connection. In the applications of interest to us, p is the cocycle obstructing a certain lift, such as a lift of the Spin-structure of a manifold to a Spin^c-structure, or to a String-structure, or to a Fivebrane-structure, while α is the class interpreted as providing the twist, which in our applications however arises itself as the obstruction of a lift of the gauge bundle through the Whitehead tower of $U(n)$.

To fill this with a bit more life, consider the following example. If we take $\hat{F}$ to be the cocycle corresponding to a vector bundle with connection living in the category of vector bundles with connection and isomorphisms between these, then $\hat{1}$ would be the trivial bundle with trivial connection and η a trivialization of $\hat{F}$. However, we could take $\hat{F}$ to live in the category of cocycles for vector bundles with connection and arbitrary bundle morphisms between them. In this case $\hat{1}$ would be the cocycle for the trivial rank-1 vector bundle and η would not necessarily be a trivialization, but just a *section* of that bundle. The heuristics which regard a section of a bundle as a possibly *twisted function* is familiar.

Generalizing, consider some n-cocycle $\hat{F}$ representing some n-bundle $\mathcal{F}$ with connection. For instance a bundle with 1-form connection and 2-form curvature for $n = 1$, or a 2-bundle/1-gerbe with 2-form connection and 3-form curvature, and similarly for higher n. Moreover, denote by $\hat{1}$ the corresponding trivial cocycle. Then a morphism from $\hat{1}$ to $\hat{F}$

$$\hat{1} \xrightarrow{\eta} \hat{F} \tag{1.1}$$

constitutes a kind of trivialization of $\hat{F}$. Depending on the precise category in which one takes the cocycles to live, this morphism may be an equivalence, in which case η would be a *coboundary* and $\hat{F}$ would be cohomologous to $\hat{1}$, or it may be something less than an equivalence.

Noticing that a function itself can be regarded as a 0-bundle with connection, we thus find the first occurrence of a general phenomenon, viewed in two equivalent ways:

- *Morphisms* $\mathcal{O} \xrightarrow{\eta} \mathcal{F}$ *from a trivial n-bundle $\mathcal{O}$ to an n-bundle $\mathcal{E}$ may themselves be regarded as* twisted $(n-1)$*-bundles.*
- *Morphisms* $\epsilon \xrightarrow{\eta} \hat{F}$ *from a trivial cocycle to an n-cocycle may themselves be regarded as twisted* $(n-1)$*-cocycles.* We say that $\hat{F}$ *twists* η and that η *is twisted by* $\hat{F}$.

We give a general formalization of higher sections and their interpretation as twisted cocycles in section 2.1.5.

The simplest example to keep in mind is the toy example where the cocycles are closed differential $(n+1)$-forms and the coboundaries are arbitrary n-forms on some space X giving rise to exact forms. In this context we have

$$(\hat{1} \xrightarrow{\eta} \hat{F}) \Leftrightarrow \left(\eta \in \Omega^n(X), \hat{F} \in \Omega^{n+1}_{\mathrm{closed}}(X) \, , \hat{F} = d\eta\right) , \tag{1.2}$$

where $\hat{1}$ now denotes the vanishing differential form (to be thought of as a trivial connection on a trivial vector bundle). We can read this as saying that η, while not closed, is a "twisted closed form", with the twisting being $\hat{F}$. In section 2.1.6 this simple idea is generalized to a definition of general nonabelian differential cohomology as curvature-twisted flat differential cohomology.

The first interesting instance of twisted cohomology occurs for line 2-bundles $\mathcal{F}$ with connection, i.e. abelian gerbes with connection. If we regard $\mathcal{F}$ as living in the 2-category of line 2-bundles and equivalences, then $\hat{1} \xrightarrow{\eta} \hat{F}$ is a trivialization of the corresponding cocycle $\hat{F}$. Such a trivialization is known to come itself from a line 1-bundle. More generally, we can take $\mathcal{F}$ to live in the 2-category of 2-vector bundles. Then (as discussed in [62]) it turns out that a morphsim η may exists even if $\hat{F}$ is not trivializable. It is well known that in this case the morphism η is what is called a *bundle gerbe module* or a *twisted vector bundle* (essentially a projective vector bundle). Twisted K-theory is about classes of such vector bundles that are twisted by a 3-class $\hat{F}$. The pattern continues this way. While less familiar, 3-bundles (2-gerbes) $\hat{F}$ may have morphisms into them which encode twisted 2-bundles (twisted 1-gerbes [1]). These are therefore twists coming from a degree 4-class. We discuss the latter twists in section 5.

There is a simple but relevant generalization of the above situation, obtained by replacing the trivial n-bundle represented by the trivial cocycle $\hat{1}$ by any other fixed n-cocycle $\hat{A}$. In that case we could say that a morphism, replacing (1.1),

$$\hat{A} \xrightarrow{\eta} \hat{F} \tag{1.3}$$

1a. is a trivialization of $\hat{F}$ *relative to* $\hat{A}$, or
1b. that $\hat{A}$ is a *twist of the* $\hat{F}$*-structure*, or
2. that $\hat{F}$ *twists* η *relative to* $\hat{A}$.

This idea is formalized in section 2.1.3.

Notice that in our toy example of closed differential forms, we have

$$(\hat{\alpha} \xrightarrow{\eta} \hat{F}) \Leftrightarrow \left(\eta \in \Omega^n(X), \hat{\alpha}, \hat{F} \in \Omega^{n+1}_{\mathrm{closed}}(X)\,, \hat{F} = \hat{\alpha} + d\eta \right) . \tag{1.4}$$

In particular this means that the (de Rham, in this case) classes $[\hat{\alpha}]$ and $[\hat{F}]$ coincide, $[\hat{\alpha}] = [\hat{F}]$, or, that while $[\hat{F}]$ no longer vanishes, it vanishes after subtracting (the relative trivialization twist) $[\hat{\alpha}]$:

$$[\hat{F}] - [\hat{\alpha}] \in H^{n+1}_{DR}(X)\,. \tag{1.5}$$

If our morphisms are less than equivalences, it need not be true in general that the classes agree entirely this way. They could for instance agree only up to torsion classes.

The most familiar relevant example of this in physics is perhaps perturbative gauge theory: the path integral in Yang-Mills theory breaks up into a sum over classes $[\hat{\alpha}]$ of gauge bundles with connection (the instanton sectors) and in each class an integral over all bundles with connections $\hat{F}$ which are *trivial relative to* $\hat{\alpha}$, hence whose connection is that of $\hat{\alpha}$ plus a globally defined 1-form on the base.

The physical examples of more relevance here arise in various anomaly cancellations in String theory. The Freed-Witten condition [26] in type IIA string theory says that the third integral Stiefel-Whitney class W_3 of a D-brane Q has to be *trivial relative to the Neveu-Schwarz field* $H_3|_Q$ restricted to the D-brane, in that the two classes agree: $W_3 = [H_3|_Q]$. The vanishing of W_3 allows the existence of a Spin^c-structure. Hence here $H_3|_Q$ is sometimes referred to as a *twist of the* Spin^c-structure, in the sense of a Spin^c-structure *relative to* $H_3|_Q$. Diagrammatically, this means that there is a coboundary $W_3(Q) \xrightarrow{\eta} H_3|_Q$

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[dr, "H_3|Q"'] & B\mathrm{Spin}(n) \arrow[d, "W_3"] \arrow[dl, phantom, "\swArrow\eta"] \\ & K(\mathbb{Z},3) \end{tikzcd} . \tag{1.6}$$

All spaces involved here can be taken to be ordinary topological spaces, all morphisms ordinary continuous maps between these and all 2-morphisms ordinary homotopies between those as long as one considers just the topological classes. On the other hand, the analogous discussion for the differential refinement of this situation requires all spaces here to be replaced with generalized smooth spaces in the sense of the models discussed in section 2.2.

The higher version of this example are the Green-Schwarz mechanism and its magnetic dual version. Recall the notion of String structures and of Fivebrane structures from [57]. In [70] the notion of twist for a String structure was considered: a space X can have a twisted String structure without having a String structure, i.e. the fractional Pontryagin class $\frac{1}{2}p_1(TX)$ of the tangent bundle can be nonzero while the modified class $\frac{1}{2}p_1(TX) + [\beta] = 0$, where $\beta : X \to K(\mathbb{Z}, 4)$ is a fixed twisting class for the String structure. The Green-Schwarz mechanism in String theory may be understood as defining a twisted String-structure on target space, with twist given by a class of the gauge bundle.

Since a String-structure is refined by a Fivebrane-structure in analogy to how String-structure itself refines a Spin-structure it is natural to consider twists of Fivebrane-structures in the above sense. In this paper we give a definition of *twisted Fivebrane structures* and show that the dual Green-Schwarz mechanism in heterotic String theory reviewed in detail in [57] provides an example. Hence the twisted Fivebrane conditions do in fact appear in string theory and M-theory and they correspond, as we will see, to anomaly cancelation conditions for the heterotic fivebrane [21] [43] and for the M-fivebrane [74] [76] [24] [19], respectively. We discuss these two cases in section 3.1 and section 3.2 in terms of topological cocycles and describe in section 5.2 and 5.4 their differential refinements.

2 Nonabelian Differential Cohomology

Here we describe the relevant cohomology theory in full generality, providing the background for the L_∞-algebraic description in section 4. Readers interested in a more comprehensive discussion may consult [58]. Readers not wishing to bother with ∞-categories may skip to the next section, which provides the differential algebraic description.

There are two formalizations of the notion of generalized cohomology: on one hand in terms of the generalized Eilenberg-Steenrod axioms, in modern language represented in terms of spectra, and on the other hand in terms of higher sheaf cohomology, in modern language phrased in terms of ∞-topoi of ∞-stacks.

The generalized Eilenberg-Steenrod axioms in particular capture K-theory and elliptic cohomology as generalized cohomology theories beyond singular integral cohomology and as such have found realizations and applications in formal high energy physics, notably in string theory and M-theory (cf. [23] [20] [41]). For applications there what really matters are differential refinements of these cohomology theories, reflecting their description of physical fields that resemble connections on fiber bundles, or rather their higher dimensional analogs. In their seminal work [31] Hopkins and Singer presented a general theory of differential refinements of Eilenberg-Steenrod type cohomology theories and Freed showed [23] how this serves to formalize and analyze global structures in string theory related to the Green-Schwarz anomaly cancellation mechanism [28] and similar effects of relevance in string theory.

Despite this success of the language of generalized differential cohomology in describing these phenomena, it may be noteworthy that generalized Eilenberg-Steenrod cohomology is intrinsically of 'abelian' nature in that it involves stable phenomenona, groups which are infinite loop spaces, whereas the Green-Schwarz mechanism at its heart is governed by differetial cocycles on nonabelian groups which admit only a single delooping and therefore should in its totality have an explanation in a nonabelian version of generalized differential cohomology.

Such a generalized nonabelian cohomology is provided by higher sheaf cohomology, often referred to as the theory of ∞-stacks. The general theory has been developed and studied in some detail by Brown, Joyal, Jardine, Toën, Lurie and others. What is missing or has found less attention in the literature is the differential refinement of this generalized nonabelian cohomology and its applications to problems as those arising in string theory. In an attempt to start filling this gap, here we consider the following.

The objects of our study in nonabelian differential cohomology are representatives for higher smooth principal bundles and their associated higher smooth vector bundles equipped *with connection and curvature*. This subsumes in particular higher gerbes with connection.

In section 2.1 we indicate the abstract setup of generalized nonabelian cohomology theory and introduce the central constructions, notably the obstruction theory for lifting and extension problems, the notion of twisted cohomology and of sections and the axiomatics for differential nonabelian cohomology.

Section 2.2 describes concrete realizations of the abstract setup.

2.1 Axiomatic nonabelian cohomology

Ordinary nonabelian cohomology in degree 1 of a topological space X with values in a discrete (and possibly nonabelian) group G is the pointed set of homotopy classes of morphisms of topological spaces from X into the classifying space BG. The content of *nonabelian cohomology* is the generalization of this statement to cohomology in higher degree. The content of general nonabelian *differential* cohomology is moreover the generalization of nonabelian cohomology to generalized spaces with extra structure, in particular with smooth structure.

In modern terminology the above situation of degree 1 nonabelian topological cohomology is stated as saying that there is an $(\infty,1)$-category Top, with objects X and BG and an $(\infty,0)$-category (an ∞-groupoid) $\mathsf{Top}(X,BG)$ of morphisms between these, with pointed set of connected components $H^1(X,G) := \pi_0\mathsf{Top}(X,BG)$. Here the degree of the cohomology is encoded in the categorical degree of the coefficient object BG, which is a homotopy 1-type. a sheaf on Top with values in ∞-groupoids it takes values in just 1-groupoids. So if G itself is discrete then the topological space representing BG is a homotopy 1-type.

Replacing the space BG by a homotopy 2-type, for instance the classifying space $BG_{(2)}$ of a discrete 2-group $G_{(2)}$, gives rise to second nonabelian cohomology $H^2(X,G_{(2)}) := \pi_0\mathsf{Top}(X,BG_{(2)})$ of X. And so on.

The ∞-categorical formulation of topological nonabelian cohomology indicates how it generalizes to nonabelian cohomology of other kinds of (generalized) spaces: one finds that the crucial property of the $(\infty,1)$-category Top of topological spaces which underlies the familiar constructions in homotopy theory is that it is an ∞-*topos*. In any ∞-topos the objects may be regarded as generalized spaces and the equivalence classes of morphisms between objects as cohomology classes. General ∞-topoi, such as those of nonabelian differential cohomology, arise as $(\infty,1)$-categories of *parameterized* $(\infty,0)$-groupoids, namely as $(\infty,1)$-categories of ∞-sheaves.

2.1.1 ∞-Sheaves

Higher smooth principal bundles *without* connection are naturally described in terms of smooth ∞-sheaves (often called ∞-stacks) as developed by Brown, Joyal, Jardine, Toën, Lurie and others.

An $(\infty,0)$-*category* – equivalently an ∞-*groupoid* – is a combinatorial model (usually realized as a Kan complex) for higher homotopies, encoding the same homotopical information as a well behaved (namely compactly generated weakly Hausdorff) topological space. The totality of $(\infty,0)$-categories together with homomorphisms between them naturally forms the $(\infty,1)$-category $(\infty,0)\mathsf{Cat}$. Here an $(\infty,1)$-category is the structure of a category (a *1-category*, for emphasis) up to higher coherent homotopies. Well behaved topological spaces themselves form an $(\infty,1)$-category, Top, and there is a suitable equivalence $(\infty,0)\mathsf{Cat} \simeq \mathsf{Top}$. Therefore studying the $(\infty,1)$-category $(\infty,0)\mathsf{Cat}$ of $(\infty,0)$-categories (= ∞-groupoids) is nothing but doing classical homotopy theory. This statement is the exact higher analog of the statement that studying the 1-category $0\mathsf{Cat} = \mathsf{Set}$ of 0-categories (= sets) is nothing but doing classical set theory.

A central point of rephrasing classical set theory and classical homotopy theory in category theoretic terms is that it provides the right framework for describing *parameterized* sets and parameterized topological spaces in the form of ∞-sheaves $\mathrm{Sh}(S) \subset [S^{\mathrm{op}}, (\infty,0)\mathsf{Cat}]$ on categories S. For suitable choices of S these ∞-presheaves are models for higher homotopies *with extra structure*. For instance if S is a category of smooth test manifolds then ∞-sheaves on S are a model for generalized smooth spaces.

An ∞-sheaf is for this purpose best thought of as a generalized space modeled as an object X characterized by

1. a rule $X(-)$ which assigns to each test space U in a given category S of test spaces a collection $X(U)$ thought of as a collection of maps from this test space into X;

2. such that this collection of maps, together with homotopies between them and homotopies between homotopies, forms an ∞-groupoid (an $(\infty,0)$-category).

More precisely this says that the generalized space X is characterized by a morphism $X(-) : S^{\mathrm{op}} \to (\infty,0)\mathsf{Cat}$ of $(\infty,1)$-categories: it is an ∞-groupoid valued presheaf on S. In order to consistently interpret such an ∞-presheaf as an assignment of probe maps to a generalized space, it must be true that mapping equivalent probes into X yields equivalent results, i.e. that $[-,X] : (\infty,0)\mathsf{Cat}^{S^{\mathrm{op}}} \to (\infty,0)\mathsf{Cat}^{S^{\mathrm{op}}}$ respects certain equivalences in a suitable sense. This is formalized by saying that $X(-)$ *satisfies descent* in that it is an ∞-sheaf (an ∞-stack).

Such ∞-sheaves naturally live in an $(\infty,1)$-category H called a (Grothendieck-Rezk-Lurie) ∞*-topos*. An ∞-topos is the general context in which general homotopy theory and general nonabelian cohomology theory is situated.

2.1.2 Cohomology

Let $\mathcal{H}$ be an $(\infty,1)$-topos. Its objects we call generalized spaces. For generalized spaces X,A, we denote by $\mathcal{H}(X,A) = \mathrm{Maps}(X,A)$ the $(\infty,0)$-category of maps from X to A. This may be addressed as the space of cocycles of X with coefficients in A: the objects in $\mathrm{Maps}(X,A)$ are the A-valued cocycles on X, the morphisms are homotopies/coboundaries between these and the higher morphisms are coboundaries between coboundaries. The connected components in $\mathrm{Map}(X,A)$ are the cohomology classes, $H(X,A) = \pi_0\mathrm{Map}(X,A)$. This are the sets of morphisms in the homotopy category H of $\mathcal{H}$.

For instance for G an ordinary abelian group and X an ordinary topological space, the choice $A = K(G,n)$ (an Eilenberg-MacLan space) yields the ordinary cohomology $H^n(X,G) = H(X,K(G,n)) = \pi_0\mathcal{H}(X,A)$.

If A is pointed in that it is equipped with a morphism $* \xrightarrow{\mathrm{pt}_A} A$ then $\mathcal{H}(X,A)$ is naturally pointed with point $X \longrightarrow * \xrightarrow{\mathrm{pt}_A} A$ the trivial A-cocycle on X. In particular if A is the delooping, $A = \mathbf{B}G$, of a group-like object G in $\mathcal{H}$ (an ∞-group) and if $g : X \to \mathbf{B}G$ is a cocycle, then the homotopy fiber of g, i.e. the $(\infty,1)$-categorical or homotopy pullback $P \to X$ of the point of A in

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] \\ X \arrow[r, "g"] & \mathbf{B}G \end{tikzcd} \tag{2.1}$$

is the G-principal bundle classified by the cocycle g. The symbol $\lrcorner$ indicates that a square is a $(\infty,1)$-categorical or homotopy pullback.

For $* \xrightarrow{\mathrm{pt}_A} A$ and $* \xrightarrow{\mathrm{pt}_B} B$ two pointed objects in $\mathcal{H}$ and $k : A \longrightarrow B$ a morphism of pointed objects, the homotopy fiber of the homotopy fiber is the object $\Omega_* B$ of based loops in B:

$$\begin{tikzcd} \Omega_* B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \hat{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d, "\mathrm{pt}_B"] \\ * \arrow[r, "\mathrm{pt}_A"] & A \arrow[r, "k"] & B \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} \Omega_* B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d, "\mathrm{pt}_B"] \\ * \arrow[r, "\mathrm{pt}_B"] & B \end{tikzcd} . \tag{2.2}$$

Continuing the formation of homotopy fibers yields the fibration sequence

$$\cdots \longrightarrow \Omega_* \hat{A} \longrightarrow \Omega_* A \xrightarrow{\bar{\Omega}_* k} \Omega_* B \longrightarrow \hat{A} \longrightarrow A \xrightarrow{k} B \ . \tag{2.3}$$

By the defining universal property of the pullback

$$\begin{tikzcd} \mathcal{H}(X,\hat{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "\mathrm{pt}_{H(X,B)}"] \\ \mathcal{H}(X,A) \arrow[r, "\delta := k_*"] & \mathcal{H}(X,B) \end{tikzcd} \tag{2.4}$$

this yields for each object X in H a long exact sequence in cohomology, namely a fibration sequence

$$\cdots \longrightarrow \mathcal{H}(X,\Omega_*\hat{A}) \longrightarrow \mathcal{H}(X,\Omega_* A) \longrightarrow \mathcal{H}(X,\Omega_* B) \longrightarrow \mathcal{H}(X,\hat{A}) \longrightarrow \mathcal{H}(X,A) \xrightarrow{\delta} \mathcal{H}(X,B) \tag{2.5}$$

of ∞-groupoids and the corresponding exact sequence

$$\cdots \longrightarrow H(X,\Omega_*\hat{A}) \longrightarrow H(X,\Omega_* A) \longrightarrow H(X,\Omega_* B) \longrightarrow H(X,\hat{A}) \longrightarrow H(X,A) \xrightarrow{\delta} H(X,B) \tag{2.6}$$

of pointed sets of cohomology classes.

2.1.3 Obstructions and twisted cohomology

This means in particular that the obstruction to lifting an A-cocycle $g \in \mathcal{H}(X,A)$ to a $\hat{A}$-cocycle is the class of the image $\delta g \in \mathcal{H}(X,B)$, and that the difference between two lifts lies in $H(X,\Omega_* A)$. If the obstruction class does not vanish but is equivalent to some nontrivial cocycle $c \in \mathcal{H}(X,B)$ one may regard g as an element in $[c]$*-twisted* $\hat{A}$*-cohomology* $\mathcal{H}^{[c]}(X,\hat{A})$:

Definition 1 (twisted cohomology) *Given a fibration sequence* $\hat{A} \longrightarrow A \xrightarrow{k} B$ *in an* $(\infty,1)$*-category* $\mathcal{H}$ *and given a cocycle* $c \in \mathcal{H}(X,B)$ *with class* $[c] \in H(X,B)$ *we say the object* $\mathcal{H}^{[c]}(X,\hat{A})$ *defined up to equivalence as the pullback*

$$\begin{tikzcd} \mathcal{H}^{[c]}(X,\hat{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "c"] \\ \mathcal{H}(X,A) \arrow[r, "\delta := k_*"] & \mathcal{H}(X,B) \end{tikzcd}$$

is the <u>$[c]$*-twisted* $\hat{A}$*-cohomology*</u> *of* X.

More explicitly, cocycles in twisted cohomology $\mathcal{H}^{[c]}(X,\hat{A})$ are homotopy commutative diagrams

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[d, "c"] \\ A \arrow[r, "k"] & B \end{tikzcd} \tag{2.7}$$

consisting of an A-cocycle g and a coboundary from the obstruction cocycle $\delta g = k \circ g$ to c. Collecting these twisted cohomologies for all possible twists by forming the colimit over $\mathcal{H}(X,B)$ yields the total twisted $\hat{A}$-cohomology with respect to the fibration sequence induced by $k : A \to B$, which we write

$$\mathcal{H}^{[\cdot]}(X,\hat{A}) := \int^{c \in \mathcal{H}(X,B)} \mathcal{H}^{[c]}(X,\hat{A}) \tag{2.8}$$

when the choice of k is understood. Notice that we have two canonical morphisms c and λ out of twisted cohomology,

$$\begin{tikzcd}
\mathcal{H}^{[\cdot]}(X,\hat{A}) \arrow[r, "c"] \arrow[d, "\lambda"] & \mathcal{H}(X,B) \\
\mathcal{H}(X,A) &
\end{tikzcd}$$

sending a twisted $\hat{A}$-cocycle to its twist and to its underlying A-cocycle, respectively.

Of particular interest is the case where the twisting cocycle c is itself the obstruction $c = k' \circ g'$ to a lift of an A'-cocycle g' through another fibration sequence $\cdots \longrightarrow \hat{A}' \longrightarrow A' \xrightarrow{k'} B$ based at the same object B:

$$\begin{tikzcd}
\mathcal{H}^{[c]}(X,\hat{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {*} \arrow[d, "g'"] \arrow[dd, bend left=60, "c"] \\
& \mathcal{H}(X,A') \arrow[d, "\delta'"] \\
\mathcal{H}(X,A) \arrow[r, "k_*"] & \mathcal{H}(X,B)
\end{tikzcd} . \tag{2.9}$$

In this case we may interpret $\mathcal{H}^{[c]}(X,\hat{A})$ as the collection of those A-cocycles whose obstruction cancels that of the given A'-cocycle g'.

This notion of twisted cohomology underlies

- the notion of sections of a cocycle, in section 2.1.5;
- the notion of non-flat differential cohomology in section 2.1.6 and section 4;
- the notion of twisted Spin−, String− and Fivebrane−structures in section 3.

For varying g' the situation is encoded in *bi-twisted* cohomology

Definition 2 (bi-twisted cohomology) *Given two fibration sequences of the form* $\hat{A} \longrightarrow A \xrightarrow{k} B$ *and* $\hat{A}' \longrightarrow A' \xrightarrow{k'} B$ *we say that cohomology* $H(X, A \times_B A')$ *with coefficients in the fiber product*

$$\begin{tikzcd}
A \times_B A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "k'"] \\
A \arrow[r, "k"] & B
\end{tikzcd}$$

is <u>*bi-twisted* $\hat{A}$*-*$\hat{A}'$*-cohomology*</u>*.*

For any fixed A'-cocycle $c \in \mathcal{H}(X,A')$ the $[c]$-twisted $\hat{A}$-cohomology is extracted from the $\hat{A}$-$\hat{A}'$-cohomolgy by pulling back along c:

$$\begin{tikzcd}
\mathcal{H}^{[c]}(X,\hat{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {*} \arrow[d, "g'"] \\
\mathcal{H}(X, A \times_B A') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{H}(X,A') \arrow[d, "\delta'"] \\
\mathcal{H}(X,A) \arrow[r, "\delta"] & \mathcal{H}(X,B)
\end{tikzcd} . \tag{2.10}$$

More explicitly, a cocycle $(\delta g \to \delta g') \in H(X, A \times_B A')$ in bi-twisted cohomology is a homotopy commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[d, bend right, "\delta g"'] \arrow[d, bend left, "\delta g'"] \arrow[dr, "g'"] & \\ A \arrow[r, "k"] & B & A' \arrow[l, "k'"'] \end{tikzcd} \tag{2.11}$$

consisting of

- an A-cocycle $g \in \mathcal{H}(X, A)$, and an A'-cocycle $g' \in \mathcal{H}(X, A')$ with obstructions $\delta g, \delta g' \in H(X, B)$, respectively;
- together with a B-coboundary $\delta g \to \delta g'$.

This notion of bi-twisted cohomology is the central ingredient of our interpretation of the Green-Schwarz mechanism and its siblings in section 3 and section 5.

2.1.4 Equivariant and relative cohomology

For G an ∞-group, i.e. $G \simeq \Omega_* \mathbf{B}G$, an action ρ of G on an object X in $\mathcal{H}$ is given by an object $X//G$, called the corresponding *action ∞-groupoid*, sitting in a fibration sequence

$$X \xrightarrow{i_\rho} X//G \xrightarrow{p_\rho} \mathbf{B}G \tag{2.12}$$

in $\mathcal{H}$. Equivalently $X//G$ can be thought of as the Borel construction $\mathbf{E}G \times_G X$.

A *G-equivariant structure* on a cocycle $g \in \mathcal{H}(X, A)$ with respect to this action is a choice of extension of g through i_ρ, i.e. a choice of dashed morphism in the homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "i_\rho"] & A \\ X//G \arrow[ur, dashed, "\underline{g}"'] & \end{tikzcd} . \tag{2.13}$$

More generally, for $i : X \longrightarrow \underline{X}$ a morphism of domain objects and $p : A \longrightarrow B$ a morphism of codomains, a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & A \arrow[d] \\ \underline{X} \arrow[r, "\underline{g}"] & B \end{tikzcd} \tag{2.14}$$

may be addressed as a B-cocycle $\underline{g}$ on $\underline{X}$ relative to an A-cocycle g on X.

We can rephrase this in terms of *relative* cohomology, which we use in particular in section 2.1.6 for the formulation of differential cohomology:

This can be understood as a relative cocycle which is an object in the $(\infty, 0)$-category $\mathcal{H}_{\mathrm{rel}}\left(\begin{smallmatrix} X \\ \downarrow \\ \underline{X} \end{smallmatrix}, \begin{smallmatrix} A \\ \downarrow \\ B \end{smallmatrix} \right)$ of morphisms in the arrow $(\infty, 1)$-category $\mathcal{H}_{\mathrm{rel}} := [\mathcal{I}, \mathcal{H}]$. (Here $\mathcal{I}$ denotes the directed interval $(\infty, 1)$-category $\mathcal{I} = \{a \to b\}$ with nerve the weak Kan complex $N(\mathcal{I}) = \Delta^1$). Noticing the natural equivalences

$$\mathcal{H}(X, A) \simeq \mathcal{H}_{\mathrm{rel}}\left(\begin{smallmatrix} X \\ \downarrow \\ \underline{X} \end{smallmatrix}, \begin{smallmatrix} A \\ \downarrow \\ * \end{smallmatrix} \right) \qquad \text{and} \qquad \mathcal{H}(\underline{X}, A) \simeq \mathcal{H}_{\mathrm{rel}}\left(\begin{smallmatrix} X \\ \downarrow \\ \underline{X} \end{smallmatrix}, \begin{smallmatrix} A \\ \downarrow = \\ A \end{smallmatrix} \right) \tag{2.15}$$

the obstruction to the extension of a cocycle in $\mathcal{H}(X, A)$ through $X \longrightarrow \underline{X}$ is equivalently conceived as the obstruction to the lift of a relative cocycle in $\mathcal{H}_{\mathrm{rel}}(\begin{array}{c} X \\ \downarrow \\ \underline{X} \end{array}, \begin{array}{c} A \\ \downarrow \\ * \end{array})$ through the canonical morphism

$$\left(\begin{array}{c} A \\ \downarrow{=} \\ A \end{array}\right) \to \left(\begin{array}{c} A \\ \downarrow \\ * \end{array}\right). \tag{2.16}$$

If A is de-loopable in that $A \simeq \Omega_* K$ for some pointed object K, then one shows that this morphism is the homotopy fiber in H_{rel} of a morphism

$$\left(\begin{array}{c} A \\ \downarrow \\ * \end{array}\right) \to \left(\begin{array}{c} * \\ \downarrow \\ \mathbf{B}A \end{array}\right) \tag{2.17}$$

given by the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathbf{E}A \arrow[d] \\ \mathbf{E}A \arrow[r] & \mathbf{B}A \end{tikzcd} \tag{2.18}$$

in $\mathcal{H}$ that is determined by the universal A-fibration with $\mathbf{E}A \simeq *$.

The corresponding long exact sequence in cohomology

$$\cdots \longrightarrow \mathcal{H}(\underline{X}, \Omega_* A) \longrightarrow \mathcal{H}(X, \Omega_* A) \longrightarrow \mathcal{H}_{\mathrm{rel}}(\begin{array}{c} X \\ \downarrow \\ \underline{X} \end{array}, \begin{array}{c} * \\ \downarrow \\ A \end{array}) \longrightarrow \mathcal{H}(\underline{X}, A) \longrightarrow \mathcal{H}(X, A) \xrightarrow{\delta} \mathcal{H}_{\mathrm{rel}}(\begin{array}{c} X \\ \downarrow \\ \underline{X} \end{array}, \begin{array}{c} * \\ \downarrow \\ \mathbf{B}A \end{array}) \tag{2.19}$$

controls the extension problem: the obstruction to extending an A-cocycle $g \in \mathcal{H}(X, A)$ on X to an A-cocycle on $\underline{X}$ is the class of $\delta g \in \mathcal{H}(\begin{array}{c} X \\ \downarrow \\ \underline{X} \end{array}, \begin{array}{c} * \\ \downarrow \\ \mathbf{B}A \end{array})$. If this class vanishes the freedom in the possible extensions is given by $\mathcal{H}(\begin{array}{c} X \\ \downarrow \\ \underline{X} \end{array}, \begin{array}{c} * \\ \downarrow \\ A \end{array})$.

2.1.5 Associated bundles and sections

For $V \xrightarrow{i_\rho} V//G \xrightarrow{p_\rho} \mathbf{B}G$ a fibration sequence encoding an action ρ of the group-object G on a space V as in section 2.1.4 and for $g : X \to \mathbf{B}G$ a G-cocycle on a space X, we call the fibration $E \to X$ obtained as the homotopy pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p_E"] \arrow[dr, phantom, very near start, "\lrcorner"] & V//G \arrow[d, "p_\rho"] \\ X \arrow[r, "g"] & \mathbf{B}G \end{tikzcd} \tag{2.20}$$

the *V-bundle ρ-associated* to the G-principal bundle $P \to X$

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & * \arrow[d] \\ X \arrow[r, "g"] & \mathbf{B}G \end{tikzcd} \tag{2.21}$$

classified by g. A *section* σ of $E \to X$ is a homotopy commutative diagram

$$\begin{tikzcd} & E \arrow[dr, "p_E"] & \\ X \arrow[ur, "\sigma"] \arrow[rr, "="'] & & X \end{tikzcd} . \tag{2.22}$$

By the universal property of the homotopy pullback which defines E this is the same as a lift of the cocycle g through the morphism p_ρ, i.e. a homotopy commutative diagram

$$\begin{tikzcd} & V//G \arrow[d, "p_\rho"] \\ X \arrow[ur, "\sigma"] \arrow[r, "g"] & \mathbf{B}G \end{tikzcd} , \tag{2.23}$$

where by abuse of notation we denote the lift by the same symbol as the section it corresponds to. Comparing with section 2.1.3 this means that we can identify the collection $\Gamma(E)$ of sections σ of the bundle $E \to X$ which is ρ-associated to the bundle $P \to X$ classified by $g \in H(X, \mathbf{B}G)$ as the $[g]$-twisted $V//G$-cohomology on X:

$$\begin{tikzcd} \Gamma(E) := H^{[g]}(X, V//G) \arrow[r] \arrow[d] & * \arrow[d, "g"] \\ H(X, V//G) \arrow[r, "(p_\rho)_*"] & H(X, \mathbf{B}G) \end{tikzcd} . \tag{2.24}$$

Remark. When comparing the notion of sections with the general discussion of twisted cohomology, notice that the fibration sequence $V \xrightarrow{i_\rho} V//G \xrightarrow{p_\rho} \mathbf{B}G$ through which the section is a lift extends one more step to the right. If it does extend as $\mathbf{B}G \xrightarrow{k} B$ then $k \circ g$ is the obstruction for $E \to X$ to admit any section. But in applications one is often interested in associated bundles which always admit at least the trivial section $X \longrightarrow * \xrightarrow{\mathrm{pt}_{X//G}} X//G$, such as higher vector bundles. These can typically be obtained from a morphism $\mathbf{B}G \xrightarrow{\rho} n\mathrm{Vect}$ to an object $n\mathrm{Vect}$, which does not live in the $(\infty,1)$-category H but in an $(\infty,2)$-category H' into which H embeds $H \hookrightarrow H'$.

