Lectures Notes

Partial Differential Equations I

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Partial Differential Equations I: Table of Contents

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Chapter 1

Basics, examples, classification

In these notes we stick to the basic conventions $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Terminology (for **general PDEs**). *A **partial differential equation (PDE)** is an **equation for a function** u **in two or more** (real) **variables**:*

$$F(x, u(x), \mathrm{D}u(x), \mathrm{D}^2u(x), \ldots, \mathrm{D}^{m-1}u(x), \mathrm{D}^m u(x)) = 0_{\mathbb{R}^M} \qquad \text{for all } x \in \Omega \tag{$*$}$$

or, in short-hand notation,

$$F(\,\cdot\,, u, \mathrm{D}u, \mathrm{D}^2u, \ldots, \mathrm{D}^{m-1}u, \mathrm{D}^m u) \equiv 0_{\mathbb{R}^M} \qquad \text{in } \Omega\,.$$

Here we denote ...

- *by $m \in \mathbb{N}$ (if chosen minimal[1]) the **order of the PDE** ($*$),*
- *by Ω an arbitrary open set in $\mathbb{R}^n$,*
- *by $u \colon \Omega \to \mathbb{R}^N$ the **unknown function** (by $\mathrm{D}u(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^N) = \mathbb{R}^{N\times n}$ its first derivative at the point x, regarded as linear mapping $\mathbb{R}^n \to \mathbb{R}^N$ or $(N\times n)$-matrix, and more generally by $\mathrm{D}^k u(x) \in \mathcal{L}^k_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^N)$ its k-th derivative at x, regarded as symmetric k-linear mapping $(\mathbb{R}^n)^k \to \mathbb{R}^N$),*
- *by $n \in \mathbb{N}$ the **number of (independent) variables**, here generally $\boldsymbol{n \geq 2}$,*
- *by $N \in \mathbb{N}$ the **number of unknown** (component) **functions**,*
- *by $M \in \mathbb{N}$ the **number of** (component) **equations**,*
- *by $F \colon \Omega \times \mathbb{R}^N \times \mathcal{L}(\mathbb{R}^n, \mathbb{R}^N) \times \mathcal{L}^2_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^N) \times \ldots \times \mathcal{L}^{m-1}_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^N) \times \mathcal{L}^m_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^N) \to \mathbb{R}^M$ the given **structure function of the PDE** ($*$).*

*For $N = 1$ the unknown is a **scalar/single function**, otherwise a **vector function**. In case $M = 1$ we speak of a **scalar/single PDE**, otherwise of a **system of M PDEs**. Finally, we call $u \colon \Omega \to \mathbb{R}^N$ a **solution of/to the PDE** ($*$) if ($*$) holds for u.*

[1] Minimality of m means that the PDE can see a difference in the m-th derivative only. In precise terms, this means that there exist functions u, v, and a point $x_0 \in \Omega$ with $\mathrm{D}^k u(x_0) = \mathrm{D}^k v(x_0)$ for $k = 0, 1, 2, \ldots, m{-}1$ but still with $F(x_0, u(x_0), \mathrm{D}u(x_0), \mathrm{D}^2u(x_0), \ldots, \mathrm{D}^m u(x_0)) = 0 \neq F(x_0, v(x_0), \mathrm{D}v(x_0), \mathrm{D}^2v(x_0), \ldots, \mathrm{D}^m v(x_0))$.

We emphasize that the word **'partial'** in the term 'partial differential equation' **signifies the occurrence of partial derivatives** $\partial^\alpha u = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_n^{\alpha_n}u$ with $\alpha \in \mathbb{N}_0^n$ (which are the components of $\mathrm{D}^k u$ with $k = |\alpha| = \alpha_1+\alpha_2+\ldots+\alpha_n \le m$) and is used to **distinguish** these equations **from ordinary differential equations (ODEs)** for functions of a single variable.

Terminology (for **linear PDEs**)**.** *The PDE* ($*$) *above is termed* ***linear*** *if the structure function* F *is an affine function of the* u, $\mathrm{D}u$, D^2u, $\ldots$, $\mathrm{D}^{m-1}u$, $\mathrm{D}^m u$ *variables, that is if it takes the form*

$$\sum_{i=1}^N \sum_{|\alpha|\le m} a_\alpha^{ij}(x)\partial^\alpha u_i(x) = f^j(x) \qquad \text{for all } x\in\Omega \text{ and } j=1,2,\ldots,M{-}1,M \tag{$**$}$$

with ***coefficients*** $a_\alpha^{ij}\colon \Omega \to \mathbb{R}$ *and* ***inhomogeneities*** $f^j\colon \Omega\to\mathbb{R}$*. The PDE* ($**$) *has* ***constant coefficients*** *if all coefficients* a_α^{ij} *are constant functions*; *and it is* ***homogeneous*** *if all inhomogeneities* f^j *vanish.*

While linear PDEs are clearly a very basic type, a lot of advanced PDE theory has nowadays been developed for cases which are not truly linear but linear in the highest-order derivatives at least. Though not all authors use the same terminology for such equations, there is *some* agreement that **quasilinear PDEs** are equations of the type

$$\sum_{i=1}^N \sum_{|\alpha|=m} A_\alpha^{ij}(\,\cdot\,,u,\mathrm{D}u,\mathrm{D}^2u,\ldots,\mathrm{D}^{m-1}u)\partial^\alpha u_i = G^j(\,\cdot\,,u,\mathrm{D}u,\mathrm{D}^2u,\ldots,\mathrm{D}^{m-1}u)$$

and **semilinear PDEs** are equations of the somewhat more special type

$$\sum_{i=1}^N \sum_{|\alpha|=m} a_\alpha^{ij}(\,\cdot\,)\partial^\alpha u_i = G^j(\,\cdot\,,u,\mathrm{D}u,\mathrm{D}^2u,\ldots,\mathrm{D}^{m-1}u)\,.$$

PDEs which are not even quasilinear are generally known as **fully non-linear PDEs**.

Examples (of PDEs and PDE systems)**.** Consider an open set $\Omega\subset\mathbb{R}^n$ with $n\in\mathbb{N}_{\ge 2}$.

(1) The **Cauchy-Riemann system**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

is a first-order ($m{=}1$) linear system of $M{=}2$ PDEs for $N{=}2$ functions $u,v\colon\Omega\to\mathbb{R}^N$ in $n{=}2$ variables. It corresponds to the case of the structure function $F(z,w,\ell) = (\ell_{11}{-}\ell_{22},\ell_{12}{+}\ell_{21})$ for $(z,w,\ell)\in\Omega{\times}\mathbb{R}^2{\times}\mathbb{R}^{2\times 2}$ in ($*$).

If we identify $\mathbb{R}^2$ with $\mathbb{C}$, solutions $(u,v)\colon\Omega\to\mathbb{R}^2$ of the Cauchy-Riemann system turn out to be precisely the holomorphic functions $h\colon\Omega\to\mathbb{C}$. Thus, the Cauchy-Riemann system can be viewed as the underlying PDE system in complex analysis (at least in case of a single complex variable). The Cauchy integral and the Poisson integral yield explicit formulas for solutions.

(2) The **Laplace equation**

$$\operatorname{div}(\nabla u) \equiv 0$$

and the **Poisson equation**

$$\operatorname{div}(\nabla u) = f$$

with non-vanishing $f\colon \Omega \to \mathbb{R}$, respectively, are scalar (M=1) second-order (m=2) linear PDEs for a single (N=1) function $u\colon \Omega \to \mathbb{R}$ in an arbitrary number n of variables. On the left-hand side these equations involve the **Laplace operator** Δ, defined by

$$\Delta u := \operatorname{div}(\nabla u) = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_{n-1}^2} + \frac{\partial^2 u}{\partial x_n^2} = \sum_{i=1}^n \partial_i^2 u = \operatorname{trace}(\nabla^2 u)$$

(where $\mathrm{D}^2 u(x)$ is represented, in this scalar case, by the Hessian $\nabla^2 u(x) \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$). The Poisson equation corresponds to the case of the structure function $F(x,u,\ell,q) = \operatorname{trace}(q) - f(x)$ for $(x,u,\ell,q) \in \Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n}_{\mathrm{sym}}$ in ($*$), and clearly the choice $f \equiv 0$ yields the Laplace equation.

Solutions of the Laplace equation are known as **harmonic functions** and will be of central interest in this lecture. For n=2 there is strong connection to (1), as harmonic functions of two variables arise as the real and imaginary parts of holomorphic functions.

The Poisson equation in $\Omega = \mathbb{R}^3$ serves as a model equation in electrostatics, which determines the electric potential u corresponding to the charge distribution f.

(3) **(Linear) Transport Equations** take the form

$$\frac{\partial u}{\partial t} + b \cdot \nabla_x u + cu \equiv 0\,,$$

where the variables $(t,x) \in \Omega \subset \mathbb{R}\times\mathbb{R}^{n-1}$ are split into a single 'time' variable t and $(n-1)$ 'space' variables x. Here, the non-vanishing time-dependent vector field $b\colon \Omega \to \mathbb{R}^{n-1}$ and the coefficient $c\colon \Omega \to \mathbb{R}$ are considered as given, and the equations are scalar (M=1) first-order (m=1) linear PDEs for a single (N=1) function $u\colon \Omega \to \mathbb{R}$ in an arbitrary number n of variables. They correspond to the case of the structure function $F(z,u,\ell) = \ell_0 + b(z)\cdot\ell' + c(z)u$ for $(z,u,\ell) \in \Omega\times\mathbb{R}\times\mathbb{R}^n$, $\ell = (\ell_0,\ell') \in \mathbb{R}\times\mathbb{R}^{n-1}$ in ($*$).

The case of constant b and c is discussed in the exercise class.

Linear transport equations in $n = 1{+}3$ time-space variables model the mass transport in a velocity field b.

(4) The **Heat Equation** or **Diffusion Equation**

$$\frac{\partial u}{\partial t} - \Delta_x u \equiv 0$$

and the **Wave Equation** (for $n = 2$ sometimes called Equation of the Vibrating String)

$$\frac{\partial^2 u}{\partial t^2} - \Delta_x u \equiv 0\,,$$

respectively, involve once more time-space-split variables $(t,x) \in \Omega \subset \mathbb{R}\times\mathbb{R}^{n-1}$. These equations are scalar (M=1) second-order (m=2) linear PDEs for a single (N=1) function

$u\colon \Omega \to \mathbb{R}$ in an arbitrary number n of variables, and they correspond to the case of the structure functions $F(z,u,\ell,q) = \ell_0 - \sum_{i=1}^{n-1} q_{ii}$ and $F(z,u,\ell,q) = q_{00} - \sum_{i=1}^{n-1} q_{ii}$ for $(z,u,\ell,q) \in \Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n}_{\mathrm{sym}}$, $\ell = (\ell_0,\ell') \in \mathbb{R}\times\mathbb{R}^{n-1}$, $q = (q_{ij})_{i,j=0,1,2,\ldots,n-1}$ in $(*)$.

The heat/diffusion equation and the wave equation in $n = 1{+}3$ time-space variables constitute basic physical models for the free propagation of heat/particles and waves/oscillations, respectively.

(5) The ***p*-Laplace Equation** with parameter $p \in [1,\infty)$

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) \equiv 0$$

is considered a scalar ($M{=}1$) second-order ($m{=}2$) quasilinear PDE for a single ($N{=}1$) function $u\colon \Omega \to \mathbb{R}$, though, strictly taken, the equation does not possess the form $(*)$. However, by expanding the divergence it is brought into this form with the structure function $F(x,u,\ell,q) = |\ell|^{p-2}\operatorname{trace}(q) + (p{-}2)|\ell|^{p-4}\sum_{i,j=1}^n q_{ij}\ell_i\ell_j$ for $(x,u,\ell,q) \in \Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n}_{\mathrm{sym}}$.

In the only linear case $p = 2$ (which is also the only semilinear case), the p-Laplace equation reduces to the Laplace equation from (2). In the general case, it plays the role of a natural model case for quasilinear equations.

(6) The **Monge-Ampère Equation**

$$\det(\nabla^2 u) = f$$

with right-hand side $f\colon \Omega \to \mathbb{R}$ is a scalar ($M{=}1$) second-order ($m{=}2$) fully non-linear PDE for a single ($N{=}1$) function $u\colon \Omega \to \mathbb{R}$ in an arbitrary number n of variables. It corresponds to the case of the structure function $F(x,u,\ell,q) = \det(q) - f(x)$ for $(x,u,\ell,q) \in \Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n}_{\mathrm{sym}}$ in $(*)$.

The Monge-Ampère Equation is a basic model case among fully non-linear equations and has important applications in convex geometry and optimal transportation (of measures).

(7) **Many more PDEs** and systems of PDEs occur in **physics** or **geometry**. Famous linear examples are the Schrödinger equation, the Dirac equation, the equations of linear elasticity, and the Maxwell equations. Famous non-linear examples are the Navier-Stokes equations, the Einstein field equations, the minimal surface equation, and the Yamabe equation. Typically one has (as indeed it stands in all the mentioned examples) $M = N$ and $m \in \{1,2,4\}$.

In the course of this lecture, a couple of explicit integral formulas for solutions of model PDEs will be shown. Nevertheless, in more general PDE theory such formulas are available only in very rare cases and the focus of interest is more on the following basic questions and related general principles:

(1) **Existence:** Does a solution exist? **Uniqueness:** Is a solution unique?

(2) **Stability** (usually asked only with uniqueness at hand)**:** Does the solution depend on the structure function/the coefficients/the data in a continuous way? Is the solution stable under (small) perturbations of the structure function/the coefficients/the data?

(3) **Regularity:** Do (higher) derivatives of solutions necessarily exist? Are all solutions smooth functions?

Generally we may hope for **positive answers** to these basic questions **only** ...