2.1.6 Differential cohomology

We conceive differential cohomology as relative cohomology, section 2.1.4, with respect to the inclusion of (generalized) spaces X into their *fundamental path ∞-groupoids* $\Pi(X)$. For instance for $\mathcal{H}$ the $(\infty,1)$-categories of simplicial presheaves on the category $S = \mathsf{Diff}$ of smooth connected manifolds, $\Pi : \mathsf{Diff} \to [\mathsf{Diff}^{\mathrm{op}}, \mathsf{SSet}]$ would be the smooth singular simplicial complex $\Pi(U) : V \mapsto \mathsf{Diff}(V \times \Delta^\bullet)$, where Δ^n is the standard n-simplex regarded as a smooth manifold.

In general this is formalized by specifying a functor

$$\Pi : S \to \mathcal{H} \tag{2.25}$$

from the category S of test spaces to the $(\infty,1)$-category *equivariantandrelativecohomology* of ∞-sheaves on S that models the assignment to each test space U of its fundamental ∞-groupoid $\Pi(U)$. One thinks of the k-cells of $\Pi(U)(V)$ as V-families of k-dimensional paths in U. What is formally required of Π is that

- the left Kan extension along the embedding $S \hookrightarrow \mathrm{Sh}(S) = \mathcal{H}$ to an ∞-functor $\Pi : \mathcal{H} \to \mathcal{H}$ exists;

- and comes equipped with a natural transformation $i_\Pi : \mathrm{Id}_{\mathcal{H}} \Rightarrow \Pi$.

The components $i_\Pi(X) : X \to \Pi(X)$ of this natural transformation can be regarded as objects of $\mathcal{H}_{\mathrm{rel}}$ and differential cohomology on X is nothing but cohomology relative to $i_\Pi(X)$. More precisely, the A-cohomology $\mathcal{H}(\Pi(X), A)$ of $\Pi(X)$ is to be addressed as *flat* differential A-cohomology of X, corresponding to A-principal bundles with flat connection. Moreover, in the case that A is de-loopable, so that $\mathbf{B}A$ exists, the obstruction to lifting an A-cocycle on X to a flat differential A-cocycle, hence the obstruction to equipping an A-principal bundle with a flat connection, is, by the discussion in section 2.1.4, a cocycle in

$$\mathcal{H}_{\mathrm{dR}}(X, \mathbf{B}A) := \mathcal{H}\left(\begin{array}{c} X \\ \downarrow \\ \Pi(X) \end{array} , \begin{array}{c} * \\ \downarrow \\ \mathbf{B}A \end{array} \right) . \tag{2.26}$$

By definition these cocycles are flat differential $\mathbf{B}A$-cocycle with the property that they become trivial when regarded as bare A-cocycles. Hence these are *trivial* A-principal bundles with flat connection, and can be identified with closed differential form data on X. The obstruction $F_g := \delta g \in \mathcal{H}_{\mathrm{dR}}(X, \mathbf{B}A)$ of an A-cocycle $g \in \mathcal{H}(X, A)$ to lift to a flat differential cocycle is hence *curvature*. Differential cohomology with non-flat connections may therefore be realized as *twisted* flat differential cohomology $\bar{\mathcal{H}}^{[F]}(X, A)$, where the twist $[F]$ is the corresponding curvature class. General differential A-cohomology is then $\bar{\mathcal{H}}^{[\cdot]}(X, A)$, in the notation of section 2.1.3. The two canonical projections F and λ

$$\begin{tikzcd} \bar{\mathcal{H}}^{[\cdot]}(X, A) \arrow[r, "F"] \arrow[d, "\lambda"'] & \mathcal{H}_{\mathrm{dR}}(X, \mathbf{B}A) \\ \mathcal{H}(X, A) & \end{tikzcd}$$

send a differential A-cocycle to its curvature and to its underlying A-cocycle, respectively.

cohomology	**interpretation**
$\bar{\mathcal{H}}(X, A) := \mathcal{H}(\Pi(X), A)$	flat differential A-cohomology: A-principal bundles with flat connection
$\mathcal{H}_{\mathrm{dR}}(X, \mathbf{B}A) := \mathcal{H}\left(\begin{array}{c} X \\ \downarrow \\ \Pi(X) \end{array} , \begin{array}{c} * \\ \downarrow \\ \mathbf{B}A \end{array} \right)$	closed $\mathbf{B}A$-valued differential forms on X
$\begin{array}{ccc} \bar{\mathcal{H}}^{[F]}(X, A) & \longrightarrow & * \\ \downarrow & \Vert & \downarrow F \\ \mathcal{H}(X, A) & \xrightarrow{\delta} & \mathcal{H}_{\mathrm{dR}}(X, \mathbf{B}A) \end{array}$	F-twisted flat differential A-cohomology: A-principal bundles with connection and curvature/characteristic form F.
$\bar{\mathcal{H}}(X, A) := \int^{F \in H_{\mathrm{dR}}(X, \mathbf{B}A)} \bar{\mathcal{H}}^{[F]}(X, A)$	differential A-cohomology: A-principal bundles with connection and curvature

Table 1: **Concept formulation for differential nonabelian cohomology.** The notion of *flat* differential cohomology on X, which is just nonabelian cohomology of fundamental ∞-groupoids $\Pi(X)$, is the basic concept. The general mechanism of twisted nonabelian cohomology is used to conceive non-flat differential cocycles as *curvature-twisted* flat differential cocycles.

By the discussion in section 2.1.3 a non-flat differential cocycle $(g, \nabla) \in \bar{\mathcal{H}}^{[F]}(X, A)$ is a diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & A \arrow[r] \arrow[d] & \mathbf{E}A \arrow[d] \\ \Pi(X) \arrow[r, "\nabla"] \arrow[rr, bend right, "F"'] & \mathbf{E}A \arrow[r] & \mathbf{B}A \end{tikzcd} \tag{2.27}$$

with $g \in \mathcal{H}(X, A)$ the underlying A-cocycle and $F \in \bar{\mathcal{H}}_{\mathrm{dR}}(X, \mathbf{B}A)$ the given curvature datum. The morphism ∇ realizes the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & A \arrow[d] \\ \Pi(X) \arrow[r] & * \end{tikzcd} \tag{2.28}$$

in its weakly equivalent form as above, using $\mathbf{E}A \simeq *$. This involves an arbitrary choice which is the choice of *connection* that interpolates between the cocycle g and its curvature class F. An equivalently re-organized version of the above diagram with labels indicating the interpretation of its parts is shown below:

$$\begin{tikzcd} & X \arrow[rr, "g"] \arrow[d] \arrow[ddl, "="'] & & A \arrow[d] \arrow[ddl] & \text{underlying } A\text{-cocycle} \; . \\ & \Pi(X) \arrow[rr, "\nabla"] \arrow[dd, "="] & & \mathbf{E}A \arrow[dd] & \text{connection} \\ X \arrow[rr] \arrow[dr] & & \mathbf{E}A \arrow[dr] & & \\ & \Pi(X) \arrow[rr, "F"] & & \mathbf{B}A & \text{curvature} \end{tikzcd} \tag{2.29}$$

Pulling this diagram back along the fibration $P \to X$ that is classified by the cocycle g turns it into a structure manifestly generalizing the definition of an Ehresmann connection on a principal bundle in terms of differential form data on the total space of the bundle, compare the diagrammtic description of Ehresmann connections in [56]. To see this consider the *vertical path groupoid* $\Pi_{\mathrm{vert}}(P)$ of the map $P \to X$ defined as the homotopy pullback

$$\begin{tikzcd} \Pi_{\mathrm{vert}}(P) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ \Pi(P) \arrow[r] & \Pi(X) \end{tikzcd} \; . \tag{2.30}$$

We may call

$$H_{\mathrm{dR}}^{\mathrm{vert}}(P, A) := H\left(\begin{array}{c} P \\ \downarrow \\ \Pi_{\mathrm{vert}}(P) \end{array} , \begin{array}{c} * \\ \downarrow \\ A \end{array} \right) \tag{2.31}$$

closed vertical differential A-valued forms on P. Using this in the above diagram for the differential cocycle after pulling that back along $P \to X$ yields the following expression in terms of differential forms on P:

$$
\begin{tikzcd}
P \arrow[rrr] \arrow[dr] \arrow[drr] \arrow[dd, "="'] & & & * \arrow[dr] & \\
& \Pi_{\mathrm{vert}}(P) \arrow[r] \arrow[dd] & X \arrow[rr, "g"] \arrow[dd] & & A \arrow[dd] \\
P \arrow[rrr] \arrow[dr] \arrow[dd] & & & * \arrow[dr] & \\
& \Pi(P) \arrow[r] \arrow[dd] & \Pi(X) \arrow[rr, "\nabla"] \arrow[dd, "="'] & & \mathbf{E}A \arrow[dd] \\
X \arrow[rrr] \arrow[dr] & & & * \arrow[dr] & \\
& \Pi(X) \arrow[r, "="'] & \Pi(X) \arrow[rr, "F"] & & \mathbf{B}A
\end{tikzcd}
\tag{2.32}
$$

connection form restricted to fibers

connection form on the total space

curvature form on base space

Notice that the connection form itself in $H\left(\begin{array}{c}P\\ \downarrow\\ \Pi(P)\end{array}, \begin{array}{c}*\\ \downarrow\\ \mathbf{E}A\end{array}\right)$ is a flat $\mathbf{E}A$-valued form which may be addressed simply as a possibly non-flat A-valued form paired with its curvature.

In section 4 we consider this realization of differential cocycles as systems of generalized differential forms for the model described in section **??** and discuss how it leads to the notion of L_∞-connections described in [56].

2.2 Models

For constructing and handling ∞-topoi it is convenient in practice, if not necessary, to encode them in terms of ordinary 1-categories equipped with suitable extra structure.

The canonical tool for this purposes is the structure of a Quillen model category on the given 1-category, which is essentially a 1-category equipped with the information which of its 1-morphisms (then called, respectively, weak equivalences, fibrations and cofibrations) are to be regarded as 1-categorical shadows of morphisms in an $(\infty, 1)$-category that are, respectively, isomorphisms, epimorphisms and monomorphisms in an ∞-categorical sense.

Every Quillen model category is canonically enriched in simplicial sets and its full subcategory of fibrant-cofibrant objects is enriched in Kan complexes, i.e. in $(\infty, 0)$-categories, and hence constitutes an $(\infty, 1)$-category. In a body of work by Joyal, Jardine [36] , Toën [67] [68], Lurie [44]and others, the model category structure on the 1-category of (pre)sheaves on some Grothendieck site S with values in simplicial sets has been developed and studied as a realization of ∞-topoi of ∞-shaves (i.e. ∞-stacks) on S.

Another 1-categorical axiom system for encoding homotopy theory that is less rich and hence less rigid than a full model category structure is that of a *category of fibrant objects* as introduced by K. Brown in [7]. Here essentially all the axioms of a model category concerning cofibrations are droppped. The earliest model of the homotopy theory of simplicial sheaves is given in terms of the structure of a category of fibrant objects on simplicial sheaves that are stalkwise Kan complexes in [7].

For various practical purposes the Brown-structure on the category of simplicial sheaves is more tractable than Joyal's full model, as the fibrant objects in the full model category structure are the globally Kan-valued presheaves that satisfy descent, hence already the fully ∞-stackified presheaves.

So let in all of the following $\mathcal{C}$ be a Brown category, i.e. a category of fibrant objects in the sense of [7]. Concretely, for our applications later on $\mathcal{C}$ is the category of fibrant objects given by locally Kan simplicial shaves on Diff.

2.2.1 Kan simplicial enrichment of a Brown category

In practice, $(\infty, 1)$-categories are often obtained and handled in terms of 1-categories $\mathcal{C}$ equipped with extra structure on their collections of 1-morphisms which encodes the structure of higher morphisms. This is much like the way that a single map $X \to Y^{\Delta^1}$ of topological spaces encodes, for $\Delta^1 = \mathcal{I}$ the interval, two maps together with a homotopy between them. In the process of *simplicial localization* systems of such 1-morphisms arrange to form a (weak) simplicial enrichment of the original 1-category, thus realizing the corresponding ∞-category. Simplicial localizations exist quite generally for homotopical 1-categories [**?**]. If the 1-category in question is equipped with the full structure of a *model category* a choice of *framing* [**?**] provides a particularly useful simplicial localization.

A *framing* is essentially an assinment to each object Y of $\mathcal{C}$ and each $n \in \mathbb{N}$ of an object denoted Y^{Δ^n}, which behaves like the collection of maps from the standard n-simplex into Y. This way 1-morphisms into Y^{Δ^n} encode n-fold homotopies between $n+1$ different $(n-1)$-fold homotopies between maps into Y. These n-fold homotopies constitute n-morphisms of a corresponding ∞-category.

In the following we consider the notion of a *1-category with higher directed homotopies* which is similar to that of a framing on a model category but exists in its own right. It always naturally induces a simplicial enrichment in which all cells have adjoints in a certain sense. For the purpose of describing nonabelian differential cohomology we find it useful to consider such systems of higher homotopies on a *category of fibrant objects* [**?**], a concept similar to but different from that of a model category. We show that categories of fibrant objects equipped compatibly with higher homotopies are naturally enriched in Kan complexes.

We use the following notation for simplicial entities:

- Δ is the simplex category, the category of totally ordered finite non-empty sets $[n] := \{ 0 \to 1 \to \cdots \to n \}$ and order-preserving maps;
  - $\delta^i : [n-1] \to [n]$ is the unique injection whose image does not contain $i \in [n]$;
  - $\sigma^i : [n+1] \to [n]$ is the unique surjection such that $i \in [n]$ has two elements in its preimage
- $\Delta_+ \subset \Delta$ is the subcategory of injective maps;
- $\Delta_- \subset \Delta$ is the subcategory of surjective maps;
- $\Delta^n \in \mathsf{SSet}$ is the standard simplicial n-simplex, the simplicial set represented by $[n] \in \Delta$;
  - i_n is the single nondegenerate n-simplex in Δ^n;
  - $\Lambda^n_i \in \mathsf{SSet}$ is the ith horn of Δ_n;
- $\mathsf{SSet} := [\Delta^{\mathrm{op}}, \mathsf{Set}]$ is the category of simplicial sets which we take as equipped with the following canonical extra structure:
  - $(\mathsf{SSet}, \otimes, I)$ is SSet regarded as a symmetric monoidal category with tensor product $\otimes = \times$ the cartesian product and tensor unit $I = \Delta^0 = \mathrm{pt}$ the 0-simplex;
- $\mathrm{Cells}(K)$ for $K \in \mathsf{SSet}$ a simplicial set is the poset of non-degenerate sub-cell inclusions in K: the category whose objects are morphisms $c : \Delta^n \to K$ in SSet such that $c(i_n)$ is a nondegenerate cell in S and whose morphisms $c \to c'$ are commuting triangles $\begin{tikzcd} \Delta^n \arrow[rr, "f"] \arrow[dr, "c"'] & & \Delta^{n'} \arrow[dl, "c'"] \\ & K & \end{tikzcd}$ with $f \in \Delta_+$;
- $F : \mathrm{Cells}(K) \to \mathsf{SSet}$ for $K \in \mathsf{SSet}$ is the forgetful functor which sends $\left(\begin{tikzcd} \Delta^n \arrow[rr, "f"] \arrow[dr, "c"'] & & \Delta^{n'} \arrow[dl, "c'"] \\ & K & \end{tikzcd} \right) \mapsto$ $\Delta^n \xrightarrow{f} \Delta^{n'}$.

A basic fact about simplicial sets that will play a role is

Lemma 1 *Every simplicial set $K \in \mathsf{SSet}$ is the colimit over its poset of non-degenerate cells:* $K = \int^{c\in\mathrm{Cells}(K)} F(c) = \int^{(\Delta^n \to K)} \Delta^n$.

Proof. See for instance lemma 3.1.4 in [**?**]. $\square$

The main structure we use to induce simplicially enriched categories (and thereby eventually higher-categories) from 1-categorical data is the the structure of a category equipped with higher directed homtopies.

Definition 3 (category with higher directed homotopies) *A <u>category with higher directed homotopies</u> $\mathcal{C}$ is*

- *a category with finite limits;*
- *equipped with a functor $[-,-] : \Delta^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$;*
- *such that*
  - *there is a natural isomorphism $[\Delta^0, -] \simeq \mathrm{Id}_{\mathcal{C}}$;*
  - *for all $Y \in \mathcal{C}$ the left Kan extension of $[-,Y]^{\mathrm{op}} : \Delta \to \mathcal{C}^{\mathrm{op}}$ along the Yoneda embedding $\Delta \to \mathsf{SSet}$ exists;*
  - *for all $K, L \in \mathsf{SSet}$ and $X \in \mathcal{C}$ there is an isomorphism $[K \times L, X] \simeq [K,[L,X]]$, natural in all three variables.*

.

A morphism in a category $\mathcal{C}$ with directed homotopies of the form $X \longrightarrow Y^{\Delta^n}$ for $n \geq 1$ we call a *directed n-fold homotopy* between the ordered collection of directed $(n-1)$-fold homotopies $X \longrightarrow Y^{\Delta^n} \xrightarrow{Y^{\delta^i}} Y^{\Delta^{n-1}}$ of which there are $n+1$. A 0-fold directed homotopy is just a morphism $X \to Y$. More in detail, we naturally have for all $n \in \mathbb{N}$ and objects $Y \in \mathcal{C}$ functors $[-,Y] : \mathrm{Cells}(\Delta^n)^{\mathrm{op}} \to \mathcal{C}$ which by abuse of notation we also denote by Y^{Δ^n}. For low n the image of these functors looks like

- $[\mathrm{Cells}(\Delta^0)^{\mathrm{op}}, Y] = \{Y\}$
- $[\mathrm{Cells}(\Delta^1)^{\mathrm{op}}, Y] = \left\{ Y \xleftarrow{Y^{\delta^0}} Y^{\Delta^1} \xrightarrow{Y^{\delta^1}} Y \right\}$
- $$[\mathrm{Cells}(\Delta)^{\mathrm{op}}, Y] = \left\{ \begin{tikzcd} & & Y & & \\ & Y^{\Delta^1} \arrow[ur, "Y^{\delta^0}"] \arrow[dl, "Y^{\delta^1}"'] & Y^{\Delta^2} \arrow[l, "Y^{\delta^2}"'] \arrow[r, "Y^{\delta^0}"] \arrow[d, "Y^{\delta^1}"] & Y^{\Delta^1} \arrow[ul, "Y^{\delta^1}"'] \arrow[dr, "Y^{\delta^0}"] & \\ Y & & Y^{\Delta^1} \arrow[ll, "Y^{\delta^1}"] \arrow[rr, "Y^{\delta^0}"'] & & Y \end{tikzcd} \right\}.$$

An n-fold directed homotopy $X \longrightarrow Y^{\Delta^n}$ extends uniquely to a natural transformation which is denoted by the same symbols when we take X to denote the functor on $\mathrm{Cells}(\Delta^n)^{\mathrm{op}}$ constant on X. The component of this natural transformation on a given inclusion $(\Delta^{n'} \hookrightarrow \Delta^n) \in \mathrm{Cells}(\Delta^n)^{\mathrm{op}}$ is the corresponding boundary n'-fold directed homotopy.

For evaluating $[-,Y]$ on horns, notice that

Lemma 2 *The Kan extension $[-,Y]^{\mathrm{op}} : \mathsf{SSet} \to \mathcal{C}^{\mathrm{op}}$ preserves colimits.*

Proof. Standard fact about the universal property of the Yoneda embedding. □

Corollary 2.1 *For $K \in \mathsf{SSet}$ and $Y \in \mathcal{C}$, the value of $[-, Y]$ on K is the limit in $\mathcal{C}$ of $[-, Y]$ over the cells of K: $Y^K = \int_{(c:\Delta^k \to K) \in \mathrm{Cells}(K)} Y^{\Delta^k}$.*

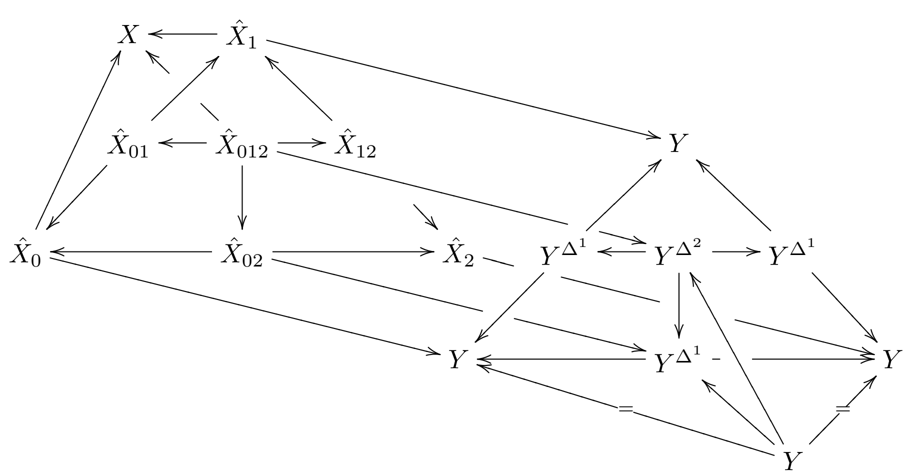
Proof. By lemma 1 we have $[K, Y] = [\int^{(c:\Delta^k \to K) \in \mathrm{Cells}(K)} \Delta^k, Y]$. By lemma 2 and noticing that under $(\cdot)^{\mathrm{op}}$ colimits turn into limits this yields $\cdots = \int_{(c:\Delta^k \to K) \in \mathrm{Cells}(K)} [\Delta^k, Y]$. □

Definition 4 (multispans) *For two objects X, Y in a category $\mathcal{C}$ with higher directed homotopies and for $n \in \mathbb{N}$, $n \geq 1$, an <u>n-multispan</u> from X to Y is*

- *a functor $\hat{X} : \mathrm{Cells}(\Delta^n)^{\mathrm{op}} \to \mathcal{C}$;*
- *natural transformations p, g in* $$\begin{tikzcd} \hat{X} \arrow[r, "g"] \arrow[d, "p"] & Y^{\Delta^n} & Y \arrow[l, "Y^{\delta}"'] \\ X & & \end{tikzcd}.$$

Here the transformation $Y^{\Delta^n} \xleftarrow{Y^{\delta}} Y$ is included just for cosmetic reasons. For low n, multispans look as follows.

- For $n = 0$ an n-multispan between X and Y is just an ordinary span
$$X \longleftarrow \hat{X} \longrightarrow Y$$
between X and Y.
- For $n = 1$ an n-multispan between X and Y is a diagram of the form
$$\begin{tikzcd} & X_0 \arrow[r] \arrow[dl] & Y & \\ X & \hat{X}_{01} \arrow[l] \arrow[u] \arrow[d] \arrow[r] & Y^{\Delta^1} \arrow[u, "Y^{\delta^0}"] \arrow[d, "Y^{\delta^1}"'] & Y \arrow[l] \arrow[ul, "="'] \arrow[dl, "="] \\ & X_1 \arrow[r] \arrow[ul] & Y & \end{tikzcd}$$
- For $n = 2$ an n-multispan between X and Y is a diagram of the form



where arrows $\hat{X}_{01} \to Y^{\Delta^1}$ and $\hat{X}_{12} \to Y^{\Delta^1}$ analogous to the displayed $\hat{X}_{02} \to Y^{\Delta^1}$ have been suppressed for readability.

The object $\hat{X}$ in a multispan as above is called its *correspondence object*. There is no essential information in the choice of the correspondence object $\hat{X}$ of a multispan *within isomorphism* and we shall in the following identify two multispans if they differ only by an isomorphism of their correspondence object. For emphasis the isomorphism class of a multispan is denoted by angular brackets:

$$\left[\begin{array}{ccc} \hat{X}_1 & \xrightarrow{g_1} & Y^{\Delta^n} \\ \downarrow{\scriptstyle p_1} & & \\ X & & \end{array}\right] = \left[\begin{array}{ccc} \hat{X}_2 & \xrightarrow{g_2} & Y^{\Delta^n} \\ \downarrow{\scriptstyle p_2} & & \\ X & & \end{array}\right] \quad \Leftrightarrow \quad \exists\, f \in \mathrm{Isomorphisms}(\mathcal{C}^{\mathrm{Cells}(\Delta^n)^{\mathrm{op}}}) \;:\; \begin{tikzcd} \hat{X}_1 \arrow[r, "g_1"] \arrow[d, "p_1"'] \arrow[dr, "f"] & Y^{\Delta^n} \\ X & \hat{X}_2 \arrow[l, "p_2"] \arrow[u, "g_2"'] \end{tikzcd} .$$

Definition 5 (enrichment by higher directed homotopies) *For $\mathcal{C}$ a category with directed homotopies, for all objects $X, Y \in \mathcal{C}$ let $\hat{\mathcal{C}}(X,Y) \in \mathsf{SSet}$ be the simplicial set which in degree n is the set of n-multispans between X and Y up to isomorphism of their correspondence object,*

$$\hat{\mathcal{C}}(X,Y)_n := \left\{ \left[\begin{array}{ccc} \hat{X} & \xrightarrow{g} & Y^{\Delta^n} \\ \downarrow{\scriptstyle p} & & \\ X & & \end{array}\right] \right\}$$

and whose face and degeneracy maps are given by postcomposition with the corresponding morphisms $[\Delta^{n'} \to \Delta^n, Y]$. For X, Y, Z three objects let a composition morphism

$$\circ_{X,Y,Z} : \hat{\mathcal{C}}(X,Y) \times \hat{\mathcal{C}}(Y,Z) \to \hat{\mathcal{C}}(X,Z)$$

in SSet be given on pairs of elements in degree n by

$$\left(\left[\begin{array}{ccc} \hat{X} & \xrightarrow{g_1} & Y^{\Delta^n} \\ \downarrow{\scriptstyle p_1} & & \\ X & & \end{array}\right], \left[\begin{array}{ccc} \hat{Y} & \xrightarrow{g_2} & Z^{\Delta^n} \\ \downarrow{\scriptstyle p_2} & & \\ Y & & \end{array}\right] \right) \mapsto \left[\begin{tikzcd} g_1^* \hat{Y}^{\Delta^n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \hat{Y}^{\Delta^n} \arrow[r, "{[\Delta^n, g_2]}"] \arrow[d, "{[\Delta^n, p_2]}"] & Z^{\Delta^n \times \Delta^n} \arrow[r] & Z^{\Delta^n} \\ \hat{X} \arrow[r, "g_1"] \arrow[d, "p_1"] & Y^{\Delta^n} & & \\ X & & & \end{tikzcd} \right],$$

where the morphism on the top right is $[\ \Delta^n \xrightarrow{\mathrm{Id} \times \mathrm{Id}} \Delta^n \times \Delta^n, Z\ ]$.

This is well-defined as the pullback $g_1^* \hat{Y}^{\Delta^n}$ is unique up to isomorphism. In low dimensions the above composition operation looks as follows.

- For $n = 0$ composition is the familiar composition of spans by pullback

$$\begin{tikzcd} & & \hat{X} \times_Y \hat{Y} \arrow[dl] \arrow[dr] & & \\ & \hat{X} \arrow[dl, "p_1"'] \arrow[dr, "g_1"] & & \hat{Y} \arrow[dl, "p_2"'] \arrow[dr, "g_2"] & \\ X & & Y & & Z \end{tikzcd}$$

Proposition 1 *The construction in definition 5 yields a* SSet*-enriched category $\hat{\mathcal{C}}$ with objects those of $\mathcal{C}$. Its underlying* Set*-category $\hat{\mathcal{C}}_0$ is the 1-category* $\mathrm{Span}(\mathcal{C})$ *of (isomorphism classes of) spans in $\mathcal{C}$. Hence there is a canonical faithful functor $\mathcal{C} \hookrightarrow \hat{\mathcal{C}}_0$ of ordinary (*Set*-enriched) categories. Under the canonical inclusion* Set $\hookrightarrow$ SSet *these can be regarded as* SSet*-enriched categories and as such we have a sequence of inclusions*

$$\mathcal{C} \hookrightarrow \hat{\mathcal{C}}_0 \hookrightarrow \hat{\mathcal{C}}\,.$$

Proof. The elements of $\hat{\mathcal{C}}_0(X,Y)$ are the morphisms of simplicial sets $\mathrm{pt} \to \hat{\mathcal{C}}(X,Y)$, hence the 0-cells of $\hat{\mathcal{C}}(X,Y)$:

$$\hat{\mathcal{C}}_0(X,Y) \simeq \hat{\mathcal{C}}(X,Y)_0\,.$$

By construction these are the isomrphism classes of spans in $\mathcal{C}$ and their composition is the composition of spans by pullback. The canonical inclusion regards a morphism $X \to Y$ in $\mathcal{C}$ as the span $\begin{tikzcd} X \arrow[r] \arrow[d, "="] & Y \\ X \end{tikzcd}$ □

We now consider further structure on $\mathcal{C}$ and the further structure induced by this on $\hat{\mathcal{C}}$.

Definition 6 (category of fibrant objects with compatible higher homotopies) *A <u>category of fibrant objects with compatible higher homotopies</u> is a category $\mathcal{C}$*

- *equipped with the structure of a* category of fibrant objects *in the sense of [7];*
- *and equipped with the structure of a* category with higher directed homotopies*;*
- *such that the following three compatibility conditions are satisfied:*
  - *for all objects $Y \in \mathcal{C}$ the functor $[-,Y] : \Delta^{\mathrm{op}} \to \mathcal{C}$ sends all morphisms in Δ_-^{op} to weak equivalences; and all morphisms in Δ_+^{op} to acyclic fibrations;*
  - *the morphisms* $Y^{\Delta^n} \xrightarrow{Y^\sigma \times \cdots \times Y^\sigma} \underbrace{Y \times \cdots \times Y}_{n+1}$ *are fibrations;*
  - *for every morphism $X \to Y^{\Lambda^n_k}$ there exists an acylic fibration $\hat{X} \xrightarrow{\simeq} X$ and a dashed morphism in the diagram*

$$\begin{tikzcd} & & Y^{\Delta^n} \arrow[d] \\ \hat{X} \arrow[urr, dashed] \arrow[r, two heads, "\simeq"'] & X \arrow[r] & Y^{\Lambda^n_k} \end{tikzcd}\,.$$

The first condition encodes that Y^{Δ^n} behaves like the mapping space for Δ^n a *contractible* space. With that the second condition ensures that Y^{Δ^n} provides a factorization of $Y \xrightarrow{\mathrm{Id} \times \cdots \times \mathrm{Id}} \underbrace{Y \times \cdots \times Y}_{n+1}$ into a weak equivalence followed by a fibration, so that in particular

Lemma 3 *In a category $\mathcal{C}$ of fibrant objects with compatible higher homotopies for every object $Y \in \mathcal{C}$ the factorization $Y \xrightarrow[\simeq]{Y^\sigma} Y^{\Delta^1} \xrightarrow{Y^{\delta^0} \times Y^{\delta^1}} Y \times Y$ exhibits Y^{Δ^1} as a path space object for Y in the sense of [7].*

The third condition says that all objects in $\mathcal{C}$ are *locally Kan* with respect to the choice of higher homotopies.

Definition 7 *For $\mathcal{C}$ a category of fibrant objects with compatible higher homotopies, let its enrichment $\hat{\mathcal{C}}$ in* SSet *be given as in definition 5 but with the extra condition that*

- *the functor* $\hat{X} : \mathrm{Cells}(\Delta^n)^{\mathrm{op}} \to \mathcal{C}$ *sends all morphisms to acyclic fibrations;*
- *the components of the natural transformation* $p : \hat{X} \to X$ *are all acyclic fibrations.*

Proposition 2 *For* $\mathcal{C}$ *a* category of fibrant objects with compatible higher homotopies *and for* X, Y *any two objects, the simplicial hom-set* $\mathcal{C}(X, Y)$ *from definition 7 is a Kan complex.*

We display the proof explicitly for degree $n = 2$:

A horn in degree 2 is a diagram of the form

$$
\begin{tikzcd}
 & \hat{X}_1 \arrow[drrr] & & & & \\
 & & \hat{X}_{12} \arrow[ul, two heads, "\simeq"] \arrow[d, two heads, "\simeq"] \arrow[drrr] & & Y & \\
\hat{X}_0 \arrow[drrr] & \hat{X}_{02} \arrow[l, two heads, "\simeq"'] \arrow[r, two heads, "\simeq"] \arrow[drrr] & \hat{X}_2 \arrow[drrr] & & & Y^{\Delta^1} \arrow[ul, two heads, "\simeq"'] \arrow[d, two heads, "\simeq"] \\
 & & & Y & Y^{\Delta^1} \arrow[l, two heads, "\simeq"'] \arrow[r, two heads, "\simeq"] & Y
\end{tikzcd}
.
$$

We form the limit over the back and front wedges. By lemma 2.1 this yields $Y^{\Lambda^2_i}$ at the Y-end of the diagram. Since in a category of fibrant objects acyclic fibrations are stable under pullback, the limit $\hat{X}_{012}$ at the X-end has acyclic fibrations emanating.

$$
\begin{tikzcd}
 & \hat{X}_1 \arrow[drrr] & & & & \\
 & \hat{X}'_{012} \arrow[r, two heads, "\simeq"] \arrow[d, two heads, "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] & \hat{X}_{12} \arrow[ul, two heads, "\simeq"] \arrow[d, two heads, "\simeq"] \arrow[drrr] & & Y & \\
\hat{X}_0 \arrow[drrr] & \hat{X}_{02} \arrow[l, two heads, "\simeq"'] \arrow[r, two heads, "\simeq"] \arrow[drrr] & \hat{X}_2 \arrow[drrr] & & Y^{\Lambda^2_i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{\Delta^1} \arrow[ul, two heads, "\simeq"'] \arrow[d, two heads, "\simeq"] \\
 & & & Y & Y^{\Delta^1} \arrow[l, two heads, "\simeq"'] \arrow[r, two heads, "\simeq"] & Y
\end{tikzcd}
.
$$

By the limit property of $Y^{\Lambda^2_i}$ there is a morphism $\hat{X}'_{012} \to Y^{\Lambda^2_i}$ fitting into this diagram

$$
\begin{tikzcd}
 & \hat{X}_1 \arrow[drrr] & & & & \\
 & \hat{X}'_{012} \arrow[r, two heads, "\simeq"] \arrow[d, two heads, "\simeq"] \arrow[drrr] & \hat{X}_{12} \arrow[ul, two heads, "\simeq"] \arrow[d, two heads] \arrow[drrr] & & Y & \\
\hat{X}_0 \arrow[drrr] & \hat{X}_{02} \arrow[l, two heads, "\simeq"'] \arrow[r, two heads, "\simeq"] \arrow[drrr] & \hat{X}_2 \arrow[drrr] & & Y^{\Lambda^2_i} \arrow[r] \arrow[d] & Y^{\Delta^1} \arrow[ul, two heads, "\simeq"'] \arrow[d, two heads, "\simeq"] \\
 & & & Y & Y^{\Delta^1} \arrow[l, two heads, "\simeq"'] \arrow[r, two heads, "\simeq"] & Y
\end{tikzcd}
.
$$

By assumption on a category of fibrant objects with compatible higher homotopies, there is a cover $\hat{X}_{012} \xrightarrow{\simeq} \hat{X}'_{012}$ on which this morphism factors through Y^{Δ^2}:

$$
\begin{tikzcd}
& \hat{X}_1 \arrow[rrrrd] & & & & & & \\
& \hat{X}_{012} \arrow[r, two heads, "\simeq"] \arrow[d, two heads, "\simeq"] \arrow[rrrrd] & \hat{X}_{12} \arrow[ul, "\simeq"] \arrow[dr] \arrow[rrrrd] & & & Y & & \\
\hat{X}_0 \arrow[rrrd] & \hat{X}_{02} \arrow[l, two heads, "\simeq"] \arrow[rr, two heads, "\simeq"] \arrow[rrrrd] & & \hat{X}_2 \arrow[rrrrd] & & Y^{\Delta^2} \arrow[r, two heads, "\simeq"] \arrow[d, two heads, "\simeq"] & Y^{\Delta^1} \arrow[ul, "\simeq"] \arrow[dr, two heads, "\simeq"] & \\
& & & Y & & Y^{\Delta^1} \arrow[ll, two heads, "\simeq"] \arrow[rr, two heads, "\simeq"] & & Y
\end{tikzcd}
.
$$

From this the diagram is completed by setting

$$
\begin{tikzcd}
& & \hat{X}_1 \arrow[rrrrd] & & & & & & \\
& \hat{X}_{012} \arrow[r, equal] \arrow[ur, "\simeq"] \arrow[dl, "\simeq"] & \hat{X}_{012} \arrow[r, two heads, "\simeq"] \arrow[d, two heads, "\simeq"] \arrow[rrrrd] & \hat{X}_{12} \arrow[ul, "\simeq"] \arrow[dr] \arrow[rrrrd] & & & Y & & \\
\hat{X}_0 \arrow[rrrrd] & & \hat{X}_{02} \arrow[ll, two heads, "\simeq"] \arrow[rr, two heads, "\simeq"] \arrow[rrrrd] & & \hat{X}_2 \arrow[rrrrd] & Y^{\Delta^1} \arrow[ur, "\simeq"] \arrow[dl, two heads, "\simeq"] & Y^{\Delta^2} \arrow[l, two heads, "\simeq"] \arrow[r, two heads, "\simeq"] \arrow[d, two heads, "\simeq"] & Y^{\Delta^1} \arrow[ul, "\simeq"] \arrow[dr, two heads, "\simeq"] & \\
& & & & Y & & Y^{\Delta^1} \arrow[ll, two heads, "\simeq"] \arrow[rr, two heads, "\simeq"] & & Y
\end{tikzcd}
.
$$

In any $\mathcal{V}_0$-enriched category $\hat{\mathcal{C}}$ for $\mathcal{V}_0$ closed symmetric monoidal, the composition operation on Hom-objects induces a functor $\hom_{\hat{\mathcal{C}}} : \hat{\mathcal{C}}_0^{\mathrm{op}} \times \hat{\mathcal{C}}_0^{\mathrm{op}} \to \mathcal{V}_0$, where $\hat{\mathcal{C}}_0$ is the ordinary (Set-enriched) category underlying $\hat{\mathcal{C}}$.

Let from now on $\hat{\mathcal{C}}$ be as in definition 7.

Lemma 4 *For $\hat{\mathcal{C}}$ the SSet-enriched category from definition 7 the functor*

$$
\hat{\mathcal{C}}_0^{\mathrm{op}} \times \mathcal{C} \hookrightarrow \hat{\mathcal{C}}_0^{\mathrm{op}} \times \hat{\mathcal{C}}_0 \xrightarrow{\hom_{\hat{\mathcal{C}}}} \mathsf{SSet}
$$

(with the first morphism the inclusion from proposition 1) respects weak equivalences in the second argument in that it sends weak equivalences in $\mathcal{C}$ to weak equivalences in SSet.

Proof. For X any object in $\mathcal{C}$ and for $f : Y \xrightarrow{\simeq} Z$ a weak equivalence in $\mathcal{C}$ the morphism of simplicial sets

$$
\hom_{\hat{\mathcal{C}}}(X, f) =: \hat{\mathcal{C}}(X, f) : \hat{\mathcal{C}}(X, Y) \to \hat{\mathcal{C}}(X, Z)
$$

acts by proposition 1 and definition 5 by sending

$$
\left(\left[\begin{tikzcd} \hat{X} \arrow[r, "g"] \arrow[d, "p"] & Y^{\Delta^n} \\ X & \end{tikzcd} \right], \left[\begin{tikzcd} Y \arrow[r, "f", "\simeq"'] & Z \arrow[r, "Z^\sigma"] & Z^{\Delta^n} \end{tikzcd} \right] \right) \mapsto \left[\begin{tikzcd} \hat{X} \arrow[r, "g"] \arrow[d, "p"] & Y^{\Delta^n} \arrow[r, "f^{\Delta^n}", "\simeq"'] & Z^{\Delta^n} \\ X & & \end{tikzcd} \right] .
$$

We show that this is a weak equivalence of simplicial sets by giving an element $\bar{f} \in \hat{\mathcal{C}}(Z, Y)_0$ and exhibiting $\hat{\mathcal{C}}(X, \bar{f}) : \hat{\mathcal{C}}(X, Z) \to \hat{\mathcal{C}}(X, Y)$ as a homotopy inverse of $\hat{\mathcal{C}}(X, f)$. To that end let $\bar{f}$ be the span given by the isomorphism class of the outer left boundary of the pullback diagram

$$\bar{f} := \left[\begin{tikzcd} \mathbf{E}_f Z \arrow[r, "\simeq"] \arrow[d, two heads, "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, two heads, bend right=50, "\simeq"'] & Y \arrow[d, two heads, "\simeq", "f"'] \\ Z^{\Delta^1} \arrow[r, two heads, "d_0", "\simeq"'] \arrow[d, two heads, "\simeq", "d_1"'] & Z \\ Z & \end{tikzcd}\right].$$

Here the total left morphism is an acyclic fibration by the factorization lemma of [7], which is discussed in detail in section 2.2.3. The composite $\bar{f} \circ f$ is the span on the top left outside of

$$\bar{f} \circ f = \left[\begin{tikzcd} Q \arrow[r, two heads, "\simeq"] \arrow[dd, two heads, "\simeq"] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathbf{E}_f Z \arrow[r, "\simeq"] \arrow[d, two heads, "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, two heads, bend right=40, "\simeq"'] & Y \arrow[d, two heads, "\simeq", "f"'] \\ & Z^{\Delta^1} \arrow[r, two heads, "d_0", "\simeq"'] \arrow[d, two heads, "\simeq", "d_1"'] & Z \\ Y \arrow[r, "f", "\simeq"'] & Z & \end{tikzcd}\right].$$

By lemma 2 in section 1 of [7] there is a path object $Y^{\mathcal{I}}$ fitting into this diagram as follows:

$$\begin{tikzcd} Q \arrow[rr, "\simeq"] \arrow[dr, dashed] \arrow[ddd, two heads, "\simeq"] & & \mathbf{E}_f Z \arrow[r, "\simeq"] & Y \arrow[dd, two heads, "\simeq", "f"'] \\ & Y^{\mathcal{I}} \arrow[urr, two heads, "d_0"'] \arrow[dr] \arrow[ddl, two heads, "d_1"'] & & \\ & & Z^{\Delta^1} \arrow[r, two heads, "d_0", "\simeq"'] \arrow[d, two heads, "\simeq", "d_1"'] & Z \\ Y \arrow[rr, "f", "\simeq"'] & & Z & \end{tikzcd},$$

with the dashed morphism the unique one from the pullback property of Q. By proposition 1 ii) following this in [7] there is a refinement $\hat{Q} \overset{\simeq}{\twoheadrightarrow} Q$ of Q such that generic path object $Y^{\mathcal{I}}$ can be replaced with the particular path object Y^{Δ^1}

$$\begin{tikzcd} \hat{Q} \arrow[r, two heads, "\simeq"] \arrow[dr] \arrow[dd, two heads, "\simeq"] & Q \arrow[r, two heads, "\simeq"] & \mathbf{E}_f Z \arrow[r, "\simeq"] & Y \\ & Y^{\Delta^1} \arrow[urr, two heads, "d_0"'] \arrow[dl, two heads, "d_1"'] & & \\ Y & & & \end{tikzcd}.$$

This constitutes a 1-cell in $\hat{\mathcal{C}}(Y,Y)$ that connects $\bar{f}\circ f$ with Id_Y:

$$
(\bar{f}\circ f \Rightarrow \mathrm{Id}_f) := \left[
\begin{tikzcd}
 & Q \arrow[dl, "\simeq"'] \arrow[r, "\simeq"] & \mathbf{E}_f Z \arrow[r, "\simeq"] & Y \\
Y & \hat{Q} \arrow[l, two heads, "\simeq"'] \arrow[u, two heads, "\simeq"] \arrow[d, two heads, "\simeq"] \arrow[rr] & & Y^{\Delta^1} \arrow[u, "d_0"'] \arrow[d, "d_1"] \\
 & Y \arrow[ul, "="] \arrow[rr, "="] & & Y
\end{tikzcd}
\right]
$$

By postcomposition this induces the desired simplicial homotopy

$$
\hat{\mathcal{C}}(X,\bar{f}\circ f) \to \hat{\mathcal{C}}(X,\mathrm{Id}_f) : \hat{\mathcal{C}}(X,Y) \to \hat{\mathcal{C}}(X,Y)\,.
$$

$\square$

Proposition 3 *For all objects X, Y of $\mathcal{C}$ there is a weak equivalence of simplicial sets*

$$
\hat{\mathcal{C}}(X,Y) \longrightarrow \hat{\mathcal{C}}(X,Y)
$$

whose image contains only such spans $\left[\begin{tikzcd} \hat{X} \arrow[r] \arrow[d, two heads] & Y^{\Delta^n} \\ X & \end{tikzcd}\right]$ *for which* $\hat{X} : \mathrm{Cells}(\Delta^n)^{\mathrm{op}} \to \mathcal{C}$ *is a constant functor.*

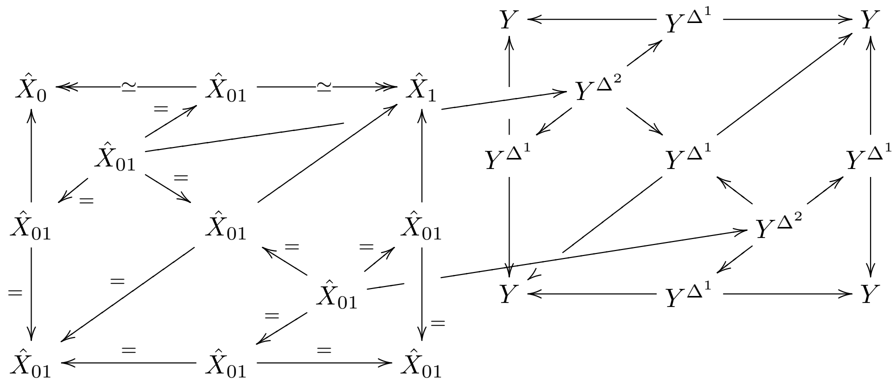
Proof. Consider the simplicial homotopy η

$$
\begin{tikzcd}
 & \hat{\mathcal{C}}(X,Y) \\
\hat{\mathcal{C}}(X,Y) \arrow[ur, "\mathrm{Id}"] \arrow[r, "\eta"] \arrow[dr] & {[\Delta^1, \hat{\mathcal{C}}(X,Y)]} \arrow[u, "d_0"'] \arrow[d, "d_1"] \\
 & \hat{\mathcal{C}}(X,Y)
\end{tikzcd}
$$

defined by .... For instance the value of η on a 1-cell

$$
\begin{tikzcd}
 & \hat{X}_0 \arrow[dl] \arrow[r] & Y & \\
\hat{X} & \hat{X}_{01} \arrow[l] \arrow[u] \arrow[d] \arrow[r] & Y^{\Delta^1} \arrow[u, "Y^{\delta^0}"] \arrow[d, "Y^{\delta^1}"'] & Y \arrow[l] \arrow[ul, "="'] \arrow[dl, "="] \\
 & X_1 \arrow[ul] \arrow[r] & Y &
\end{tikzcd}
$$

is the 1-cell



in $[\Delta^1, \hat{\mathcal{C}}(X,Y)]$, i.e. an element of $[\Delta^1, \hat{\mathcal{C}}(X,Y)]_1 \simeq \mathrm{Hom}_{\mathsf{SSet}}(\Delta^1 \times \Delta^1, \hat{\mathcal{C}}(X,Y))$. $\square$

Proposition 4 *The functor* $[-,-] : \mathsf{SSet}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ *extends to a* powering *of the* SSet*-enriched category* $\mathcal{C}$ *over* SSet*: for all* $K \in \mathsf{SSet}$ *and* $X, Y \in \mathcal{C}$ *we have an isomorphism*

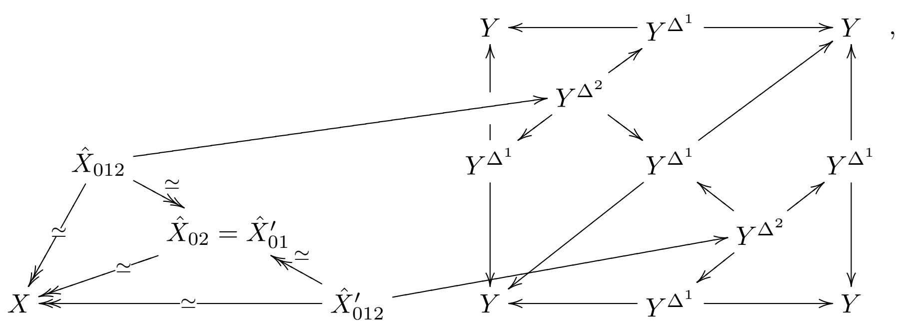
$$\hat{\mathcal{C}}(X, Y^K) \simeq [K, \hat{\mathcal{C}}(X,Y)]$$

in SSet*, natural in all three variables.*

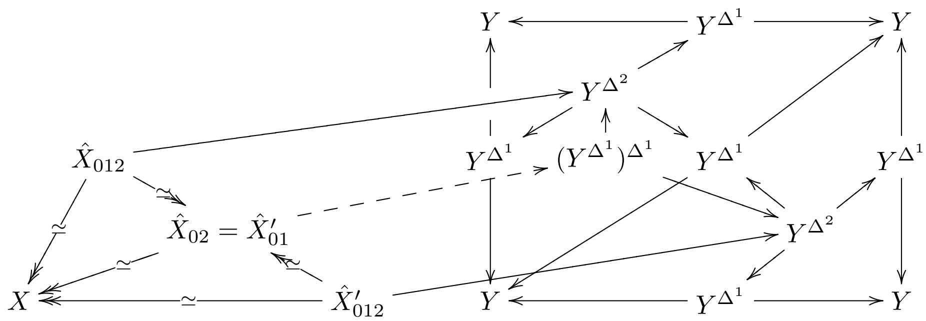
Proof. By definition 5 we have $\hat{\mathcal{C}}(X, Y^K)_n := \left\{ \left[\begin{array}{c} \begin{tikzcd} \hat{X} \arrow[r, "g"] \arrow[d, "p"] & (Y^K)^{\Delta^n} \\ X & \end{tikzcd} \end{array} \right] \right\}$. On the other hand we compute using lemma 1 and corollary 2.1 the right hand as

$$[K, \hat{\mathcal{C}}(X,Y)]_n = \mathrm{Hom}_{\mathsf{SSet}}(K \times \Delta^n, \hat{\mathcal{C}}(X,Y)) = \mathrm{Hom}_{\mathsf{SSet}}\Big(\int^{c:\Delta^k \to K \times \Delta^n} \Delta^n, \hat{\mathcal{C}}(X,Y)\Big) = \int_{c:\Delta^k \to K \times \Delta^n} \hat{\mathcal{C}}(X,Y))_k \,.$$

Consider this for the simple case $K = \Delta^1$ and $n = 1$. The elements of $[K, \hat{\mathcal{C}}(X,Y)]_n$ then are pairs of triangular spans that fit together as



where we suppressed some arrows for readability. The pullback over $Y^{\Delta^2} \to Y^{\Delta^1} \leftarrow Y^{\Delta^2}$ in the middle on the right is $Y^{\Delta^1 \times \Delta^1} = (Y^{\Delta^1})^{\Delta^1}$. By its universal property we get the dashed morphism



□

2.2.2 Cohomology

Let $\mathrm{Ho}_{\mathcal{C}}$ be the *homotopy category* of $\mathcal{C}$, i.e. the category universal with the property that there is a functor $\mathcal{C} \to \mathrm{Ho}_{\mathcal{C}}$ which sends weak equivalences to isomorphisms.

Proposition 5 *Let X be an object with trivial homotopies (any two homotopic maps into X are already equal). The morphisms in the homotopy category of $\mathcal{C}$ are in bijection with the cohomology classes in the above sense:*

$$\mathrm{Ho}_{\mathcal{C}}(X, A) \simeq H(X, A) := \pi_0(\hat{\mathcal{C}}(X, A))\,.$$

Proof. Use the characterization of $\mathrm{Ho}_{\mathcal{C}}$ from theorem 1 in part 1 of [7] (using acyclic fibrations instead of just weak equivalences as in remark 2 there):

- write $\pi\mathcal{C}$ for the category obtained from $\mathcal{C}$ by identifying morphisms $f \sim g : X \to Y$ precisely if pulled back along some acyclic fibration they become homotopic with respect to some path space object $Y^{\mathcal{I}}$ of Y:

$$f \sim g : X \to Y \quad \Leftrightarrow \quad \exists\ \hat{X} \overset{\simeq}{\twoheadrightarrow} X\ ,\ \hat{X} \longrightarrow Y^{\mathcal{I}}\ :\quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle\simeq} & & \uparrow{\scriptstyle\simeq\, d_0} \\ \hat{X} & \longrightarrow & Y^{\mathcal{I}} \\ \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq\, d_1} \\ X & \xrightarrow[g]{} & Y \end{array}$$

- then $\mathrm{Ho}_{\mathcal{C}}(X, A) \simeq \int^{\{\hat{X} \overset{\simeq}{\twoheadrightarrow} X\}} \pi\mathcal{C}(X, A)$, where the colimit is in $\pi\mathcal{C}$ over the category whose objects are acyclic fibrations over X in $\pi\mathcal{C}$ and whose morphisms are commuting triangles in $\pi\mathcal{C}$.

To see that this set $\mathrm{Ho}_{\mathcal{C}}(X, A)$ is in natural bijection with the connected components of the simplicial set $\hat{\mathcal{C}}(X, A)$ first observe that if two morphisms $f, g : X \to Y$ are homotopic with respect to any given one path space object $Y^{\mathcal{I}}$ of Y in $\mathcal{C}$, then there exists an acyclic fibration $\hat{X} \overset{\simeq}{\twoheadrightarrow} X$ such that pulled back along it the two morphisms become homotopic with respect to our fixed path space object Y^{Δ^1} from lemma 3:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X \arrow[u, "="] \arrow[d, "="'] \arrow[r] & Y^{\mathcal{I}} \arrow[u, two heads, "\simeq d_0"'] \arrow[d, two heads, "\simeq d_1"] \\ X \arrow[r, "g"'] & Y \end{tikzcd} \quad \Rightarrow \quad \exists\ \hat{X} \twoheadrightarrow X \ ,\ \hat{X} \longrightarrow Y^{\Delta^1} \ : \quad \begin{tikzcd} X \arrow[r, "f"] & Y \\ \hat{X} \arrow[u, two heads, "\simeq"] \arrow[d, two heads, "\simeq"'] \arrow[r] & Y^{\Delta^1} \arrow[u, two heads, "\simeq Y^{\delta^0}"'] \arrow[d, two heads, "\simeq Y^{\delta^1}"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

This is a special case of proposition 1 ii) in part I of [7].

Using this we can characterize the set $\mathrm{Ho}_{\mathcal{C}}(X, A)$ as a double quotient by equivalence classes as follows:

$$\mathrm{Ho}_{\mathcal{C}}(X, A) = ((\bigsqcup_{\hat{X} \to X} \mathcal{C}(\hat{X}, A))_{\sim_{\mathrm{h}}})_{\sim_{\mathrm{r}}} \,,$$

where two morphism $\hat{X} \rightrightarrows A$ are related by $\sim_{\mathrm{h}}$ if they become homotopic with respect to A^{Δ^1} on a refinement of $\hat{X}$, and where two $\sim_{\mathrm{h}}$-classes $[\ \hat{X}_1 \longrightarrow A\ ]_h$, $[\ \hat{X}_2 \longrightarrow A\ ]_h$ for different domains $\hat{X}_1$ and $\hat{X}_2$ are related by $\sim_{\mathrm{r}}$ if they coincide on a joint refinement $\hat{X}$ of $\hat{X}_1$ and $\hat{X}_2$.

This can be reformulated as a quotient by a single equivalence relation

$$\mathrm{Ho}_{\mathcal{C}}(X, A) = ((\bigsqcup_{\hat{X} \to X} \mathcal{C}(\hat{X}, A))_{\simeq_{\mathrm{rh}}})$$

where two morphisms $\hat{X}_1 \to A$ and $\hat{X}_2 \to A$ are related by $\sim_{rh}$ if on a joint refinement $\hat{X}$ of $\hat{X}_1$ and $\hat{X}_2$ they become homotopic with respect to A^{Δ_1}.

To show the desired bijection, we will now

1. define a map of sets $f : \pi_0(\hat{\mathcal{C}}(X, A)) \to \mathrm{Ho}_{\mathcal{C}}(X, A)$
2. show that it is injective on classes;
3. show that it is surjective on classes.

1. For $[\ X \overset{\simeq}{\twoheadleftarrow} \hat{X} \xrightarrow{g} A\ ]_{\pi_0}$ the connected component in $\hat{\mathcal{C}}(X, A)$ of the cocycle g, let its image in $\mathrm{Ho}_{\mathcal{C}}(X, A)$ be the $\sim_{\mathrm{rh}}$-class represented by this cocycle. This is well defined on classes, since by construction of $\hat{\mathcal{C}}(X, A)$ every other cocycle $X \overset{\simeq}{\twoheadleftarrow} \hat{X}' \xrightarrow{g'} A$ in the same connected component as g is homotopic to g on a joint refinement of their domain in that there is a diagram

$$\begin{tikzcd} \hat{X} \arrow[r, "g"] & A \\ \hat{X}'' \arrow[u, two heads, "\simeq"'] \arrow[d, two heads, "\simeq"] \arrow[r] & A^{\Delta^1} \arrow[u] \arrow[d] \\ \hat{X}' \arrow[r, "g'"'] & A \end{tikzcd} \quad .$$

But this means that both define the same $\sim_{\mathrm{rh}}$-class.

2. Suppose the above map sends two elements $[\, X \overset{\simeq}{\twoheadleftarrow} \hat{X} \xrightarrow{g} A \,]_{\pi_0}$ and $[\, X \overset{\simeq}{\twoheadleftarrow} \hat{X}' \xrightarrow{g'} A \,]_{\pi_0}$ to the same $\sim_{\mathrm{rh}}$-class. Then by the above both g and g' are in the same $\sim_{\mathrm{rh}}$-class. This means that there is a joint refinement $\hat{X}''$ of $\hat{X}$ and $\hat{X}'$ such that pulled back to $\hat{X}''$ both become homotopic with respect to A^{Δ^1}. But this means they are connected by a 1-cell in $\hat{\mathcal{C}}(X, A)$, hence that the two π_0-classes which we started with are equal. So the map is injective.

3. Surjectivity is clear.

□

2.2.3 Universal bundles and the factorization lemma

A central lemma in the theory of categories of fibrant objects is the *factorization lemma* [7].

Lemma 5 (factorization lemma) *For every morphism $f : C \longrightarrow B$ in a category $\mathcal{C}$ of fibrant objects factors there is an object $\mathbf{E}_f B$ such that f factors as*

$$\begin{tikzcd} & \mathbf{E}_f B \arrow[dr, "p_f"] & \\ C \arrow[ur, "\sigma_f", "\simeq"'] \arrow[rr, "f"] & & B \end{tikzcd},$$

with p_f a fibration and σ_f a weak equivalence that is a right inverse:

$$\begin{tikzcd} \mathbf{E}_f B \arrow[r, "\simeq"] \arrow[rr, bend right, "="'] & C \arrow[r, "\sigma_f", "\simeq"'] & \mathbf{E}_f B \end{tikzcd}.$$

We recall the proof of this lemma in two steps and then interpret it in the context of homotopy fiber products and universal bundles.

Definition 8 (universal bundles) *For $f : C \longrightarrow B$ a morphism we say the morphism $p_f : \mathbf{E}_f B \longrightarrow B$ defined as the composite vertical morphism in the pullback diagram*

$$\begin{tikzcd} \mathbf{E}_f B \arrow[r, two heads, "\simeq"] \arrow[d] \arrow[dd, bend right, "p_f"'] & C \arrow[d, "f"] \\ {[\mathcal{I}, B]} \arrow[r, two heads, "d_0", "\simeq"'] \arrow[d, two heads, "\simeq", "d_1"'] & B \\ B & \end{tikzcd}$$

is the <u>universal B-bundle relative to f</u>.

Lemma 6 *The morphism p_f is a fibration, $\mathbf{E}_f B \overset{p_f}{\twoheadrightarrow} B$.*

Proof. The pullback diagram in definition 8 can be refined to the double pullback diagram

$$\begin{tikzcd} \mathbf{E}_f B \arrow[r, two heads] \arrow[d] \arrow[dd, bend right, "p_f"'] & C \times B \arrow[r, two heads, "\mathrm{pr}_1"] \arrow[d, "f \times \mathrm{id}"] & C \arrow[d, "f"] \\ {[\mathcal{I}, B]} \arrow[r, two heads, "d_0 \times d_1"] \arrow[d, "d_1"] & B \times B \arrow[r, two heads, "\mathrm{pr}_1"] \arrow[dl, "\mathrm{pr}_2"] & B \\ B & & \end{tikzcd}$$

which exhibits p_f as the composite of two fibrations.

□

Lemma 7 *The morphism* $\mathbf{E}_f B \xrightarrow{\simeq} C$ *has a section* $\sigma_f : C \xrightarrow{\simeq} \mathbf{E}_f B$ *and its composite with* p_f *is* f

$$\begin{tikzcd}
\mathbf{E}_f B \arrow[d, "p_f"'] & C \arrow[l, "\sigma_f"', "\simeq"] \arrow[dl, "f"] \\
B
\end{tikzcd} .$$

Proof. The section is the morphism induced via the universal property of the pullback by the section σ of $[\mathcal{I}, B] \overset{d_0}{\twoheadrightarrow} B$:

$$\begin{tikzcd}
C \arrow[r, "\sigma_f"] \arrow[rr, bend left, "\mathrm{Id}"] \arrow[d, "f"] & \mathbf{E}_f B \arrow[r] \arrow[d] & C \arrow[d, "f"] \\
B \arrow[r, "\simeq", "\sigma"'] \arrow[rr, bend left, "\mathrm{Id}"] \arrow[dr, "\mathrm{Id}"'] & {[\mathcal{I}, B]} \arrow[r, "\simeq", "d_1"'] \arrow[d, two heads, "\simeq", "d_0"'] & B \\
& B &
\end{tikzcd}$$

□

Lemmas 6 and 7 together constitute the *factorization lemma* 5.

Universal bundles can be understood as a way to realize homotopy limits $\mathrm{holim}_D F := \mathbb{R}\mathrm{lim}_D F$ by ordinary limits evaluated on fibrant replacements.

Definition 9 (homotopy fiber product) *The* <u>*homotopy fiber product*</u> *of a diagram* $D \longrightarrow B \xleftarrow{f} C$ *is the pullback*

$$D \times_{B^{\mathcal{I}}} C \;\; := \;\; \lim \left(\begin{tikzcd}
& & C \arrow[d, "f"] \\
& {[\mathcal{I}, B]} \arrow[r, "d_0"'] \arrow[d, "d_1"] & B \\
D \arrow[r] & B &
\end{tikzcd} \right) .$$

In the context of topological spaces this is definition 2.1.10 in [44].

Lemma 8 *The homotopy fiber product is the fiber product with a universal bundle:*

$$D \times_{B^{\mathcal{I}}} C = D \times_B \mathbf{E}_f B \, .$$

Proof. The homotopy fiber product can be expressed as two consecutive pullbacks

$$\begin{tikzcd}
D \times_{B^{\mathcal{I}}} C \arrow[r] \arrow[dd] & \mathbf{E}_f B \arrow[r, two heads, "\simeq"] \arrow[d] \arrow[dd, bend right, "p_f"'] & C \arrow[d, "f"] \\
& {[\mathcal{I}, B]} \arrow[r, two heads, "\simeq", "d_0"'] \arrow[d, two heads, "\simeq", "d_1"'] & B \\
D \arrow[r] & B &
\end{tikzcd} ,$$

where the right pullback is a universal bundle in the sense of definition 8. □

Corollary 2.2 *The two projection morphisms out of a homotopy fiber product are fibrations:*

$$\begin{tikzcd} D \times_{C^{\mathcal{I}}} B \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d] \\ D \arrow[r] & B \end{tikzcd} .$$

Proof. By lemma 6 the morphism p_f in the above proof is a fibration, hence so is its pullback pr_1 : $D \times_{B^{\mathcal{I}}} C \xrightarrow{\simeq} D$. By the symmetry of the situation the same argument applies to pr_2 : $D \times_{B^{\mathcal{I}}} C \xrightarrow{\simeq} C$
□

Corollary 2.3 *For* $\begin{tikzcd} C \arrow[rr, "f", "\simeq"'] \arrow[dr] & & C' \arrow[dl] \\ & B & \end{tikzcd}$ *a weak equivalence, the induced morphism*

$$D \times_{B^{\mathcal{I}}} f \ : \ D \times_{B^{\mathcal{I}}} B \xrightarrow{\simeq} D \times_{B^{\mathcal{I}}} B'$$

is a weak equivalence.

Proof. By corollary 2.2 the morphism is the pullback of a weak equivalence along a fibration:

$$\begin{tikzcd} D \times_{B^{\mathcal{I}}} C \arrow[r] \arrow[d, "\simeq"] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d, "\simeq", "f"'] \\ D \times_{B^{\mathcal{I}}} C' \arrow[r, two heads] & C' \end{tikzcd} .$$

□

Definition 10 (fibrant replacement diagrams) *A* <u>*fibrant replacement diagram*</u> *for a pullback diagram* $D \to B \leftarrow C$ *is a weakly equivalent diagram* $\begin{tikzcd} D \arrow[r] \arrow[d, "\simeq"] & B \arrow[d, "\simeq"] & C \arrow[l] \arrow[d, "\simeq"] \\ D' \arrow[r] & B' & C' \arrow[l, two heads] \end{tikzcd}$ *such that* $C' \twoheadrightarrow B'$ *is a fibration, as indicated.*

Lemma 9 *For* $D \longrightarrow B \xleftarrow{f} C$ *any diagram the universal bundle diagram* $D \longrightarrow B \overset{p_f}{\twoheadleftarrow} \mathbf{E}_f B$ *is a fibrant replacement diagram.*

Proof. By lemma 7 we have a weak equivalence of diagrams

$$\begin{tikzcd} D \arrow[r] \arrow[d, "="] & B \arrow[d, "="] & C \arrow[l, "f"'] \arrow[d, "\simeq", "\sigma_f"'] \\ D \arrow[r] & B & \mathbf{E}_f B \arrow[l, two heads, "p_f"'] \end{tikzcd} .$$

By lemma 6 this is a fibrant replacement diagram in that p_f is a fibration, as indicated. □

Corollary 2.4 *If the ambient category $\mathcal{F}_0$ of fibrant objects extends to the structure of a model category, the homotopy fiber product of a pullback diagram as above is weakly equivalent to the homotopy limit* $\mathbb{R}\mathrm{lim}$ *of the diagram.*

Proof. By example 4.2 of [**?**] the homotopy limit is weakly equivalent to the ordinary limit of any fibrant replacement diagram. The claim follows by lemma 9. □

Definition 11 (monoid of loops) *The monoid of loops* $\Omega_{\mathrm{pt}} B$ *of a pointed object* $\mathrm{pt} \xrightarrow{\mathrm{pt}_B} B$ *is the homotopy fiber product of the point with itself over* B*:*

$$\Omega_{\mathrm{pt}} B := \mathrm{pt} \times_{B^{\mathcal{I}}} \mathrm{pt} \ .$$

Notice that the monoid of loops

- is the fiber of the universal B-bundle over the point.
- is the fiber of $[\mathcal{I}, B] \xrightarrow{d_0 \times d_1} B \times B$ with $B \times B$ equipped with its canonical point $\mathrm{pt} \xrightarrow{\mathrm{pt}_B \times \mathrm{pt}_B} B \times B$, i.e. the pullback

$$\begin{tikzcd} \Omega_{\mathrm{pt}} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{I}, B]} \arrow[d, "d_0 \times d_1"] \\ \mathrm{pt} \arrow[r, "\mathrm{pt}_B \times \mathrm{pt}_B"] & B \times B \end{tikzcd} .$$

This shows that $\Omega_{\mathrm{pt}} B$ is naturally equipped with the structure of an A_∞-monoid induced from the structure of the interval object.

Definition 12 *There is a natural action* $\rho : \mathbf{E}_{\mathrm{pt}} \times \Omega_{\mathrm{pt}} B \to \mathbf{E}_{\mathrm{pt}} B$ *of the monoid of loops on the universal bundle, induced from the co-category structure on* $\mathcal{I}$*.*

Lemma 10 *This action is a morphism of bundles*

$$\begin{tikzcd} \mathbf{E}_{\mathrm{pt}} B \times \Omega_{\mathrm{pt}} B \arrow[rr, "\rho"] \arrow[dr, "p \circ p_1"'] & & \mathbf{E}_{\mathrm{pt}} B \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

2.2.4 Cocycles and bundles

Recall that a cocycle on X with values in A is a span $\begin{tikzcd} \hat{X} \arrow[r] \arrow[d, two heads, "\simeq"] & A \\ X & \end{tikzcd}$. Composition of such cocycles is by pullback of their spans

$$\begin{tikzcd} g_1^* \hat{B} \arrow[r] \arrow[d, two heads, "\simeq"] & \hat{B} \arrow[r, "g_2"] \arrow[d, two heads, "\simeq"] & C \\ \hat{A} \arrow[r, "g_1"] \arrow[d, two heads, "\simeq"] & B & \\ A & & \end{tikzcd} .$$

which is associative and unital up to isomorphism of spans.