(A) in **case $M = N$**, that is, if the **number of** (component) **equations equals** the **number of unknown** (component) **functions**,

(B) if we **add boundary conditions**, that is, if we prescribe u and/or some of its derivatives on $\partial\Omega$ (where they are defined e.g. after continuous extension from Ω to $\overline{\Omega}$).

As a general **rule of thumb**, it often makes sense to impose $\boldsymbol{\frac{1}{2}mN}$ (real-valued) **boundary conditions**.

Despite the common questions (1), (2), (3) and the common general principles (A), (B), there is **no successful common theory of *all* PDEs**. Indeed, different types of PDEs exhibit a very different behavior, and thus such a common theory cannot be reasonably expected. Rather **different** (classes of) **PDEs require their own theories** and notions of (generalized) solutions. We do not attempt to survey or compare the various known theories and approaches but only mention the general guiding principles that linear PDEs are usually simpler than non-linear ones and that problems with small values of m, n, M, N tend to be simpler than problems with large values of these numbers.

Classification (of **scalar linear PDEs**). Consider an open set $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}_{\geq 2}$.

(1) The general scalar linear **first-order** PDE on Ω reads

$$\sum_{i=1}^{n} b^i \partial_i u + cu = f \qquad \text{or equivalently } b \cdot \nabla u + cu = f$$

with given non-vanishing vector field $b = (b^1, \ldots, b^n) \colon \Omega \to \mathbb{R}^n$, given coefficient $c \colon \Omega \to \mathbb{R}$, and given inhomogeneities $f \colon \Omega \to \mathbb{R}$. In principle, this PDE reduces to ODEs by a general method, the **method of characteristics**, which we now roughly describe:

Under suitable regularity assumptions on the vector field b, one considers, for $x \in \Omega$, the **flux lines** $\gamma_x \colon I_x \to \Omega$ of b, that is the maximal **solutions of** the **ODE initial value problem**

$$\begin{aligned} \gamma_x'(t) &= b(\gamma_x(t)) \qquad (t \in I_x)\,, \\ \gamma_x(0) &= x \end{aligned}$$

on the maximal existence interval I_x around 0. One commonly thinks of each γ_x as the time-t-parametrized trajectory of a particle, which moves in the velocity field b and passes through the point x at time $t = 0$. Anyway, the chain rule and the ODE give $\frac{\mathrm{d}}{\mathrm{d}t} u(\gamma_x(t)) = \gamma_x'(t) \cdot \nabla u(\gamma_x(t)) = b(\gamma_x(t)) \cdot \nabla u(\gamma_x(t))$, and in view of this formula the **PDE reduces to** the **ODEs**

$$\frac{\mathrm{d}}{\mathrm{d}t} u(\gamma_x(t)) + c(\gamma_x(t)) u(\gamma_x(t)) = f(\gamma_x(t)) \qquad (t \in I_x)$$

along the flux lines γ_x. In well-behaved cases this allows to determine all PDE solutions by prescribing values on a hypersurface which intersects each (equivalence class of reparametrized) flux lines in exactly one point and by solving the ODEs above along the flux lines.

The method of characteristics vastly simplifies in case of a constant field b, since then the flux lines γ_x are just (constant-speed parametrized) line segments. The treatment of cases with irregular b and the extension of the method to non-linear first-order PDEs, however, turn out to be much more involved and are partially topics of ongoing mathematical research.

(2) The general scalar linear **second-order** PDE on Ω reads

$$Lu := \sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j u + \sum_{i=1}^{n} b^i\partial_i u + cu = f \qquad (***)$$

with given coefficients $a^{ij}, b^i, c\colon \Omega \to \mathbb{R}$ (among which at least one a^{ij} does not vanish) and given inhomogeneities $f\colon \Omega \to \mathbb{R}$. In view of $\partial_i\partial_j u = \partial_j\partial_i u$ (at least for C^2 functions u), we can and do **assume** the **symmetry** condition $\boldsymbol{a^{ij} = a^{ji}}$. Moreover, we call ...

- the operator L, casually defined in $(***)$, a (linear) **partial differential operator (PDO)**,
- the polynomial $p(x,\xi) := \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j + \sum_{i=1}^{n} b^i(x)\xi_i + c(x)$ in $\xi \in \mathbb{R}^n$ the **symbol** of the PDO L (where, by the way, in this notation we can formally write $L = p(x,\partial) = p(x,\partial_1,\partial_2,\ldots,\partial_n)$),
- the PDO $L_0 := \sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j$ the **principal part** (and the polynomial $p_0(x,\xi) := \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j$ in $\xi \in \mathbb{R}^n$ the principal part symbol) of the PDO L.

An alternative, sometimes very convenient form of scalar linear second-order PDEs is the **divergence form**

$$\sum_{i,j=1}^{n} \partial_i(\tilde{a}^{ij}\partial_j u) + \sum_{i=1}^{n} \partial_i\big(\hat{b}^i u\big) + \sum_{i=1}^{n} \tilde{b}^i\partial_i u + \tilde{c}u = f \qquad (****)$$

with coefficients $\tilde{a}^{ij}, \hat{b}^i, \tilde{b}^i, \tilde{c}\colon \Omega \to \mathbb{R}$ (among which at least one $\tilde{a}^{ij}$ does not vanish) and inhomogeneity $f\colon \Omega \to \mathbb{R}$. Here, the name 'divergence form' stems from the possibility to express the first two terms on the left-hand side as a divergence (namely the divergence of the vector field whose i-th component is $\sum_{j=1}^{n} \tilde{a}^{ij}\partial_j u + \hat{b}^i u$).

In case of C^1 coefficients, the forms $(***)$ and $(****)$ turn out to be essentially equivalent — with coinciding principal part coefficients $a^{ij} = \tilde{a}^{ij}$ if symmetry is assumed. More precisely, by straightforward computations with the product rule one verifies: If a PDE is given in the form $(***)$ with $a^{ij} \in \mathrm{C}^1(\Omega)$, it can be brought in the form $(****)$ with $\tilde{a}^{ij} = a^{ij}$, $\hat{b}^i \equiv 0$, $\tilde{b}^i = b^i - \sum_{j=1}^{n} \partial_j a^{ji}$, $\tilde{c} = c$. Conversely, if a PDE is given in the form $(****)$ with $\tilde{a}^{ij}, \hat{b}^i \in \mathrm{C}^1(\Omega)$, it can be brought in the form $(***)$ with $a^{ij} = \tilde{a}^{ij}$, $b^i = \hat{b}^i + \tilde{b}^i + \sum_{j=1}^{n} \partial_j\tilde{a}^{ji}$, $c = \tilde{c} + \sum_{i=1}^{n} \partial_i\hat{b}^i$.

One usually classifies scalar linear second-order PDEs at hand of definiteness properties of the symmetric matrix $A(x) := (a^{ij}(x))_{i,j=1,2,\ldots,n} \in \mathbb{R}^{n\times n}$: The PDEs $(***)$, $(****)$ and the PDO L, respectively, are called ...

(a)
- negatively elliptic if $A(x)$ is a positive matrix (i.e. $A(x)$ has only positive eigenvalues) for all $x \in \Omega$,
- positively elliptic if $A(x)$ is a negative matrix (i.e. $A(x)$ has only negative eigenvalues) for all $x \in \Omega$,
- **elliptic** if it is either positively elliptic or negatively elliptic,
- **uniformly elliptic** if there exists a constant $\lambda \in \mathbb{R}_{>0}$ such that either $\xi \cdot A(x)\xi \geq \lambda|\xi|^2$ holds for all $x \in \Omega$, $\xi \in \mathbb{R}^n$ or $\xi \cdot A(x)\xi \leq -\lambda|\xi|^2$ holds for all $x \in \Omega$, $\xi \in \mathbb{R}^n$.

As **prototype elliptic equations** we will study the **Laplace and Poisson equations**.

(b) **parabolic** if — possibly after change of variables — the PDE takes the form

$$\frac{\partial u}{\partial t} - L(t)u = f \qquad \text{in } \Omega\,,$$

once more with time-space-split variables $(t, x') \in \Omega \subset \mathbb{R} \times \mathbb{R}^{n-1}$ and with a t-dependent 1-parameter family of negatively elliptic PDOs $L(t) = p(t, x', \partial'_x)$ which involve only derivatives with respect to x' but none with respect to t. In this case, the matrix $A(x)$ (which corresponds to the principal part $-L(t)$ of $L = \frac{\partial}{\partial t} - L(t)$) has one zero eigenvalue and $(n-1)$ negative eigenvalues.

As a **prototype parabolic equation** we will study the **heat equation**.

(c) **hyperbolic** if — possibly after change of variables — the PDE takes the form

$$\frac{\partial^2 u}{\partial t^2} + b_0 \frac{\partial u}{\partial t} - L(t)u = f \qquad \text{in } \Omega\,,$$

with coefficient b_0, time-space-split variables, and negatively elliptic PDOs $L(t) = p(t, x', \partial'_x)$ as in (2b). In this case, the matrix $A(x)$ (which corresponds to the principal part $\frac{\partial^2}{\partial t^2} - L(t)$ of $L = \frac{\partial^2}{\partial t^2} + b_0 \frac{\partial}{\partial t} - L(t)$) has one positive eigenvalue and $(n-1)$ negative eigenvalues.

As a **prototype hyperbolic equation** we will study the **wave equation**.

In the next chapters we will discuss, one by one, the prototype equations of elliptic, parabolic, and hyperbolic type.

Chapter 2

The Laplace equation and the Poisson equation

In this chapter, we investigate (scalar solutions of) the (scalar) **Laplace equation**

$$\Delta u \equiv 0$$

and the (scalar) **Poisson equation**

$$\Delta u = f$$

in an open (and often also bounded) set Ω in $\mathbb{R}^n$, $n \in \mathbb{N}_{\geq 2}$ with the previously mentioned Laplace operator $\Delta := \operatorname{div} \nabla = \sum_{i=1}^n \partial_i^2 = \operatorname{trace}(\nabla^2)$ and given a non-vanishing inhomogeneity $f \colon \Omega \to \mathbb{R}$. As discussed in the previous chapter, the equations then give rise to a uniquely solvable, well-behaved problem only if combined with a boundary condition. The simplest such conditions, which also turn out to be relevant in typical applications, are the **Dirichlet boundary condition**

$$u|_{\partial\Omega} = \varphi$$

with prescribed $\varphi \colon \partial\Omega \to \mathbb{R}$ and the **Neumann boundary condition**

$$\partial_\nu u|_{\partial\Omega} = \psi$$

with prescribed $\psi \colon \partial\Omega \to \mathbb{R}$. In the latter condition, $\nu \colon \partial\Omega \to \mathbb{R}^n$ denotes the outward unit normal field of Ω (defined only in case that $\partial\Omega$ is sufficiently smooth), and the normal derivative $\partial_\nu u(x)$ at $x \in \partial\Omega$ is nothing but the directional derivative $\partial_{\nu(x)} u(x) = \nu(x) \cdot \nabla u(x)$.

However, we postpone the detailed treatment of boundary value problems (that is the combination of PDE and boundary condition) to later sections in this chapter. First we record a basic definition and discuss specific symmetric solutions:

Definition (**harmonic function**). *Consider an open set Ω in $\mathbb{R}^n$. We say that a function $u \in \mathrm{C}^2(\Omega)$ is **(classically) harmonic** if it solves the Laplace equation in Ω, that is, if $\Delta u \equiv 0$ holds in Ω.*

In the following sections we will see examples of harmonic functions.

2.1 The fundamental solution

Here we consider a rotationally symmetric (scalar C^2) function u on $\mathbb{R}^n \setminus \{0\}$, that is, a function $u \in \mathrm{C}^2(\mathbb{R}^n \setminus \{0\})$ which satisfies

$$u(x) = g(|x|) \qquad \text{for } x \in \mathbb{R}^n \setminus \{0\}$$

with some C^2 function $g\colon (0,\infty) \to \mathbb{R}$. By a computation (to be discussed in the exercise class), for the Laplace operator on such u, we obtain the formula

$$\Delta u(x) = g''(|x|) + \frac{n-1}{|x|} g'(|x|) \qquad \text{for } x \in \mathbb{R}^n \setminus \{0\}\,.$$

If u is, in addition, harmonic on $\mathbb{R}^n \setminus \{0\}$, we infer that g satisfies the scalar linear second-order ODE

$$g''(r) + \frac{n-1}{r} g'(r) = 0 \qquad \text{for } r \in (0,\infty)\,,$$

which can also be seen as a scalar linear *first*-order ODE *for* g'. By a basic ODE formula (or alternatively by rewriting the equation as $\frac{\mathrm{d}}{\mathrm{d}r}(r^{n-1}g'(r)) = 0$), we deduce that solutions of the ODE are characterized by $g'(r) = ar^{1-n}$ for $r \in (0,\infty)$ with constant $a \in \mathbb{R}$ or equivalently by

$$g(r) = \begin{cases} cr^{2-n}+d & \text{if } n \geq 3 \\ c(\log r)+d & \text{if } n = 2 \end{cases} \qquad \text{for } r \in (0,\infty)$$

with two constants $c, d \in \mathbb{R}$ (where, here and in what follows, log always denotes the natural logarithm). Thus, we have shown that the **rotationally symmetric harmonic functions on $\mathbb{R}^n \setminus \{0\}$** are precisely the function u of the form

$$u(x) = \begin{cases} c|x|^{2-n}+d & \text{if } n \geq 3 \\ c(\log |x|)+d & \text{if } n = 2 \end{cases} \qquad \text{for } x \in \mathbb{R}^n \setminus \{0\}$$

with constants $c, d \in \mathbb{R}$. In the case $c \neq 0$ these functions exhibit an isolated singularity at the origin and cannot be extended continuously to the whole space $\mathbb{R}^n$. (Thus, as a side benefit we have also shown that the only rotationally symmetric harmonic functions on *all of* $\mathbb{R}^n$ are the constant functions obtained in case $c = 0$).