Definition 13 (bundles obtained from cocycles) *Given a cocycle* $X \overset{\simeq}{\twoheadleftarrow} \hat{X} \xrightarrow{g} B$ *into a pointed object* $\mathrm{pt} \xrightarrow{\mathrm{pt}_B} B$ *the corresponding* B*-bundle* $p : g^*\mathbf{E}_{\mathrm{pt}}B \longrightarrow X$ *is the pullback*

$$
\begin{tikzcd}
g^*\mathbf{E}_{\mathrm{pt}}B \arrow[r] \arrow[d, two heads] \arrow[dd, bend right, two heads, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E}_{\mathrm{pt}}B \arrow[d, two heads] \\
\hat{X} \arrow[r, "g"] \arrow[d, two heads, "\simeq"] & B \\
X
\end{tikzcd}
$$

This bundle inherits an action $\rho : (g^*\mathbf{E}_{\mathrm{pt}}B) \times \Omega_{\mathrm{pt}}B \to g^*\mathbf{E}_{\mathrm{pt}}B$ of the monoid of loops from the commutativity of

$$
\begin{tikzcd}
g^*\mathbf{E}B \times \Omega_{\mathrm{pt}}B \arrow[r] \arrow[d, "p_1"] & \mathbf{E}B \times \Omega_{\mathrm{pt}}B \arrow[r, "\rho"] \arrow[d, "p_1"] & \mathbf{E}B \arrow[ddl] \\
g^*\mathbf{E}B \arrow[r] \arrow[d] & \mathbf{E}B \arrow[d] & \\
\hat{X} \arrow[r, "g"] & B &
\end{tikzcd}
$$

Lemma 11 *This induced action is still a morphism of bundles*

$$
\begin{tikzcd}
(g^*\mathbf{E}_{\mathrm{pt}}B) \times \Omega_{\mathrm{pt}}B \arrow[rr, "\rho"] \arrow[dr, "p \circ p_1"'] & & g^*\mathbf{E}_{\mathrm{pt}}B \arrow[dl, "p"] \\
& X &
\end{tikzcd}
$$

Lemma 12 *If* X *is pointed and the cocycle* $X \overset{\simeq}{\twoheadleftarrow} \hat{X} \xrightarrow{g} B$ *respects the point, then the fiber of* $g^*\mathbf{E}_{\mathrm{pt}}B$ *over the point is the loop monoid* $\Omega_{\mathrm{pt}}B$

$$
\begin{tikzcd}
\Omega_{\mathrm{pt}}B \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & g^*\mathbf{E}_{\mathrm{pt}}B \arrow[d, two heads] \\
\mathrm{pt}_X \arrow[r, "\mathrm{pt}"] & X
\end{tikzcd}
$$

Definition 14 (fiber bundle) *We say the morphism* $P \longrightarrow X$ *equipped with an action of* $\Omega_{\mathrm{pt}}B$ *is a* B-*<u>fiber bundle</u> if there is a cocycle* $X \overset{\simeq}{\twoheadleftarrow} \hat{X} \xrightarrow{g} B$ *and a weak equivalence*

$$
\begin{tikzcd}
g^*\mathbf{E}_{\mathrm{pt}}B \arrow[rr, "\simeq"] \arrow[dr] & & P \arrow[dl] \\
& X &
\end{tikzcd}
$$

respecting the $\Omega_{\mathrm{pt}}B$*-action on both sides.*

Definition 15 (trivial bundle) *The <u>trivial B-bundle</u> over an object* X *is the pullback of* $\mathbf{E}_{\mathrm{pt}}B$ *along the trivial cocycle: the one that factors through the point of* B*.*

Lemma 13 *The trivial* B*-bundle over* X *is the product of* X *with the monoid of loops* $\Omega_{\mathrm{pt}}B$*:*

$$
X \times \Omega_{\mathrm{pt}}B \xrightarrow{p_1} X \ .
$$

Proof. By lemma **??** we have
$$\begin{tikzcd}
X \times \Omega_{\mathrm{pt}} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_{\mathrm{pt}} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E}_{\mathrm{pt}} B \arrow[d] \\
X \arrow[r] & \mathrm{pt} \arrow[r, "\mathrm{pt}_B"] & B
\end{tikzcd}.$$
□

Proposition 6 (fiber bundle trivializes over itself) *Every fiber bundle $P \to X$ is trivializable after pulled back to its own total space.*

Proof. For $X \overset{\simeq}{\twoheadleftarrow} \hat{X} \longrightarrow B$ a cocycle characterizing the bundle $P \to X$ we obtain the pullback diagram

$$\begin{tikzcd}
P \arrow[dd] & g^* \mathbf{E}_{\mathrm{pt}} B \arrow[l, two heads, "\simeq"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E}_{\mathrm{pt}} B \arrow[r, two heads] \arrow[d] & \mathrm{pt} \arrow[d] \\
& g^* B^{\mathcal{I}} \arrow[r] \arrow[d, two heads, "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\mathcal{I}} \arrow[r, two heads, "d_0", "\simeq"'] \arrow[d, two heads, "\simeq"', "d_1"] & B \\
X & \hat{X} \arrow[l, two heads, "\simeq"'] \arrow[r, "g"] & B &
\end{tikzcd}.$$

The cocycle g pulled back to P is represented by the morphism from $g^* \mathbf{E}_{\mathrm{pt}} B$ to the B at the bottom. The right part of the diagram says that this is homotopic to a map factoring through the point. □

2.2.5 Homotopy fiber sequences

As described in section **??** in the Kan-enriched category $\hat{\mathcal{C}}$ there is a canonical notion of homotopy limits and in particular of homotopy pullbacks. In order to get a good control over these we define a notion of homotopy pullback in $\mathcal{C}$ and show in proposition 10 that this definition is compatible with homotopy pullbacks in $\hat{\mathcal{C}}$ under the inclusion $\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$.

Definition 16 (homotopy pullback [44]) *A commutative diagram* $\begin{tikzcd} W \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & B \end{tikzcd}$ *is a* <u>*homotopy pullback square*</u>*, denoted* $\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\rrbracket", very near start] & C \arrow[d] \\ D \arrow[r] & B \end{tikzcd}$*, if the induced composite morphism* $W \dashrightarrow D \times_{B^{\mathcal{I}}} C$ *to the homotopy fiber product from definition 9 is a weak equivalence:*

$$\begin{tikzcd}
W \arrow[r, dashed] \arrow[rr, bend right, "\simeq"'] & D \times_B C \arrow[r, dashed] & D \times_{B^{\mathcal{I}}} C
\end{tikzcd}.$$

Proposition 7 *If* $D \longrightarrow\!\!\!\!\!\rightarrow B$ *is a fibration then the ordinary pullback is a homotopy pullback*

$$\begin{tikzcd}
W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
D \arrow[r, two heads] & B
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
W \arrow[r] \arrow[d] \arrow[dr, phantom, "\rrbracket", very near start] & C \arrow[d, "f"] \\
D \arrow[r] & B
\end{tikzcd}.$$

Proof. Recall from lemma 8 that $D \times_{B^{\mathcal{I}}} B = D \times_B \mathbf{E}_f B$. Consider the double pullback square

$$\begin{tikzcd}
D \times_B C \arrow[r] \arrow[d, dashed, "\simeq"] \arrow[dd, bend right] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\sigma_f"', "\simeq"] \arrow[dd, bend left, "f"] \\
D \times_{B^{\mathcal{I}}} C \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E}_f B \arrow[d, "p_f"'] \\
D \arrow[r, two heads] & B
\end{tikzcd}$$

constructed using the morphism σ_f from lemma 7, where the bottom square is due to lemma 8. Using that in a category of fibrant objects fibrations are preserved under pullback (by definition) the middle horizontal morphism is a fibration. Using that weak equivalences are preserved under pullback along fibrations (by lemma 2, p. 428 in [7]) the dashed morphism is a weak equivalence. $\square$

Proposition 8 *The pasting composition of two homotopy pullback diagrams*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d] \\
C \arrow[r] & D \arrow[r] & F
\end{tikzcd}$$

is itself a homotopy pullback diagram

Proof. Consider the pasting composite of the two homotopy fibrations involved:

$$\begin{tikzcd}
A \arrow[d, "\simeq"] \arrow[drrr] & & & & & \\
C \times_{D^{\mathcal{I}}} B \arrow[rrr] \arrow[d, dashed, "\simeq"] \arrow[dr, "\simeq"] \arrow[dddd, bend right] & & & B \arrow[d, "\simeq"] \arrow[drr] & & \\
C \times_{F^{\mathcal{I}}} E \arrow[r, "\simeq"] \arrow[ddd] \arrow[dr, phantom, very near start, "\lrcorner"] & C \times_{D^{\mathcal{I}}} D \times_{F^{\mathcal{I}}} E \arrow[rr, two heads] \arrow[dd, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & & D \times_{F^{\mathcal{I}}} E \arrow[rr, two heads] \arrow[dd, two heads] & & E \arrow[d] \\
& & & & {[\mathcal{I}, F]} \arrow[r] \arrow[d] & F \\
& \mathbf{E}_f D \arrow[r] \arrow[d, "\simeq"] \arrow[rr, bend left, two heads, "p_f"] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\mathcal{I}, D]} \arrow[r, two heads, "d_0"'] \arrow[d, "d_1"] & D \arrow[r] & F & \\
C \arrow[ur, "\sigma_f", "\simeq"'] \arrow[r, "\mathrm{Id}"] \arrow[urrr, bend right, "f"'] & C \arrow[r, "f"] & D & & &
\end{tikzcd},$$

where in the bottom left we have inserted the universal D-bundle $\mathbf{E}_f D$ relative to f together with the section morphism σ_f to factor the left vertical morphisms to C through $\mathbf{E}_f D$. This is as in the proof of lemma 7 and serves to show that the morphism $C \times_{F^{\mathcal{I}}} E \xrightarrow{\simeq} C \times_{D^{\mathcal{I}}} (D \times_{F^{\mathcal{I}}} E)$ is a weak equivalence, being the pullback of the weak equivalence σ_f by a fibration, which itself is the pullback of the fibration $D \times_{F^{\mathcal{I}}} E \twoheadrightarrow D$ given by corollary 2.2.

By corollary 2.3 the morphism $C \times_{D^{\mathcal{I}}} B \xrightarrow{\simeq} C \times_{D^{\mathcal{I}}} (D \times_{F^{\mathcal{I}}}) E$ is a weak equivalence. It then follows by two-out-of-three that the dashed morphism is a weak equivalence, which finally gives the required weak equivalence $A \xrightarrow{\simeq} C \times_{F^{\mathcal{I}}} E$. $\square$

Definition 17 (homotopy fibration sequence) *A sequence of morphisms*

$$\cdots \xrightarrow{f_{n+1}} C_{n+1} \xrightarrow{f_n} C_n \longrightarrow \cdots$$

of pointed objects is a homotopy fibration sequence if every morphism is the homotopy kernel of its succesor, in that it fits into a homotopy pullback diagram of the form

$$\begin{tikzcd}
C_{n+2} \arrow[r] \arrow[d, "f_{n+1}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{pt} \arrow[d, "\mathrm{pt}_{C_n}"] \\
C_{n+1} \arrow[r, "f_n"] & C_n
\end{tikzcd}.$$

Proposition 9 *Every homotopy fiber product*

$$
\begin{tikzcd}
X \times_{B^{\mathcal{I}}} \mathrm{pt} =: P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d, "\mathrm{pt}_B"] \\
X \arrow[r] & B
\end{tikzcd}
$$

induces a long homotopy fibration sequence to the left, of the form

$$
\begin{tikzcd}
\cdots \arrow[r] & \Omega_{\mathrm{pt}}\Omega_{\mathrm{pt}}P \arrow[d, "\Omega_{\mathrm{pt}} i"] & & \\
& \Omega_{\mathrm{pt}}P \arrow[d, "\Omega_{\mathrm{pt}}"] & & \\
& \Omega_{\mathrm{pt}}X \arrow[r, "\overline{\Omega g}"] & \Omega_{\mathrm{pt}}B \arrow[d] & \\
& & P \arrow[d, "p"] & \\
& & X \arrow[r, "g"] & B
\end{tikzcd} .
$$

Proof. The homotopy fibration sequence is constructed by the pasting of homotopy pullbacks:

$$
\begin{tikzcd}
\vdots \arrow[r] & \Omega_{\mathrm{pt}}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d] & \\
\arrow[r] & \Omega_{\mathrm{pt}}B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d] \\
& \mathrm{pt} \arrow[r] & X \arrow[r, "g"] & B
\end{tikzcd} ,
$$

where we identify the pullback objects appearing here (up to weak equivalence) by using lemma 8 for identifying every two consecutive pullback diagrams with their total pullback diagram.

In the first step this yields

$$
\begin{tikzcd}
\mathrm{pt} \times_{X^{\mathcal{I}}} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d] \\
\mathrm{pt} \arrow[r] & X \arrow[r, "g"] & B
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
\Omega_{\mathrm{pt}}B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d] \\
\mathrm{pt} \arrow[r] & B
\end{tikzcd}
$$

by definition 11. Similarly, in the next step we have

$$
\begin{tikzcd}
\Omega_{\mathrm{pt}}B \times_{P^I} \mathrm{pt} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d] \\
\Omega_{\mathrm{pt}}B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d] \\
\mathrm{pt} \arrow[r] & X
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
\Omega_{\mathrm{pt}}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d] \\
\mathrm{pt} \arrow[r] & X
\end{tikzcd} ,
$$

where again the homotopy pullback is identified with the loop monoid, only that now the orientation of the loops appears in the opposite order, so that the induced morphism of loop monoids is

$$
\overline{\Omega_{\mathrm{pt}} g} : \ \Omega_{\mathrm{pt}}X \longrightarrow \Omega_{\mathrm{pt}}B \ .
$$

And so on. $\square$

There is a canonical notion of homotopy limits, fibration sequences, etc. for Kan-enriched categories. The next proposition asserts that these notions in $\mathcal{C}$ discussed so far are compatible under the embedding $\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$ from proposition 1.

Proposition 10 *For every object $X \in \mathcal{C}$ the nameKan-valued hom induced on $\mathcal{C}$ from the embedding $\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$*

$$\mathcal{C} \hookrightarrow \hat{\mathcal{C}} \xrightarrow{\hom_{\mathcal{C}}} \mathsf{SSet}$$

sends homotopy pullbacks in the sense of definition **??**

This says that for $D \times_{B^{\mathcal{I}}} C$ a homotopy fiber product in $\mathcal{C}$, definition 9, and $X \in \mathcal{C}$ any object, the internal hom object is $\mathcal{C}(X, D \times_{B^{\mathcal{I}}} C) = \mathcal{C}(X, B) \times_{\mathcal{C}(X,B)^{\Delta^1}} \mathcal{C}(X, C)$.

This would follow directly from the statements

- $\mathcal{C}(X, -) : \mathcal{C} \to \mathsf{SSet}$ preserves ordinary limits;
- $\mathcal{C}(X, B^{\Delta^1}) = \mathcal{C}(X, B)^{\Delta^1}$.

The first statement seems to be the statement below (3.14) in [**?**], though this may require care with the internal notion of limit used there. The second statement is a special case of proposition 4.

2.2.6 Extension and lifting problem

Given a cocycle $g : X \to A$ it is of interest to ask if

- it *lifts* through a given morphism into its coefficient object;
- it *extends* through a given morphism out of its domain object.

$$\begin{tikzcd}
 & \overline{A} . \arrow[d] \\
X \arrow[r, "g"] \arrow[ur, dashed, "\overline{g}"] \arrow[d, hook] & A \\
\underline{\hat{X}} \arrow[ur, dashed, "\underline{g}"'] &
\end{tikzcd}$$

In section 2.1.4 we conceived the extension problem as a lifting problem on relative cohomology classes. To model this in terms of a Brown category, recall from section 5 of [7] that for $\mathcal{C}$ a category of fibrant objects, there is canonically the structure of a category of fibrant objects on $\mathrm{Tow}(\mathcal{C})$, the category whose objects are sequences of fibrations

$$A = (\cdots \twoheadrightarrow A_i \xrightarrow{p_i} A_{i-1} \twoheadrightarrow \cdots)$$

for $i \in \mathbb{Z}$ such that $A_k = e$ for all $k \leq b_A$ for some $b_A \in \mathbb{Z}$. Morphisms $f : A \to B$ are given by collections of morphisms $f_i : A_i \to B_i$ such that the obvious diagrams commute. The canonical structure of a category of fibrant objects on $\mathrm{Tow}(\mathcal{C})$ regards a morphism f as a weak equivalence if all components f_i are weak equivalences in $\mathcal{C}$ and regards a morphism f as a fibration if all the induced morphisms $A_i \to A_{i-1} \times_{B_{i-1}} B_i$ are fibrations in $\mathcal{C}$.

Definition 18 *For $\mathcal{C}$ a category of fibrant objects, let* $\mathrm{Rel}(\mathcal{C})$ *be the category whose objects are fibrations* $A = (\ A_1 \overset{p}{\twoheadrightarrow} A_0\ )$ *in $\mathcal{C}$, and whose morphisms $f : A \to B$ are commuting diagrams* $\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, two heads, "p_A"] & B_1 \arrow[d, two heads, "p_B"] \\ A_0 \arrow[r, "f_0"] & B_0 \end{tikzcd}$ *. Such a morphism is called a* weak equivalence *if f_1, f_2 are weak equivalences in $\mathcal{C}$. It is called a* fibration *if the induced morphism $A_1 \to A_0 \times_{B_0} B_1$ is a fibration in $\mathcal{C}$.*

Corollary 2.5 *The category* $\mathrm{Rel}(\mathcal{C})$ *with these fibrations and weak equivalences is a category of fibrant objects in the sense of [7].*

Proof. This is a special case of section 5 in [7]. $\square$

Write $\widehat{\mathrm{Rel}(\mathcal{C})}$ for the simplicially enriched version of $\mathrm{Rel}(\mathcal{C})$ according to proposition 2 and write

$$H\left(\begin{array}{cc} X & \overline{A} \\ \downarrow, & \downarrow \\ \underline{X} & A \end{array}\right) := \pi_0 \widehat{\mathrm{Rel}(\mathcal{C})}\left(\begin{array}{cc} X & \overline{A} \\ \downarrow, & \downarrow \\ \underline{X} & A \end{array}\right)$$

for the corresponding cohomologies.

Proposition 11 *For all morphisms $i : X \to \underline{X}$ and objects A in $\mathcal{C}$ we have natural isomorphisms*

$$H\left(\begin{array}{cc} X & A \\ \downarrow, & \downarrow \\ \underline{X} & \mathrm{pt} \end{array}\right) \simeq H(X, A)\,, \qquad H\left(\begin{array}{cc} X & A \\ \downarrow, & \downarrow\simeq \\ \underline{X} & A \end{array}\right) \simeq H(\underline{X}, A)$$

Proof. That's at least the idea. Should be clear. But this is a little bit more technical than it may seem. May have to tune the notion of homotopy monomorphism here. $\square$

It follows that the extension problem $\begin{tikzcd} X \arrow[r] \arrow[d, "i"] & A \\ \underline{X} \arrow[ur, two heads] & \end{tikzcd}$ in ordinary cohomology is the lifting problem

$$\begin{tikzcd}
 & & A \arrow[dr, "\simeq"] \arrow[dd, "="] & \\
 & & & A \arrow[dd] \\
X \arrow[uurr, dashed] \arrow[rr] \arrow[dr] & & A \arrow[dr] & \\
 & \underline{X} \arrow[uurr, dashed] \arrow[rr] & & \mathrm{pt}
\end{tikzcd}$$

in relative cohomology.

The next proposition asserts that this lifting problem admits a good obstruction theory in the sense of definition **??** when A is once deloopable.

Proposition 12 *For every pointed object K in $\mathcal{C}$ there is in* $\widehat{\mathrm{Rel}(\mathcal{C})}$ *a homotopy fiber sequence*

$$\begin{tikzcd}
\Omega_{\mathrm{pt}} K \arrow[d, two heads, "\simeq"'] & & \Omega_{\mathrm{pt}} K \arrow[d] & & \mathbf{E}_{\mathrm{pt}} K \arrow[d, two heads, "p_K"] \\
\Omega_{\mathrm{pt}} K & \arrow[r] & \mathrm{pt} & \arrow[r] & K
\end{tikzcd}$$

Proof. The second morphism is the span in $\mathrm{Rel}(\mathcal{C})$ that is given in $\mathcal{C}$ by the diagram

$$
\begin{tikzcd}
\mathbf{E}_{i_K}\mathbf{E}_{\mathrm{pt}}K \arrow[rr, "p_{i_K}"] \arrow[dr, "p_{i_K}"'] \arrow[dd, two heads, "\simeq"] & & \mathbf{E}_{\mathrm{pt}}K \arrow[dr, two heads, "p_K"] & \\
& \mathbf{E}_{\mathrm{pt}}K \arrow[rr, two heads, "p_K"] \arrow[dd, two heads, "\simeq"] & & K \\
\Omega_{\mathrm{pt}}K \arrow[dr] & & & \\
& \mathrm{pt} & &
\end{tikzcd}
.
$$

We compute the homotopy kernel of that morphism using the homotopy fiber product from definition 9 in $\mathrm{Rel}(\mathcal{C})$

$$
\begin{tikzcd}
Q_1 \arrow[rrrr] \arrow[dr, two heads, "\simeq"] \arrow[dddd] & & & & \mathrm{pt} \arrow[dr, "="] \arrow[dd] & \\
& Q_0 \arrow[rrrr] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dddd] & & & & \mathrm{pt} \arrow[dd] \\
& & (\mathbf{E}_{\mathrm{pt}}K)^{\Delta^1} \arrow[rr, two heads, "d_1", "\simeq"'] \arrow[dr, "p_K^{\Delta^1}"'] \arrow[dd, two heads, "\simeq d_0"] & & \mathbf{E}_{\mathrm{pt}}K \arrow[dr, two heads, "p_K"] & \\
& & & K^{\Delta^1} \arrow[rr, two heads, "\simeq"] \arrow[dd, two heads, "\simeq"] & & K \\
\mathbf{E}_{i_K}\mathbf{E}_{\mathrm{pt}}K \arrow[rr, "p_{i_K}"] \arrow[dr, "p_{i_K}"'] \arrow[dd, two heads, "\simeq"] & & \mathbf{E}_{\mathrm{pt}}K \arrow[dr, two heads, "p_K"] \arrow[dd, two heads, "\simeq"] & & & \\
& \mathbf{E}_{\mathrm{pt}}K \arrow[rr, two heads, "p_K"] \arrow[dd, two heads, "d_0", "\simeq"'] & & K & & \\
\Omega_{\mathrm{pt}}K \arrow[rr] \arrow[dr] & & \mathrm{pt} & & & \\
& \mathrm{pt} & & & &
\end{tikzcd}
.
$$

Here Q_1 is in particular the limit in $\mathcal{C}$ over the rear face of the diagram, which is manifestly a weakly equivalent replacement for the diagram $\Omega_{\mathrm{pt}}K \longrightarrow \mathrm{pt} \longleftarrow \mathrm{pt}$. Therefore Q_1 is weakly equivalent to $\Omega_{\mathrm{pt}}K$. Similarly the front diagram manifestly computes the homotopy limit of the diagram $\mathrm{pt} \xrightarrow{\mathrm{pt}_K} K \xleftarrow{\mathrm{pt}_K} \mathrm{pt}$, which is $\Omega_{\mathrm{pt}}K$, by definition. Finally the morphism $Q_1 \xrightarrow{\simeq} Q_0$ is an acyclic fibration, as it is the pullback of the acyclic fibration $\mathrm{Id}_{\mathrm{pt}}s$.

□

2.2.7 Smooth ∞-groupoids

(... to be filled in ...)
- ω-nerve takes us from strict ∞-groupoid valued sheaves to Kan simplicial sheaves
- yields large repository of examples, one for each smooth crossed complex of groupoids,
- especially recall smooth model for $\mathbf{B}$String, $\mathbf{B}$Fivebrane, and so on and so forth

2.2.8 Differential cohomology

We now consider the explicit realization of the fundamental ∞-groupoid ∞-functor Π from section 2.1.6 on which the formalization of differential cohomology is based. We do this again in the context of the model provided by the Brown category $\mathsf{SPSh} := \mathsf{SPSh}(\mathrm{Diff})$ of locally Kan simplicial presheaves on smooth manifolds.

The functor Π here is induced by Kan extension from the *smooth* (parameterized over $S = \mathrm{Diff}$) version of the familiar singular simplicial complex functor. For $S = \mathrm{Top}$ this Π is essentially the same construction as considered for instance in section 7.1 of [44], where its left adjoint is considered, which plays the role of geometric realization. Here we instead look at the right adjoint of Π, which we identify with the functor that forms cohomology of flat differential forms in the sense of section 2.1.6.

The following makes sense for S any cartesian monoidal site equipped with a cosimplicial object $\Delta^\bullet : \Delta \to S$. For instance $S = \mathsf{Top}$ or $S = \mathrm{Diff}$ with Δ^n the standard n-simplex manifold. We write $\mathsf{SPSh}(S)$, or just SPSh for short, for the SSet-enriched category of SSet-valued presheaves on S.

Definition 19 (smooth fundamental ∞-groupoid) *Let $\Pi : S \to \mathsf{SPSh}$ be given by $\Pi := S(- \times \Delta^\bullet, -)$, so that for $U \in S$ we have $\Pi(U)_n : V \mapsto S(V \times \Delta^n, U)$. By abuse of notation, we write $\Pi : \mathsf{SPSh} \to \mathsf{SPSh}$ for the (left) Kan extension*

$$\mathrm{Lan}\,\Pi : \mathrm{SPSh} \to \mathrm{SPSh}$$

of Π through the chain of canonical inclusions

$$S \hookrightarrow \mathsf{PSh}(S) \hookrightarrow \mathsf{SPSh}(S)\,,$$

where the first morphism is the Yoneda embedding and the second regards presheaves as constant simplical presheaves.

Notice that in the case of $S = \mathsf{Top}$ we have $\Pi(U) : \mathrm{pt} \mapsto \mathrm{Sing}(U)$, the singular simplicial complex of U.

Lemma 14 *For any functor $\Pi : S \to \mathsf{SPSh}$ the left Kan extension $\Pi : \mathsf{SPSh} \to \mathsf{SPSh}$ is given on any $X \in \mathsf{SPSh}$ by the following equivalent coend formulas*

$$\Pi(X) = \int^{U\in S} X(U) \times \Pi(U) = \int^{[n]\in\Delta} \Delta^n \times \Pi(X_n)\,,$$

where on the left $\Pi(U)$ is the application of the original Π, whereas on the right $\Pi(X_n)$ is the application of the Kan extension along $S \hookrightarrow \mathsf{PSh}(S)$.

Proof. In the context of SSet-enriched category theory the left Kan extension along $S \hookrightarrow \mathsf{SSh}$ is computed by the coend

$$\Pi(X) = \int^{U\in S} \mathsf{SSh}(U, X) \cdot \Pi(U) = \int^{U\in S} X(U) \cdot \Pi(U)\,,$$

where the operation in the integrand is the tensoring of simplicial presheaves over simplicial sets. Inserting in this expression the standard decomposition of simplicial sets into their cells, $X(U) = \int^{[n]\in\Delta} \Delta^n \times X(U)_n$, where now the operation in the integrand is the tensoring of simplicial sets over sets, yields

$$\cdots = \int^{U\in S}\int^{[n]\in\Delta} \Delta^n \cdot X_n(U) \cdot \Pi(U)\,.$$

Exchanging the coends using the Fubini theorem yields

$$\cdots = \int^{[n]\in\Delta} \Delta^n \cdot \int^{U\in S} X_n(U) \cdot \Pi(U)\,.$$

The inner integrand is now manifestly the left Kan extension along the Yoneda embedding $S \hookrightarrow \mathsf{Sh}$

$$\cdots = \int^{[n]\in\Delta} \Delta^n \cdot \int^{U\in S} \mathsf{Sh}_S(U, X_n) \cdot \Pi(U) = \int^{[n]\in\Delta} \Delta^n \times \Pi(X_n)$$

□

Notice that $\int^{[n]\in\Delta} \Delta^n \times \Pi(X_n)$ is the *diagonal presheaf* obtained from the bisimplicial presheaf $\Pi(X) : ([n],[m]) \mapsto \Pi(X_m)_n$.

The category SPSh is enriched over SSet and the enrichment extends to an enriched hom-functor

$$\mathsf{SPSh}(S)(-,-) : \mathsf{SPSh}^{\mathrm{op}} \times \mathsf{SPSh}^{\mathrm{op}} \to \mathsf{SSet}\,.$$

In particular for every $A \in \mathsf{SPSh}(S)$ there is simplicial presheaf

$$\mathrm{H}_{\mathrm{dR}}(-,A) := \mathsf{SPSh}(S)(\Pi(-),A) : S^{\mathrm{op}} \to \mathsf{SSet}$$

and this depends functorially on A.

Lemma 15 *The functor $\Pi : \mathsf{SPSh}(S) \to \mathsf{SPSh}(S)$ if left adjoint to the functor $A \mapsto H_{\mathrm{dR}}(-,A)$:*

$$\Pi \dashv H_{\mathrm{dR}}\,.$$

Proof. We demonstrate the Hom-isomorphism that characterizes the adjunction: Start with the first coend description of $\Pi(X)$ from above

$$\mathrm{Hom}_{\mathsf{SPSh}}(\Pi(X),A) \simeq \mathrm{Hom}_{\mathsf{SPSh}}(\int^{U\in S} \Pi(U)\cdot X(U), A)\,.$$

Then use the continuity of the Hom-functor to pass it through the coend and obtain the following end:

$$\cdots \simeq \int_{U\in S} \mathrm{Hom}_{\mathsf{SPSh}}(\Pi(U)\cdot X(U), A)\,.$$

The defining property of the tensoring operation implies that this is equivalent to

$$\simeq \int_{U\in S} \mathrm{Hom}_{\mathsf{SSet}}(X(U), \mathrm{SPSh}(\Pi(U),A))\,.$$

But this is the end-formula for SSet-object of natural transformations between simplicial presheaves:

$$\cdots \simeq \mathrm{Hom}_{\mathsf{SPSh}}(X, \mathrm{SPSh}(\Pi(-),A))\,.$$

By definition this is the desired right hand of the hom isomorphism

$$\cdots \simeq \mathrm{Hom}_{\mathsf{SPSh}}(X, H_{\mathrm{dR}}(-,A))\,.$$

□

3 Twisted Topological Structures

In this section we show that the following phenomena in string theory define twisted nonabelian cocycles in the sense of section 2. See also [10] [17] or the review in the introduction of [57].

Three kinds of anomalies in string theory.

1. The Freed-Witten anomaly: This is a global worldsheet anomaly of type II string theory in the presence of D-branes and a nontrivial H_3-field. The statement for the cancellation of the anomaly is that a D-brane Q can wrap a cycle $Q \to X$ in a ten-dimensional spacetime X only if [26]

$$W_3(Q) + [H_3]|_Q = 0 \in H^3(X^{10}; \mathbb{Z}) \; , \tag{3.1}$$

where $W_3(Q)$ is the third integral Stiefel-Whitney class of TQ. When $[H_3] = 0$ is trivial in cohomology, i.e. $H_3 = dB_2$, the Freed-Witten condition states that the D-brane must be Spin^c.

2. The Green-Schwarz anomaly: This is an anomaly in heterotic and type I string theory, i.e. a string theory coupled to a gauge theory, with an H_3-field. The cancellation of the anomaly is via the Green-Schwarz anomaly cancellation mechanism [28] which amounts to canceling a gravitational anomaly, coming from the coupling of fermions to gravity in the supergravity part, with a gauge anomaly, coming from the coupling of fermions to the gauge field in the gauge bundle E. The process requires the following condition to hold

$$ch_2(E) - p_1(TX) = 0 \in H^4(X; \mathbb{Z}) \; . \tag{3.2}$$

This formula in cohomology is trivialized by H_3, i.e. at the level of differential forms the expression with representatives in place of classes is equal to dH_3. Mathematically (cf. [23]), the above two contributions correspond to the Pfaffian line bundle Pfaff and an electric charge line bundle $\mathcal{L}_e$, and the statement is that the anomaly line bundle Pfaff $\otimes \mathcal{L}_e$ needs to be trivialized. The local (global) anomaly is the curvature (holonomy) of this line bundle.

3. The dual Green-Schwarz anomaly: This is also an anomaly in heterotic and type I string theory, but now with an H_7-field in the dual formulation of the theory [13]. The cancellation of the anomaly is via the dual of the above Green-Schwarz anomaly cancellation mechanism [63] [27]. The process requires the following condition to hold

$$\frac{1}{48} p_2(X) - \mathrm{ch}_4(E) + \frac{1}{48} p_1(X) ch_2(E) - \frac{1}{64} p_1(X)^2 = 0 \in H^8(X; \mathbb{Z}) \; . \tag{3.3}$$

This formula in cohomology is trivialized by H_7, i.e. at the level of differential forms the expression with representatives in place of classes is equal to dH_7. Mathematically (cf. [23]), the statement is that the anomaly line bundle Pfaff $\otimes \mathcal{L}_m$, where $\mathcal{L}_m$ is the magnetic charge line bundle, needs to be trivialized.

The vanishing of the first fractional Pontrjagin class $\frac{1}{2}p_1(X)$ of a Spin-manifold X is also known as the condition for X to admit a String structure [38], i.e. a lifting of the structure group on the tangent bundle from Spin(n) to String(n). Notice that in homotopy theory and in physics the class $\frac{1}{2}p_1 \in H^4(X, \mathbb{Z})$, which is well-defined on a Spin manifold, is sometimes called λ.

We make the following definitions, which originate in [70] and which we already interpreted in section 2 in terms of twisted nonabelian cohomology.

Definition 20 *An α-twisted String structure (or a String structure relative to α) on a Spin manifold M with classifying map $f : M \to B\mathrm{Spin}(n)$ is a cocycle $\alpha : M \to K(\mathbb{Z}, 4)$ and a homotopy η:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "\alpha"'] & B\mathrm{Spin}(n) \arrow[d, "\frac{1}{2}p_1"] \arrow[dl, phantom, "\swarrow\eta"] \\ & K(\mathbb{Z},4) \end{tikzcd} \; . \tag{3.4}$$

If α is trivial (i.e. factors through a point) then this reduces to an ordinary String-structure. Analogously for twisted Fivebrane-structures:

Definition 21 *An α-twisted Fivebrane structure (or a Fivebrane structure relative to α) on a String manifold M with classifying map $f : M \to B\mathrm{String}(n)$ is a cocycle $\alpha : M \to K(\mathbb{Z}, 8)$ and a homotopy η:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "\alpha"'] & B\mathrm{String}(n) \arrow[d, "\frac{1}{6}p_2"] \arrow[dl, phantom, "\Swarrow\eta"] \\ & K(\mathbb{Z}, 8) \end{tikzcd} . \tag{3.5}$$

If α is trivial (i.e. factors through a point) then this reduces to an ordinary Fivebrane-structure.

Notation. We fix once and for all connections ω and A on the Spin bundles and the gauge bundles respectively. The corresponding curvatures are F_ω and F_A, respectively. We will use these to give differential form representatives of the corresponding characteristic classes. We will use the convention of writing a class with argument the curvature form to indicate the differential form representative of the class, written with argument the corresponding bundle. For instance, $p_i(TM) = p_i(M)$ means the cohomology class while $p_i(F_\omega)$ will mean the differential $4i$-form representative.

(Twisted) Fivebrane cobordism. Recall that Spin cobordism $\Omega_*^{\mathrm{spin}} = \Omega_*^{\langle 4 \rangle}$ refers to cobordism of spaces equipped with Spin structure and String cobordism $\Omega_*^{\mathrm{String}} = \Omega_*^{\langle 8 \rangle}$ refers to cobordism of spaces equipped with a String structure. For spaces X with Fivebrane structure we can also define Fivebrane cobordism $\Omega_*^{\mathrm{Fivebrane}}(X) = \Omega_*^{\langle 9 \rangle}(X)$ in a similar manner. Given a manifold X with a twisting $\beta : X \to K(\mathbb{Z}, 8)$, one can form a cobordism category, in analogy to the String case [70], called the β-twisted Fivebrane cobordism over (X, β), whose objects are compact smooth String manifolds over X with a β-twisted Fivebrane structure. We call the corresponding cobordism group $\Omega_*^{Fivebrane}(X, \beta)$ the *β-twisted Fivebrane cobordism group of X.*

3.1 Twisted String Structures

Relative trivialization on branes. As the example of the twisted Spin^c-structures, discussed in the introduction, already indicates, in string theory such structures usually arise on branes M sitting in an ambient space X, $\iota : M \to X$, and the twist is by the restriction $\alpha := \beta|_M := \iota^*\beta$

$$\begin{tikzcd} M \arrow[d, "\iota"'] \arrow[dr, "\alpha = \iota^*\beta"] & \\ X \arrow[r, "\beta"'] & K(\mathbb{Z}, n) \end{tikzcd} \tag{3.6}$$

of a class of the ambient space to the brane. Since this special case of twisted structures is important in applications, we state it as separate definition:

Definition 22 *A β-twisted String structure on a brane $\iota : M \to X$ with Spin structure classifying map $f : M \to B\mathrm{Spin}(n)$ is a cocycle $\beta : X \to K(\mathbb{Z}, 4)$ and a homotopy η:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\iota"'] & B\mathrm{Spin}(n) \arrow[d, "\frac{1}{2}p_1"] \arrow[dl, Rightarrow, "\eta"] \\ X \arrow[r, "\beta"'] & K(\mathbb{Z}, 4) \end{tikzcd} . \tag{3.7}$$

This is essentially the definition also given in [70]. This situation arises with X being the 11-dimensional M-theory target space and $M = \partial X$, its 10-dimensional boundary, being the target for the heterotic string.

3.1.1 A refinement for further divisibility of $\frac{1}{2}p_1$

We have given above the definition of a twisted String structure, essentially following [70], see definitions 20 and 22.

In version 1 of the eprint [70], it was also parenthetically noted that this is related to Witten's quantization condition. In fact, it is essentially so, because of the extra factor of $\frac{1}{2}$ in front of $\frac{1}{2}p_1$ in Witten's formula in [73]. In this section we exhilineq a space in which $\frac{1}{4}p_1 = \frac{1}{2}\lambda$ is an obstruction, thus obtaining the flux quantization condition in M-theory exactly, as well as providing a further example from string theory.

In order to characterize $\frac{1}{4}p_1$, we consider a Spin structure on a space Y and consider the following diagram

$$\begin{tikzcd}
Y \arrow[r, "x"] \arrow[dr] & \mathcal{F}^{\langle 4 \rangle} \arrow[r] \arrow[d] & K(\mathbb{Z}, 4) \arrow[d, "\times 2"] & \\
& (BO)\langle 4 \rangle = B\mathrm{Spin} \arrow[r, "\frac{1}{2}p_1"] \arrow[d] & K(\mathbb{Z}, 4) \arrow[r] & K(\mathbb{Z}_2, 4) \arrow[d, "="] \\
& (BO)\langle 2 \rangle = BSO \arrow[rr, "w_4"] & & K(\mathbb{Z}_2, 4),
\end{tikzcd} \tag{3.8}$$

where x is our class $\frac{1}{4}p_1$ which naturally lives not in $(BO)\langle 4 \rangle = B\mathrm{Spin}$ but rather in the desired space $\mathcal{F}^{\langle 4 \rangle}$ (the $\langle 4 \rangle$ is for $BO\langle 4 \rangle$). The above diagram specifies $\mathcal{F}^{\langle 4 \rangle}$. Thus, we have

Observation 1 *The class $\frac{1}{4}p_1$ is the obstruction to lifting an $\mathcal{F}^{\langle 4 \rangle}$ bundle, where $\mathcal{F}^{\langle 4 \rangle} = BO\langle \frac{1}{4}p_1 \rangle$ is defined by diagram (3.8), to a String bundle.*

This observation is analogous to proposition 2 in [57] for the Fivebrane case, where there we were considering the comparison of $\frac{1}{48}p_2$ to the obstruction to Fivebrane structure given by $\frac{1}{6}p_2$.

3.1.2 The Green-Schwarz anomaly and the M-theory C-field

Example I: The Green-Schwarz formula. We consider the first setting where twisted String structures make an appearance. Anomaly cancelation in heterotic string theory is governed by the Green-Schwarz mechanism [28]. Consider a ten-dimensional Spin manifold M, on which there is also a vector bundle E with rank 16 structure group G, which is either $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$. E is part of the data of a (super)Yang-Mills (SYM) theory and has characteristic classes built out of the curvature F. Both the tangent bundle TM and the gauge vector bundle a priori have degree four classes $\lambda(M) = \frac{1}{2}p_1(TM)$ and $\lambda(E) = \frac{1}{2}p_1(E)$, coming from pullbacks from $B\mathrm{Spin}(10)$ and BG, respectively. The anomaly cancelation condition is given by

$$\frac{1}{2}p_1(M) - \frac{1}{2}p_1(E) = 0. \tag{3.9}$$

Inspecting this formula we can immediately identify it as a twisted String structure with a twist given by $-\lambda(E) = -\frac{1}{2}p_1(E)$. Therefore we immediately have

Proposition 13 *The Green-Schwarz anomaly cancelation condition defines a twisted String structure.*

Example II: (Heterotic) M-theory. We next consider the second setting where twisted String structures appear. The low energy limit of M-theory is eleven-dimensional supergravity (Sugra). The dimensional reduction of the latter corresponds to ten-dimensional supergravity, which in turn is the low energy limit of superstring theory. If the process of taking a boundary is done carefully, one can actually recover also the coupling to Yang-Mills theory. Then taking a high energy limit leads to heterotic string theory. This is the subject of heterotic M-theory, and the process is depicted in this diagram

$$\begin{tikzcd} \mathrm{M}-\mathrm{theory} \arrow[r, "\text{low energy}", "\ell_1"'] \arrow[d, "\partial_2"] & D=11\ \mathrm{Supergravity} \arrow[d, "\partial_1"] \\ \mathrm{Heterotic\ String} \arrow[r, "\text{low energy}", "\ell_2"'] & D=10\ \mathrm{Supergravity}+\mathrm{SYM}\ . \end{tikzcd} \tag{3.10}$$

In [32], Horava and Witten carefully studied the map ∂_1 and gave arguments on how to extend towards the strong coupling limit, i.e. going along ℓ_1^{-1} and ℓ_2^{-1}. The result is a modification of the usual Green-Schwarz cancelation condition for $G = E_8 \times E_8$

$$\frac{1}{4}p_1(M) - \frac{1}{2}p_1(E) = 0. \tag{3.11}$$

The extension of this to the eleven-dimensional bulk, i.e. roughly towards the upper left corner of diagram (3.10) leads, by certain locality arguments, to the analogous condition to (3.11) but now for the eleven-dimensional spacetime Y [73]

$$\frac{1}{4}p_1(Y) - \frac{1}{2}p_1(E) = 0, \tag{3.12}$$

where, with an obvious abuse of notation, E in equation (3.11) is the restriction of E in equation (3.12). From (3.11), (3.12), and proposition 13 we get

Proposition 14 *The anomaly cancelation condition in heterotic M-theory and the flux quantization condition in M-theory each define a twisted String structure on* $\mathcal{F}^{\langle 4\rangle} = BO\langle \frac{1}{4}p_1\rangle$.

3.2 Twisted Fivebrane Structures

In section 3.1 we interpreted the conditions on degree four classes in heterotic string theory and in M-theory as obstructions to twisted String structures. On the other hand, in [57] we showed that the dual fields give rise to Fivebrane structures, provided some additional terms are set to zero. In this section we show that these dual fields give rise to twisted Fivebrane structures, which we defined at the beginning of section 3. In doing so, we also remedy some of the caveats raised in [57].

We consider the obvious generalization of definition 22 to the Fivebrane case:

Definition 23 *A β-twisted Fivebrane structure on a brane $\iota : M \to X$ with String structure classifying map $f : M \to B\mathrm{String}(n)$ is a cocycle $\beta : X \to K(\mathbb{Z}, 8)$ and a homotopy η:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\iota"'] & B\mathrm{String}(n) \arrow[d, "\frac{1}{6}p_2"] \arrow[dl, Rightarrow, "\eta"] \\ X \arrow[r, "\beta"'] & K(\mathbb{Z},8) \end{tikzcd} . \tag{3.13}$$

The above definition reduces to the usual definition of untwisted String structure of a space M upon setting X to a point. Note that setting α to zero follows from setting X to a point.

Recall [57] that we have $\frac{1}{6}p_2 : B\mathrm{String} \to K(\mathbb{Z},8)$ as the classifying map of the principal $K(\mathbb{Z},7)$-bundle $B\mathrm{Fivebrane} \to B\mathrm{String}$, represented by the generator of $H^8(B\mathrm{String},\mathbb{Z})$.

Remarks.
1. Two β-twisted Fivebrane structures η and η' on M are called equivalent if there is a homotopy between η and η'.
2. From the definition, given a String manifold M and a space X with a Fivebrane twisting $\beta : X \to K(\mathbb{Z},8)$, then M admits a β-twisted Fivebrane structure if and only if there is a continuous map $\iota : M \to X$ such that

$$\frac{1}{6}p_2(M) + \iota^*([\beta]) = 0 \tag{3.14}$$

in $H^8(M,\mathbb{Z})$.
3. If $\iota^*([\beta]) + \frac{1}{6}p_2(M) = 0$, then the set of equivalence classes of β-twisted Fivebrane structures on M are in one-to-one correspondence with elements in $H^7(M,\mathbb{Z})$.

3.2.1 A refinement for further divisibility of $\frac{1}{6}p_2$

We first consider the case of heterotic string theory. We start at the level of differential forms and then refine to the integral case. The condition

$$dH_3 = \frac{1}{2}p_1(F_\omega) \tag{3.15}$$

from the Green-Schwarz anomaly cancelation condition [28] in heterotic string theory on M, in the absence of gauge bundles, i.e. for E the trivial vector bundle, says that $\frac{1}{2}p_1(M)$ is exact. Since $[dH_3] = 0$, this implies that M lifts to $B\mathrm{String}$ and the set of lifts is labeled by H_3. Similarly, the condition

$$dH_7 = \frac{1}{48}p_2(F_\omega) \tag{3.16}$$

appears in two theories: In type IIA string theory with String structure and trivial Ramond-Ramond fields, and in heterotic string theory with String structure and with a trivial gauge bundle. This condition (3.16), being a trviality condition on $\frac{1}{48}p_2(M)$, implies that M lifts to $B\mathrm{Fivebrane}$ and the set of lifts is labeled by H_7. What we are interested in is the case where the fractional Pontrjagin classes are not trivialized, but are rather shifted by a nontrivial class, which we interpret as a twist.

Remark. The two equations (3.15) and (3.16) can be thought of as expressions in differential integral cohomology, with $\hat{H}_3$ and $\hat{H}_7$ the differential cochains for the Neveu-Schwarz field and its dual, and $\frac{1}{2}p_1(M)$ and $\frac{1}{48}p_2(M)$ the differential cocycles of the heterotic fivebrane magnetic charge and heterotic string electric charge, respectively. In fact, in the heterotic theory, the fields H_3 and H_7 should be thought of as being in differential K-theory for the heterotic string.

Recall that in [56] [57] the classes encountered in the anomaly expressions do not involve quite the obstruction $\frac{1}{6}p_2$, but rather involve $\frac{1}{48}p_2$. The extra division by 8 was explained in [57], where it was interpreted as living in a space $\mathcal{F}$ rather than on $B\mathrm{String}$ and the corresponding maps were given. Here we change the notation suggestively and we label the space as follows: $\mathcal{F}^{\langle 8\rangle} = B\mathrm{String}^{\mathcal{F}} := BO\langle \frac{1}{48}p_2\rangle$. Then we have the following definition.

Definition 24 *A β-twisted $\mathcal{F}^{\langle 8\rangle}$-structure is defined by a homotopy η in the diagram*

$$\begin{tikzcd} M \arrow[r, "\nu"] \arrow[d, "\iota"'] & \mathcal{F}^{\langle 8\rangle} \arrow[d, "\frac{1}{48}p_2"] \arrow[dl, Rightarrow, "\eta"] \\ X \arrow[r, "\beta"'] & K(\mathbb{Z},8) \end{tikzcd} \tag{3.17}$$

The obstruction in this case that would replace (3.14) is given by the following.

Observation 2 *The condition for a twisted Fivebrane structure obtained by lifting an $\mathcal{F}^{\langle 8\rangle}$ structure is given by*

$$\frac{1}{48}p_2(M) + \iota^*([\beta]) = 0. \tag{3.18}$$

Another description of the fractional classes. We will use the path space of the Eilenberg-MacLane spaces to provide an alternative, but related, description of the fractional obstructions. The path space $PK(\mathbb{Z}, m)$ is a contractible space, and so it has trivial homotopy groups. Then, from the long exact sequence on homotopy of the path fibration

$$\Omega K(\mathbb{Z}, m) \longrightarrow PK(\mathbb{Z}, m) \longrightarrow K(\mathbb{Z}, m) \tag{3.19}$$

we get that $\pi_{i-1}\left(\Omega K(\mathbb{Z}, m)\right) \simeq \pi_i(K(\mathbb{Z}, m))$, so that $\Omega K(\mathbb{Z}, m)$ is the Eilenberg-MacLane space $K(\mathbb{Z}, m-1)$. As in [40], denote, for $m \equiv 0 \bmod 4$, by $B_{d,m} \to BO\langle m\rangle$ the pullback of the fibration (3.19) via a map $\phi : BO\langle m\rangle \to K(\mathbb{Z}, m)$ such that the induced map $\pi_* : \pi_m\left(BO\langle m\rangle\right) \cong \mathbb{Z} \to \pi_m(K(\mathbb{Z}, m)) = \mathbb{Z}$ is multiplication by d. This determines ϕ up to homotopy. The long exact sequence on homotopy of the fibration

$$B_{d,m} \longrightarrow BO\langle m\rangle \xrightarrow{\ \phi\ } K(\mathbb{Z}, m) \tag{3.20}$$

shows that the induced map

$$\mathbb{Z} \cong \pi_m(B_{d,m}) \longrightarrow \pi_m\left(BO\langle m\rangle\right) \cong \mathbb{Z} \tag{3.21}$$

is multiplication by d.

Remarks

1. For the String structure, we have $m = 4$. Then we have the diagram

$$\begin{tikzcd} & B_{d,4} \arrow[d] & PK(\mathbb{Z},4) \arrow[d] & K(\mathbb{Z},3) \arrow[l] \ . \\ X \arrow[ur] \arrow[r] & BO\langle 4\rangle \arrow[r, "\phi"] & K(Z,4) & \end{tikzcd} \tag{3.22}$$

Various fractions of the String structure correspond to various choices of d. In the examples we saw that $d = 2$ was special.

2. For the Fivebrane structure, we have $m = 8$. Then we have the diagram

$$\begin{tikzcd} & B_{d,8} \arrow[d] & PK(\mathbb{Z},8) \arrow[d] & K(\mathbb{Z},7) \arrow[l] \ . \\ X \arrow[ur] \arrow[r] & BO\langle 8\rangle \arrow[r, "\phi"] & K(Z,8) & \end{tikzcd} \tag{3.23}$$

In the examples in this case, the value $d = 8$ play a special role.

3.2.2 The dual Green-Schwarz anomaly and the dual M-theory C-field

We have defined (see section 3.2.1) the notion of twisted Fivebrane structures and $\mathcal{F}^{\langle 9\rangle}$ structures. In this section we show that such structures appear in String theory and M-theory and they are in fact conditions for cancelation of anomalies. We will consider the dual version of the Green-Schwarz mechanism and the dual field in M-theory. Note that one of the two main examples in [57] was type IIA string theory. In that theory the one-loop term on a String manifold is given simply by $\frac{1}{48}p_2$, i.e. without a twist. Therefore, type IIA string theory does not need the twisted structure we define in this paper.

Example I: The dual Green-Schwarz formula. We now consider ten-dimensional heterotic and type I string theories, whose low energy limit is type I supergravity theory coupled to superYang-Mills theory with structure group $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$. In [57] the main example of a Fivebrane structure came from the dual formulation [63] [27] of the Green-Schwarz anomaly cancelation mechanism [28], using the dual H-field H_7 of [13]. The expression is given by

$$dH_7 = 2\pi \left[ch_4(F_A) - \frac{1}{48} p_1(F_\omega) ch_2(F_A) + \frac{1}{64} p_1(F_\omega)^2 - \frac{1}{48} p_2(F_\omega) \right] . \tag{3.24}$$

In order to define a Fivebrane structure we assume we already have a String structure, and so we require $\frac{1}{2}p_1(TM) = 0$. Then the expression (3.24) becomes

$$dH_7 = 2\pi \left[ch_4(F_A) - \frac{1}{48} p_2(F_\omega) \right] . \tag{3.25}$$

In [57] we had to find ways to get rid of the extra terms to isolate the non-decomposable terms. In the twisted formalism in this paper we see that the presence of such terms amounts to a part of the twist and that it does not matter how many terms we have as long as they have the same total degree and hence provide a map to $K(\mathbb{Z}, 8)$. Indeed, if we can define

$$[\beta] := -ch_4(E) : \; M \xrightarrow{!} K(\mathbb{Z}, 8) \;\; , \tag{3.26}$$

i.e. require factorization

$$\begin{tikzcd} M \arrow[rr] \arrow[dr, dotted, "{[\beta]}"'] & & K(\mathbb{Q}, 8) \\ & K(\mathbb{Z}, 8) \arrow[ur, hook] & \end{tikzcd} \;\; , \tag{3.27}$$

then we can reinterpret expression (3.25) as $\frac{1}{48}p_2(TM) + [\beta] = 0$, since $[dH_7] = 0$, the cohomology class of an exact form.

We discuss the validity of the map in (3.26). The Chern character is in general not an integral expressions, but rather

$$ch : K^0(X) \to H^{\mathrm{even}}(X; \mathbb{Q}). \tag{3.28}$$

One way out of this is to first define a rational version of the twist, for which the map in (3.26) is replaced by a map from M to the rational Eilenberg-MacLane space

$$[\beta] := -ch_4(E) : M \to K(\mathbb{Q}, 8), \tag{3.29}$$

which gives that indeed $ch_4(E)$ is in general in $[M, K(\mathbb{Q}, 8)] = H^8(M, \mathbb{Q})$. Hence

Definition 25 *A rational Fivebrane twist on M is a map from M to $K(\mathbb{Q}, 8)$, i.e. an element of $H^8(M; \mathbb{Q})$.*

However, we can also give conditions under which the map in (3.26) is valid. The degree four Chern character is given by

$$ch_4 = \frac{1}{24} \left(c_1^4 - 4c_1^2 c_2 + 4c_1 c_3 + 2c_2^2 - 4c_4 \right) , \tag{3.30}$$

The Chern classes are integral classes and so the Chern character is a priori integral up to a factor of 24.

We describe this a follows. The Chern character is not integral in BU but it should be integral in some lift, say $\mathcal{BU}$, of BU. Then we ask: when can we lift to this new space? This is given in terms of the following diagram

$$
\begin{tikzcd}
 & & K(\mathbb{Z}_{24},7) \arrow[d] & \\
 & \mathcal{BU} \arrow[r, "ch_4"] \arrow[d] & K(\mathbb{Z},8) \arrow[d, "\times 24"] & \\
M \arrow[r] \arrow[ur, dotted, "f"] & BU \arrow[r, "24ch_4"] & K(\mathbb{Z},8) \arrow[r] & K(\mathbb{Z}_{24},8)\ .
\end{tikzcd}
\tag{3.31}
$$

The right-most factor $K(\mathbb{Z}_{24}, 8)$ represents the obstruction: there is a class k in $H^8(M;\mathbb{Z}_{24})$ which measures this obstruction. The top-most factor $K(\mathbb{Z}_{24}, 7)$ represents the different labeling of lifts f to the new space $\mathcal{BU}$. If we take connected covers of BU rather than BU itself in the diagram then we have that the space $\mathcal{BU}$ is isomorphic to another space in which $\frac{1}{6}c_4$, instead of ch_4, is integral. The relevance of the unitary groups here is because they provide the adjoint representation for our structure groups and this is the representation relevant for Yang-Mills theory. For E_8, the adjoint representation is $\mathrm{ad} : E_8 \to SU(248)$, so that the adjoint representation of $G = E_8 \times E_8$ is

$$(\mathrm{ad}, \mathrm{ad}) : E_8 \times E_8 \to SU(248) \times SU(248) \hookrightarrow SU(496) \tag{3.32}$$

Note that the above general discussion can be simplified. For both structure groups $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ we have $c_1(E) = 0$, so that in this case

$$ch_4(E) = \frac{1}{12}(c_2(E)^2 - 2c_4(E)) \tag{3.33}$$

We now consider two cases. First, that , in addition, $c_2(E) = 0$. In this case, the formula for the Chern character $ch_4(E)$ further simplifies to

$$ch_4(E) = -\frac{1}{6}c_4(E). \tag{3.34}$$

Here what we have really done is lifted the unitary group to its connected cover $BU\langle 8\rangle$. Indeed let us consider the result from [64] where the mod p (p an odd prime) cohomology of the connective cover $BU\langle 2n\rangle$ was calculated. From that result and the result of Stong [66] for $p = 2$, the following divisibility result was deduced for all primes p in [64]. Let $c_k \in H^{2k}(BU;\mathbb{Z})$ be the universal Chern class in BU, then the Chern class $r_n^*(c_k)$ in $BU\langle 2n\rangle$ where $r_n : BU\langle 2n\rangle \to BU$ be the canonical projection is divisible by [64]

$$\prod_p p^q \tag{3.35}$$

where q is the least integer part of $\frac{(n-1)-\sigma_p(k-1)}{p-1}$, with $\sigma_p(n) = \sum a_i$ the sum of the coefficients in the unique decomposition of the integer n as $n = a_0 + a_p + \cdots + a_k p^k$, with $a_i < p$. Applying this result for $n = 4$, $p = 2, 3$, and using $\sigma_2(3) = 2$, $\sigma_3(3) = 1$, we get that $r_4^*(c_4)$ is divisible by

$$2^{\frac{2-\sigma_2(3)}{1}} \cdot 3^{\frac{3-\sigma_3(3)}{2}} = 6\ . \tag{3.36}$$

We will give an example where this occurs and where the expression (3.34) is integral.

Example: Consider a complex vector bundle E on the eight-sphere S^8. For ten-manifold we can simply take $S^8 \times \mathbb{R}^2$ for example. The index of the Dirac operator on S^8 coupled to the vector bundle E is given by the evaluation of the twisted $\widehat{A}$-genus $\widehat{A}(S^8, E) := \Big(ch(E) \cdot \widehat{A}(S^8)\Big)[S^8]$ on the fundamental class $[S^8]$ of S^8

$$\mathrm{Index} D_E = \Big(\widehat{A}(TS^8) \cdot ch(E)\Big) = ch(E)[S^8], \tag{3.37}$$

as $\widehat{A}(TS^8) = 1$, since spheres have stably trivial tangent bundles. Since S^8 is a Spin manifold, the index should be an integer. This then gives the requirement

$$ch_4(E)[S^8] = -\frac{1}{6}c_4(E)[S^8] \in \mathbb{Z}. \tag{3.38}$$

Recall that we have refined the Fivebrane structure and its twisted version to include the division by 8 in definition 24. Given equation (3.18), the discussion leading to (3.36) then proves the following

Proposition 15 *The right hand side of the dual formula for the Green-Schwarz anomaly cancelation condition on a String 10-manifold M is the image in rational cohomology of the sum of integral classes representing the obstruction to defining a twisted Fivebrane structure, with the integral twist given by $ch_4(E)$, the fourth Chern character of the gauge bundle E, which itself is lifted to $BU\langle 8\rangle$.*

We can actually view the above result as providing a characterization of when the dual Green-Schwarz cocycle is integral. We have a sufficient result that this is so when the bundles are lifted from the String case to the Fivebrane case.

In section 5 we will give a more complete result which takes into account the differential refinements we discussed in section 2.

Remarks. 1. We can define *complex*- String and Fivebrane structures, as is implicitly done in [57], as the lifts of $BU\langle 4\rangle$ to $BU\langle 8\rangle$ and of $BU\langle 8\rangle$ to $BU\langle 10\rangle$, respectively. These can also be twisted leading to *twisted complex*- String and Fivebrane structures, in a similar way as in the real case. The twist in proposition 15 is an example of a twisted complex Fivebrane structure for the complex vector bundle corresponding to the gauge bundle.
2. In proposition 15 and the discussion around it we took the point of view that the natural bundle (i.e. the lift of the tangent bundle leading to Spin then String and so on) is the one that is being twisted by the gauge bundle. Of course we could have taken another point of view where the natural bundle acts as a twist for the corresponding gauge bundle. However, we prefer the first point of view here because the natural bundles seem to be, in a sense, more intrinsic and hence should come first in the order of giving structures.

In what follows we put the above discussion in the context of the discussion of nonabelian cohomology of section 2. Later in section 5 we consider the differential case.

We consider bi-twisted cohomology in the sense of definition 2 with respect to the two fibration sequences

$$\begin{tikzcd} \mathbf{B}U\langle 10\rangle \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {*} \arrow[d] \\ \mathbf{B}U\langle 8\rangle \arrow[r] & \mathbf{B}^8\mathbb{Z} \end{tikzcd}, \qquad \begin{tikzcd} \mathbf{B}\mathrm{Fivebrane} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {*} \arrow[d] \\ \mathbf{B}\mathrm{String}^F \arrow[r, "\frac{1}{48}p_2"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{B}^8\mathbb{Z} \arrow[d] \\ \mathbf{B}\mathrm{String} \arrow[r, "\frac{1}{6}p_2"] & \mathbf{B}^8\mathbb{Z} \end{tikzcd}$$

and relate it to the condition known as the *dual Green-Schwarz anomaly cancellation mechanism* in dual magnetic heterotic string theory.

Before proceeding, notice that

Lemma 16 *The pullback of the cohomology class $\frac{1}{6}c_4 : \mathbf{B}U \to \mathbf{B}^8\mathbb{R}$ to the universal 7-connected cover $\mathbf{B}U\langle 8\rangle$ of $\mathbf{B}U$ is integral:*

$$\begin{tikzcd} \mathbf{B}U\langle 8\rangle \arrow[r] \arrow[d, "\frac{1}{6}c_4"] & \mathbf{B}U \arrow[d, "\frac{1}{6}c_4"] \\ \mathbf{B}^8\mathbb{Z} \arrow[r] & \mathbf{B}^8\mathbb{Z} \end{tikzcd}$$

Proof. This follows from a theorem of Singer. ... □

Definition 26 (gauge-twisted Fivebrane structure) *On a space X let $E \in H(X, \mathbf{B}U)$ be a unitary cocycle to be called the* gauge bundle *which has a lift $\hat{E} \in H(X, \mathbf{B}U\langle 8\rangle)$. By lemma 16 this implies that its class $\frac{1}{6}c_4(E)$ is integral. Then in the sense of definition 1 and definition 2 we say that the space of gauge twisted* Fivebrane*-structures $H^{[E]}(X, \mathbf{B}\mathrm{Fivebrane})$ on X with gauge twist E is the $\mathbf{B}\mathrm{Fivebrane}^F$-$\mathbf{B}U\langle 10\rangle$-bitwisted cohomology whose $\mathbf{B}U\langle 10\rangle$-twist is $[\mathrm{ch}_4(E)]$, i.e. the homotopy pullback*

$$\begin{tikzcd}
H^{[E]}(X, \mathbf{B}\mathrm{Fivebrane}) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & {*} \arrow[d, "\hat{E}"] \arrow[dr, "E"] & \\
H(X, \mathbf{B}\mathrm{Fivebrane} \times_{\mathbf{B}^8\mathbb{Z}} \mathbf{B}U\langle 10\rangle) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & H(X, \mathbf{B}U\langle 8\rangle) \arrow[r] \arrow[d, "\frac{1}{6}c_4"] & H(X, \mathbf{B}U) \arrow[d, "\frac{1}{6}c_4"] \\
H(X, \mathbf{B}\mathrm{String}^F) \arrow[r, "\frac{1}{48}p_2"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & H(X, \mathbf{B}^8\mathbb{Z}) \arrow[r] \arrow[d, "\times 8"] & H(X, \mathbf{B}^8\mathbb{R}) \\
H(X, \mathbf{B}\mathrm{String}) \arrow[r, "\frac{1}{6}p_2"] & H(X, \mathbf{B}^8\mathbb{Z}) &
\end{tikzcd}.$$

Definition 27 (dual Green-Schwarz anomaly cancellation) *For X an oriented space and $E \to X$ a complex vector bundle on X the dual Green-Schwarz anomaly cancellation condition is the requirement that the following equation holds in $H^8(X, \mathbb{R})$:*

$$\frac{1}{48}p_2(X) - \mathrm{ch}_4(E) + \frac{1}{48}p_1(X)ch_2(E) - \frac{1}{64}p_1(X)^2 = 0\,.$$

Proposition 16 (dual Green-Schwarz and twisted Fivebrane**-structure)** *If X has* String^F*-structure and E has complex* String*-structure in that we have lifts of classifying maps of bundles*

$$\begin{tikzcd}
 & \mathbf{B}\mathrm{String}(n)^F \arrow[d] \\
X \arrow[ur, "\hat{g}_{TX}"] \arrow[r, "g_{TX}"'] & \mathbf{B}\mathrm{SO}(n)
\end{tikzcd}\;,\qquad
\begin{tikzcd}
 & \mathbf{B}U\langle 8\rangle \arrow[d] \\
X \arrow[ur, "\hat{g}_E"] \arrow[r, "g_E"'] & \mathbf{B}U
\end{tikzcd}$$

then the dual Green-Schwarz anomaly cancellation condition from definition 27 is equivalent to the condition that X has $[E]$-twisted Fivebrane*-structure lifting $\hat{g}_{TX}$.*

Proof. By assumption of $\mathrm{String}^{\mathcal{F}}$-structure the class $\frac{1}{2}p_1(X)$ vanishes. Therefore the mixed terms $p_1(X)(\cdots)$ in the dual Green-Schwarz condition vanish. Similarly, using the assumption of complex String-structure one finds that $\mathrm{ch}_4(E) = \frac{1}{6}c_4(E)$ as in the discussion leading to equation (3.36). It follows that the dual Green-Schwarz condition says in this case that the outer diagram in

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left, "\hat{g}_E"] \arrow[ddr, bend right, "\hat{g}_{TX}"'] & & \\
 & \mathbf{B}\mathrm{Fivebrane} \times_{\mathbf{B}^8\mathbb{Z}} \mathbf{B}U\langle 10\rangle \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{B}U\langle 8\rangle \arrow[d, "\frac{1}{6}c_4"] \\
 & \mathbf{B}\mathrm{String}^{\mathcal{F}} \arrow[r, "\frac{1}{48}p_2"] & \mathbf{B}^8\mathbb{Z}
\end{tikzcd}$$

commutes up to homotopy. By definition of $\mathbf{B}\mathrm{Fivebrane} \times_{\mathbf{B}^8\mathbb{Z}} \mathbf{B}U\langle 10\rangle$ this is the case precisely if $\hat{g}_{TX}$ and $\hat{g}_E$ have a commong lift given by the dashed morphism in the above diagram. This lift is by definition the $[E]$-twisted Fivebrane-structure lifting $\hat{g}_{TX}$. □

Example II: The dual field in M-theory. Now we consider M-theory, via its low energy limit, namely eleven-dimensional supergravity. The M-theory C-field C_3 is a degree three 'potential' whose curvature form is the degree four field strength G_4. Its dual is obtained in the following way. The equation of motion for C_3 is obtained from varying the action

$$S(C_3) = \int_Y \left[G_4 \wedge *G_4 + \frac{1}{6} G_4 \wedge G_4 \wedge C_3 - I_8 \wedge C_3 \right] \tag{3.39}$$

on an eleven-dimesional manifold Y to obtain

$$d * G_4 = -\frac{1}{2} G_4 \wedge G_4 + I_8. \tag{3.40}$$

Here I_8 is the one-loop term [69] [22] given in terms of the Pontrjagin classes of the tangent bundle TY to Y

$$I_8 = \frac{p_2(TY) - \frac{1}{2}(\frac{1}{2}p_1(TY))^2}{48}, \tag{3.41}$$

and $*$ is the Hodge duality operator in eleven dimensions.

The integral lift of (3.40) leads to a class defined in [19]

$$\begin{aligned} [G_8] &= \left[\frac{1}{2} G_4^2 - I_8 \right] \\ &= \frac{1}{2} a(a-\lambda) + \frac{7\lambda^2 - p_2}{48}, \end{aligned} \tag{3.42}$$

where $\lambda = \frac{1}{2}p_1$, and a is the degree four class of an E_8 bundle coming from Witten's shifted quantization condition for G_4 [73]

$$[G_4] = a - \frac{1}{2}\lambda = a - \frac{1}{4} p_1. \tag{3.43}$$

In [76] Witten interpreted the vanishing of a certain torsion class θ on the M-fivebrane worldvolume as a necessary condition for the decoupling of the 5-brane from the ambient space ("the bulk"). Hence the vanishing of θ meant that the fivebrane can have a well-defined partition function. Consider the embedding $\iota : W \hookrightarrow Y$ of the fivebrane with six-dimensional worldvolume W into eleven-dimensional spacetime Y. Consider the ten-dimensional unit sphere bundle $\pi : X \to W$ of W with fiber S^4 associated to the normal bundle $N \to W$ of the embedding ι. Then it was shown in [19] that the integration of G_8 over the fiber of X gives exactly the torsion class θ on the fivebrane worldvolume

$$\theta = \pi_*(G_8) \in H^4(W; \mathbb{Z}). \tag{3.44}$$

Therefore, the vanishing of G_8 is a necessary condition for the existence of a non-zero partition function [19].