Among the rotationally symmetric harmonic functions found above one singles out a specific one by setting $d = 0$ and by a particular choice of c, which will be explained in the remark below:

Definition (fundamental solution). *The function $F\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, given by*

$$F(x) := \begin{cases} -\frac{1}{n(n-2)\omega_n}|x|^{2-n} & \text{if } n \geq 3 \\ \frac{1}{2\pi}\log|x| & \text{if } n = 2 \end{cases} \qquad \text{for } x \in \mathbb{R}^n \setminus \{0\}\,,$$

is called the ***fundamental solution of the Laplace equation*** *in $\mathbb{R}^n$. Here, $\omega_n := \mathscr{L}^n(\mathrm{B}_1)$ denotes the volume (in the sense of Lebesgue measure) of the unit ball $\mathrm{B}_1 := \{x \in \mathbb{R}^n : |x| < 1\}$.*

Remark (on the choice of the constant c)**.** In order to arrive at the definition of F, the constant $c \in \mathbb{R}$ in the preceding considerations has been fixed such that the **flux of the gradient vector field** ∇F through every sphere $\mathrm{S}_r := \{x \in \mathbb{R}^n : |x| = r\}$ with center at the origin **amounts to 1**, that is (as it will be verified in the exercise class)

$$\int_{\mathrm{S}_r} \nu \cdot \nabla F \,\mathrm{d}\mathcal{H}^{n-1} = 1 \qquad \text{for all } r \in (0,\infty)$$

with the outward unit normal field $\nu(x) = \frac{x}{|x|}$ to the ball $\mathrm{B}_r := \{x \in \mathbb{R}^n : |x| < r\}$ and the $(n{-}1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$.

In view of the divergence theorem, this formula for the flux and the fact that F is harmonic on $\mathbb{R}^n \setminus \{0\}$ can be reasonably summarized by the here-only-heuristic equation **“$\boldsymbol{\Delta F = \delta_0}$” on $\boldsymbol{\mathbb{R}^n}$** with the Dirac measure δ_0 at the origin (and in some more advanced sense this equation can actually be given a rigorous meaning). Remembering the interpretation of the Poisson equation in electrostatics, we also express this fact by saying that the **fundamental solution** yields the **electric potential** of a single **unit point charge** at the origin.

Addendum on surface measures and surface integration

The **$\boldsymbol{k}$-dimensional spherical Hausdorff measure $\boldsymbol{\mathcal{H}^k}$** assigns to every subset A of $\mathbb{R}^n$ a non-negative number which **measures the $\boldsymbol{k}$-dimensional area** of A (where the ‘k-dimensional area’ is to be thought of as length, surface area, volume, and higher-dimensional extensions for $k=1$, $k=2$, $k=3$, and $k \geq 4$, respectively). This concept differs from the more well-known Lebesgue measure insofar that the k-dimensional measurement now concerns sets A in $\mathbb{R}^n$, typically with $n > k$, and is no longer restricted to sets in the ambient space $\mathbb{R}^k$ of the same dimension. The definition of $\mathcal{H}^k$ (which, by the way, makes sense even for subsets A of an arbitrary metric space) rests on coverings with sets of arbitrarily small diameter and reads as follows:

Definition (**Hausdorff measure**)**.** *For $n \in \mathbb{N}$ and $k \in [0,\infty)$, the k-dimensional Hausdorff measure $\mathcal{H}^k$ is the set function on the power set of $\mathbb{R}^n$ given, for arbitrary $A \subset \mathbb{R}^n$, by*

$$\mathcal{H}^k(A) := \lim_{\delta \searrow 0} \left(\inf \left\{ \sum_{i=1}^\infty \omega_k \left(\frac{\operatorname{diam} C_i}{2} \right)^k \,:\, C_i \subset \mathbb{R}^n \,,\, A \subset \bigcup_{i=1}^\infty C_i \,,\, \operatorname{diam} C_i < \delta \right\} \right) \in [0,\infty]\,,$$

where $\operatorname{diam} C := \sup_{x,y \in C} |y-x|$ denotes the diameter of a set $C \subset \mathbb{R}^n$ (with understanding $\operatorname{diam} \emptyset = 0$) and where we rely on the convention $\omega_k := \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}$ (which involves the Γ-function and reduces for $k \in \mathbb{N}$ to the earlier definition of ω_k as volume of the k-dimensional unit ball).

In order to understand this definition one should think of $\omega_k \big(\frac{\operatorname{diam} C_i}{2}\big)^k$ as the **area of a $\boldsymbol{k}$-dimensional disc** with the same diameter as C_i. With this interpretation it then becomes plausible that $\sum_{i=1}^\infty \omega_k r_i^k$ may yield a good approximation to the k-dimensional area of A — at least in those cases where the above infimum is almost attained with each intersection $C_i \cap A$ being similar to a disc with the same small diameter as C_i itself.

Properties (of Hausdorff measures $\mathcal{H}^k$ on $\mathbb{R}^n$)**.** We record, for $n \in \mathbb{N}$, $k \in [0,\infty)$, in brief summary and without proofs:

- **σ-subadditivity:** $\mathcal{H}^k(\bigcup_{i=1}^\infty A_i) \le \sum_{i=1}^\infty \mathcal{H}^k(A_i)$ for arbitrary $A_i \subset \mathbb{R}^n$.
- **σ-additivity:** $\mathcal{H}^k(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mathcal{H}^k(A_i)$ for *disjoint* Borel sets $A_i \subset \mathbb{R}^n$.
- special cases:
  - $\mathcal{H}^0$ is the counting measure, i.e. $\mathcal{H}^0(A)$ is the possibly infinite number of elements of A,
  - $\mathcal{H}^n$ equals the Lebesgue measure $\mathscr{L}^n$ on $\mathbb{R}^n$,
  - $\mathcal{H}^k$ with $k > n$ vanishes on all subsets of $\mathbb{R}^n$.
- **measures of balls and spheres:**
  - $\mathcal{H}^k(\mathrm{B}_r^V(x)) = \omega_k r^k$ for a ball $\mathrm{B}_r^V(x) = \{y \in V : |y-x| < r\}$ in a k-dimensional affine subspace V of $\mathbb{R}^n$ with radius $r \in \mathbb{R}_{>0}$ and center $x \in V$ *in this subspace* (specifically of course $\mathscr{L}^n(\mathrm{B}_r(x)) = \omega_n r^n$ for $\mathrm{B}_r(x) := \mathrm{B}_r^n(x) := \mathrm{B}_r^{\mathbb{R}^n}(x)$),
  - $\mathcal{H}^{n-1}(\mathrm{S}_r(x)) = n\omega_n r^{n-1}$ for a sphere $\mathrm{S}_r(x) = \{y \in \mathbb{R}^n : |y-x| = r\}$ in $\mathbb{R}^n$ with radius $r \in \mathbb{R}_{>0}$ and center $x \in \mathbb{R}^n$.
- **scaling:** $\mathcal{H}^k(rA) = r^k\mathcal{H}^k(A)$ for all $A \subset \mathbb{R}^n$ and $r \in \mathbb{R}_{>0}$.
- **translation invariance and rotation/orthogonal invariance:** $\mathcal{H}^k(x+T(A)) = \mathcal{H}^k(A)$ for all $A \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $T \in \mathcal{O}(\mathbb{R}^n)$.
- **Lipschitz bound:** $\mathcal{H}^k(f(A)) \le L^k\mathcal{H}^k(A)$ for $A \subset \mathbb{R}^n$ and a Lipschitz map $f\colon A \to \mathbb{R}^N$ with Lipschitz constant $\le L$.
- **$\mathcal{H}^k$-integration** is a special case of the integration with respect to arbitrary measures and yields a notation of (unoriented) **k-dimensional surface integrals**.
- For a Borel set $A \subset \mathbb{R}^k$, a one-to-one mapping $T \in \mathrm{C}^1(U,\mathbb{R}^n)$ on a neighborhood U of A in $\mathbb{R}^k$, and a Borel function $g\colon T(A) \to \mathbb{R}^N$, the **area formula** asserts

  $$\int_{T(A)} g \,\mathrm{d}\mathcal{H}^k = \int_A (g \circ T)\,\mathrm{J}T\,\mathrm{d}x$$

  with the Jacobian $\mathrm{J}T(x) := \sqrt{\det(\mathrm{D}T(x)^*\mathrm{D}T(x))}$. This formula reduces the computation of $\mathcal{H}^k$-integrals and, in the case $g \equiv 1$, of $\mathcal{H}^k$-measures to the computation of volume integrals (i.e. integrals with respect to the Lebesgue measure) for which many standard tools are available. In concrete situations, the formula is applied after parametrizing a k-dimensional surface or parts thereof in the form $T(A)$.
- For *every* $A \subset \mathbb{R}^n$ there exists a uniquely determined $d \in [0,n]$ such that $\mathcal{H}^k(A) = \infty$ holds for all $k \in [0,d)$ and $\mathcal{H}^k(A) = 0$ holds for all $k \in (d,n]$ (while $\mathcal{H}^d(A)$ can be zero, a finite positive number, or infinity). This number d is called the **Hausdorff dimension** of A and is compatible with usual intuitive ideas of dimension for 'nice' sets A. Still it is well known (and not even too difficult to prove) that d can be non-integer (!) and that in fact, for every $n \in \mathbb{N}$ and every $d \in [0,n]$, there exists a set $A \subset [0,1]^n$ of Hausdorff dimension d.

2.2 Harmonic polynomials

Some **basic examples of harmonic polynomials** in $x \in \mathbb{R}^n$ are ...

- all **affine functions** $\sum_{i=1}^n b^i x_i + c$ with $b^i, c \in \mathbb{R}$ (or, in other words, all polynomials of degree ≤ 1),
- the degree-two polynomials $x_i x_j$ and $x_i^2 - x_j^2$ with $i \neq j$ in $\{1, 2, \ldots, n\}$,
- the degree-three polynomial $x_i^3 - 3x_i x_j^2$ with $i \neq j$ in $\{1, 2, \ldots, n\}$.

Here, harmonicity can be checked easily by direct computation of the Laplacian. Similar examples of higher degree can also be given; compare with the exercises.

Definition (spaces of homogeneous polynomials)**.** *For $\alpha \in \mathbb{N}_0^n$, we write p_α for the mononomial given by $p_\alpha(x) := x^\alpha$. For $k \in \mathbb{N}_0$, we then introduce the space*

$$\mathcal{P}_k := \left\{ \sum_{|\alpha|=k} c_\alpha p_\alpha \,:\, c_\alpha \in \mathbb{R} \right\}$$

of degree-k homogeneous polynomials on $\mathbb{R}^n$ and the space

$$\mathcal{H}_k := \{ h \in \mathcal{P}_k \,:\, \Delta h \equiv 0 \text{ on } \mathbb{R}^n \}$$

of degree-k homogeneous harmonic polynomials on $\mathbb{R}^n$. Moreover, for $k \in \mathbb{N}_{\geq 2}$, we set

$$\mathcal{Q}_k := \{ q \in \mathcal{P}_k \,:\, q(x) = |x|^2 p(x) \text{ for all } x \in \mathbb{R}^n \text{ with some } p \in \mathcal{P}_{k-2} \} \,,$$

and we agree on the convention $\mathcal{Q}_1 := \mathcal{Q}_0 := \{0\}$.

Theorem. *For all $k \in \mathbb{N}_0$, we have $\mathcal{P}_k = \mathcal{H}_k \oplus \mathcal{Q}_k$ and* $\dim \mathcal{H}_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$.

The *proof* will be treated in the exercise class. □

Corollary (**solvability of the Dirichlet problem**; case of **polynomial boundary data**)**.** *Every polynomial on $\mathbb{R}^n$ coincides at S_1 with a harmonic polynomial. Equivalently, whenever the **boundary datum** φ is (the restriction to S_1 of) a **polynomial** on $\mathbb{R}^n$, then the **Dirichlet problem for harmonic functions***

$$\begin{aligned} \Delta h &\equiv 0 \qquad \text{in } \mathrm{B}_1 \,, \\ h &= \varphi \qquad \text{at } \mathrm{S}_1 \end{aligned}$$

***posseses a** (polynomial) **solution** h.*

Proof. The corollary is proved by induction on the degree k of the polynomial φ: In the case $k \leq 1$ the claim holds trivially with $h = \varphi$, since φ itself is harmonic. For the inductive step, we consider a polynomial φ of degree $k \in \mathbb{N}_{\geq 2}$ and write $\varphi = p + \widetilde{\varphi}$ with $p \in \mathcal{P}_k$ and a polynomial $\widetilde{\varphi}$ of degree $\leq k-1$. By the preceding theorem we can further decompose $p(x) = h(x) + |x|^2 \widetilde{p}(x)$ for $x \in \mathbb{R}^n$ with $h \in \mathcal{H}_k$ and $\widetilde{p} \in \mathcal{P}_{k-2}$. As $\widetilde{p} + \widetilde{\varphi}$ has degree $\leq k-1$, the inductive hypothesis yields a harmonic polynomial $\widetilde{h}$ on $\mathbb{R}^n$ which coincides with $\widetilde{p} + \widetilde{\varphi}$ at S_1. For $x \in \mathrm{S}_1$, that is $|x| = 1$, we now observe

$$h(x) + \widetilde{h}(x) = h(x) + \widetilde{p}(x) + \widetilde{\varphi}(x) = h(x) + |x|^2 \widetilde{p}(x) + \widetilde{\varphi}(x) = p(x) + \widetilde{\varphi}(x) = \varphi(x) \,.$$

Thus, the harmonic polynomial $h + \widetilde{h}$ coincides with φ at S_1, and the induction is complete. □

Remark. The hypotheses that the **boundary datum** be **polynomial** is **by no means necessary for existence** of solutions to the Dirichlet problem. In fact, we will soon extend the above result to general (continuous) boundary data φ.