We now proceed with the interpretation. Since we have Fivebrane structures in mind, we assume that Y already admits a String structure, i.e. that $\frac{1}{2}p_1(Y) = 0$. Then, from (3.42) we see that the class $G_8(Y)$ simplifies to

$$G_8(Y) = \frac{1}{2}a^2 - \frac{1}{48} p_2(Y). \tag{3.45}$$

The class a is an integral class of an E_8 bundle and hence defines a map to $K(\mathbb{Z}, 4)$. Then the square of a defines a map to $K(\mathbb{Z}, 8)$, and hence defines a twist for us. As we also have the class $\frac{1}{48}p_2$, then we have a twist for the modified fivebrane structure. Thus we have the following.

Proposition 17 *The integral class in M-theory dual to G_4 defines an obstruction to a twisted Fivebrane structure lifted from an $\mathcal{F}^{\langle 8 \rangle}$-structure. This is the obstruction to having a well-defined partition function for the M-fivebrane.*

Necessity of the Fivebrane condition? The Fivebrane condition is stronger than simply the requirement of the one-loop term I_8 to vanish. For the former we require the obstructions $\frac{1}{2}p_1$ and $\frac{1}{6}p_2$ vanish *separately*, whereas for the latter we only require the combination to vanish. This has been studied in [35] [34] [71]. For instance, following [71], a Riemannian 8-dimensional spin manifold M^8 is said to be *doubly supersymmetric* if and only if the tangent bundle TM^8 and the spinor bundles $\Delta_+ M^8$ and $\Delta_- M^8$ are associated with a principal G-fiber bundle such that there exist G-invariant isomorphisms between any two of the three bundles TM^8, the positive and negative chirality spinor bundles Δ_+ or Δ_-, i.e. $TM^8 = \Delta_+ M^8 = \Delta_- M^8$. If M^8 is doubly supersymmetric,

$$w_1 = w_2 = 0, \qquad e = 0, \qquad 4p_2 = p_1^2 \, , \tag{3.46}$$

where e is the Euler class. Then this implies for the signature $sgn(M^8) = 16\hat{A}[M^8]$. In particular, $sgn(M^8) \equiv 0 \mod 16$. One example is $PSU(3)$-structure for which

$$\begin{aligned} w_i &= 0 \ \ (i \neq 4), \qquad w_4^2 = 0 && (3.47)\\ e &= 0, \qquad\qquad\qquad p_1^2 = 4p_2 . && (3.48) \end{aligned}$$

In particular, all Stiefel-Whitney numbers vanish. A second example is a differentiable 8-fold M^8 with an odd topological generalized Spin(7)-structure for which

$$\chi(M^8) = 0, \qquad p_1(M^8)^2 - 4p_2(M^8) = 0. \tag{3.49}$$

The 7-sphere admits a Spin structure and therefore admits a generalized G_2-structure. The tangent bundle of the 8-sphere is stably trivial and therefore all the Pontrjagin classes vanish. Since the Euler class is non-trivial, there exists no generalized $Spin(7)$-structure on an 8-sphere. However, equation (3.49) is automatically satisfied for manifolds of the form $M^8 = S^1 \times N^7$ with N^7 Spin.

4 L_∞-Connections

The discussion in section 2.1.6, specified to smooth ∞-groupoids in in section 2.2, describes smooth non-abelian differential cocycles in terms of a system of morphisms between smooth ∞-groupoids. We now indicate in section 4.2 how to extract from such a differential cocycle a collection of simplicial differential forms with values in L_∞-algebras of the form considered in [56]. A complete derivation of this step is relegated to [58].

After recalling details of L_∞-algebroids and L_∞-connections we introduce the L_∞-algebraic analogs of the constructions in section 2.1.4 and 2.1.5: representations of L_∞-algebras and associated L_∞-connections.

4.1 Review of L_∞-algebras

The tools we employ are varied, so we provide in this section a review of the essential L_∞-algebra notions that we need. All L_∞-algebras will be of finite type, i.e. finite-dimensional in each degree. By"quasi-free" DGCAs we mean those that are free as GCAs (Graded Commutative Algebras).

4.1.1 L_∞-algebras and L_∞-algebroids

Lie algebras are defined as structures on vector spaces and L_∞-algebras as structures on graded vector spaces. Both generalize to modules over commutative algebras, for example, over $A := C^\infty(X)$, the algebra of smooth functions over a manifold X.

Remark on grading. Grading conventions can be a nuisance when dealing with differential graded algebras. Here we shall take the grading convention of the deRham complex as fundamental and choose our conventions such that the Chevalley-Eilenberg algebra of the tangent Lie algebroid TX of a smooth space X coincides with the deRham complex $\mathrm{CE}(TX) = \Omega^\bullet(X)$ with the correct grading. This implies that Chevalley-Eilenberg algebras of general L_∞-algebroids are taken to be $\mathbb{N}$-graded and hence the L_∞-algebras themselves which are dual to them are taken to be $\mathbb{Z}$-graded and concentrated in non-positive degree. To make the pattern more obvious we will say that a $\mathbb{Z}$-graded complex concentrated in non-positive degree is $-\mathbb{N}$-graded.

Definition 28 *An* L_∞-algebroid $(X, \mathfrak{g})$ *is a smooth space* X *and a* $-\mathbb{N}$*-graded cochain complex* $\mathfrak{g}$ *of finite rank* $A := C^\infty(X)$*-modules together with a degree +1 derivation*

$$d : \wedge_A^\bullet \mathfrak{g}^* \to \wedge_A^\bullet \mathfrak{g}^* \,, \tag{4.1}$$

linear over the ground field (not necessarily over A*) on the free (over* A*) graded-symmetric tensor algebra generated from the* $\mathbb{N}$*-graded dual* $\mathfrak{g}^*$ *(over* A*), such that* $d^2 = 0$*. The quasi-free (over* A*) differential graded-commutative algebra*

$$\mathrm{CE}_A(\mathfrak{g}) := (\wedge_A^\bullet \mathfrak{g}^*, d) \tag{4.2}$$

defined this way we call the Chevalley-Eilenberg algebra *of the* L_∞*-algebroid* $(A, \mathfrak{g})$*.*

Remark (types of L_∞-algebroids). We have the following special cases:

- For $X = \mathrm{pt}$ and $\mathfrak{g}$ concentrated in degree 0 we have $\mathrm{CE}(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}, d_{\mathfrak{g}})$ where $\wedge^\bullet \mathfrak{g}$ is the Grassmann algebra on the vector space $\mathfrak{g}$ and $d_{\mathfrak{g}}$ is the Chevalley-Eilenberg differential uniquely corresponding to the structure of a Lie algebra on $\mathfrak{g}$.
- For $X = \mathrm{pt}$ and $\mathfrak{g}$ in arbitrary (non-positive) we have an arbitrary L_∞*-algebra* (of finite type).
- For arbitrary X and $\mathfrak{g}$ concentrated in degree 0 (being finitely generated and projective as a module over $C^\infty(X)$) this is equivalent to the usual definition of Lie algebroids as vector bundles $E \to X$ with anchor map [47] $\rho : E \to TX$: we have $\mathfrak{g} = \Gamma(E)$ and the anchor is encoded as $d_{\mathfrak{g}}|_{C^\infty(X)} : f \mapsto \rho(\cdot)(g)$.
- If $\mathfrak{g}$ is concentrated in degree 0 and $-(n-1)$, then it is called *2-stage* in homotopy theory.
- If $\mathfrak{g}$ is concentrated in degrees 0 through $-(n-1)$, then we speak of a *Lie n-algebra*.
- If $X = \mathrm{pt}$ and $d : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$ then $\mathfrak{g}$ is a dg-Lie algebra, where the co-unary part of $d_{\mathfrak{g}}$ is dual to the differential on the chain complex underlying $\mathfrak{g}$.

More generally, if A is any commutative associative algebra, we speak of a *Lie-Rinehart* algebra [50] [33].

4.1.2 L_∞-algebra valued differential forms and twisting forms

Recall, for instance from [56], that, for $\mathfrak{g}$ any L_∞-algebra, differential form data on a space X with values in $\mathfrak{g}$ is a GCA morphism (*not necessarily* respecting the differentials) from $\mathrm{CE}(\mathfrak{g})$ into forms on X:

$$\Omega^\bullet(X, \mathfrak{g}) := \mathrm{Hom}_{\mathrm{GCA}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(X)) \,. \tag{4.3}$$

Later we will need to work entirely within homomorphisms of differential graded algebras. This is accomplished by passing from the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ to the Weil algebra $W(\mathfrak{g})$, which can be defined as the DGCA which is universal with the property that $\Omega^\bullet(X, \mathfrak{g})$ is isomorphic, up to homotopy, to $\mathrm{Hom}_{\mathrm{DGCA}}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(X))$.

The space of GCA homomorphisms is a subspace of the space of linear maps of graded vector spaces from CE($\mathfrak{g}$) to $\Omega^\bullet(\mathfrak{g})$ and, since CE($\mathfrak{g}$) is freely generated as a GCA and of finite type, this is isomorphic to the space of grading preserving homomorphisms

$$\mathrm{Hom}_{\mathrm{Vect}[\mathbb{Z}]}(\mathfrak{g}^*, \Omega^\bullet(X)) \tag{4.4}$$

of linear grading-preserving maps from the graded vector space $\mathfrak{g}^*$ of dual generators to $\Omega^\bullet(X)$, with $\mathfrak{g}^*$ still regarded as being in positive degree. By the usual relation in Vect[$\mathbb{Z}$] for $\mathfrak{g}$ of finite type, this is isomorphic to the space of elements of total degree degree 1 in forms tensored with $\mathfrak{g}$:

$$\Omega^\bullet(X, \mathfrak{g}) \simeq (\Omega^\bullet(X) \otimes \mathfrak{g})_0 \,. \tag{4.5}$$

(Recall that $\mathfrak{g}$ is $-\mathbb{N}$-graded, i.e. in non-positive degree by definition.)

If instead we do consider DGCA homomorphisms from CE($\mathfrak{g}$) into forms, we find *flat* L_∞-algebra valued forms

$$\Omega^\bullet(X, \mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCA}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(X)) \,. \tag{4.6}$$

The inclusion

$$\Omega^\bullet_{\mathrm{flat}}(X, \mathfrak{g}) = \mathrm{Hom}_{\mathrm{DGCA}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(X)) \hookrightarrow \Omega^\bullet(X) \otimes \mathfrak{g} \tag{4.7}$$

realizes flat L_∞-algebra valued forms as elements $A \in \Omega^\bullet(X) \otimes \mathfrak{g}$ of forms of total degree 0 with the special property that they satisfy a flatness constraint of the form

$$dA + \partial A + [A \wedge A] + [A \wedge A \wedge A] + \cdots = 0 \,, \tag{4.8}$$

where d and $\wedge$ are the operations in $A \in \Omega^\bullet(X) \otimes \mathfrak{g}$ and where $[\cdot, \cdot, \cdots]$ are the n-ary brackets in the L_∞-algebra and ∂ is the differential in the chain complex $\mathfrak{g}$. For $\mathfrak{g}$ a dg-Lie algebra only the binary bracket is present and A is an ordinary Maurer-Cartan element:

$$DA + [A \wedge A] = 0 \;, \tag{4.9}$$

where $D = d + \partial$.

This equation of course has a long and honorable history in various guises. When the algebra is that of differential forms on a Lie group, it is called the *Maurer-Cartan equation*. In deformation theory, it is the *integrability equation*. In mathematical physics, especially in the Batalin-Vilkovisky formalism, it is known as the *Master Equation*. At present, the name *Maurer-Cartan equation* seems to have the upper hand.

Remark (twisting cochains). There is an obvious well-known generalization of the above where the DGCA $\Omega^\bullet(X)$ is replaced by any other DGCA (A, d_A). Then by the above reasoning DGCA homomorphisms

$$(A, d_A) \longleftarrow \mathrm{CE}(\mathfrak{g}) \tag{4.10}$$

correspond to certain "flat" elements τ of degree 1 in the tensor product $\tau \in A \otimes \mathfrak{g}$, where the flatness condition is again

$$d_A \tau + d_{\mathfrak{g}} \tau + [\tau \wedge \tau] + \cdots \,. \tag{4.11}$$

See for instance definition 3.1 in [29]. This provides an example of a twisted tensor product:

$$(A \tilde{\otimes} \mathfrak{g}, d_A \otimes 1 + 1 \otimes d_{\mathfrak{g}} + \tau \wedge),$$

the untwisted differential d being $d_A \otimes 1 + 1 \otimes d_L$. The twist provided by τ can be considered as a twisting cochain $\tau : C \to L$ where C is a dg coalgebra such that $A := \mathrm{Hom}(C, \mathbb{C})$ is the dg algebra dual to C. Without assuming flatness, a similar element in $A \hat{\otimes} \mathfrak{g}$ is what K.T. Chen calls a *connection* [15]. Chen saw that his condition for flatness becomes that of a twisting cochain. For our purposes (see section 4.3), particularly important examples are given by extensions of Lie algebras by abelian lie algebras and their Chevalley-Eilenberg complexes. There the twisted differential $d_{\mathfrak{g}_\mu}$ can be written as $d_{\mathfrak{g}} \otimes 1 + \mu \partial_b$.

4.1.3 L_∞-algebra connections

The definition of L_∞-algebraic connections from [56] is a generalization of ordinary connections on ordinary principal bundles as follows. For $\mathfrak{g}$ a Lie algebra, G a Lie group integrating it and $\pi : P \to X$ a principal G-bundle, an Ehresmann connection on P is a $\mathfrak{g}$-valued 1-form on P, $A \in \Omega^1(P, \mathfrak{g},)$ which satisfies two conditions:

1. <u>First Ehresmann condition:</u> A restricts to the canonical flat $\mathfrak{g}$-valued 1-form on the fibers.
2. <u>Second Ehresmann condition:</u> A is equivariant with respect to the G-action on P.

Cartan observed that this could be expressed in terms of a morphism of graded-commutative algebras on which there is the action of a Lie group (though only the action of the Lie algebra $\mathfrak{g}$ is necessary). For each vector $x \in \mathfrak{g}$, there are derivations called 'infinitesimal transformation' $\mathcal{L}(x)$(today usually known as the Lie derivative) and 'interior product' $\imath(x)$ satisfying the relations:

1. $\mathcal{L}$ is a Lie morphism
2. $\imath([x, y] = \mathcal{L}(x)\imath(y) - \imath(y)\mathcal{L}(x)$
3. $\mathcal{L}(x) = \imath(x)d + d\imath(x)$

A Cartan connection $\Omega^\bullet(P) \xleftarrow{A} \mathrm{CE}(\mathfrak{g})$ is then defined as respecting the operations $i(x)$ and $\mathcal{L}(x)$ for all $x \in \mathfrak{g}$, but not necessarily respecting d. In formulas, for $x \in \mathfrak{g}$ and $a \in \mathrm{CE}(\mathfrak{g})$

1. <u>First Cartan-Ehresmann condition:</u> $\imath(x)A(a) = A(\imath(x)a)$
2. <u>Second Cartan-Ehresmann condition:</u> $\mathcal{L}(x)A(a) = A(\mathcal{L}(x)a)$.

If we extend A to a morphism not just of graded-commutative algebras, but to a morphism of differential graded commutative algebras $\Omega^\bullet(P) \xleftarrow{A} \mathrm{W}(\mathfrak{g})$, then we can express these two conditions in terms of diagrams as follows.

Let $\Omega^\bullet_{\mathrm{vert}}(P)$ denote the *quotient* of $\Omega^\bullet(P)$ by the image $\pi^*\Omega^\bullet(X)$.

The first Cartan-Ehresmann condition says that the following square of DGCA morphisms commutes

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(P) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\ \Omega^\bullet(P) \arrow[u] & \mathrm{W}(\mathfrak{g}). \arrow[l, "A"] \arrow[u] \end{tikzcd} \tag{4.12}$$

The relevance of the second Cartan condition is that it ensures that plugging the curvature of the 1-form A into an invariant polynomial of the Lie algebra yields a *basic* form on P which comes from pulling back a form on X. This is equivalent to saying that the following square of DGCA morphisms commutes:

$$\begin{tikzcd} \Omega^\bullet(P) & \mathrm{W}(\mathfrak{g}) \arrow[l, "A"'] \\ \Omega^\bullet(X) \arrow[u] & \mathrm{inv}(\mathfrak{g}). \arrow[l, "\{P_i\}"'] \arrow[u] \end{tikzcd} . \tag{4.13}$$

Here $\{P_i\}$ denotes the set of images in $\Omega^\bullet(X)$ of the generators of $\mathrm{inv}(\mathfrak{g})$: the diagram says that these are, under the Chern-Weil homomorphism, the characteristic forms obtained from the curvature F_A of the connection form A corresponding to the indecomposable invariant polynomials on $\mathfrak{g}$.

The advantage of these two diagrams is that they have an immediate generalization from Lie algebras to arbitrary L_∞-algebras, which is the content of the following definition. In particular, the second diagram allows us to generalize characteristic forms of L_∞-algebra valued forms without having to deal with equivariance of total spaces of higher bundles, which is a delicate issue: in this approach equivariance is not mentioned but instead the crucial consequence of equivariance, the descent of characteristic forms down to a base space, is encoded in a definition.

Definition 29 (L_∞ Cartan-Ehresmann connection [56]) *For $\mathfrak{g}$ an L_∞-algebra and $Y \to X$ a smooth surjection, we say that a pair of commutative diagrams*

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\
\Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "(A,F_A)"'] \arrow[u, two heads] \\
\Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{P_i\}"'] \arrow[u, hook]
\end{tikzcd} \tag{4.14}$$

is a principal $\mathfrak{g}$-Cartan-Ehresmann connection descent object.

Often we just say "$\mathfrak{g}$-connection" or even just "connection" for such objects.

Remark. There is described in [58] the procedure of how such $\mathfrak{g}$-connection descent objects may be integrated, if they satisfy certain integrability conditions, to cocycles in cohomology with coefficients in ∞-category valued sheaves. Such "nonabelian (differential) cohomology" classifies higher bundles (with connection). Therefore we can think of the above L_∞-algebra connection descent objects as *real approximations* to nonabelian differential cohomology. This is discussed in [58].

4.2 L_∞-connections from nonabelian differential cocycles

We indicate here how definition 29 of an L_∞-algebra connection is obtained from the general description of differential cocycles in section 2.1.6.

Consider first X an ordinary manifold and $\Pi(X)$ its smooth fundamental path ∞-groupoid from section **??**. For G an ordinary Lie group and $\mathbf{B}G$ the smooth groupoid with a single object and G worth of morphisms, cocycles in

$$H_{\mathrm{dR}}(X, \mathbf{B}G) := H_{\mathrm{rel}}\left(\begin{array}{c} X \\ \downarrow \\ \Pi(X) \end{array} , \begin{array}{c} * \\ \downarrow \\ \mathbf{B}G \end{array} \right) \tag{4.15}$$

correspond to morphisms of smooth groupoids from the 1-coskeleton groupoid $\Pi_1(X)$ of $\Pi(X)$ to $\mathbf{B}G$. In [60] it is shown that such morphisms are in bijection with flat $\mathfrak{g}$-valued differential forms $A \in \Omega^1(X, \mathfrak{g})$, $F_A := dA + [A, A] = 0$ on X, where $\mathfrak{g}$ is the Lie algebra of G: the functor $\mathrm{tra}_A : \Pi_1(X) \to \mathbf{B}G$ defined by such a form sends a morphism in $\Pi(X)$, which is a γ path in X, to its *parallel transport* $P\exp(\int_\gamma A) \in G$. Noticing in turn that flat $\mathfrak{g}$-valued differential forms on X are in turn in canonical bijection with morphisms of differential graded commutative algebras from the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ of $\mathfrak{g}$ to the deRham algebra of differential forms $\Omega^\bullet(X)$ on X, we find in total a bijection

$$H_{\mathrm{dR}}(X, \mathbf{B}G) \simeq \mathrm{Hom}(\Pi(X), \mathbf{B}G) \simeq \mathrm{Hom}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(X))\,. \tag{4.16}$$

One categorical dimension higher one finds a similar situation: let now G be a strict Lie 2-group and let $\mathbf{B}G$ be the corresponding strict one-object 2-groupoid. The analog of the Lie algebra of this is an L_∞-algebra $\mathfrak{g}$ that is concentrated in the lowest two degrees. There is a notion of Chevalley-Eilenberg algebra for this, too, which we again denote by $\mathrm{CE}(\mathfrak{g})$. The details are recalled in section 4.1. Now cocycles

$$H_{\mathrm{dR}}(X, \mathbf{B}G) := H_{\mathrm{rel}}\left(\begin{array}{c} X \\ \downarrow \\ \Pi(X) \end{array} , \begin{array}{c} * \\ \downarrow \\ \mathbf{B}G \end{array} \right) \tag{4.17}$$

correspond to morphisms of smooth 2-groupoids from the 2-coskeleton groupoid $\Pi_2(X)$ of $\Pi(X)$ to $\mathbf{B}G$. As shown in [61], such morphisms are in bijection with differential form data that may, using the discussion in [56], again be identified to be in bijection with morphisms of of differential graded algebras and we again have

$$H_{\mathrm{dR}}(X, \mathbf{B}G) \simeq \mathrm{Hom}(\Pi(X), \mathbf{B}G) \simeq \mathrm{Hom}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(X)) \,. \tag{4.18}$$

This pattern continues [58]: to every smooth ∞-groupoid G there is associated an L_∞-algebra $\mathfrak{g}$ such that the abstractly defined cocycles in $H_{\mathrm{dR}}(X, \mathbf{B}G)$ for X a smooth manifold correspond to the morphisms of differential algebras from $\mathrm{CE}(\mathfrak{g})$ to $\Omega^\bullet(X)$. Moreover, if X is instead an arbitrary smooth ∞-groupoid itself, regarded as a smooth simplicial space, this statement remains true if we interpret $\Omega^\bullet(X)$ as the differential algebra of *simplicial* forms on X. For degree ≤ 2 this follows along the lines of the discussion in [62], details are relegated to [58].

In conclusion, this means that the system of abstract differential forms from section 2.1.6 yields the system of morphisms of differential graded algebras from definition 29.

4.3 String-like Lie n-algebras

The main applications of our general theory are specific examples of L_∞-algebras: the String Lie 2-algebra $\mathfrak{g}_\mu$ and its generalization to higher String-like extensions, especially the Fivebrane Lie 6-algebra considered in [56, 57]. There is a straightforward generalization of the String algebra in which μ is of arbitrary odd degree. The String-like extensions were originally considered in [4]. In terms of the DGCA language of definition 28 they read as follows:

Definition 30 (String-like extensions) *For $\mathfrak{g}$ an ordinary Lie algebra and μ a Lie algebra cocycle of degree $(n+1)$, the* String-like extension *$\mathfrak{g}_\mu$ is the Lie n-algebra determined by its Chevalley-Eilenberg algebra as*

$$\mathrm{CE}(\mathfrak{g}_\mu) := \left(\wedge^\bullet(\underbrace{\mathfrak{g}^*}_{1} \oplus \underbrace{\langle b \rangle}_{n}), d|_{\mathfrak{g}^*} = d_{\mathfrak{g}} \,, db = \mu \right) \,. \tag{4.19}$$

In particular, for $\mathfrak{g}$ a semisimple Lie algebra with invariant bilinear form $\langle \cdot, \cdot \rangle$ and μ_3 a multiple of its canonical 3-cocycle, $\mu_3 = \langle \cdot, [\cdot, \cdot] \rangle$, we call $\mathfrak{g}_{\mu_3}$ the *String Lie 2-algebra* of $\mathfrak{g}$.

Furthermore, for $\mathfrak{g} = \mathfrak{so}(n)$ we have the 3-cocycle μ_3 and 7-cocycle μ_7. We call $(\mathfrak{g}_{\mu_3})_{\mu_7}$ the *Fivebrane Lie 6-algebra*. This will be used in section 3.2.

Remarks.

1. This is a shifted central extension of L_∞-algebras $b^{n-1}u(1) \to \mathfrak{g}_\mu \to \mathfrak{g}$.

2. The differential $d_{\mathfrak{g}_\mu}$ is a twisted differential

$$d_{\mathfrak{g}_\mu} = d_{\mathfrak{g}} + \mu \wedge \frac{\partial}{\partial b} \tag{4.20}$$

of the kind which we will interpret in terms of *twisting cochains* in section **??** and, essentially equivalently, in terms of representations of L_∞-algebras as described in proposition 19 below.

3. The finite dimensional but weak Lie 2-algebra $\mathfrak{g}_{\mu_3}$ is equivalent [**?**] to the strict but infinite-dimensional Lie 2-algebra $(\hat{\Omega}\mathfrak{g} \to P\mathfrak{g})$.

4. As discussed in [**?**, **?**, 58] $\mathfrak{g}_{\mu_3}$ integrates in various ways to the String Lie 2-group. JIM: WHAT are the various ways??

5. Topologically, using rational homotopy theory, one can geometrically interpret the qDGCAs with which we are dealing, as models for the DGCA of differential forms on certain spaces. Recall the following basic facts from rational homotopy theory:

- In general, the cohomology of a given qDGCA represents the real cohomology of a space. If the given qDGCA is minimal, i.e. if there are no linear terms in the differential, then the homology of the space of generators is isomorphic to the dual $\mathrm{Hom}(\pi_*, \mathbb{R})$ of the homotopy groups of the space.

  A generator of degree n represents a basis element of $\pi_n \otimes \mathbb{R}$.

- For $2n+1$ odd, the DGCA $\mathrm{CE}(b^{2n}\mathfrak{u}(1)) = (\wedge^\bullet(\underbrace{\langle b\rangle}_{n}), d = 0)$ represents the $2n+1$-sphere S^{2n+1}, whose only non-torsion homotopy group is $\pi_{2n+1}(S^{2n+1})$.

- For $2n$ even, the DGCA $\mathrm{CE}(b^{2n-1}\mathfrak{u}(1)) = (\wedge^\bullet(\underbrace{\langle b\rangle}_{n}), d = 0)$ represents not the $2n$-sphere S^{2n}, but the loop space, ΩS^{2n+1} whose only non-torsion homotopy group is $\pi_{2n}(\Omega S^{2n+1})$.

- The $2n$-sphere S^{2n} is instead represented by the DGCA $(\wedge^\bullet(\underbrace{\langle b\rangle}_{2n} \oplus \underbrace{\langle c\rangle}_{4n-1}), db = 0\,, dc = b\wedge b)$, where the second generator c is such that it trivializes the unwanted cocycles $b\wedge b$, $b\wedge b\wedge b$ etc, so that the only remaining nontrivial cocycle is b itself. Notice that indeed for $2n$, the non-torsion homotopy groups of the $2n$-sphere are $\pi_{2n}(S^{2n})$ and $\pi_{4n-1}(S^{2n})$.

Thus the string-like extension $\mathfrak{g}_\mu$ can be realized in terms of the differential forms of a fibration

$$\Omega S^{2n+1} \to \hat{G} \to G \tag{4.21}$$

where $\mathfrak{g}$ is the Lie algebra of the semisimple Lie group G. Moreover, at least up to homotopy, this can also be realized as a fibration

$$\hat{G} \to G \to S^{2n+1}. \tag{4.22}$$

In fact, in the sense of real or rational homotopy, G has the homotopy type of a product of odd dimensional spheres. Since μ is indecomposable, it is represented by one of the spheres; in other words, this second fibration splits and we have:

Proposition 18 *For $\mathfrak{g}$ a semisimple Lie algebra, the cohomology of $\mathrm{CE}(\mathfrak{g}_\mu)$ is that of $\mathrm{CE}(\mathfrak{g})$ modulo the class of μ:*

$$H^\bullet(\mathrm{CE}(\mathfrak{g}_\mu)) \simeq H^\bullet(\mathrm{CE}(\mathfrak{g}))/[\mu]\,. \tag{4.23}$$

Proof. A general element of degree k in $\mathrm{CE}(\mathfrak{g}_\mu)$ is of the form

$$\omega = \omega_k + b\wedge\omega_{k-n} + \frac{1}{2} b\wedge b\wedge\omega_{k-2n} + \cdots + \frac{1}{[k/n]!}\underbrace{b\wedge\cdots\wedge b}_{[k/n]}\wedge\omega_{k-[k/n]n} \tag{4.24}$$

for $\omega_i \in \wedge^i\mathfrak{g}^*$ and where $[k/n]$ is the largest integer smaller than k/n. Now we assume that ω is $d_{\mathfrak{g}_\mu}$-closed and deduce the implications of that. The cochain ω being $d_{\mathfrak{g}_\mu}$-closed is equivalent to a list of equations

$$d_{\mathfrak{g}}\omega_{k-rn} + \mu\wedge\omega_{k-(r+1)n} = 0 \tag{4.25}$$

for all $0 \le r \le [k/n]$, where we take $\omega_r = 0$ for $r < 0$. We will solve this system recursively, starting with the lowest degree component. Notice that (4.25) says that all $\mu\wedge\omega_{k-(r+1)n}$ are $d_{\mathfrak{g}}$-exact so that their $d_{\mathfrak{g}}$-cohomology class $[\mu\wedge\omega_{k-(r+1)n}] = [\mu]\wedge[\omega_{k-r+1}n] = 0$. By a classical result the cohomology of $\mathrm{CE}(\mathfrak{g})$ for $\mathfrak{g}$ semisimple is a free graded-commutative algebra on odd generators, one of which is $[\mu]$. It follows that there is $\alpha_{k-(r+2)n-1}$ such that $[\omega_{k-(r+1)n}] = [\mu\wedge\alpha_{k-(r+2)n-1}]$. This in turn means that there is $\alpha_{k-(r+1)n-1}$ such that

$$\omega_{k-(r+1)n} = d_{\mathfrak{g}}\alpha_{k-(r+1)n-1} + \mu\wedge\alpha_{k-(r+2)n-1}\,. \tag{4.26}$$

Define

$$\kappa_k := \omega_k - \mu \wedge \alpha_{k-n-1} - d_{\mathfrak{g}} \alpha_{k-1} \tag{4.27}$$

and notice that by equation (4.25) κ_k is a $d_{\mathfrak{g}}$-cocycle. Then observe that by repeatedly using (4.26) we can write ω_k identically as

$$\omega = \kappa_k + d_{\mathfrak{g}_\mu} \left(\alpha_{k-1} + b \wedge \alpha_{k-n-1} + \frac{1}{2} b \wedge b \wedge \alpha_{k-2n-1} + \cdots \right) . \tag{4.28}$$

This means that every $d_{\mathfrak{g}_\mu}$-closed element ω_k in $\mathrm{CE}(\mathfrak{g}_\mu)$ is the sum of a $d_{\mathfrak{g}}$-closed element κ_k from $\mathrm{CE}(\mathfrak{g})$ plus a $d_{\mathfrak{g}_\mu}$-exact element. But the image of $d_{\mathfrak{g}_\mu}$ in the restriction to $\mathrm{CE}(\mathfrak{g})$ is the space of all $d_{\mathfrak{g}}$-cocycles plus the ideal generated by μ. Therefore the map $H^\bullet(\mathrm{CE}(\mathfrak{g}_\mu)) \to H^\bullet(\mathrm{CE}(\mathfrak{g}))/[\mu]$ which on representatives sends $\omega_k \mapsto \kappa_k$ is an isomorphism. This completes the proof of the proposition. □

Remark. The above lemma can be also proved using the Wang sequence for bundles over spheres by using the bigrading by the $CE(\mathfrak{g})$ degree and the power of b.

Furthermore, we can handle such cycles together. For example, for $\mathfrak{g} = \mathfrak{so}(n)$ we have the 3-cocycle μ_3 and 7-cocycle μ_7. We call $(\mathfrak{g}_{\mu_3})_{\mu_7}$ the *Fivebrane Lie 6-algebra.* This will be used in section 3.2.

4.4 Associated L_∞-connections

To discuss the twisted structures that are of use to us in the context of L_∞-connections, we need the following concepts in addition to the material covered in [56] and reviewed in section 4.1.

4.4.1 L_∞-algebra representations on cochain complexes

Remark. Cochain complexes in non-positive degree are sometimes referred to as ∞-vector spaces.

Definition 31 (representations of L_∞-algebroids) *A representation of an L_∞-algebroid $(A, \mathfrak{g})$ on a cochain complex V of finite rank A-modules is an L_∞-algebroid $(A, \mathfrak{g}, V)_\rho$ whose Chevalley-Eilenberg algebra* $\mathrm{CE}_\rho(\mathfrak{g}, V)$ *is an extension of* $\mathrm{CE}(\mathfrak{g})$ *by* $\wedge_A^\bullet V^*$

$$\begin{tikzcd} \wedge_A^\bullet V^* & \mathrm{CE}_{A,\rho}(\mathfrak{g}, V) \arrow[l, two heads] & \mathrm{CE}_A(\mathfrak{g}) \arrow[l, hook'] \arrow[ll, bend left, "0"] \end{tikzcd} \tag{4.29}$$

where $\mathrm{CE}_\rho(\mathfrak{g}, V) = (\wedge_A^\bullet(\mathfrak{g}^* \oplus V^*), d)$.

This means that the differential is

$$\begin{aligned} d|_{\mathfrak{g}^*} &= d_{\mathfrak{g}} \\ d|_{V^*} &= d_V + d_\rho \, , \end{aligned} \tag{4.30}$$

where

$$d_\rho : V^* \to \mathfrak{g}^* \wedge_A (\wedge_A^\bullet(\mathfrak{g}^* \oplus V^*)) \tag{4.31}$$

encodes the action of $\mathfrak{g}$ on V.

Remarks.
1. In roughly this latter form, the definition appears in [9], where it is called a *superconnection.* Indeed, in cases where the L_∞-algebroid in question is similar to a tangent Lie algebroid of some space, its representations behave like (flat) connections on that space. Of more relevance to our present purposes are the representations of L_∞-algebras in full generality.

2. Notice that the definition can also be phrased as follows. An L_∞-algebroid $(A, \mathfrak{g}, V)_\rho$ is an L_∞-algebroid $(A, \mathfrak{g} \oplus V)$ of a special form: V is an abelian subalgebroid and an ideal and $(A, \mathfrak{g})$ is an L_∞-subalgebroid, just as not all ordinary Lie algebra structures on $\mathfrak{g} \oplus V$ come from V as a representation of $\mathfrak{g}$. In fact, there exist L_∞-algebroids $(A, \mathfrak{g} \oplus V)$ *not* of this form, occurring for example in the Batalin-Fradkin-Vilkovisky construction for constrained Hamiltonian systems involving structure *functions* (see [39]). Of course, for the case of structure constants, the BRST complex is an example of the module case above.
3. One might expect that a representation of an L_∞-algebra $\mathfrak{g}$ should be an L_∞ morphism to an L_∞-algebra $\mathrm{end}(V)$ of endomorphisms of the complex V. In fact, for V an ordinary (or graded or differential graded) vector space, $\mathrm{end}(V)$ is an ordinary (respectively graded or differential graded) Lie algebra. For V a cochain complex, there is the definition of L_∞-actions – sh-representations [65] – by Lada and Markl [42] in coalgebra language. The above definition captures that but retains the DGCA perspective on representations.