2.3 Consequences of the divergence theorem

Definition (**Gauss-Green domain**)**.** *For us, a domain is a non-empty, open, connected set. We call a bounded domain G in $\mathbb{R}^n$ a Gauss-Green domain if we have $\mathcal{H}^{n-1}(\partial G) < \infty$ and there exists a Borel unit vector field ν_G on ∂G, then called outward unit normal field to G, such that the* ***divergence theorem***

$$\int_G \operatorname{div} V \,\mathrm{d}x = \int_{\partial G} V \cdot \nu_G \,\mathrm{d}\mathcal{H}^{n-1}$$

holds for all $V \in \mathrm{C}^1(G,\mathbb{R}^n) \cap C^0(\overline{G},\mathbb{R}^n)$ such that the left-hand integral exists in $[-\infty,\infty]$. (Here, the last-mentioned assumption means that either $(\operatorname{div} V)_+$ or $(\operatorname{div} V)_-$ has to be in $\mathrm{L}^1(G)$. If *the divergence theorem holds, then the right-hand integral exists in $\mathbb{R}$ and thus a posteriori $\operatorname{div} V$ itself and therefore both $(\operatorname{div} V)_+$ and $(\operatorname{div} V)_-$ are in $\mathrm{L}^1(G)$ for all V admissible here.)*

Remarks (on Gauss-Green domains)**.**

(1) A standard version of the divergence theorem applies on bounded C^1 domains G (for fields V as above). Thus, bounded C^1 domains are Gauss-Green domains in the preceding sense.

(2) The divergence theorem is also valid on cubes, cuboids, half-balls, triangles, and similar domains with corners or cusps. Thus, these non-smooth domains are Gauss-Green domains as well.

(3) All domains mentioned so far are also contained in a general class of Gauss-Green domains, sufficient for most purposes, namely the bounded domains G with $\mathcal{H}^{n-1}(\partial G) < \infty$ which are **$\mathbf{C}^1$-smooth near $\mathcal{H}^{n-1}$-*almost* every boundary point** in ∂G.

The next result is closely related to the divergence theorem, but is indeed valid without any smoothness assumption on the underlying domain Ω:

Lemma (on **integration by parts**)**.** *Consider an open set Ω in $\mathbb{R}^n$ and $u, v \in \mathrm{C}^1(\Omega)$ such that either $\operatorname{spt} u$ or $\operatorname{spt} v$ is a compact subset of Ω. Then, for $i = 1, 2, \ldots, n$, we have*

$$\int_\Omega (\partial_i u)\, v \,\mathrm{d}x = -\int_\Omega u\, (\partial_i v) \,\mathrm{d}x\,.$$

Definition (**support**)**.** *We define the support $\operatorname{spt} u$ of a function $u\colon \Omega \to \mathbb{R}^N$ on $\Omega \subset \mathbb{R}^n$ as the closure of $\{x \in \Omega : u(x) \neq 0\}$* ***in $\mathbb{R}^n$***.

Proof of the lemma. We assume, without loss of generality, that $K := \operatorname{spt} u$ is compact in Ω and choose a Gauss-Green domain (e.g. a large ball) G such that $K \subset G$. Then, by taking $V := uve_i$ on $\overline{G} \cap \Omega$ and $V := 0$ on $\overline{G} \setminus K$ we obtain a well-defined vector field $V \in \mathrm{C}^1(\overline{G})$, and we get

$$\int_\Omega \partial_i(uv) \,\mathrm{d}x = \int_G \operatorname{div} V \,\mathrm{d}x = \int_{\partial G} V \cdot \nu_G \,\mathrm{d}\mathcal{H}^{n-1} = 0$$

The claim then follows by using $\partial_i(uv) = (\partial_i u)\, v + u\, (\partial_i v)$ and by rearranging terms. □

Applications (of the divergence theorem)**.** In what follows we suppose that G is a Gauss-Green domain with outward unit normal field $\nu = \nu_G$.

(1) The divergence theorem for the vector field $v\nabla u$ gives **Green's first identity**

$$\int_G \nabla u \cdot \nabla v \,\mathrm{d}x + \int_G v \Delta u \,\mathrm{d}x = \int_{\partial G} v\, \partial_\nu u \,\mathrm{d}\mathcal{H}^{n-1}$$

for $u \in \mathrm{C}^2(G) \cap \mathrm{C}^1(\overline{G})$ and $v \in \mathrm{C}^1(\overline{G})$ such that $\int_G v\Delta u \,\mathrm{d}x$ exists in $[-\infty,\infty]$.

(2) Specifically **for harmonic functions** $u \in \mathrm{C}^2(G) \cap \mathrm{C}^1(\overline{G})$, we have the conclusions

$$\int_{\partial G} \partial_\nu u \,\mathrm{d}\mathcal{H}^{n-1} = 0 \qquad \text{and} \qquad \int_{\partial G} u\,\partial_\nu u \,\mathrm{d}\mathcal{H}^{n-1} = \int_G |\nabla u|^2 \,\mathrm{d}x \geq 0\,,$$

which follow by choosing $v \equiv 1$ and $v = u$ in Green's first identity.

(3) As important consequences of Green's first identity we obtain the following twin uniqueness theorems:

Theorem (**uniqueness** of solutions to the **Dirichlet problem for Poisson's equation**). *For each $f \in \mathrm{C}^0(G)$ and each $\varphi \in \mathrm{C}^0(\partial G)$, the Dirichlet problem for Poisson's equation*

$$\begin{aligned} \Delta u &\equiv f \quad && \text{on } G\,, \\ u &= \varphi \quad && \text{on } \partial G \end{aligned}$$

has at most one solution $u \in \mathrm{C}^2(G) \cap \mathrm{C}^1(\overline{G})$.

It will become evident in Section 2.4 that this first uniqueness statement remains also valid for slightly less regular solutions $u \in \mathrm{C}^2(G) \cap \mathrm{C}^0(\overline{G})$.

Theorem (**uniqueness** of solutions to the **Neumann problem for Poisson's equation**). *For each $f \in \mathrm{C}^0(G)$ and each $\psi \in \mathrm{C}^0(\partial G)$, solutions $u \in \mathrm{C}^2(G) \cap \mathrm{C}^1(\overline{G})$ to the Neumann problem for Poisson's equation*

$$\begin{aligned} \Delta u &\equiv f \quad && \text{on } G\,, \\ \partial_\nu u &= \psi \quad && \text{on } \partial G \end{aligned}$$

are unique up to additive constants.

Proof of both theorems. Consider two solutions u_1 and u_2 of the respective problem. Then $h := u_1 - u_2$ is harmonic on G with $h|_{\partial G} \equiv 0$ and $\partial_\nu h|_{\partial G} \equiv 0$, respectively. By the second identity in (2), we infer

$$\int_G |\nabla h|^2 \,\mathrm{d}x = \int_{\partial G} h\,\partial_\nu h \,\mathrm{d}\mathcal{H}^{n-1} = 0\,.$$

Thus, ∇h vanishes everywhere on G, and $u_1 - u_2 = h$ is constant on $\overline{G}$. In the Neumann case this completes the proof. In the Dirichlet case, taking into account $h|_{\partial G} \equiv 0$, it even follows that the constant is zero and u_1 equals u_2. □

(4) The divergence theorem for the vector field $v\nabla u - u\nabla v$ gives **Green's second identity**

$$\int_G (v\Delta u - u\Delta v) \,\mathrm{d}x = \int_{\partial G} (v\,\partial_\nu u - u\,\partial_\nu v) \,\mathrm{d}\mathcal{H}^{n-1}$$

for $u, v \in \mathrm{C}^2(G) \cap \mathrm{C}^1(\overline{G})$ such that the left-hand integral exists in $[-\infty, \infty]$.

2.4 The mean value property and the maximum principle

Notation. *Consider a measure space* $(\Omega,\mathscr{A},\mu)$*. Then, for* $A\in\mathscr{A}$ *with* $0<\mu(A)<\infty$ *and* $h\in \mathrm{L}^1_\mu(A,\mathbb{R}^N)$*, we call*

$$\fint_A h\,\mathrm{d}\mu := \frac{1}{\mu(A)}\int_A h\,\mathrm{d}\mu \in \mathbb{R}^N$$

the mean value (integral) or the integral mean of h *on* A*.*

Theorem (mean value property). *Consider an open set* $\Omega\subset\mathbb{R}^n$*, a harmonic function* $h\in \mathrm{C}^2(\Omega)$*, and an arbitrary ball* $\overline{\mathrm{B}_r(a)}\subset\Omega$*. Then, for the mean values on the ball* $\mathrm{B}_r(a)=\{x\in\mathbb{R}^n : |x{-}a|<r\}$ *and the sphere* $\mathrm{S}_r(a)=\partial\mathrm{B}_r(a)=\{x\in\mathbb{R}^n : |x{-}a|=r\}$*, we have*

$$h(a)=\fint_{\mathrm{B}_r(a)} h\,\mathrm{d}x=\fint_{\mathrm{S}_r(a)} h\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

Proof. For arbitrary $\varrho\in(0,r)$, by the change of variables $x=a{+}\varrho\omega$ and the corresponding integral transformation (which in turn follows from the invariance and scaling properties of the Hausdorff measure) we have

$$\fint_{\mathrm{S}_\varrho(a)} h(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)=\fint_{\mathrm{S}_1} h(a{+}\varrho\omega)\,\mathrm{d}\mathcal{H}^{n-1}(\omega)\,.$$

By differentiation of this equation, exchange of derivative and integral (here justified, since ∇h is locally bounded on Ω), and the chain rule we then infer

$$\frac{\mathrm{d}}{\mathrm{d}\varrho}\fint_{\mathrm{S}_\varrho(a)} h(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)=\fint_{\mathrm{S}_1}\frac{\mathrm{d}}{\mathrm{d}\varrho}h(a{+}\varrho\omega)\,\mathrm{d}\mathcal{H}^{n-1}(\omega)=\fint_{\mathrm{S}_1}\omega\cdot\nabla h(a{+}\varrho\omega)\,\mathrm{d}\mathcal{H}^{n-1}(\omega)\,.$$

Moreover, by the reverse change of variables, the divergence theorem, and the harmonicity of h we deduce

$$\fint_{\mathrm{S}_1}\omega\cdot\nabla h(a{+}\varrho\omega)\,\mathrm{d}\mathcal{H}^{n-1}(\omega)=\fint_{\mathrm{S}_\varrho(a)}\frac{x{-}a}{\varrho}\cdot\nabla h(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)=\frac{1}{\mathcal{H}^{n-1}(\mathrm{S}_\varrho(a))}\int_{\mathrm{B}_\varrho(a)}\Delta h(x)\,\mathrm{d}x=0\,.$$

Combining the last two chains of equations, we can conclude that the continuous mapping $\varrho\mapsto\fint_{\mathrm{S}_\varrho(a)}h\,\mathrm{d}\mathcal{H}^{n-1}$ has zero derivative on $(0,r)$ and thus is constant on $(0,r]$. In addition, continuity of h at a implies $\big|\fint_{\mathrm{S}_\varrho(a)}h\,\mathrm{d}\mathcal{H}^{n-1}{-}h(a)\big|\le\sup_{\mathrm{S}_\varrho(a)}|h{-}h(a)|\xrightarrow[\varrho\searrow 0]{}0$ and hence

$$\lim_{\varrho\searrow 0}\fint_{\mathrm{S}_\varrho(a)}h\,\mathrm{d}\mathcal{H}^{n-1}=h(a)\,.$$

Therefore, the constant value of $\varrho\mapsto\fint_{\mathrm{S}_\varrho(a)}h\,\mathrm{d}\mathcal{H}^{n-1}$ is indeed equal to $h(a)$, and the claim is verified for *spherical* means.

With the help of spherical coordinates the mean value property on balls can now be deduced as follows:

$$\begin{aligned}\fint_{\mathrm{B}_r(a)}h\,\mathrm{d}x&=\frac{1}{\omega_n r^n}\int_0^r\int_{\mathrm{S}_\varrho(a)}h\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}\varrho\\&=\frac{1}{\omega_n r^n}\int_0^r n\omega_n\varrho^{n-1}h(a)\,\mathrm{d}\varrho=\frac{n}{r^n}\int_0^r\varrho^{n-1}\,\mathrm{d}\varrho\,h(a)=h(a)\,.\end{aligned}$$

This completes the proof. □

Remarks and Definitions. Consider an open set Ω in $\mathbb{R}^n$.

(1) A function $u \in \mathrm{C}^2(\Omega)$ is called ***sub*harmonic**[1] on Ω if $\Delta u \geq 0$ holds on Ω.

For subharmonic u on Ω and $\overline{\mathrm{B}_r(a)} \subset \Omega$, an inspection of the above proof reveals that the mean values on both $\mathrm{B}_\varrho(a)$ and $\mathrm{S}_\varrho(a)$ are non-decreasing functions of $\varrho \in (0, r]$ and that the **mean value *in*equality**

$$u(a) \leq \fint_{\mathrm{B}_r(a)} u \,\mathrm{d}x \leq \fint_{\mathrm{S}_r(a)} u \,\mathrm{d}\mathcal{H}^{n-1}$$

is valid. If additionally it is $\Delta u(a) > 0$, the mean values are even strictly increasing functions, and also the mean value inequality holds in the strict form

$$u(a) < \fint_{\mathrm{B}_r(a)} u \,\mathrm{d}x < \fint_{\mathrm{S}_r(a)} u \,\mathrm{d}\mathcal{H}^{n-1}\,.$$

In the same way, ***super*harmonic** functions u are defined by the inequality $\Delta u \leq 0$ and satisfy the reverse mean value inequality.

(2) In fact, the respective form of mean value inequality even characterizes sub- and superharmonic functions, respectively, i.e. the converse to the assertions in (1) also holds. For the subharmonic case this spells out as follows: If we have $u \in \mathrm{C}^2(\Omega)$ and either $u(a) \leq \fint_{\mathrm{B}_r(a)} u \,\mathrm{d}x$ or $u(a) \leq \fint_{\mathrm{S}_r(a)} u \,\mathrm{d}\mathcal{H}^{n-1} = u(a)$ holds for every ball $\overline{\mathrm{B}_r(a)} \subset \Omega$, then u is subharmonic. Clearly, the combination of the assertions on sub- and superharmonicity implies that the **mean value property characterizes harmonic functions**.