One might expect that L_∞ representations are L_∞ morphisms to $end(V)$. The following example shows that this is a special case of definition 31.

Example: ordinary Lie representations. Let $\mathfrak{g}$ be a Lie algebra with basis $\{t_a\}$ and dual basis $\{\underbrace{t^a}_{\deg=+1}\}$ of $\mathfrak{g}^*$. Let V be a vector space with basis $\{v_i\}$ and let $\rho : \mathfrak{g} \otimes V \to V$ be an ordinary Lie representation of $\mathfrak{g}$ on V with components $\{\rho^i{}_{ja}\}$. Then

$$\mathrm{CE}_\rho(\mathfrak{g}, V) = (\wedge^\bullet(\mathfrak{g}^* \oplus V^*), d_\rho) \tag{4.32}$$

with the differential given by the dual of the representation map $d_\rho|_{V^*} := \rho^*$ is the corresponding qDGCA. In terms of components relative to the chosen basis this reads

$$d_\rho : v_i \mapsto \rho^j{}_{ia} v_j \wedge t^a \tag{4.33}$$

This is nothing but the standard Chevalley-Eilenberg complex of the $\mathfrak{g}$-module V, but expressed in terms of bases. It can be regarded as a DGCA instead of just as a complex by thinking of V as a trivial/Abelian Lie algebra.

Representation in terms of Lie algebras of endomorphisms. For a Lie alg $\mathfrak{g}$, you want $End(\mathfrak{g})$

Let V be a finite-dimensional vector space and $\mathrm{end}(V)$ its Lie algebra of endomorphisms. Then a representation is a linear map $\rho : \mathfrak{g} \otimes V \to V$ which satisfies

$$\rho([X, Y]) = \rho(X) \circ \rho(Y) \pm \rho(Y) \circ \rho(X) \tag{4.34}$$

for $X, Y \in \mathfrak{g}$. To see how this is a special case of the above general definition, choose a basis $\{v_i\}$ of V inducing a basis of $\mathrm{end}(V)$ is $\{\omega_i{}^j\}$ with dual basis $\{\underbrace{\omega^i{}_j}_{\deg=1}\}$ and Chevalley-Eilenberg differential

$$d\omega^i{}_j = -\sum_k \omega^i{}_k \wedge \omega^k{}_j\,. \tag{4.35}$$

Given any L_∞-algebra $\mathfrak{g}$ with basis $\{\underbrace{t^a}_{\deg=1}, \underbrace{\cdots}_{\deg>1}\}$ for $\mathfrak{g}^*$, a morphism $\rho : \mathfrak{g} \to \mathrm{end}(V)$ is a DGCA morphism $\rho^* : \mathrm{CE}(\mathrm{end}(V)) \to \mathrm{CE}(\mathfrak{g})$ given on basis elements by $\rho^* : \omega^i{}_j \mapsto \rho^i{}_{ja} t^a$ and satisfying

$$\rho^i{}_{ja}(d_{\mathfrak{g}} t^a) = -\rho^i{}_{kb} \rho^k{}_{jc} t^b \wedge t^c\,. \tag{4.36}$$

It can be directly checked that the data encoded in ρ^* is equivalent to the twisted differential on $\wedge^\bullet(V^* \oplus \mathfrak{g}^*)$ given by $d_\rho : v^i \mapsto \rho^i{}_{ja} v^j \wedge t^a$ since its nilpotency requires that

$$d_\rho d_\rho : v^i \mapsto \rho^i{}_{kb} \rho^k{}_{jc} t^c \wedge t^b + \rho^i{}_{ja}(d_{\mathfrak{g}} t^a)\,, \tag{4.37}$$

which vanishes by (4.36).

The adjoint representation. For $\mathfrak{g}$ any L_∞-algebra, there is a representation of $\mathfrak{g}$ on itself given by the adjoint representation.

Definition 32 (adjoint representation for L_∞-algebras) *Let $\mathfrak{g}$ be any L_∞-algebra so that, by our conventions $\mathfrak{g}^*$ is concentrated in positive degree. Let $V_\mathfrak{g}$ be the underlying cochain complex of $\mathfrak{g}$ shifted down by one such that it is concentrated in non-positive degree.*

$$\mathrm{CE}_{\rho_{\mathrm{ad}}}(\mathfrak{g},\mathfrak{g}) := \left(\wedge^\bullet(\mathfrak{g}^*\oplus V_\mathfrak{g}^*), d_\rho\right) \tag{4.38}$$

with $d_\rho|_{V_\mathfrak{g}^*} = \sigma^{-1}\circ d_\mathfrak{g}\circ\sigma$, where $\sigma : V_\mathfrak{g}^* \to \mathfrak{g}^*$ is the canonical isomorphism of cochain complexes which shifts degrees up by one, σ^{-1} is its inverse and both are extended as graded derivations to $\wedge^\bullet(\mathfrak{g}^*\oplus V_\mathfrak{g}^*)$.

One checks that $(d_\rho)^2 = 0$ by noticing that while σ and σ^{-1} are not inverses as graded derivations, they satisfy $\sigma\circ\sigma^{-1}|_{\wedge^n\mathfrak{g}^*} = n\mathrm{Id}$.

Remark. In terms of higher brackets as in [42] the adjoint representation is given by

$$\rho : X_1\otimes\cdots X_n\otimes Y \mapsto [X_1,\cdots,X_n,Y] \tag{4.39}$$

for $X_i, i = 1,\cdots,n$ and Y in $\mathfrak{g}$.

Remark. Notice that the construction of the adjoint representation of the L_∞-algebra $\mathfrak{g}$ essentially analogous to the construction of the Weil algebra $\mathrm{W}(\mathfrak{g})$, only that here the shift operation is down in degree, where for the Weil algebra it goes up in degree.

Example: ordinary adjoint representation. Let $\mathfrak{g}$ be an ordinary Lie algebra with basis $\{t_a\}$ and structure constants $\{C^a{}_{bc}\}$. Write $\{\underbrace{t^a}_{\deg=+1}\}$ for the corresponding dual basis elements and $\{\underbrace{\chi^a}_{\deg=0}\}$ for the corresponding basis elements of $V_\mathfrak{g}^*$. Then we have $d_\rho\chi^a = \sigma^{-1}(d_\mathfrak{g}t^a) = \sigma^{-1}(-\frac{1}{2}C^a{}_{bc}t^b\wedge t^c) = C^a{}_{bc}t^b\chi^c$.

Definition 33 (extended standard representation of $b^{2k}\mathfrak{u}(1)$) *Define the extended representation by its CE-algebra as*

$$\mathrm{CE}_\rho(b^{2k}\mathfrak{u}(1), \bigoplus_r \mathbb{R}[2kr]) := \left(\wedge^\bullet\left(\bigoplus_r \underbrace{\langle v_{2rk}\rangle}_{\deg=2rk} \oplus \underbrace{\langle h\rangle}_{\deg=2k+1}\right), d\right) \tag{4.40}$$

with $d : v_{2kr}\mapsto v_{2k(r-1)}\wedge h$ and $d : h\mapsto 0$.

Notice that also the "twisted" Chevalley-Eilenberg algebras arising from the String-like extensions in definition 30 are examples of representations:

Proposition 19 $\mathrm{CE}(\mathfrak{g}_\mu)$ *from definition 30 is a representation of $\mathfrak{g}$ on the shifted 1-dimensional vector space $\mathbb{R}[n]$ such that*

$$(\Lambda^\bullet\langle b\rangle, d=0) \longleftarrow \mathrm{CE}(\mathfrak{g}_\mu) = (\Lambda^\bullet(\mathfrak{g}^*\oplus\langle b\rangle, d) \longleftarrow \mathrm{CE}(\mathfrak{g})\ . \tag{4.41}$$

Proof. This obviously satisfies the axioms of a representation. □

Remark. This is of course just another way of saying that $\mathfrak{g}_\mu$ is entirely governed by the Lie algebra cocycle μ. It is well known from the theory of higher groups that such cocycles can be regarded as higher representations on shifted vector spaces (see [6] and references within).

4.4.2 Sections, covariant derivatives, and morphisms of L_∞-connections

Here we give the L_∞-algebraic version of the constructions in section 2.1.5.

Definition 34 (sections and covariant derivatives) *Let $\mathfrak{g}$ be an L_∞-algebra, let $(\rho, \mathfrak{g}, V)$ be a representation of $\mathfrak{g}$ and consider the principal $\mathfrak{g}$-Cartan-Ehresmann connection (4.14). Then a* section of the ρ-associated connection *is an extension of this diagram through the extension defining the representation in that it is a choice of the dotted arrows in*

$$
\begin{tikzcd}
 & \mathrm{CE}_\rho(\mathfrak{g}, V) \arrow[dl, dotted, "{(s, A_{\mathrm{vert}})}"'] & \\
\Omega^\bullet_{\mathrm{vert}}(Y) & & \mathrm{CE}(\mathfrak{g}) \arrow[ul, hook] \arrow[ll, "A_{\mathrm{vert}}"'] \\
 & \mathrm{W}_\rho(\mathfrak{g}, V) \arrow[uu] \arrow[dl, dotted, "{(s, \nabla_A s, A, F_A)}"'] & \\
\Omega^\bullet(Y) \arrow[uu] & & \mathrm{W}(\mathfrak{g}) \arrow[ul, hook] \arrow[ll, "{(A, F_A)}"'] \arrow[uu] \\
 & \mathrm{inv}_\rho(\mathfrak{g}, V) \arrow[uu, hook] \arrow[dl, dotted] & \\
\Omega^\bullet(X) \arrow[uu, hook] & & \mathrm{inv}(\mathfrak{g}) \arrow[ul, hook] \arrow[ll] \arrow[uu, hook]
\end{tikzcd}
\quad .
$$

Here
- s is the section itself, the image of V in $\Omega^\bullet(Y)$.
- $\nabla_A s$ is its covariant derivative.

Example: ordinary vector bundles. Let $\mathfrak{g}$ be an ordinary Lie algebra with Lie group G, let V be a vector space (a chain complex concentrated in degree 0) and ρ an ordinary representation of $\mathfrak{g}$ on V, let $Y := P$ a principal G-bundle and (A, F_A) an ordinary Cartan-Ehresmann connection on P. Then the dotted morphism in

$$
\begin{tikzcd}
 & \mathrm{CE}_\rho(\mathfrak{g}, V) \arrow[dl, dotted, "{(s, A_{\mathrm{vert}})}"'] & \\
\Omega^\bullet_{\mathrm{vert}}(P) & & \mathrm{CE}(\mathfrak{g}) \arrow[ul, hook] \arrow[ll, "A_{\mathrm{vert}}"']
\end{tikzcd}
\tag{4.42}
$$

is dual to a V-valued function on the total space of the bundle (not on base space!) $s : P \to V$, which is covariantly constant along the fibers in that the covariant derivative

$$\nabla_A s := ds + (\rho \circ A)s \tag{4.43}$$

vanishes when evaluated on vertical vectors, where $(\rho \circ A)s$ denotes the action of A on the section s using the representation ρ. This means that s descends to a section of the associated vector bundle $P \times_G V$. The covariant derivative 1-form $\nabla_A s$ of the section s is one component of the extension in the middle part of our diagram

$$
\begin{tikzcd}
& \mathrm{W}_\rho(\mathfrak{g}, V) \arrow[dl, dotted, "{(s, \nabla_A s, A, F_A)}"'] & \\
\Omega^\bullet(Y) & & \mathrm{W}(\mathfrak{g}) \arrow[ul, hook] \arrow[ll, "A_{\mathrm{vert}}"]
\end{tikzcd}
\quad . \tag{4.44}
$$

The equation

$$
\nabla_A \nabla_A s = (\rho \circ F_A) \wedge s \tag{4.45}
$$

is the Bianchi identity for $\nabla_A s$. If s is everywhere non-vanishing, this says that the curvature F_A of our bundle is covariantly exact on Y. In the case that $\mathfrak{g} = \mathfrak{u}(1)$ it follows that F_A is an exact 2-form on Y and the choice of the non-vanishing section amounts to a trivialization of the bundle.

Opposite L_∞-algebras. We would like to describe morphisms of L_∞-connections following the description of morphisms between vector bundles $E_1 \to E_2$ in terms of a section of the tensor product bundle $E_1^* \otimes E_2$. If E_1 is a G-associated bundle and E_2 a G'-associated bundle, then $E_1^* \otimes E_2$ is a $G^{\mathrm{op}} \times G'$-associated bundle, for G^{op} the group G equipped with the opposite product $g_1 \cdot_{\mathrm{op}} g_2 = g_2 \cdot g_1$. On the level of Lie algebras passing to the opposite corresponds to change the Lie bracket by a sign. The following definition generalizes this from Lie algebras to L_∞-algebras.

Definition 35 (opposite L_∞-algebra) *For $\mathfrak{g}$ any L_∞-algebra with Chevalley-Eilenberg algebra* $\mathrm{CE}(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}^*, d_{\mathfrak{g}})$ *we define the* opposite L_∞-algebra $\mathfrak{g}^{\mathrm{op}}$ *to have the same underlying vector space* $\mathrm{CE}(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}^*, d_{\mathfrak{g}^{\mathrm{op}}})$ *but the differential of the Chevalley-Eileneberg algebra is $d_{\mathfrak{g}^{\mathrm{op}}} := (-1)^{N+1} \circ d_{\mathfrak{g}}$, where N is the operator which counts* word length *in the free graded algebra $\wedge^\bullet \mathfrak{g}^*$.*

This implies that the structure constants of $\mathfrak{g}^{\mathrm{op}}$ are those of $\mathfrak{g}$ equipped with a sign if they have an even number of input arguments. There is a canonical morphism of L_∞-algebras (hence of their CE DGCAs) $\mathrm{CE}(\mathfrak{g}) \longleftarrow \mathrm{CE}(\mathfrak{g}^{\mathrm{op}})$, which sends each generator to its negative.

Definition 36 *We say a morphism from a $\mathfrak{g}_1$-connection to a $\mathfrak{g}_2$-connection is a representation ρ of $\mathfrak{g}_1^{\mathrm{op}} \oplus \mathfrak{g}_2$ and a section of the $\mathfrak{g}_1^{\mathrm{op}} \oplus \mathfrak{g}_2$-connection canonically induced by the given $\mathfrak{g}_1$-connection and $\mathfrak{g}_2$-connection.*

$$
\begin{tikzcd}
& \mathrm{CE}_\rho(\mathfrak{g}_1^{\mathrm{op}} \oplus \mathfrak{g}_2, V) \arrow[dl, dotted, "{(\sigma, A_{\mathrm{vert}})}"'] & \\
\Omega^\bullet_{\mathrm{vert}}(Y) & & \mathrm{CE}(\mathfrak{g}_1^{\mathrm{op}} \oplus \mathfrak{g}_2) \arrow[ul, hook] \arrow[ll, "A_{2\,\mathrm{vert}} - A_{1\,\mathrm{vert}}"] \\
& \mathrm{W}_\rho(\mathfrak{g}_1^{\mathrm{op}} \oplus \mathfrak{g}_2, V) \arrow[uu, two heads] \arrow[dl, dotted, "{(\sigma, \nabla_A \sigma, A, F_A)}"'] & \\
\Omega^\bullet(Y) \arrow[uu, two heads] & & \mathrm{W}(\mathfrak{g}_1^{\mathrm{op}} \oplus \mathfrak{g}_2) \arrow[uu, two heads] \arrow[ul, hook] \arrow[ll, "{(A_2, F_{A_2}) - (A_1, F_{A_1})}"] \\
& \mathrm{inv}_\rho(\mathfrak{g}_1^{\mathrm{op}} \oplus \mathfrak{g}_2, V) \arrow[uu, hook] \arrow[dl, dotted] & \\
\Omega^\bullet(X) \arrow[uu, hook] & & \mathrm{inv}(\mathfrak{g}_1^{\mathrm{op}} \oplus \mathfrak{g}_2) \arrow[uu, hook] \arrow[ul, hook] \arrow[ll]
\end{tikzcd}
\quad .
$$

Example: morphisms of $b^{n-1}\mathfrak{u}(1)$-connections. Let $\mathfrak{g}_1 = \mathfrak{g}_2 = b^{(n-1)}\mathfrak{u}(1)$. The standard representation of $b^{(n-1)\mathfrak{u}(1)}$ from definition **??** naturally extends to a representation of $b^{(n-1)}\mathfrak{u}(1) \oplus b^{(n-1)}\mathfrak{u}(1)^{\mathrm{op}}$. Let σ be the shift operator which shifts the degree up by 1. The Weil-algebra $\mathrm{W}_\rho(b^{n-1}\mathfrak{u}(1), V)$ of $\mathrm{W}_\rho(b^{n-1}\mathfrak{u}(1), V)$ looks as follows:

$$\mathrm{W}_\rho(h^{n-1}\mathfrak{u}(1), V) = (\wedge^\bullet(\underbrace{\langle v_0\rangle}_{\deg=0} \oplus \underbrace{\langle \sigma v_0\rangle}_{\deg=1} \oplus \underbrace{\langle v_{n-1}\rangle}_{\deg=n-1} \oplus \underbrace{\langle \sigma v_{n-1}\rangle}_{\deg=n} \oplus \underbrace{\langle h\rangle}_{\deg=n} \oplus \underbrace{\langle \sigma h\rangle}_{\deg=n+1}, d)) \tag{4.46}$$

with

$$\begin{aligned} dv_0 &= \sigma v_{n-1} \\ d\sigma v_0 &= 0 \\ dv_{n-1} &= v_0 \wedge h + \sigma v_{n-1} \\ d\sigma v_{n-1} &= -\sigma v_0 \wedge h - v_0 \wedge \sigma h \,. \end{aligned} \tag{4.47}$$

JIM; I JUST SKIMMED THIS PAGE - NEED TO DOUBLE CHECK IT LATER

The above standard representation of $b^{n-1}\mathfrak{u}(1)$ has a straightforward generalization for the case that n is odd. The case n even does not occur because the differentials do not square to zero. The connection itself is

$$\Omega^\bullet(Y) \xleftarrow{(A_2,F_2)-(A_1,F_2)} W(b^{n-1}\mathfrak{u}(1)^{\mathrm{op}} \oplus b^{n-1}\mathfrak{u}(1)) \;, \tag{4.48}$$

given by forms $H_1, H_2 \in \Omega^n(Y)$ being the images of the generator $\underbrace{b}_{\deg=n}$ and $dH_1, dH_2 \in \Omega^{n+1}(Y)$ the images of the generator $\underbrace{\sigma b}_{\deg=n+1}$, for $b^{n-1}\mathfrak{u}(1)$ and its opposite, respectively. As we extend this morphism through the twisted DGCA of the standard representation definition **??**

$$\begin{tikzcd} & \mathrm{W}_\rho(\mathfrak{g}_1^{\mathrm{op}} \oplus \mathfrak{g}_2, V) \arrow[dl, dotted, "{(s,\nabla_A s,A,F_A)}"'] \\ \Omega^\bullet(Y) & & \mathrm{W}(\mathfrak{g}_1^{\mathrm{op}} \oplus \mathfrak{g}_2) \arrow[ul, hook] \arrow[ll, "{(A_2,F_{A_2})-(A_1,F_{A_1})}"] \end{tikzcd} \tag{4.49}$$

we pick up forms which are the images of the other generators appearing in equations (4.47):

$$\begin{aligned} v_0 &\mapsto s_0 \in \Omega^0(Y) \\ \sigma v_0 &\mapsto ds_0 \in \Omega^1(Y) \\ v_{n-1} &\mapsto s_{n-1} \in \Omega^{n-1}(Y) \\ \sigma v_{n-1} &\mapsto \nabla\sigma_{n-1} := ds_{n-1} + s_0 \wedge (H_2 - H_1) \,. \end{aligned} \tag{4.50}$$

For instance, in the case of the Green-Schwarz mechanism we have the two $b^2\mathfrak{u}(1)$ Chern-Simons 3-connections as described in [56] with 3-form connections $H_1 = \mathrm{CS}(\omega_{\mathfrak{so}(n)})$ and $H_2 = \mathrm{CS}(A_{\mathfrak{e}_8})$. The above section of the difference of these two connections then is to be interpreted itself a twisted 2-connection with connection 2-form σ_2 and curvature 3-form $H_3 := \nabla\sigma_2$ which satisfies the twisted Bianchi identity

$$dH_3 = \langle F_\omega \wedge F_\omega\rangle - \langle F_A \wedge F_A\rangle \,. \tag{4.51}$$

This, and its magnetic dual version, is discussed in more detail in section **??**.

Another example is given by the degree two F_2 and the degree zero component F_0 of the Ramond-Ramond (RR) fields in type IIA string theory. Consider the case $n = 3$ so that $\mathfrak{g}_1 = \mathfrak{g}_2 = b^2\mathfrak{u}(1)$. The curvature F_2 is twisted by the Neveu-Schwarz field H_3

$$dF_2 + H_3 \wedge F_0 = 0, \tag{4.52}$$

where F_0, also known as the cosmological constant in this theory, satisfies $dF_0 = 0$. We thus have the identification of σ_2 with F_2 and σ_0 with F_0. Equation (4.52) then says that F_2 is covariantly constant with respect to ∇.

4.4.3 Twisted L_∞-connections

Let $\mathfrak{g}$ be some L_∞-algebra. In [56] we had discussed that the obstruction to lifting a $\mathfrak{g}$-connection (see definition 4.14) through a String-like central extension

$$0 \longrightarrow b^{n-1}\mathfrak{u}(1) \longrightarrow \mathfrak{g}_\mu \longrightarrow \mathfrak{g} \longrightarrow 0 \tag{4.53}$$

is the $b^n\mathfrak{u}(1)$-connection obtained by canonically completing this diagram to the right as shown in figure 2. The obstruction is given by starting from the top-rightmost entry in the big square in figure 2 and continuing all the way horizontally to the left.

ordinary $\mathfrak{g}$-connection	twisted lift to $\mathfrak{g}_\mu$-connection	obstructing $b^n\mathfrak{u}(1)$-connection	**obstruction interpretation**
ordinary $\mathfrak{g}$-connection	twisted $\mathfrak{g}_\mu$-connection	twisting $b^n\mathfrak{u}(1)$-connection	**twisting interpretation**
ordinary $\mathfrak{g}$-connection	twisted $\mathfrak{g}_\mu$-connection	magnetic charge	**charge interpretation**

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] & \mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[l] & \mathrm{CE}(b^n\mathfrak{u}(1)) \arrow[l] \\
\Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "(A,F_A)"'] \arrow[u, two heads] & \mathrm{W}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[l, "f^{-1}"'] \arrow[u, two heads] & \mathrm{CE}(b^n\mathfrak{u}(1)) \arrow[l] \arrow[u, two heads] \\
\Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{P_i\}"'] \arrow[u, hook] & \mathrm{inv}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[l] \arrow[u, hook] & \mathrm{inv}(b^n\mathfrak{u}(1)) \arrow[l] \arrow[u, hook]
\end{tikzcd}$$

Cartan-Ehresmann L_∞-connection

Figure 2: **Obstructing $b^n\mathfrak{u}(1)$ $(n+1)$-connections and "twisted" $\mathfrak{g}_\mu$ n-connections** are two aspects of the same mechanism: the $(n+1)$-connection is the obstruction to "untwisting" the n-connection. The n-connection is "twisted by" the $(n+1)$-connection. There may be many non-equivalent twisted n-connections corresponding to the same twisting $(n+1)$-connections. We can understand these as forming a collection of *n-sections* of the $(n+1)$-connection.

The construction crucially involves first forming the lift of the $\mathfrak{g}$-connection to a $(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$-connection, where $(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$ is the "weak cokernel" or "homotopy quotient" of the injection of $b^{n-1}\mathfrak{u}(1)$ into $\mathfrak{g}_\mu$. This lift through the homotopy quotient always exists, since the homotopy quotient is in fact equivalent to just $\mathfrak{g}$. But performing the lift to the homotopy quotient also extracts the failure of the underlying attempted lift to $\mathfrak{g}_\mu$ proper. This failure may be projected out under

$$(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \longrightarrow\!\!\!\!\!\rightarrow b^n\mathfrak{u}(1) \tag{4.54}$$

to yield the $b^n\mathfrak{u}(1)$-connection which obstructs the lift. It is the morphism denoted f^{-1} in figure 2 which picks up the information about the twist/obstruction. This was constructed in proposition 40 of [56]. However, the $(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$-connection itself deserves to be considered in its own right: this is just the L_∞-connection version of "twisted bundles" or "gerbe modules". In particular, the obstruction problem can also be read the other way round: given a $b^n\mathfrak{u}(1)$-bundle, we may ask for which $\mathfrak{g}$-bundles it is the obstruction to lifting these to a $\mathfrak{g}_\mu$-bundle. In string theory, this is actually usually the more natural point of of view:

- given the Kalb-Ramond background field (a $b\mathfrak{u}(1)$-connection) pulled back to the worldvolume of a D-brane, the "twisted $U(H)$-bundles" corresponding to it are the "Chan-Paton bundles" supported on that D-brane;
- given the supergravity 3-form field (a $b^2\mathfrak{u}(1)$-connection) pulled back to the end-of-the-world 9-branes, the "twisted $BU(1)$-2-bundle" corresponding to it is the Kalb-Ramond field, with the twist giving the failure of its 3-form curvature to close $dH_3 = G_4$.

5 Twisted Differential Structures

We now use the tools from section 4 to explicitly derive the L_∞-algebra valued differential form data that is defined by a differential refinement, according to section 2.1.6, of the twisted nonabelian cocycles considered in section 3. This will explicitly derive the higher form fields known in string theory together with their familiar twisted Bianchi identites.

WARNING: AT THE MOMENENT this section ignores all prefactors in front of deRham representatives of classes, it doesn't matter here anyway.

5.1 Twisted $\mathfrak{u}(n)$ 1-connections

As a warmup for the following two sections, we describe twisted 1-bundles with connection in terms of their L_∞-algebraic formulation and rederive in this language the familiar fact that their Chern character is closed in H_3-twisted de Rham cohomology, where H_3 is the curvature 3-form of the twisting 2-bundle.

In [1] twisted bundles and twisted gerbes are conceived of in terms of local transition data, using a nonabelian variant of Deligne-cohomology notation. Twisted bundles appear in the middle of section 3, while twisted gerbes are described in section 4 of that paper. It is not hard to see that their equation in between equations (55) and (56) expresses the idea which we emphasize here: that twisted n-bundles are potentially twisted lifts, i.e. obstructions to lifts, through $b^{n-1}\mathfrak{u}(1)$-extensions

Consider the extension of Lie algebras

$$0 \longrightarrow \mathfrak{u}(1) \longrightarrow \mathfrak{u}(k) \longrightarrow \mathfrak{pu}(k) \longrightarrow 0 \tag{5.1}$$

where $\mathfrak{pu}(k)$ denotes the Lie algebra of the projective unitary group $PU(k)$. It is the same as the Lie algebra of $SU(k)$, but we write $\mathfrak{pu}(k)$ to remind us that we would like to integrate to $PU(k)$ eventually. $PU(k)$-bundles and the corresponding twisted $U(k)$-bundles model the Chan-Paton bundles on D-branes and give classes in twisted K-theory [75] [37] [12] [11].

The weak quotient Lie 2-algebra $(\mathfrak{u}(1) \to \mathfrak{u}(k))$**.** We describe in detail the Lie 2-algebra arising as the weak (or homotopy) quotient of $\mathfrak{u}(k)$ by $\mathfrak{u}(1)$. Let, as usual, $\{t^0, t^a\}$ be a basis of $\mathfrak{u}(k)^*$ regarded as being in degree 1, with t^0 dual to the generator of the center $= \mathfrak{u}(1)$. Let $\{C^a{}_{bc}\}$ be the structure constants of $\mathfrak{u}(k)$ in that basis. Then the Chevalley-Eilenberg DGCA of $(\mathfrak{u}(1) \to \mathfrak{u}(k))$ is

$$\mathrm{CE}(\mathfrak{u}(1) \to \mathfrak{u}(k)) = \left(\wedge^\bullet (\underbrace{\mathfrak{u}(k)^*}_{1} \oplus \underbrace{\langle b \rangle}_{2}) \right) \tag{5.2}$$

with the differential defined on the generators as

$$\begin{aligned} dt^0 &= -b \\ dt^a &= -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c \\ db &= 0\,. \end{aligned} \tag{5.3}$$

The Weil algebra is

$$\mathrm{W}(\mathfrak{u}(1) \to \mathfrak{u}(k)) = \left(\wedge^\bullet (\underbrace{\mathfrak{u}(k)^*}_{1} \oplus \underbrace{\langle b \rangle}_{2} \oplus \underbrace{\mathfrak{u}(k)^*}_{2} \oplus \underbrace{\langle c \rangle}_{3}) \right) \tag{5.4}$$

with differential given by

$$\begin{aligned} dt^0 &= -b + r^0 \\ dt^a &= -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c + r^a \\ db &= -c\,, \end{aligned} \tag{5.5}$$

where $\{r^0, r^a\} = \sigma\{t^0, t^a\}$ is the induced basis on $\{\mathfrak{u}(k)^*\}$ in degree 2. Finally, the algebra of invariant polynomials is

$$\mathrm{W}(\mathfrak{u}(1) \to \mathfrak{u}(k))_{\mathrm{basic}} = \left(\left(\wedge^\bullet (\underbrace{\langle c \rangle}_{3} \oplus \underbrace{\langle r^0 \rangle}_{2} \oplus \underbrace{\langle c_2 \rangle}_{4} \oplus \underbrace{\langle c_3 \rangle}_{6} \oplus \cdots) \right), d \right)\,, \tag{5.6}$$

where the differential vanishes on all the c_i and on c and satisfies $dr^0 = c$. Under the inclusion

$$\mathrm{W}(\mathfrak{u}(1) \to \mathfrak{u}(k)) \hookleftarrow \mathrm{W}(\mathfrak{u}(1) \to \mathfrak{u}(k))_{\mathrm{basic}}\,, \tag{5.7}$$

c maps to c, r^0 to r^0 and the c_i to the corresponding Chern polynomial forms $c_2 \mapsto (c_2)_{ab} r^a \wedge r^b$, $c_2 \mapsto (c_3)_{abc} r^a \wedge r^b \wedge r^c$, etc. Notice that r^0 is the polynomial corresponding to the would-be first Chern class c_1. This r_0 is the non-closed invariant polynomial which will give rise to twisted de Rham cohomology. By the general princple, a twisted $\mathfrak{u}(k)$-connection now is a Cartan-Ehresmann connection with structure Lie 2-algebra $\mathfrak{g} = (\mathfrak{u}(1) \to \mathfrak{u}(k))$:

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{u}(1) \to \mathfrak{u}(k)) \arrow[l, "A_{\mathrm{vert}}"'] \\
\Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{u}(1) \to \mathfrak{u}(k)) \arrow[l, "{(A,F_A)}"'] \arrow[u, two heads] \\
\Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{u}(1) \to \mathfrak{u}(k)) \arrow[l, "\{P_i\}"'] \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{array}{l}
F_A = (F^0 = dA^0 + B,\ F^a = dA^a + [A \wedge A]^a) \\
dF^0 = H_3
\end{array}
\tag{5.8}
$$

is given by a DGCA homomorphisms $\mathrm{W}(\mathfrak{u}(1) \to \mathfrak{u}(k))_{\mathrm{basic}} \to \Omega^\bullet(X)$. This is a collection consisting of a closed 3-form $c \mapsto H_3 \in \Omega^3_{\mathrm{closed}}(X)$, a 2-form $r^0 \mapsto u \in \Omega^2(X)$, and a series of closed even forms coming from the c_i. The Chern character of this connection for the product $U(n) = SU(n) \times U(1)$ is as usual the combination

$$\mathrm{ch}(F_A) := \mathrm{tr}\exp(F + c_1) = e^{c_1}\mathrm{tr}\exp(F) \tag{5.9}$$

of the c_i. The only difference to an ordinary $\mathfrak{u}(k)$-connection is that now no longer are all of the c_i closed, but that $dc_1 = H_3$. Hence

$$d\mathrm{ch} = H_3 \wedge c\,, \tag{5.10}$$

which says that

The Chern character of a twisted $\mathfrak{u}(k)$-connection lives in H_3-twisted (periodic) de Rham cohomology

$$\mathrm{ch}(F_A) \in H^\bullet_{\mathrm{dR}}(X, H_3)\,. \tag{5.11}$$

Here twisted de Rham cohomology is the cohomology of the $\mathbb{Z}_2$-graded complex $\Omega^{\mathrm{even}}(X) \otimes \Omega^{\mathrm{odd}}(X)$, equipped with the differential $d_{H_3} = d + H_3\wedge$.

Interpretation in terms of sections of 3-connections. We can reinterpret the twisted cohomology part of the situation in terms of sections of associated 3-connections as a generalization of the mechanism in the example below definition 36. Let $\mathfrak{g} := b^2\mathfrak{u}(1)$. The extended standard representation of $b^2\mathfrak{u}(1)$ from definition 33 comes with a Weil algebra given by the obvious generalization of that defined in equations (4.47). Interpret the closed globally defined 3-form as a flat $b^2\mathfrak{u}(1)$-connection

$$\Omega^\bullet(Y) \xleftarrow{(H_3, dH_3 = 0)} W(b^2\mathfrak{u}(1)^{\mathrm{op}} \oplus b^{n-1}\mathfrak{u}(1)) \tag{5.12}$$

and consider a section of this connection via the extended standard representation in definition 33 of $b^2\mathfrak{u}(1)$. As we extend the connection morphism through the twisted DGCA of the extended standard representation

$$
\begin{tikzcd}
& \mathrm{W}_\rho(b^2\mathfrak{u}(1), \oplus_r \mathbb{R}[2r]) \arrow[dl, dotted, "{(\{c_{2r}\},\{\nabla c_{2r}\}s, H_3)}"'] \\
\Omega^\bullet(Y) & & \mathrm{W}(b^2\mathfrak{u}(1)) \arrow[ll, "{(H_3, dH_3=0)}"] \arrow[ul, hook]
\end{tikzcd}
\tag{5.13}
$$

we pick up forms which are the images of the other generators appearing in equations (4.47):

$$v_{2r} \mapsto c_{2r} \in \Omega^{2r}(Y) \tag{5.14}$$

$$\sigma v_{2r} \mapsto \nabla c_{2r} := d_{H_3} c_{2r} := dc_{2r} + H_3 \wedge c_{2(r-1)}\,. \tag{5.15}$$

The H_3-twisted de Rham differential now appears as the covariant derivative of a section of the associated cochain complex.

which is associated via the extended standard representation of $b^2\mathfrak{u}(1)$ to the L_∞-connection obtained from interpreting the globally defined 3-form H_3 as the connection 3-form on a trivial flat 3-bundle. This means that we are interpreting the $d + H_3$ twist at 2 different but closely related levels:

- the twisted 1-connection which is a morphism *into* the 2-connection has $(d+H_3)$-closed Chern character.
- Regarding the untwisted H_3 2-connection as itself being twisted, but by the trivial twist given by a flat 3-connection the interpretation of H_3 changes from that of a 3-form curvature to a 3-form connection. The covariant derivative of this 3-form connection with respect to the extended standard representation of $b^2\mathfrak{u}(1)$ is again $\nabla = d + H_3$.