Proof of the statement for the subharmonic case. Assumme that the statement is false, that is $u(a) \leq \fint_{\mathrm{B}_r(a)} u \,\mathrm{d}x$ or $u(a) \leq \fint_{\mathrm{S}_r(a)} u \,\mathrm{d}\mathcal{H}^{n-1}$ for every ball $\overline{\mathrm{B}_r(a)} \subset \Omega$, but still $\Delta u(x_0) < 0$ for some $x_0 \in \Omega$. Then, by continuity of Δu, we have $\Delta u \leq 0$ on $\mathrm{B}_{2\delta}(x_0) \subset \Omega$ for some sufficiently small $\delta > 0$. Hence, u is *super*harmonic in $\mathrm{B}_{2\delta}(x_0)$ with $\Delta u(x_0) < 0$, and (1) yields $u(x_0) > \fint_{\mathrm{B}_\delta(x_0)} u \,\mathrm{d}x > \fint_{\mathrm{S}_\delta(x_0)} u \,\mathrm{d}\mathcal{H}^{n-1}$. This contradicts the initial assumption on the mean values and thus completes the proof of the claim. □

Theorem (weak maximum principle). *Consider a bounded open set Ω in $\mathbb{R}^n$ and a **subharmonic** function $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$. Then we have the bound*

$$u \leq \max_{\partial\Omega} u \qquad \text{in } \Omega$$

(*or, clearly equivalent,* $\sup_\Omega u \leq \max_{\partial\Omega} u$).

Theorem (strong maximum principle). *Consider a domain Ω in $\mathbb{R}^n$. If a **subharmonic** function $u \in \mathrm{C}^2(\Omega)$ **attains its** global **maximum** in Ω, **then** u is **constant** on Ω.*

Remarks (on the maximum principles)**.**

(1) Roughly speaking, the weak maximum principle asserts that the maximum is attained at the boundary, and the strong maximum principle asserts that it is attained *only* at the boundary (apart from the case of constants).

[1] The reason for this choice of terminology will be explained later in this section; see Remark (2) on the comparison principle.

(2) Boundedness of Ω is essential for the above form of the weak maximum principle (but compare with the later remarks on Phragmén-Lindelöf principles). Connectedness of Ω is essential for the strong maximum insofar that otherwise u would merely need to be constant *on those connected components* of Ω where the maximum is attained.

(3) Clearly, for ***super*****harmonic** functions u, the weak **minimum principle** $u \geq \min_{\partial\Omega} u$ on Ω and the analogous strong minimum principle hold true. Specifically, **for harmonic functions** h, both **maximum and minimum principles** hold, and particularly this implies the maximum modulus estimate $|h| \leq \max_{\partial\Omega} |h|$ for harmonic h on bounded Ω.

1st proof of the weak maximum principle. The boundedness of Ω implies that $\partial\Omega$ is compact and $\max_{\partial\Omega} u \in \mathbb{R}$ exists. We fix an arbitrary $M \in \mathbb{R}$ with $M > \max_{\partial\Omega} u$ and introduce the auxiliary function $v := (u-M)_+^2$ (with the usual abbreviation $f_+ := \max\{f, 0\}$). Since v is the composition of $u-M$ and the C^1 function $x \mapsto x_+^2$ on the real line, the chain rule gives $v \in \mathrm{C}^1(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ with $\nabla v = 2(u-M)\nabla u$ on $\{u \geq M\}$ and $\nabla v \equiv 0$ on $\{u \leq M\}$. Moreover, the definition of v and the choice of M imply $\operatorname{spt} v \subset \{x \in \overline{\Omega} : u(x) \geq M\} \subset \Omega$. Using boundedness of Ω once more, we deduce that $\operatorname{spt} v$ and $\{u \geq M\}$ are compact subsets of Ω. All in all, using v 'as a test function' for the subharmonicity of u and integrating by parts, we arrive at

$$0 \leq \int_\Omega v \Delta u \,\mathrm{d}x = -\int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}x = -2 \int_{\{u>M\}} (u-M)|\nabla u|^2 \,\mathrm{d}x \,.$$

From the resulting inequality we conclude $\nabla u \equiv 0$ in $\{u > M\}$, and hence u equals some constant $> M$ in every connected component of the open set $\{u > M\}$. However, each such component, as it is also open and contained in a compact subset of Ω, posses boundary points in which the value of u is $\leq M$. Hence, the existence of any connected component would lead to discontinuity of u at its boundary and would thus result in a contradiction. This leaves $\{u > M\} = \emptyset$ as sole possibility and yields $u \leq M$ in Ω. Finally, sending $M \searrow \max_{\partial\Omega} u$, we arrive at the claim. □

2nd proof of the weak maximum principle. We first assume that even $\Delta u > 0$ holds in Ω and prove that there is no maximum point for u in Ω (that is, in the case $\Delta u > 0$ we even prove the strong maximum principle). Indeed, if $x_0 \in \Omega$ is such a maximum point, the well-known second-order necessary criterion for extremal points asserts that the Hessian $\nabla^2 u(x_0)$ is semi-negative, i.e. has only eigenvalues ≤ 0, and in conclusion we get $\Delta u(x_0) = \operatorname{trace}(\nabla^2 u(x_0)) \leq 0$. This contradicts the initial assumption and proves the absence of maximum points. Under the assumption that Ω is bounded, however, u does posses a maximum on the compactum $\overline{\Omega}$, and thus we have shown $u < \max_{\partial\Omega} u$ in Ω.

Now we merely assume that u is subharmonic. For arbitrary positive ε, we introduce an auxiliary function u_ε by $u_\varepsilon(x) := u(x)+\varepsilon|x|^2$ for $x \in \Omega$ and record $\Delta u_\varepsilon = \Delta u + 2n\varepsilon \geq 2n\varepsilon > 0$. Thus, the first part of the reasoning applies to u_ε and yields $u_\varepsilon < \max_{\partial\Omega} u_\varepsilon$ on Ω. Using $u \leq u_\varepsilon$ on the left-hand side of this estimate and writing out the definition of u_ε on its right-hand side, we arrive at

$$u < \max_{x\in\partial\Omega} \left[u(x)+\varepsilon|x|^2\right] \leq \max_{\partial\Omega} u + \varepsilon \max_{x\in\partial\Omega} |x|^2 \qquad \text{on } \Omega \,.$$

Taking into account the boundedness of Ω, we have $\max_{x\in\partial\Omega} |x|^2 < \infty$, and sending $\varepsilon \searrow 0$ we can conclude $u \leq \max_{\partial\Omega} u$ on Ω. □

Proof of the strong maximum principle. We set $M := \sup_\Omega u$. By assumption, we have $M \in \mathbb{R}$ and $\{u = M\} \neq \emptyset$. Moreover, $\{u = M\}$ is closed in Ω. Next we demonstrate that it is also open. Indeed, for $a \in \{u = M\}$, we fix a positive radius r with $\overline{\mathrm{B}_r(a)} \subset \Omega$. Then, by the choice of a and the mean value inequality for the subharmonic function u, we get

$$M = u(a) \le \fint_{\mathrm{B}_r(a)} u \,\mathrm{d}x\,,$$

but by the choice of M we also know $u \le M$ in $\mathrm{B}_r(a)$. This is only possible if $u \equiv M$ holds in the whole ball $\mathrm{B}_r(a)$ and we thus have $\mathrm{B}_r(a) \subset \{u = M\}$. All in all, the set $\{u = M\}$ is non-empty, open, and closed in Ω. Since Ω is a domain and thus connected this leaves $\{u = M\} = \Omega$ as the only possibility. We have thus shown that u is constant with value M on Ω. □

Corollary (**refined uniqueness statement** for the Dirichlet problem)**.** *The uniqueness statement in Section 2.3, Remark (3) for the Dirichlet problem to Poisson's equation remains valid on an arbitrary bounded open set Ω in $\mathbb{R}^n$ (which replaces the Gauss-Green domain G) and for solutions $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ which are merely continuous at the boundary.*

Proof. Given two solution u_1 and u_2 to the Dirichlet problem, the weak maximum principle applies to the harmonic function $u_1{-}u_2$ and yields $u_1{-}u_2 \le \max_{\partial\Omega}(u_1{-}u_2) = 0$, thus $u_1 \le u_2$ on Ω. Exchanging the roles of u_1 and u_2, we also get $u_2 \le u_1$ on Ω. Hence u_1 and u_2 coincide. □

Corollary (**continuous dependence** for the Dirichlet problem)**.** *Consider a bounded open set Ω in $\mathbb{R}^n$, and define ℓ as the maximum width of Ω in the sense of the smallest number $\ell \in (0,\infty)$ such $\Omega \subset \big\{x \in \mathbb{R}^n : |v\cdot(x{-}a)| \le \frac12\ell\big\}$ holds for some point $a \in \mathbb{R}^n$ and some unit vector $v \in \mathbb{R}^n$. If $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ solves the Dirichlet problem*

$$\Delta u = f \;\text{ in } \Omega\,, \qquad\qquad u = \varphi \;\text{ at } \partial\Omega$$

and $\widetilde{u} \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ solves the Dirichlet problem

$$\Delta \widetilde{u} = \widetilde{f} \;\text{ in } \Omega\,, \qquad\qquad \widetilde{u} = \widetilde{\varphi} \;\text{ at } \partial\Omega\,,$$

then we have the estimate

$$\max_{\overline{\Omega}} |\widetilde{u}{-}u| \le \max_{\partial\Omega} |\widetilde{\varphi}{-}\varphi| + \tfrac18 \ell^2 \sup_{\Omega} |\widetilde{f}{-}f|\,.$$

Proof. Taking into account linearity of the Laplace operator Δ, we can assume $\widetilde{u} \equiv 0$, $\widetilde{f} \equiv 0$, $\widetilde{\varphi} \equiv 0$. Moreover, it can be checked that Δ is invariant under translations and rotations, and thus we can also assume $a = 0$, $v = \mathrm{e}_1$, that is $\Omega \subset \big({-\frac12}\ell, \frac12\ell\big) {\times} \mathbb{R}^{n-1}$. We now abbreviate $M \coloneqq \sup_\Omega |f|$ and set $w(x) \coloneqq u(x){+}\frac12 M x_1^2$. Then, in view of $\Delta w = \Delta u{+}M = f{+}M \ge 0$ in Ω, we have that w is subharmonic on Ω. By the weak maximum principle, together with the choices of w and ℓ, we get

$$\max_{\overline{\Omega}} u \le \max_{\overline{\Omega}} w \le \max_{\partial\Omega} w \le \max_{\partial\Omega} u + \tfrac12 M \max_{|x_1| \le \frac12\ell} x_1^2 = \max_{\partial\Omega} \varphi + \tfrac18 \ell^2 M\,.$$

Applying the same reasoning to $-u$ (and relying on $-f{+}M \ge 0$), we also get

$$\min_{\overline{\Omega}} u \ge \min_{\partial\Omega} \varphi - \frac18 \ell^2 M\,.$$

In conclusion we arrive at

$$\max_{\overline{\Omega}} |u| \le \max_{\partial\Omega} |\varphi| + \frac18 \ell^2 M\,.$$

This is the claim. □

Corollary (comparison principle). *Consider a bounded open set Ω in $\mathbb{R}^n$ and $u,v \in \mathrm{C}^2(\Omega)\cap \mathrm{C}^0(\overline{\Omega})$. Then, the inequalities*

$$\Delta u \ge \Delta v \text{ in } \Omega\,, \qquad\qquad u \le v \text{ at } \partial\Omega$$

imply the inequality

$$u \le v \text{ even on } \overline{\Omega}\,.$$

Proof. From $\Delta(u{-}v) = \Delta u{-}\Delta v \ge 0$ in Ω we see that $u{-}v$ is subharmonic in Ω. By the weak maximum principle we infer $u{-}v \le \max_{\partial\Omega}(u{-}v) \le 0$ and thus $u \le v$ in Ω. □

Remarks (on the comparison principle)**.**

(1) Clearly, the assumption $\Delta u \ge \Delta v$ in Ω is satisfied if u is subharmonic and v superharmonic in Ω. This is the case in typical applications of the comparison principle. Often one of the two functions is even harmonic.

(2) For a subharmonic function u in Ω the comparison principle guarantees $\boldsymbol{u \le h}$ **in $\boldsymbol{\Omega}$ for every harmonic function $\boldsymbol{h}$ which coincides with $\boldsymbol{u}$ at $\boldsymbol{\partial\Omega}$.** In view of this property the introduction of the **term 'subharmonic'** indeed **makes sense**.

Remarks (on **Phragmén-Lindelöf principles**)**.** Here we always consider an open set Ω in $\mathbb{R}^n$ and functions $u \in \mathrm{C}^2(\Omega)$.

(1) In general, **on *unbounded* $\boldsymbol{\Omega}$ the weak maximum principle does not hold** in the form of the preceding theorem. A very basic counterexample is given by the unbounded harmonic function $u(x) := x_1$ on the half-space $(0,\infty)\times\mathbb{R}^{n-1}$ with zero boundary values on $\partial((0,\infty)\times\mathbb{R}^{n-1}) = \{0\}\times\mathbb{R}^{n-1}$.

However, if we regard the point $\infty_{\mathbb{R}^n}$ of infinite distance as an additional boundary point, the weak maximum principle stays valid in the following form:

$$\left.\begin{array}{c} u \text{ subharmonic in } \Omega\,, \\ \limsup\limits_{\Omega\ni x\to a} u(x) \le M \text{ at all } a \in \partial\Omega\cup\{\infty_{\mathbb{R}^n}\} \end{array}\right\} \implies u \le M \text{ in } \Omega\,. \qquad (*)$$

Proof. Assuming $\Omega \ne \emptyset$, we can find a maximizing sequence for u in Ω, that is a sequence $(x_k)_{k\in\mathbb{N}}$ in Ω with $\lim_{k\to\infty} u(x_k) = \sup_\Omega u$. It follows from the Bolzano-Weierstraß theorem, for instance, that a subsequence $\big(x_{k_\ell}\big)_{\ell\in\mathbb{N}}$ converges to a limit $a \in \overline{\Omega}\cup\{\infty_{\mathbb{R}^n}\}$.

Next we dinstiguish two cases.

We start with the case $a \in \Omega$. In this case, by continuity, we get $u(a) = \lim_{\ell\to\infty} u\big(x_{k_\ell}\big) = \sup_\Omega u$, and the strong maximum principle guarantees that u is constant $\equiv \sup_\Omega u$ in the connected component of Ω which contains a. We conclude $\sup_\Omega u \le \limsup_{\Omega\ni x\to b} u(x) \le M$ whenever this component posseses a boundary point b. It remains to deal with the situation that no such boundary point exists, which happens precisely for $\Omega = \mathbb{R}^n$. However, in that situation we get $\sup_\Omega u \le \limsup_{\Omega\ni x\to\infty_{\mathbb{R}^n}} u(x) \le M$ simply by using $\infty_{\mathbb{R}^n}$ in place of b.

Finally, we come to the (simpler, but nonetheless more relevant) case $a \in \partial\Omega\cup\{\infty_{\mathbb{R}^n}\}$. In this case we directly infer $\sup_\Omega u = \lim_{\ell\to\infty} u\big(x_{k_\ell}\big) \le \limsup_{\Omega\ni x\to a} u(x) \le M$ by the choice of $(x_k)_{k\in\mathbb{N}}$ as a maximizing sequence and the assumption for the lim sup at boundary points.

Altogether, we have shown $\sup_\Omega u \le M$ in all cases and arrive at the claim. □

(2) Somewhat surprisingly it is **often possible to weaken the assumptions** made in ($*$) **at ∞ or at other 'exceptional' boundary points**: Indeed, whenever there exist a point $a_0 \in \partial\Omega \cup \{\infty_{\mathbb{R}^n}\}$ and a superharmonic comparison function $v\colon \Omega \to (0,\infty)$ with[2] $\lim_{\Omega\ni x\to a_0} v(x) = \infty$, then the growth condition $\lim_{\Omega\ni x\to a_0} \frac{u_+(x)}{v(x)} = 0$ at a_0 suffices for the validity of the maximum principle. In other words, if a_0 and v as above exist, then the following weakened variant of ($*$) is valid:

$$\left.\begin{array}{c} u \text{ subharmonic in } \Omega\,, \\ \limsup\limits_{\Omega\ni x\to a} u(x) \le M \text{ at all } a \in (\partial\Omega \cup \{\infty_{\mathbb{R}^n}\}) \setminus \{a_0\}\,, \\ \lim\limits_{\Omega\ni x\to a_0} \dfrac{u_+(x)}{v(x)} = 0 \end{array}\right\} \implies u \le M \text{ in } \Omega\,. \qquad (**)$$

Proof. Consider an arbitary $\varepsilon > 0$. Then, under the assumptions on the left-hand side of the statement, we have $\limsup_{\Omega\ni x\to a}(u(x)-\varepsilon v(x)) \le M$ for all $a \in (\partial\Omega \cup \{\infty_{\mathbb{R}^n}\}) \setminus \{a_0\}$ and

$$\limsup_{\Omega\ni x\to a_0}(u(x)-\varepsilon v(x)) = \limsup_{\Omega\ni x\to a_0} v(x)\left(\frac{u(x)}{v(x)}-\varepsilon\right) \le \limsup_{\Omega\ni x\to a_0} v(x)\big(-\tfrac{1}{2}\varepsilon\big) = -\infty\,.$$

Therefore, ($*$) applies to the subharmonic function $u-\varepsilon v$ and yields $u-\varepsilon v \le M$ on Ω. Recalling that $\varepsilon > 0$ is arbitrary, we then infer $u \le M$ on Ω. $\square$

Assertions of the type ($**$) are known as **Phragmén-Lindelöf principles**. They can also be seen as non-existence results for subharmonic function which are unbounded only near the point a_0, but even near this point grow sufficiently slow.

The next few remarks provide concrete examples:

(3) For $n \ge 3$ and $a_0 \in \partial\Omega$, we now specialize ($**$) by using the shifted negative $v(x) = -F(x-a_0)$ of the **fundamental solution F** as a **comparison function**. In this case, v is positive and even harmonic on $\mathbb{R}^n \setminus \{a_0\} \supset \Omega$. Hence, recalling the formula for F, we get

$$\left.\begin{array}{c} u \text{ subharmonic in } \Omega\,, \\ \limsup\limits_{\Omega\ni x\to a} u(x) \le M \text{ for all } a \in (\partial\Omega)_{\neq a_0} \cup \{\infty_{\mathbb{R}^n}\}\,, \\ \lim\limits_{\Omega\ni x\to a_0} u_+(x)|x-a_0|^{n-2} = 0 \end{array}\right\} \implies u \le M \text{ in } \Omega\,.$$

In the case $n = 2$, the analogous principle with $|x-a_0|^{n-2}$ replaced by $-\log|x-a_0|$ holds only on bounded Ω. Indeed, this two-dimensional principle is deduced from ($**$) by choosing $v(x) = -F(x-a_0)+C$ there, where the constant C needs to be taken sufficiently large to keep v positive in Ω.

In particular, if a **harmonic function** h on Ω **blows up slower than the fundamental solution at an isolated boundary point** a_0 of Ω (that is, $\{a_0\}$ is relatively open in $\partial\Omega$ and $\lim_{x\to a_0} \frac{h(x)}{F(x)} = 0$), then h is in fact bounded in a neighborhood of a_0. Indeed, this simply follows by applying the above principle to $\pm h$ on a punctured ball $\mathrm{B}_r(a_0) \setminus \{a_0\}$

[2] An inspection of the following proof reveals that the assumption $\lim_{\Omega\ni x\to a_0} v(x) = \infty$ is unneccesarry in the case $M \ge 0$. However, our aim with ($**$) is indeed to allow some growth near a_0. Thus, we *want* v to be unbounded at least near a_0, and indeed $\lim_{\Omega\ni x\to a_0} v(x) = \infty$ will be satisfied in all upcoming applications.

(with r suitably small that $\overline{\mathrm{B}_r(a_0)} \subset \Omega \cup \{a_0\}$ and M larger than $\max_{\mathrm{S}_r(a_0)} |h|$). In a later section we will actually improve on this result by showing that in this situation h can be extended to a harmonic function in $\Omega \cup \{a_0\}$.

(4) The **classical Phragmén-Lindelöf principle** originates from complex analysis and concerns the case of $n = 2$ variables. It applies under the hypothesis that $\mathbf{\Omega}$ is **contained in a sector**[3] $\mathrm{D}_\alpha := \{x \in \mathbb{R}^2 \setminus \{0\} \,:\, |\operatorname{Arg}(x_1+\mathbf{i}x_2)| < \frac{1}{2}\alpha\}$ with opening angle $\alpha \in (0, 2\pi]$ and then asserts:

$$\left.\begin{array}{c} u \text{ subharmonic in } \Omega\,, \\ \limsup\limits_{\Omega \ni x \to a} u(x) \le M \text{ for all } a \in \partial\Omega\,, \\ \lim\limits_{\Omega \ni x \to \infty_{\mathbb{R}^2}} \dfrac{u_+(x)}{|x|^{\pi/\alpha}} = 0 \end{array}\right\} \implies u \le M \text{ in } \Omega\,.$$

Indeed, the growth condition in this statement is optimal. This can be seen at hand of the harmonic function h_α, defined by $h_\alpha(x) := \mathscr{Re}((x_1+\mathbf{i}x_2)^{\pi/\alpha}) = |x|^{\pi/\alpha}\cos(\frac{\pi}{\alpha}\operatorname{Arg}(x_1+\mathbf{i}x_2))$ and thus obtained as real part of a holomorphic function. Indeed h_α is positive on D_α and vanishes on $\partial\mathrm{D}_\alpha$, but $h_\alpha(x)$ equals $|x|^{\pi/\alpha}$ on the positive real axis and thus falls short — though ever so closely — of the growth condition.

On the proof. If the slightly stronger growth assumption $\lim_{\Omega \ni x \to \infty_{\mathbb{R}^2}} \frac{u_+(x)}{|x|^{\pi/\beta}} = 0$ is valid for some $\beta > \alpha$, then the claim follows along the above lines. Indeed, the harmonic function h_β considered right before satisfies $h_\beta(x) \ge \delta|x|^{\pi/\beta}$ for $x \in \mathrm{D}_\alpha$ with the fixed positive constant $\delta := \cos\frac{\pi\alpha}{2\beta}$. Thus, the growth assumption implies $\lim_{\Omega \ni x \to \infty_{\mathbb{R}^2}} \frac{u_+(x)}{h_\beta(x)} = 0$, and we can deduce the claim from ($**$) with the choice $v = h_\beta$.

In the general case a more refined argument, based on an analysis of the quantity

$$m(r) := \int_{\mathrm{S}_1 \cap \mathrm{D}_\alpha} u(rx) h_\alpha(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,,$$

is needed. We do not go through the details here, but indeed one can closely follow the reasoning described below for the case of the following Remark (5). □

(5) Another Phragmén-Lindelöf principle applies (now back to arbitrary dimension $n \ge 2$) when $\mathbf{\Omega}$ is **contained in a half-space** $(0,\infty) \times \mathbb{R}^{n-1}$. This principle says:

$$\left.\begin{array}{c} u \text{ subharmonic in } \Omega\,, \\ \limsup\limits_{\Omega \ni x \to a} u(x) \le M \text{ for all } a \in \partial\Omega\,, \\ \lim\limits_{\Omega \ni x \to \infty_{\mathbb{R}^n}} \dfrac{u_+(x)}{|x|} = 0 \end{array}\right\} \implies u \le M \text{ in } \Omega\,.$$

For $n = 2$ this is actually nothing but the case $\alpha = \pi$ of the previous Remark (4). Moreover, the basic example of the harmonic function x_1 shows that the growth condition cannot be further weakened.

Proof. W.l.o.g. we assume $M = 0$.

Following the basic approach of [1], we first give a proof of the principle in the case that Ω equals the half-space $\mathrm{H}_n := (0,\infty) \times \mathbb{R}^{n-1}$ and $u \in \mathrm{C}^2(\mathrm{H}_n) \cap \mathrm{C}^1(\overline{\mathrm{H}_n})$ is non-negative on H_n. This, in combination with the above

[3] We write $\mathbf{i}$ for the imaginary unit in $\mathbb{C}$. Moreover, for $z \in \mathbb{C} \setminus \{0\}$, we denote by $\operatorname{Arg}(z)$ the unique number in $\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$ such that $z = |z|\exp(\mathbf{i}\operatorname{Arg}(z))$.

assumptions, implies that u vanishes on $\partial\mathrm{H}_n$. Writing $\mathrm{B}_r^+ := \mathrm{B}_r \cap \mathrm{H}_n$ and $\mathrm{S}_r^+ := \mathrm{S}_r \cap \mathrm{H}_n$ for the half-balls and half-spheres in H_n, we proceed by analyzing the quantity

$$m(r) := \int_{\mathrm{S}_1^+} u(rx)x_1 \,\mathrm{d}\mathcal{H}^{n-1}(x)$$

(which, up to multiplication with a dimension-dependent constant, is a weighted mean value of u on S_r^+). As a first step, we differentiate m (where exchange of derivative and integral is possible and m turns out to be continuously differentiable, since ∇u is bounded on H_n) and use the divergence theorem[4] on B_1^+ (where the term on the boundary portion $(\partial\mathrm{B}_1^+)\setminus\mathrm{S}_1^+ \subset \partial\mathrm{H}_n$ vanishes due to the presence of x_1). In this way, we get

$$m'(r) = \int_{\mathrm{S}_1^+} x_1 \nabla u(rx)\cdot x \,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_{\mathrm{B}_1^+} \mathrm{div}_x(x_1\nabla u(rx))\,\mathrm{d}x \qquad \text{for every } r>0\,.$$

For the last integrand, via the product rule and the subharmonicity of u we get

$$\mathrm{div}_x(x_1\nabla u(rx)) = \partial_1 u(rx) + rx_1\Delta u(rx) \geq \partial_1 u(rx) = \tfrac{1}{r}\,\mathrm{div}_x(u(rx)\mathrm{e}_1)\,.$$

By this estimate and another application of the divergence theorem on B_1^+ (in view of $u\equiv 0$ on $\partial\mathrm{H}_n$ once more with vanishing boundary term on $(\partial\mathrm{B}_1^+)\setminus\mathrm{S}_1^+ \subset \partial\mathrm{H}_n$), we then arrive at

$$m'(r) \geq \frac{1}{r}\int_{\mathrm{B}_1^+} \mathrm{div}_x(u(rx)\mathrm{e}_1)\,\mathrm{d}x = \frac{1}{r}\int_{\mathrm{S}_1^+} u(rx)\mathrm{e}_1\cdot x \,\mathrm{d}\mathcal{H}^{n-1}(x) = \frac{m(r)}{r} \qquad \text{for every } r>0\,.$$

Via the quotient rule we infer $\frac{\mathrm{d}}{\mathrm{d}r}\frac{m(r)}{r} = \frac{m'(r)-\frac{m(r)}{r}}{r} \geq 0$, and thus

$$\frac{m(r)}{r} \text{ is a non-decreasing function of } r\in(0,\infty)\,.$$

In addition, the growth hypothesis for $u=u_+$ yields

$$\begin{aligned}\limsup_{r\to\infty}\frac{m(r)}{r} &= \limsup_{r\to\infty}\left(\frac{1}{r}\int_{\mathrm{S}_1^+} u(rx)x_1\,\mathrm{d}\mathcal{H}^{n-1}(x)\right)\\ &\leq \mathcal{H}^{n-1}(\mathrm{S}_1^+)\limsup_{r\to\infty}\left(\frac{1}{r}\sup_{\mathrm{S}_r^+} u(y)\right) \leq \mathcal{H}^{n-1}(\mathrm{S}_1^+)\limsup_{\mathrm{H}_n\ni x\to\infty_{\mathbb{R}^n}}\frac{u(x)}{|x|} = 0\,.\end{aligned}$$

Combining these properties of $\frac{m(r)}{r}$ we necessarily have $m(r)\leq 0$ for all $r>0$. In fact, taking into account non-negativity and continuity of u we even conclude $m(r)=0$ and $u\equiv 0$ on S_r^+ for all $r>0$. We have thus shown $u\equiv 0$ on H_n and have verified the claim in the situation at hand.