5.2 Twisted $\mathfrak{string}(n)$ 2-connections

Let X be a (generalized, smooth) space with Spin structure given by a cocycle $g \in H(X, \mathbf{B}\mathrm{Spin})$ and hence with fractional first Pontryagin class $\frac{1}{2}p_1(X) : \ X \xrightarrow{g} \mathbf{B}\mathrm{String} \xrightarrow{\frac{1}{2}p_1} U(1)$. We now work out the differential form data, according to section 4, carried by a cocycle in differential $\frac{1}{2}p_1(X)$-twisted $\mathbf{B}$String-cohomology $q \in H^{[\frac{1}{2}p_1(X)]}(X, \mathbf{B}\mathrm{String})$, i.e. the connection and curvature data and the twisted Bianchi identity of a twisted String-2-bundle. We demonstrated that this twisted Bianchi identity is relation between differential forms as appearing in the Green-Schwarz mechanism.

For that purpose let $p : \ P \twoheadrightarrow X$ be the total space of the twisted String-2-bundle concretely realized as the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dd, bend right, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E}(\mathbf{B}U(1) \to \mathrm{String}) \arrow[d] & \\ \hat{X} \arrow[r, "g_{\mathrm{tw}}"] \arrow[d, "\simeq"] & \mathbf{B}(\mathbf{B}U(1) \to \mathrm{String}) \arrow[r, two heads, "\simeq"] & \mathbf{B}\mathrm{Spin} \\ X \arrow[urr, "g"] & & \end{tikzcd} \tag{5.16}$$

for some twisted lift g_{tw} of g.

On this cover $P \to X$ the computation is essentially a special case of the general description of higher Chern-Simons connections in section 7 of [56]: there we computed the differential data of the obstruction to a differential String-lift, here we fix the obstruction and compute the nature of the cocycles twisted by it. In [56] all spaces were assumed to be ordinary smooth spaces, but all arguments go through unaltered for Kan simplicial smooth spaces such as P, with $\Omega^\bullet(P)$ taken to be the corresponding complex of simplicial forms on P. Since we shall here not further describe the total space P itself in more detail but focus on the structure of an L_∞-connection on this space, the reader can without essential loss think of P as an ordinary manifold and of $\Omega^\bullet(P)$ as ordinary differential forms on this manifold.

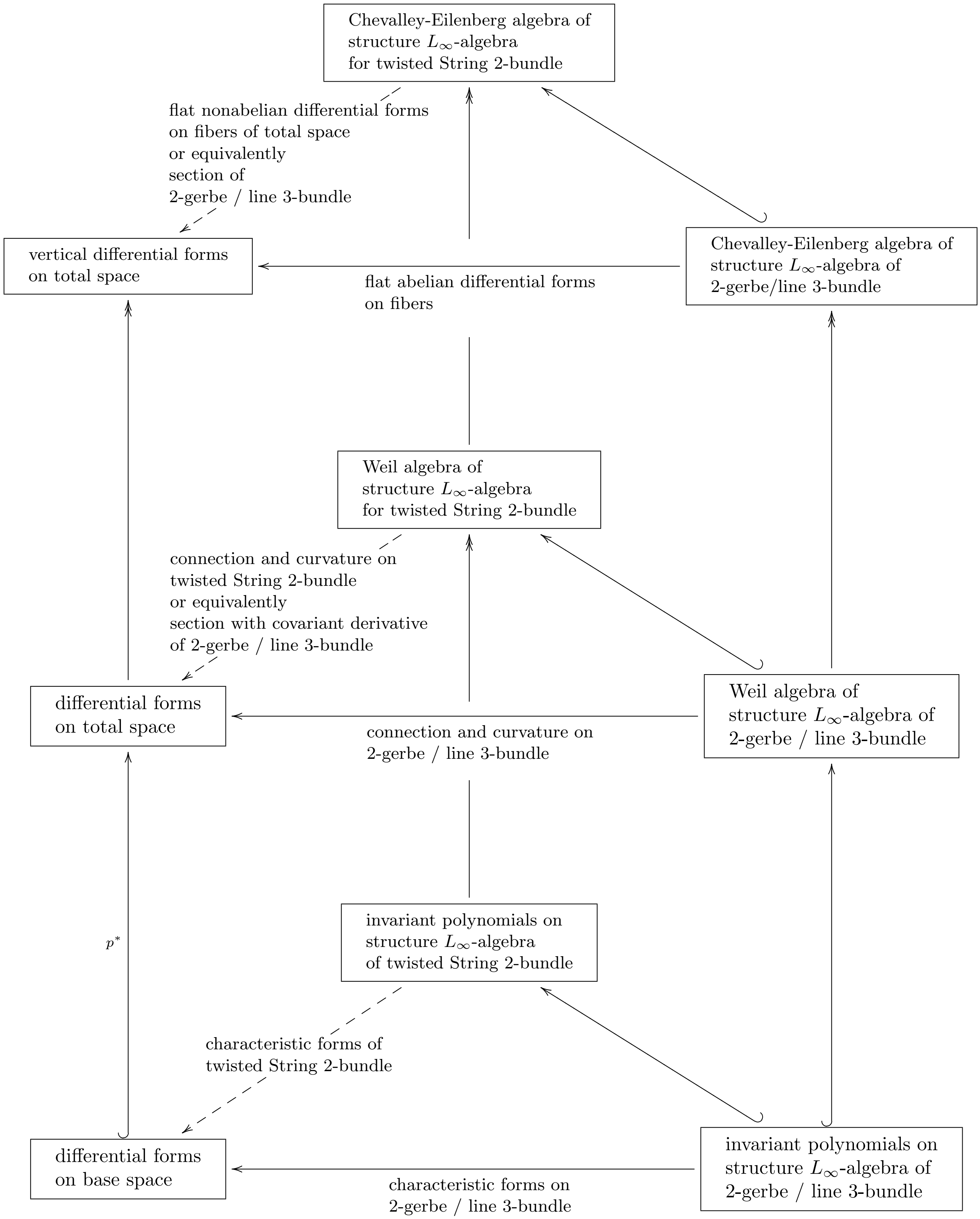
In yet another equivalent formulation of this situation, we describe the covariant derivative and its Bianchi identity of a section of a bundle associated with respect to the representation of $\mathbf{B}^2U(1)$ induced by the canonical L_∞-algebra inclusion

$$\mathrm{CE}(b\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \hookleftarrow \mathrm{CE}(b^2\mathfrak{u}(1)) \tag{5.17}$$

to a $\mathbf{B}^2U(1)$-3-bundle with local connection 3-form $C_3 \in \Omega^3(P)$ and with curvature 4-form $G_4 \in \Omega^4_{\mathrm{closed}}(X)$, By the discussion in section 4.4.2 this is a choice of dashed morphisms in the diagram

$$
\begin{tikzcd}
 & \mathrm{CE}(b\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[dl, dashed, "{\{A_{\mathrm{vert}}, B_{\mathrm{vert}}, (C_3)_{\mathrm{vert}}\}}"'] & \\
\Omega^\bullet_{\mathrm{vert}}(P) & & \mathrm{CE}(b^2\mathfrak{u}(1)) \arrow[ll, "(C_3)_{\mathrm{vert}}"'] \arrow[ul, hook] \\
 & \begin{array}{c}\mathrm{CE}(\mathrm{inn}(b\mathfrak{u}(1)) \hookrightarrow \mathrm{cs}_P(\mathfrak{g})) \\ \simeq \mathrm{W}(b\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)\end{array} \arrow[uu, two heads] \arrow[dl, dashed, "{\{A, F_A, B, \nabla B, C_3, G_4\}}"'] & \\
\Omega^\bullet(P) \arrow[uu, two heads] & & \mathrm{W}(b^2\mathfrak{u}(1)) \arrow[ll, "{(C_3, G_4)}"'] \arrow[uu, two heads] \arrow[ul, hook] \\
 & \mathrm{inv}(\mathrm{inn}(b\mathfrak{u}(1)) \hookrightarrow \mathrm{cs}_P(\mathfrak{g})) \arrow[uu] \arrow[dl, dashed, "{\{H_3, G_4, P(F_A)\}}"'] & \\
\Omega^\bullet(X) \arrow[uu, hook, "p^*"] & & \mathrm{inv}(b^2\mathfrak{u}(1)) \arrow[ll, "G_4"'] \arrow[uu, hook] \arrow[ul, hook]
\end{tikzcd}
$$

It may be helpful to recall what each of the terms in this diagram means. The following diagram is a labeled map for the above one.


Chevalley-Eilenberg algebra of structure L_∞-algebra for twisted String 2-bundle
flat nonabelian differential forms on fibers of total space or equivalently section of 2-gerbe / line 3-bundle
vertical differential forms on total space
flat abelian differential forms on fibers
Chevalley-Eilenberg algebra of structure L_∞-algebra of 2-gerbe/line 3-bundle
Weil algebra of structure L_∞-algebra for twisted String 2-bundle
connection and curvature on twisted String 2-bundle or equivalently section with covariant derivative of 2-gerbe / line 3-bundle
differential forms on total space
connection and curvature on 2-gerbe / line 3-bundle
Weil algebra of structure L_∞-algebra of 2-gerbe / line 3-bundle
p^*
invariant polynomials on structure L_∞-algebra of twisted String 2-bundle
characteristic forms of twisted String 2-bundle
differential forms on base space
characteristic forms on 2-gerbe / line 3-bundle
invariant polynomials on structure L_∞-algebra of 2-gerbe / line 3-bundle

Now, chasing the generators of the graded-commutative algebras through this diagram and recording the condition imposed by the respect of the morphisms of DGCAs for differentials, one finds that in components the commutativity of this diagram encodes the following differential form data and the following relations on that.

$$
\begin{tikzcd}
 & \boxed{\begin{array}{l} dt^a = -\frac{1}{2}C^a{}_{bc}t^b \wedge t^c \\ db = \mu - k \\ dk = 0 \end{array}} \arrow[dl, dashed, "\begin{array}{l} t^a \mapsto A^a_{\mathrm{vert}} \\ b \mapsto B_{\mathrm{vert}} \\ k \mapsto (C_3)_{\mathrm{vert}} \end{array}"'] & \\
\boxed{\begin{array}{l} F^a_{A_{\mathrm{vert}}} = 0 \\ dB_{\mathrm{vert}} = \mu_{A_{\mathrm{vert}}} - (C_3)_{\mathrm{vert}} \\ d(C_3)_{\mathrm{vert}} = 0 \end{array}} & & \boxed{dk = 0} \arrow[ul, "k \mapsto k"'] \arrow[ll, "k \mapsto (C_3)_{\mathrm{vert}}"] \\
 & \boxed{\begin{array}{l} dt^a = -\frac{1}{2}C^a{}_{bc}t^b \wedge t^c + r^a \\ dr^a = -C^a{}_{bc}t^b \wedge r^c \\ db = \mathrm{cs} + c - k \\ dc = l - P \\ dk = l \end{array}} \arrow[uu] \arrow[dl, dashed, "\begin{array}{l} t^a \mapsto A^a \\ r^a \mapsto F^a_A \\ b \mapsto B \\ c \mapsto \nabla B \\ k \mapsto C_3 \\ l \mapsto \mathcal{G}_4 \end{array}"'] & \\
\boxed{\begin{array}{l} H_3 := \nabla B = dB + C_3 - \mathrm{CS}(A, F_A) \\ dH_3 = \mathcal{G}_4 - \langle F_A \wedge F_A \rangle \\ d\mathcal{G}_4 = 0 \end{array}} \arrow[uu, "i^*"] & & \boxed{\begin{array}{l} dk = l \\ dl = 0 \end{array}} \arrow[uu, "\begin{array}{l} k \mapsto k \\ l \mapsto 0 \end{array}"'] \arrow[ul, "\begin{array}{l} k \mapsto k \\ l \mapsto l \end{array}"'] \arrow[ll, "\begin{array}{l} k \mapsto C_3 \\ l \mapsto \mathcal{G}_4 \end{array}"] \\
 & \boxed{\begin{array}{l} dc = l - P \\ dl = 0 \\ dP = 0 \end{array}} \arrow[uu] \arrow[dl, dashed, "\begin{array}{l} c \mapsto \nabla B := H_3 \\ l \mapsto \mathcal{G}_4 \\ P \mapsto \langle F_A \wedge F_A \rangle \end{array}"'] & \\
\boxed{\begin{array}{l} dH_3 = \mathcal{G}_4 - \langle F_A \wedge F_A \rangle \\ d\mathcal{G}_4 = 0 \end{array}} \arrow[uu, "p^*"] & & \boxed{dl = 0} \arrow[uu, "l \mapsto l"'] \arrow[ul, "l \mapsto l"'] \arrow[ll, "l \mapsto \mathcal{G}_4"]
\end{tikzcd}
$$

Here, as usual, $P \in W(\mathfrak{g})$ is the invariant polynomial on $\mathfrak{g}$ in transgression with with the cocycle $\mu \in \mathrm{CE}(\mathfrak{g})$. With $\{t^a\}$ a fixed chosen basis of $\mathfrak{g}^*$ in degree 1 and $\{r^a\}$ the corresponding basis in degree 2, we have $P = P_{ab} r^a \wedge r^b$ and $\mu = \mu_{abc} t^a \wedge t^b \wedge t^c$ and $\mathrm{cs} = P_{ab} t^b \wedge r^a + \frac{1}{6}\mu_{abc} t^a \wedge t^b \wedge t^c$. So we have found in particular

curvature	$H_3 := dB + C_3 - \mathrm{CS}(A, F_A)$
Bianchi identity	$dH_3 = \mathcal{G}_4 - \langle F_A \wedge F_A \rangle$

In [56] this situation was considered from a different perspective for the special case $B = 0$ and $\nabla B = 0$. There the dashed morphism was obtained as a twisted lift of a $\mathfrak{g}$-connection to a $\mathfrak{g}_\mu$-connection and the $b^2\mathfrak{u}(1)$-connection appeared as the corresponding obstruction. Here now the perspective is switched: the $b^2\mathfrak{u}(1)$-connection is prescribed and the choice of dashed morphisms is a choice of twisted $\mathfrak{g}_\mu$-connections with prescribed twist G_4.

The covariant derivative 3-form ∇B of the twisted $\mathfrak{g}_\mu$-connection which we denote by H_3 measures the difference between the prescribed $b^2\mathfrak{u}(1)$-connection and the twist of the chosen twisted $\mathfrak{g}_\mu$-connection. The Bianchi-identity

$$dH_3 = \mathcal{G}_4 - P(F_A) \tag{5.18}$$

which appears in the middle on the left says that this difference has to vanish in cohomology, as one expects. Indeed, this is the structure of the differential forms in the Green-Schwarz mechanism.

5.3 A model for the M-theory C-field

Our formalism allows for (a generalization of) three points of view regarding the description of the M-theory C-field. These are

1. as a shifted differential 2-character. This views the E_8 class a as somewhat more 'basic' and then $\frac{1}{2}\lambda$ is a shift leading to a shifted differential 2-character [19].
2. as a twisted string structure. This takes $\frac{1}{2}\lambda$ as the more 'basic' for which the E_8 class a acts as a twist.
3. we can also give a more democratic point of view by viewing both classes as twists for degree four cohomology. This is the bi-twisted point of view.

The description of the M-theory C-field is very closely related to that of the fields in heterotic string theory discussed in the previous section. In fact, one way of deriving the quantization condition (3.43) of G_4 is by comparing [73] to the heterotic theory on the boundary [32]. The condition in the latter is a trivialization of the cohomology class $\mathrm{ch}_2(E) - p_1(X)$ on a ten-manifold X. As we saw above this is equated at the level of forms to dH_3. The condition in M-theory on a Spin manifold Y is a trivialization of the cohomology class $[G_4] + \frac{1}{4}p_1 - a$, where a is the class of the E_8 bundle. At the level of forms, this is equated to dC_3.

We can already see the close similarity in the mathematical structures between the two quantization conditions. We will use this to provide a model for the C-field in twisted nonabelian differential cohomology using the case of the heterotic string from the previous section. We see that the changes we need to make to the diagrams in the previous section are simply

1. Replace $\mathcal{G}_4$ by G_4.
2. Replace H_3 by C_3.
3. Replace dB_2 by c_3.
4. Add the term $\langle F_\omega \wedge F_\omega \rangle$.

From this we conclude that the C-field in M-theory is a cocycle in the total twisted differential cohomology

$$\bar{H}^{[\cdot]}(X, \mathbf{B}\mathrm{String} \times \mathbf{B}U\langle 8\rangle) := \int^{c \in H(X, \mathbf{B}^4\mathbb{Z})} \bar{H}^{[c]}(X, \mathbf{B}\mathrm{String} \times \mathbf{B}U\langle 8\rangle)\,, \tag{5.19}$$

using the notation from section 2.1.3 and 2.1.6.

5.4 Twisted $\mathfrak{fivebrane}(n)$ 6-connections

Now we consider the connection on a twisted Fivebrane-bundle obtained from a twisted lift of a Spin-bundle. The discussion is entirely analogous to that in the previous section, only that now, more differential forms enter the picture.

Suppose a $\mathfrak{so}(n)$ connection is given and we are asking for a lift to a $\mathfrak{fivebrane}(n) \simeq (\mathfrak{so}(n)_{\mu_3})_{\mu_7}$-connection. We discussed the obstruction for that in [56]. By the general discussion in section **??**, if the obstruction does not vanish, we still get a twisted $\mathfrak{fivebrane}(n)$-connection, namely a connection with structure L_∞-algebra being

$$(b^5\mathfrak{u}(1) \hookrightarrow (b\mathfrak{u}(1) \hookrightarrow (\mathfrak{g}_{\mu_3})_{\mu_7}))\,. \tag{5.20}$$

The twisted Bianchi identity in this case is nothing but the dual Green-Schwarz formula [57] in terms of differential forms.

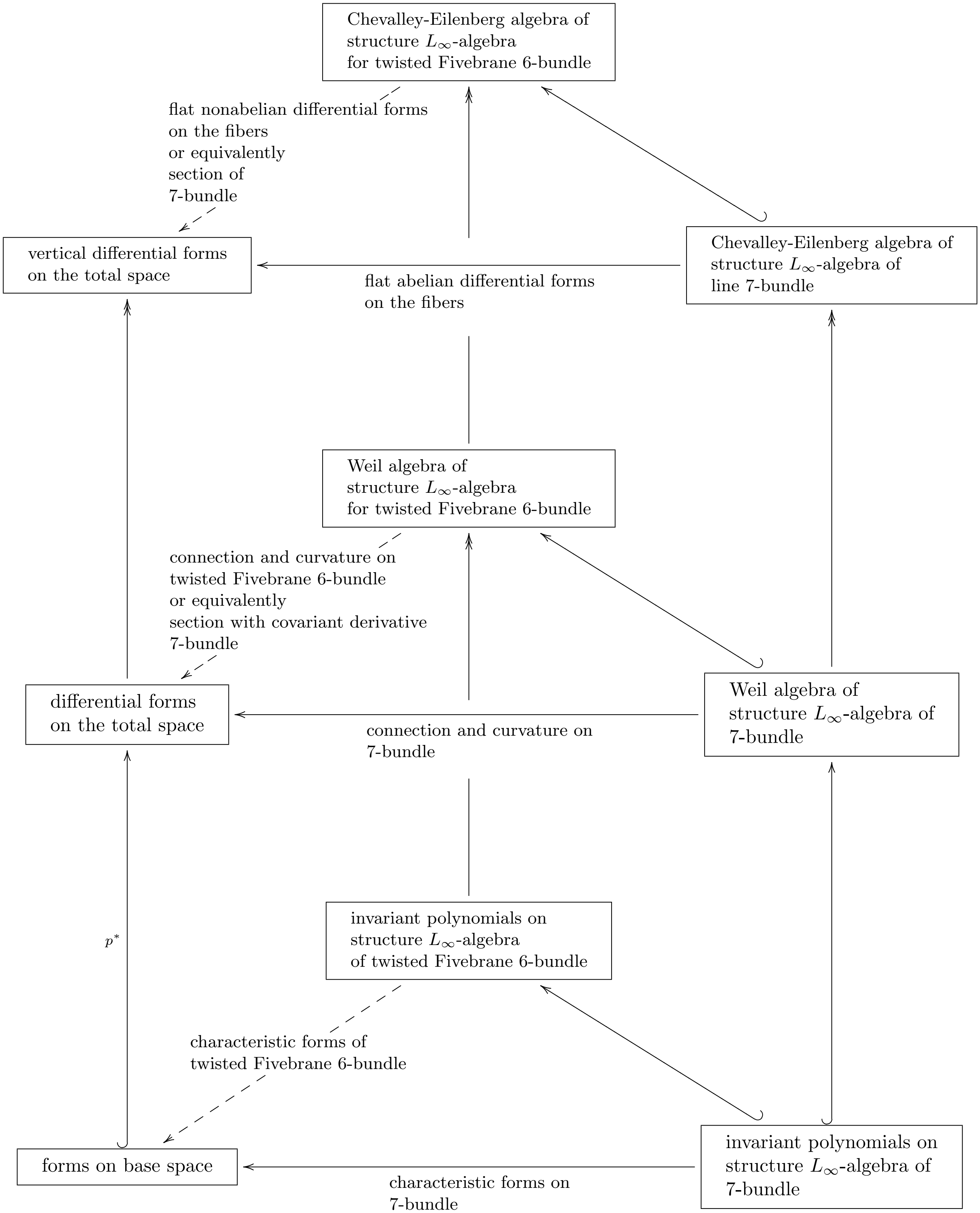
To see this, consider a section of a $b^2\mathfrak{u}(1) \oplus b^5\mathfrak{u}(1)$-connection given by a pair consisting of a connection 3- and 7-form $(C_3, C_7) \in \Omega^3(X) \times \Omega^7(X)$ with curvature 4- and 8-form $(\mathcal{G}_4, \mathcal{G}_8) \in \Omega^4_{\mathrm{closed}}(X) \times \Omega^8_{\mathrm{closed}}(X)$ with respect to the canonical inclusion

$$\mathrm{CE}((b\mathfrak{u}(1) \oplus b^5\mathfrak{u}(1)) \hookrightarrow (\mathfrak{so}(n)_{\mu_3})_{\mu_7}) \longleftarrow\!\!\!\rhook \mathrm{CE}(b^2\mathfrak{u}(1) \oplus b^5\mathfrak{u}(1)) \;\;. \tag{5.21}$$

Again by the discussion in section 4.4.2 this is a choice of dashed morphisms in the diagram

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{CE}((b\mathfrak{u}(1)\oplus b^5\mathfrak{u}(1)) \hookrightarrow (\mathfrak{so}(n)_{\mu_3})_{\mu_7}) \arrow[dl, dashed, "{\{A_{\mathrm{vert}},(B_2)_{\mathrm{vert}},(B_6)_{\mathrm{vert}},(C_3)_{\mathrm{vert}},(C_7)_{\mathrm{vert}}\}}"'] & \\
\Omega^\bullet_{\mathrm{vert}}(P) & & \mathrm{CE}(b^2\mathfrak{u}(1)\oplus b^6\mathfrak{u}(1)) \arrow[ll, "{((C_3)_{\mathrm{vert}},(C_7)_{\mathrm{vert}})}"'] \arrow[ul, hook] \\
 & \begin{array}{c}\mathrm{CE}(\mathrm{inn}(b\mathfrak{u}(1)\oplus b^5\mathfrak{u}(1)) \hookrightarrow \mathrm{cs}_{P_4+P_8}(\mathfrak{g})) \\ \simeq \mathrm{W}((b\mathfrak{u}(1)\oplus b^5\mathfrak{u}(1)) \hookrightarrow (\mathfrak{so}(n)_{\mu_3})_{\mu_7})\end{array} \arrow[uu, two heads] \arrow[dl, dashed, "{\{A,F_A,B_2,B_6,\nabla B_2,\nabla B_6,C_3,C_7,G_4,G_8\}}"'] & \\
\Omega^\bullet(P) \arrow[uu, two heads] & & \mathrm{W}(b^2\mathfrak{u}(1)\oplus b^6\mathfrak{u}(1)) \arrow[ll, "{(C_3,C_7,G_4,G_8)}"'] \arrow[ul, hook] \arrow[uu, two heads] \\
 & \mathrm{inv}(\mathrm{inn}(b\mathfrak{u}(1)\oplus b^5\mathfrak{u}(1)) \hookrightarrow \mathrm{cs}_{P_4+P_8}(\mathfrak{so}(n))) \arrow[uu, two heads] \arrow[dl, dashed, "{\{H_3,H_7,G_4,G_8,P_4(F_A),P_8(F_A)\}}"'] & \\
\Omega^\bullet(X) \arrow[uu, hook, "p^*"] & & \mathrm{inv}(b^2\mathfrak{u}(1)\oplus b^6\mathfrak{u}(1)) \arrow[ll, "{(G_4,G_8)}"'] \arrow[ul, hook] \arrow[uu, hook]
\end{tikzcd}
$$

Here is again the interpretation of the terms in this diagram:



By again chasing elements through the diagram one finds the following data:

$$
\begin{tikzcd}[column sep=small]
 & \boxed{\begin{array}{l} dt^a = -\frac{1}{2}C^a{}_{bc}t^b\wedge t^c \\ db_2 = \mu_3 - k_3 \\ db_6 = \mu_7 - k_7 \\ dk_3 = 0 \\ dk_7 = 0 \end{array}} \arrow[dl, dashed, "\begin{array}{l} t^a \mapsto A^a_{\mathrm{vert}} \\ b_2 \mapsto (B_2)_{\mathrm{vert}} \\ b_6 \mapsto (B_6)_{\mathrm{vert}} \\ k_3 \mapsto (C_3)_{\mathrm{vert}} \\ k_7 \mapsto (C_7)_{\mathrm{vert}} \end{array}"'] & \\
\boxed{\begin{array}{l} F^a_{A_{\mathrm{vert}}} = 0 \\ d(B_2)_{\mathrm{vert}} = \mu_3(A_{\mathrm{vert}}) - (C_3)_{\mathrm{vert}} \\ d(B_6)_{\mathrm{vert}} = \mu_7(A_{\mathrm{vert}}) - (C_7)_{\mathrm{vert}} \\ d(C_3)_{\mathrm{vert}} = 0 \\ d(C_7)_{\mathrm{vert}} = 0 \end{array}} & & \boxed{dk = 0} \arrow[ul, "\begin{array}{l} k_3 \mapsto k_3 \\ k_7 \mapsto k_7 \end{array}"'] \arrow[ll, "\begin{array}{l} k_3 \mapsto (C_3)_{\mathrm{vert}} \\ k_7 \mapsto (C_7)_{\mathrm{vert}} \end{array}"] \\
 & \boxed{\begin{array}{l} dt^a = -\frac{1}{2}C^a{}_{bc}t^b\wedge t^c + r^a \\ dr^a = -C^a{}_{bc}t^b\wedge r^c \\ db_2 = \mathrm{cs}_3 + c_3 - k_3 \\ db_6 = \mathrm{cs}_7 + c_7 - k_7 \\ dc_3 = l_4 - P_4 \\ dc_7 = l_8 - P_8 \\ dk_3 = l_4 \\ dk_7 = l_8 \end{array}} \arrow[uu] \arrow[dl, dashed, "\begin{array}{l} t^a \mapsto A^a \\ r^a \mapsto F^a_A \\ b_2 \mapsto B_2 \\ b_6 \mapsto B_6 \\ c_3 \mapsto \nabla B_2 \\ c_7 \mapsto \nabla B_6 \\ k_3 \mapsto C_3 \\ k_7 \mapsto C_7 \\ l_4 \mapsto \mathcal{G}_4 \\ l_8 \mapsto \mathcal{G}_8 \end{array}"'] & \\
\boxed{\begin{array}{l} H_3 := \nabla B_2 = dB_2 + C_3 - \mathrm{CS}_3(A,F_A) \\ H_7 := \nabla B_6 = dB_6 + C_7 - \mathrm{CS}_7(A,F_A) \\ dH_3 = \mathcal{G}_4 - \langle F_A\wedge F_A\rangle \\ dH_7 = \mathcal{G}_8 - \langle F_A\wedge F_A\wedge F_A\wedge F_A\rangle \\ d\mathcal{G}_4 = 0 \\ d\mathcal{G}_8 = 0 \end{array}} \arrow[uu, "i^*"] & & \boxed{\begin{array}{l} dk_3 = l_4 \\ dk_7 = l_8 \\ dl_4 = 0 \\ dl_8 = 0 \end{array}} \arrow[uu, "\begin{array}{l} k_3 \mapsto k_3 \\ k_7 \mapsto k_7 \\ l_4 \mapsto 0 \\ l_8 \mapsto 0 \end{array}"] \arrow[ul, "\begin{array}{l} k_3 \mapsto k_3 \\ k_7 \mapsto k_7 \\ l_4 \mapsto l_4 \\ l_8 \mapsto l_8 \end{array}"'] \arrow[ll, "\begin{array}{l} k_3 \mapsto C_3 \\ k_7 \mapsto C_7 \\ l_4 \mapsto \mathcal{G}_4 \\ l_8 \mapsto \mathcal{G}_8 \end{array}"] \\
 & \boxed{\begin{array}{l} dc_3 = l_4 - P_4 \\ dc_7 = l_8 - P_8 \\ dl_4 = 0 \\ dl_8 = 0 \\ dP_4 = 0 \\ dP_8 = 0 \end{array}} \arrow[uu] \arrow[dl, dashed, "\begin{array}{l} c_3 \mapsto \nabla B_2 := H_3 \\ c_7 \mapsto \nabla B_6 := H_7 \\ l_4 \mapsto \mathcal{G}_4 \\ l_8 \mapsto \mathcal{G}_8 \\ P_4 \mapsto \langle F_A\wedge F_A\rangle \\ P_8 \mapsto \langle F_A\wedge F_A\wedge F_A\wedge F_A\rangle \end{array}"'] & \\
\boxed{\begin{array}{l} dH_3 = \mathcal{G}_4 - \langle F_A\wedge F_A\rangle \\ dH_7 = \mathcal{G}_8 - \langle F_A\wedge F_A\wedge F_A\wedge F_A\rangle \\ d\mathcal{G}_4 = 0 \\ d\mathcal{G}_8 = 0 \end{array}} \arrow[uu, "p^*"] & & \boxed{\begin{array}{l} dl_4 = 0 \\ dl_8 = 0 \end{array}} \arrow[uu, "\begin{array}{l} l_4 \mapsto l_4 \\ l_8 \mapsto l_8 \end{array}"] \arrow[ul, "\begin{array}{l} l_4 \mapsto l_4 \\ l_8 \mapsto l_8 \end{array}"'] \arrow[ll, "\begin{array}{l} l_4 \mapsto \mathcal{G}_4 \\ l_8 \mapsto \mathcal{G}_8 \end{array}"]
\end{tikzcd}
$$

Here, as usual, $P_4, P_8 \in W(\mathfrak{g})$ are the invariant polynomials on $\mathfrak{g}$ in transgression with with the cocycles $\mu_3, \mu_7 \in \mathrm{CE}(\mathfrak{g})$. With $\{t^a\}$ a fixed chosen basis of $\mathfrak{g}^*$ in degree 1 and $\{r^a\}$ the corresponding basis in degree 2, we have $P_4 = P_{ab} r^a \wedge r^b$, $P_8 = P_{a_1 \cdots a_8} r^{a_1} \wedge \cdots \wedge r^{a_8}$ and $\mu_3 = \mu_{abc} t^a \wedge t^b \wedge t^c$ and $\mathrm{cs}_3 = P_{ab} t^b \wedge r^a + \frac{1}{6} \mu_{abc} t^a \wedge t^b \wedge t^c$, etc.

The covariant derivative 7-form ∇B_6 of the twisted $(\mathfrak{so}(n)_{\mu_3})_{\mu_7}$-connection which we denote by H_7 measures the difference between the prescribed $b^6\mathfrak{u}(1)$-connection and the twist of the chosen twisted $(\mathfrak{so}(n)_{\mu_3})_{\mu_7}$-connection. The Bianchi-identity

$$dH_7 = \mathcal{G}_8 - P_8(F_A) \tag{5.22}$$

which appears in the middle on the left says that this difference has to vanish in cohomology, as one expects. This is the differential form data of the dual Green-Schwarz mechanism [57].

5.5 A model for the M-theory dual C-field

Similarly to the case of the C-field, our formalism allows for (a generalization of) three points of view regarding the description dual of of the M-theory C-field. These are

1. as a shifted differential 6-character. This views the E_8 degree 8 class $\frac{1}{2}a^2$ as somewhat more 'basic' and then $\frac{1}{48}p_2$ is a shift leading to a shifted differential 6-character. This is a generalization of the case in [19] to the degree eight case.
2. as a twisted Fivebrane structure. This takes $\frac{1}{48}p_2$ as the more 'basic' for which the E_8 class $\frac{1}{2}a^2$ acts as a twist.
3. we can also give a more democratic point of view by viewing both classes as twists for degree eight cohomology cohomology. This is the bi-twisted point of view.

In this section we provide a model for the dual G_8 of the C-field in an analogous way that we did for the case of the C-field itself in section (**??**). Here again we notice the similarity in structure between the dual H_7 of the H_3 field in ten dimensions and the dual G_8 of G_4 in eleven dimensions. H_7 provides a trivialization of the dual of the Green-Schwarz anomaly formula, while G_8 is itself part of the sum of cohomology class, and hence, at the level of differential forms, it is *itself* trivialized rather than acting as trivialization. Hence, as in the degree four case, we have an extra term dC_7 that acts as a trivialization. In fact C_7 is the right hand side of the equation of motion for G_4.

We can again see the close similarity in the mathematical structures between the two quantization conditions (see equation 3.45). We will use this to provide a model for the dual of the C-field in twisted nonabelian differential cohomology using the case of the dual heterotic string from previous section. We see that the changes we need to make to the diagrams in the previous section are simply

1. Replace $\mathcal{G}_8$ by G_8.
2. Replace H_7 by C_7.
3. Replace dB_6 by c_7.
4. Add the term $\langle F_\omega \wedge F_\omega \wedge F_\omega \wedge F_\omega \rangle$.

From this we conclude, again, that the dual C-field in M-theory is a cocycle in the total twisted differential cohomology

$$\bar{H}^{[\cdot]}(X, \mathbf{B}\mathrm{Fivebrane} \times \mathbf{B}U\langle 10 \rangle) := \int^{c \in H(X, \mathbf{B}^8\mathbb{Z})} \bar{H}^{[c]}(X, \mathbf{B}\mathrm{Fivebrane} \times \mathbf{B}U\langle 10 \rangle)\,, \tag{5.23}$$

using the notation from section 2.1.3 and 2.1.6.

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