Finally, we turn to the general case and concisely describe how this case can reduced to the previously treated one. However, we warn the reader that the reduction anticipates concepts and tools discussed in detail only in later sections of these notes. This said, we recall that we now consider merely subharmonic $u\in\mathrm{C}^2(\Omega)$ in open $\Omega\subset\mathrm{H}_n$ with $\limsup_{\Omega\ni x\to a} u(x)\leq 0$ for all $a\in\partial\Omega$ and $\lim_{\Omega\ni x\to\infty_{\mathbb{R}^n}}\frac{u_+(x)}{|x|}=0$, but without boundary-regularity or non-negativity assumptions on u. We then define a non-negative function $w\in\mathrm{C}^0(\mathbb{R}^n)$ by setting $w(x):=u_+(x-\mathrm{e}_1)$ for $x\in\mathrm{e}_1+\Omega$ (where e_1 denotes the first canonical basis vector in $\mathbb{R}^n$) and $w(x):=0$ otherwise. It can be shown that w, though possibly non-differentiable at some points, is subharmonic in $\mathbb{R}^n$ in a generalized sense. Using the concept of mollification, as discussed soon, for parameters $\varepsilon\in(0,1)$, we approximate w by certain $w_\varepsilon\in\mathrm{C}^\infty(\mathbb{R}^n)$, which are still non-negative and subharmonic in $\mathbb{R}^n$ with $w_\varepsilon\equiv 0$ in $\mathbb{R}^n\setminus\mathrm{H}_n\supset\partial\mathrm{H}_n$ and $\lim_{x\to\infty_{\mathbb{R}^n}}\frac{w_\varepsilon(x)}{|x|}=0$. Now the previously proven statement applies and shows $w_\varepsilon\equiv 0$ also in H_n. As, moreover, w is the pointwise limit of w_ε for $\varepsilon\searrow 0$, we can finally deduce $w\equiv 0$ in H_n and $u\leq 0$ in Ω. □

Addendum on the technique of mollification

Next we introduce and discuss a standard technical tool in the analysis of real functions. In the subsequent section(s) it will turn out that this has important applications in the theory of PDEs.

[4] In technical detail this application of the divergence theorem requires existence of $\int_{\mathrm{B}_1^+}\mathrm{div}_x(x_1\nabla u(rx))\,\mathrm{d}x$ in $[-\infty,\infty]$. However, the requirement is met, since the subsequent estimate uncovers a lower bound for $x\mapsto\mathrm{div}_x(x_1\nabla u(rx))$ in $\mathrm{L}^1(\mathrm{B}_1^+)$.

Definitions (**mollification**)**.**

- *A* ***mollifier*** *or* ***mollification kernel*** η *on* $\mathbb{R}^n$ *is a* C^∞ *function* $\eta\colon \mathbb{R}^n \to [0,\infty)$ *such that*
$$\operatorname{spt}\eta \subset \overline{\mathrm{B}_1} \qquad \text{and} \qquad \int_{\mathbb{R}^n} \eta \,\mathrm{d}x = 1\,.$$
Occasionally one also requires that η *is rotationally symmetric, and often one agrees on a concrete choice such as* $\eta(x) := 0$ *for* $x \in \mathbb{R}^n \setminus \mathrm{B}_1$ *and* $\eta(x) := c_n \exp(-\frac{1}{1-|x|^2})$ *for* $x \in \mathrm{B}_1$*, with* $c_n := \big(\int_{\mathrm{B}_1} \exp(-\frac{1}{1-|x|^2})\,\mathrm{d}x\big)^{-1} \in (0,\infty)$.

- *Given a mollification kernel* η *on* $\mathbb{R}^n$ *we define, for* $\varepsilon > 0$*, the corresponding* ***scaled kernels*** $\eta_\varepsilon\colon \mathbb{R}^n \to [0,\infty)$ *by*
$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\Big(\frac{x}{\varepsilon}\Big) \qquad \text{for } x \in \mathbb{R}^n\,.$$
These satisfy $\operatorname{spt}\eta_\varepsilon \subset \overline{\mathrm{B}_\varepsilon}$ *and* $\int_{\mathbb{R}^n} \eta_\varepsilon \,\mathrm{d}x = 1$.

- *Given an open set* Ω *in* $\mathbb{R}^n$*,* $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$*, and a mollification kernel* η *on* $\mathbb{R}^n$*, we define, for* $\varepsilon > 0$*,* ***mollifications*** u_ε *of* u *by setting*[5]
$$u_\varepsilon(x) := (\eta_\varepsilon * u)(x) = \int_\Omega \eta_\varepsilon(x-y)u(y)\mathrm{d}y = \int_{\mathrm{B}_\varepsilon(x)} \eta_\varepsilon(x-y)u(y)\mathrm{d}y = \int_{\mathrm{B}_\varepsilon} \eta_\varepsilon(z)u(x-z)\mathrm{d}z$$
for all $x \in \mathbb{R}^n$ *with* $\overline{\mathrm{B}_\varepsilon(x)} \subset \Omega$*. Consequently,* $u_\varepsilon\colon \Omega_\varepsilon \to \mathbb{R}^N$ *is defined* (*only*) *in the subset*
$$\Omega_\varepsilon := \{x \in \mathbb{R}^n : \overline{\mathrm{B}_\varepsilon(x)} \subset \Omega\} = \{x \in \mathbb{R}^n : \operatorname{dist}(x,\mathbb{R}^n \setminus \Omega) > \varepsilon\} = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \varepsilon\}$$
of Ω*. The operators which map functions* u *to their mollifications* u_ε (*with fixed* $\varepsilon > 0$) *are also called* ***mollifiers***.

Remarks (on mollification)**.**

(1) Mollification is a **central technique** in analysis **for approximating arbitrary functions with smooth ones**.

(2) It is often illustrative to think of $u_\varepsilon(x)$ as a **weighted mean value** of u on the ball $\mathrm{B}_\varepsilon(x)$ with weight $y \mapsto \eta_\varepsilon(x-y)$. In principle, the choice $\eta = \omega_n^{-1}\mathbb{1}_{\mathrm{B}_r}$ corresponds to a uniform weight and thus gives the usual mean $u_\varepsilon(x) = \fint_{\mathrm{B}_r(x)} u\,\mathrm{d}y$, but due to the discontinuity of $\mathbb{1}_{\mathrm{B}_r}$ this choice is (at least formally) not admissible in the above definitions.

(3) For $\varepsilon \searrow 0$, the **scaled kernels** η_ε **approximate the Dirac measure at the origin** in the sense that we have $\eta_\varepsilon \geq 0$ in $\mathbb{R}^n$, $\operatorname{spt}\eta_\varepsilon \subset \overline{\mathrm{B}_\varepsilon}$, $\int_{\mathbb{R}^n} \eta_\varepsilon \,\mathrm{d}x = 1$, and $\lim_{\varepsilon\searrow 0} \eta_\varepsilon \equiv 0$ uniformly

[5] The notation $v{*}w$ is commonly used for the convolution $(v{*}w)(x) := \int_{\mathbb{R}^n} v(x{-}y)w(y)\,\mathrm{d}y$ of (suitably integrable) functions v and w defined in all of $\mathbb{R}^n$, and by a change of variables turns out to be a commutative operation. The integral expressions for u_ε are in accordance with the notion of convolution if one thinks of using an arbitrary extension of u to $\mathbb{R}^n$ and takes into account that η_ε vanishes outside B_ε. We remark that, proceeding in this way, we could indeed define u_ε in all of $\mathbb{R}^n$, but that the values are independent of the chosen extension and behave reasonably only in Ω_ε.

on $\mathbb{R}^n \setminus \mathrm{B}_\delta$ and in $\mathrm{L}^1(\mathbb{R}^n \setminus B_\delta)$ for every $\delta > 0$. With this approximation property or the mean value interpretation in mind we naturally expect the convergence

$$\lim_{\varepsilon \searrow 0} u_\varepsilon = \lim_{\varepsilon \searrow 0} \eta_\varepsilon {*} u = u$$

in a sense to be made precise. Indeed, several statement in this direction are established in the sequel.

(4) Under the above assumptions the integrals in the definition of $u_\varepsilon(x)$ exist with finite value. This is clear from boundedness of η_ε on $\mathbb{R}^n$ and the condition $\int_{\mathrm{B}_\varepsilon(x)} |u| \,\mathrm{d}x < \infty$.

(5) We have $(\mathbb{R}^n)_\varepsilon = \mathbb{R}^n$. Specifically for functions u defined on all of $\mathbb{R}^n$, the mollifications u_ε are thus defined, as a convenient feature, on the same domain $\mathbb{R}^n$.

Theorem (on **properties of mollifications**). *Consider an open set Ω in $\mathbb{R}^n$, a function $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, and a mollification kernel η on $\mathbb{R}^n$. Then the corresponding mollifications of u have the following properties.*

(I) ***Smoothness*: *We have $u_\varepsilon \in \mathrm{C}^\infty(\Omega_\varepsilon, \mathbb{R}^N)$ for all $\varepsilon > 0$.***

(II) ***Linearity*:** *For fixed $\varepsilon > 0$, the mollifier $\mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \to \mathrm{C}^\infty(\Omega_\varepsilon, \mathbb{R}^N)\,,\ u \mapsto u_\varepsilon$ is an $\mathbb{R}$-linear mapping.*

(III) ***Preservation of ($\mathbf{L}^p$) bounds*:**

(a) *In the case $N = 1$, for arbitrary $\varepsilon > 0$ and $a, b \in [-\infty, \infty]$, we have:*

$$a \le u \le b \text{ holds } \mathscr{L}^n\text{-almost everywhere in } \Omega \implies a \le u_\varepsilon \le b \text{ in } \Omega_\varepsilon\,.$$

(b) *If $u \in \mathrm{L}^p(\Omega, \mathbb{R}^N)$ holds for $p \in [1, \infty]$, then we have*

$$\|u_\varepsilon\|_{p;\Omega_\varepsilon} \le \|u\|_{p;\Omega} \qquad \text{for every } \varepsilon > 0\,.$$

Taking into account linearity this yields that the mollifier $\mathrm{L}^p(\Omega, \mathbb{R}^N) \to \mathrm{L}^p(\Omega_\varepsilon, \mathbb{R}^N)\,$, $u \mapsto u_\varepsilon$ is a contraction (i.e. is Lipschitz continuous with Lipschitz constant ≤ 1).

(IV) ***Preservation of moduli of continuity*:** *If we have*

$$|u(y) - u(x)| \le \omega(|y - x|) \qquad \text{for all } x, y \in \Omega$$

with some fixed function $\omega\colon [0, \infty) \to [0, \infty)$ (which, if it also satisfies $\omega(0+) = \omega(0) = 0$, is called a modulus of continuity for u on Ω), then, for every $\varepsilon > 0$, we also have

$$|u_\varepsilon(y) - u_\varepsilon(x)| \le \omega(|y - x|) \qquad \text{for all } x, y \in \Omega_\varepsilon\,.$$

(V) ***Convergence for $\varepsilon \searrow 0$:***

(a) *We have* $\lim_{\varepsilon\searrow 0} u_\varepsilon(x) = u(x)$ *for every Lebesgue point*[6] $x \in \Omega$ *of* u *with corresponding Lebesgue value* $u(x)$*. In particular, we have*

$$\lim_{\varepsilon\searrow 0} u_\varepsilon = u \qquad \mathscr{L}^n\text{-almost everywhere in } \Omega\,.$$

(b) *For* ***continuous*** $\boldsymbol{u}$ *on* Ω*, we have*

$$\lim_{\varepsilon\searrow 0} u_\varepsilon = u \qquad \text{locally uniformly in } \Omega\,.$$

Moreover, if u *is even uniformly continuous in* Ω*, this convergence holds even uniformly (that is, globally uniformly) on* Ω *in the sense of* $\lim_{\varepsilon\searrow 0} \sup_{\Omega_\varepsilon} |u_\varepsilon - u| = 0$*.*

(c) *For* $\boldsymbol{u \in \mathrm{L}^p(\Omega, \mathbb{R}^N)}$ *with* $p \in [1,\infty)$*, we have*

$$\lim_{\varepsilon\searrow 0} u_\varepsilon = u \qquad \text{in } \mathrm{L}^p(\Omega, \mathbb{R}^N)$$

in the more precise sense of $\lim_{\varepsilon\searrow 0} \|u_\varepsilon - u\|_{p;\Omega_\varepsilon} = 0$*. This statement* ***does not carry over to*** $\boldsymbol{p = \infty}$ *(since discontinuous functions in* $\mathrm{L}^\infty(\Omega, \mathbb{R}^N)$ *cannot be the uniform limit of their continuous mollifications).*

(VI) ***Mollification commutes with (partial) differentiation:*** *Consider a multi-index* $\alpha \in \mathbb{N}_0^n$*. If there holds* $u \in \mathrm{C}^{|\alpha|}(\Omega, \mathbb{R}^N)$*, then, for every* $\varepsilon > 0$*, we have*

$$\partial^\alpha(u_\varepsilon) = (\partial^\alpha u)_\varepsilon \qquad \text{in } \Omega_\varepsilon\,.$$

(VII) ***Gradient estimate for $\mathbf{L}^p$ approximation error:*** *In case of* $u \in \mathrm{C}^1(\Omega, \mathbb{R}^N)$ *with* $\mathrm{D}u \in \mathrm{L}^p(\Omega, \mathbb{R}^{N\times n})$ *for* $p \in [1,\infty]$ *there holds*

$$\|u_\varepsilon - u\|_{p;\Omega_\varepsilon} \le \varepsilon \|\mathrm{D}u\|_{p;\Omega} \qquad \text{for every } \varepsilon > 0\,.$$

(e.g. with operator norm on the target space $\mathbb{R}^{N\times n}$ *of* $\mathrm{D}u$*).*

Proof of (I). For compact $K \subset \Omega$, we record $K_\varepsilon := \{x \in K : \operatorname{dist}(x, \partial K) > \varepsilon\} \subset \Omega_\varepsilon$. By differentiation of the identity $u_\varepsilon(x) = \int_K \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y$ for $x \in K_\varepsilon$, we then obtain

$$\partial^\alpha(u_\varepsilon)(x) = \int_K \partial^\alpha \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y$$

for $x \in K_\varepsilon$ and all $\alpha \in \mathbb{N}_0^n$. Here, the differentiation under the integral is possible, since, for all $x \in K_\varepsilon$, we have the x-independent bound $|\partial^\alpha \eta_\varepsilon(x-y)u(y)| \le (\sup_{\mathbb{R}^n} |\partial^\alpha \eta_\varepsilon|)|u(y)|$ for the pointwisely differentiated integrand with majorant $|u| \in \mathrm{L}^1(K)$. Relying on the same bound, we also read off continuity of $\partial^\alpha u_\varepsilon$ on K. This proves $u_\varepsilon \in \mathrm{C}^\infty(K_\varepsilon, \mathbb{R}^N)$ and then, since every $x \in \Omega_\varepsilon$ is contained in the open set K_ε for some compact $K \subset \Omega$, also $u \in \mathrm{C}^\infty(\Omega_\varepsilon, \mathbb{R}^N)$. □

[6] Here, we call $x \in \Omega$ a Lebesgue point or (strong) approximate continuity point of u if there exists some $\gamma \in \mathbb{R}^N$ with $\lim_{r\searrow 0} \fint_{\mathrm{B}_r(x)} |u-\gamma|\,\mathrm{d}y = 0$. We then call γ (which is easily seen to be uniquely determined) the Lebesgue value of u at x and use notation $u(x)$ for this value. Lebesgue values are a way to define point evaluations of L^p functions in 'good' points at least. Clearly, every continuity point of u is also a Lebesgue point, but moreover a standard result from advanced measure theory (which we use without proof here) asserts for arbitrary $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ on open $\Omega \subset \mathbb{R}^n$ that $\mathscr{L}^n$-almost every point in Ω is a Lebesgue point for u and that the Lebesgue value coincides with the value of an arbitrary representative of the Lebesgue class (i.e. '$\mathscr{L}^n$-almost everywhere defined function') u at $\mathscr{L}^n$-almost every point in Ω. In this light, one also calls the function which maps the Lebesgue points of u to their Lebesgue values (and takes arbitrary values in the non-Lebesgue points) the Lebesgue representative of u.

Proof of (II). This is evident from the definition of the mollifier and the linearity of the Lebesgue integral in the integrand. □

Proof of (III). The claims in (IIIa) follow from

$$u_\varepsilon(x) = \int_\Omega \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y \le b\int_\Omega \eta_\varepsilon(x-y)\,\mathrm{d}y = b\int_\Omega \eta_\varepsilon(z)\,\mathrm{d}z = b \qquad \text{for } x\in\Omega_\varepsilon$$

and an analogous estimation which ensures $u_\varepsilon \ge a$ on Ω_ε. Using, in addition, the triangle inequality for integrals (in order to get the modulus inside), we can handle the case $p=\infty$ in (IIIb) in the same way.

For the case $p\in[1,\infty)$ in (IIIb), we use Hölder's inequality or Jensen's inequality[7] for the weighted Lebesgue measure $\eta_\varepsilon((\,\cdot\,)-y)\mathscr{L}^n$ (which is a probability measure) together with Fubini's theorem. In this manner we deduce

$$\begin{aligned}\|u_\varepsilon\|_{p;\Omega_\varepsilon}^p = \int_{\Omega_\varepsilon}|u_\varepsilon|^p\,\mathrm{d}x &= \int_{\Omega_\varepsilon}\left|\int_\Omega \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y\right|^p\mathrm{d}x\\ &\le \int_{\Omega_\varepsilon}\int_\Omega \eta_\varepsilon(x-y)|u(y)|^p\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_\Omega |u(y)|^p\int_{\Omega_\varepsilon}\eta_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y \le \int_\Omega |u|^p\,\mathrm{d}y = \|u\|_{p;\Omega}^p\end{aligned}$$

(where we have exploited $\int_{\Omega_\varepsilon}\eta_\varepsilon(x-y)\,\mathrm{d}x \le \int_{\mathbb{R}^n}\eta_\varepsilon(x-y)\,\mathrm{d}x = 1$ in the penultimate step). □

Proof of (IV). For $\varepsilon>0$ and $x,y\in\Omega_\varepsilon$, we infer

$$\begin{aligned}|u_\varepsilon(y)-u_\varepsilon(x)| &= \left|\int_{\mathrm{B}_\varepsilon}\eta_\varepsilon(z)u(y-z)\,\mathrm{d}z - \int_{\mathrm{B}_\varepsilon}\eta_\varepsilon(z)u(x-z)\,\mathrm{d}z\right|\\ &\le \int_{\mathrm{B}_\varepsilon}\eta_\varepsilon(z)|u(y-z)-u(x-z)|\,\mathrm{d}z \le \omega(|y-x|)\int_{\mathrm{B}_\varepsilon}\eta_\varepsilon\,\mathrm{d}z = \omega(|y-x|)\end{aligned}$$

from the triangle inequality and the assumption for u. □

Proof of (V). For (Va), we fix a Lebesgue point $x\in\Omega$ of u. Then, for sufficiently small $\varepsilon>0$, we have $x\in\Omega_\varepsilon$, and we can estimate

$$\begin{aligned}|u_\varepsilon(x)-u(x)| &= \left|\int_{\mathrm{B}_\varepsilon(x)}\eta_\varepsilon(x-y)u(y)\,\mathrm{d}y - u(x)\int_{\mathrm{B}_\varepsilon(x)}\eta_\varepsilon(x-y)\,\mathrm{d}y\right|\\ &\le \int_{\mathrm{B}_\varepsilon(x)}\eta_\varepsilon(x-y)|u(y)-u(x)|\,\mathrm{d}y\\ &\le \Big(\sup_{\mathbb{R}^n}\eta\Big)\frac{1}{\varepsilon^n}\int_{\mathrm{B}_\varepsilon(x)}|u-u(x)|\,\mathrm{d}y \xrightarrow[\varepsilon\searrow 0]{} 0\,,\end{aligned}$$

where in the last step we used that x is a Lebesgue point of u. This shows $\lim_{\varepsilon\searrow 0}u_\varepsilon(x) = u(x)$.

[7] Jensen's integral inequality can be stated as follows: Consider a probability measure μ on a set $\mathcal{X}$, that is, a measure μ on (a σ-algebra over) $\mathcal{X}$ with $\mu(\mathcal{X})=1$. Then, if $\Phi\colon C\to\mathbb{R}$ is convex on a convex set $C\subset\mathbb{R}^N$, there holds

$$\Phi\Big(\int_{\mathcal{X}}F\,\mathrm{d}\mu\Big) \le \int_{\mathcal{X}}\Phi(F)\,\mathrm{d}\mu$$

for all $F\in\mathrm{L}^1(\mathcal{X},\mathbb{R}^N;\mu)$ such that $F\in C$ holds μ-almost everywhere on $\mathcal{X}$.

Turning to (Vb), we first treat the case that u is uniformly continuous on Ω. Proceeding similar to the previous estimate and using uniform continuity in the end, we then infer

$$\sup_{\Omega_\varepsilon}|u_\varepsilon-u| \le \sup_{x\in\Omega_\varepsilon}\int_{\mathrm{B}_\varepsilon(x)}\eta_\varepsilon(x-y)|u(y)-u(x)|\,\mathrm{d}y \le \sup_{x,y\in\Omega,|y-x|<\varepsilon}|u(y)-u(x)| \xrightarrow[\varepsilon\searrow 0]{} 0\,.$$

This proves the claimed uniform convergence. If u is merely continuous in Ω, a standard result on continuous function on compact sets guarantees uniform continuity of u on every compact subset of Ω. It then follows from the previous consideration that the convergence $\lim_{\varepsilon\searrow 0}u_\varepsilon = u$ is uniform on each such K and hence locally uniform in Ω.

The proof of (Vc), finally, is less straightforward. Indeed, we use the density of $\mathrm{C}^0_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ in $\mathrm{L}^p(\Omega,\mathbb{R}^N)$ for $p\in[1,\infty)$ (proved typically in functional analysis classes). Given an arbitrary $\chi>0$, this density yields some $v\in\mathrm{C}^0_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ with $\|v-u\|_{p;\Omega}<\chi$, and via (II) and (IIIb) we deduce $\|v_\varepsilon-u_\varepsilon\|_{p;\Omega}<\chi$. In addition, by applying (Vb) to the continuous function v on Ω, we get

$$\|v_\varepsilon-v\|^p_{p;\Omega_\varepsilon} \le \mathscr{L}^n(\operatorname{spt}v)\sup_{\operatorname{spt}v}|v_\varepsilon-v|^p \xrightarrow[\varepsilon\searrow 0]{} 0\,.$$

Thus, using Minkowski's inequality (i.e. the triangle inequality for the L^p norm), we conclude

$$\limsup_{\varepsilon\searrow 0}\|u_\varepsilon-u\|_{p;\Omega_\varepsilon} \le \limsup_{\varepsilon\searrow 0}\|u_\varepsilon-v_\varepsilon\|_{p;\Omega_\varepsilon}+\lim_{\varepsilon\searrow 0}\|v_\varepsilon-v\|_{p;\Omega_\varepsilon}+\limsup_{\varepsilon\searrow 0}\|v-u\|_{p;\Omega}\le 2\chi\,.$$

Since $\chi>0$ was arbitrary, this proves the claim. □

Proof of (VI). The proof is similar to the argument used for (I) and uses the notation K_ε for compact $K\subset\Omega$, as introduced there. This time, however, we proceed by differentiation of the differently arranged identity $u_\varepsilon(x)=\int_{\mathrm{B}_\varepsilon}\eta_\varepsilon(z)u(x-z)\,\mathrm{d}z$ for $x\in K_\varepsilon$. We then obtain

$$\partial^\alpha(u_\varepsilon)(x)=\int_{\mathrm{B}_\varepsilon}\eta_\varepsilon(z)\partial^\alpha u(x-z)\,\mathrm{d}z=(\partial^\alpha u)_\varepsilon(x)$$

for $x\in K_\varepsilon$, where the computation is justified, since η_ε is bounded on $\mathbb{R}^n$ and $|\partial^\alpha u(x-z)|$ with $x\in K_\varepsilon$ and $z\in\mathrm{B}_\varepsilon$ is bounded by $\sup_K|\partial^\alpha u|<\infty$. Since every $x\in\Omega_\varepsilon$ is contained in some K_ε, this proves $\partial^\alpha(u_\varepsilon)=(\partial^\alpha u)_\varepsilon$ in Ω_ε. □

Proof of (VII). We first assume $p\in[1,\infty)$. Using Hölder's inequality or Jensen's inequality in the same way as in the proof of (IIIb), we infer

$$\|u_\varepsilon-u\|^p_{p;\Omega_\varepsilon}\le\int_{\Omega_\varepsilon}\int_{\mathrm{B}_\varepsilon}\eta_\varepsilon(z)|u(x-z)-u(x)|^p\,\mathrm{d}z\,\mathrm{d}x\,.$$

By the fundamental theorem of calculus and again Hölder's or Jensen's inequality, we also get

$$|u(x-z)-u(x)|^p=\left|\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}u(x-tz)\,\mathrm{d}t\right|^p\le\int_0^1|\mathrm{D}u(x-tz)z|^p\,\mathrm{d}t\le\varepsilon^p\int_0^1|\mathrm{D}u(x-tz)|^p\,\mathrm{d}t$$

for all $x\in\Omega_\varepsilon$ and $z\in\mathrm{B}_\varepsilon$. Plugging this into the previous estimate and exchanging the order of integration via Fubini's theorem, we end up with

$$\|u_\varepsilon-u\|^p_{p;\Omega_\varepsilon}\le\varepsilon^p\int_0^1\int_{\mathrm{B}_\varepsilon}\eta_\varepsilon(z)\int_{\Omega_\varepsilon}|\mathrm{D}u(x-tz)|^p\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\,.$$

By a change of variables, the innermost integral on the right-hand side equals $\|\mathrm{D}u\|^p_{p;\Omega_\varepsilon - tz}$, and in view of $\Omega_\varepsilon - tz \subset \Omega$ for all $t \in [0,1]$ and $z \in \mathrm{B}_\varepsilon$ it is thus controlled by $\|\mathrm{D}u\|^p_{p;\Omega}$. Since this bound no longer depends on t or z, the integrals decouple, and we arrive at

$$\|u_\varepsilon - u\|^p_{p;\Omega_\varepsilon} \leq \varepsilon^p \int_0^1 \mathrm{d}t \int_{\mathrm{B}_\varepsilon} \eta_\varepsilon \,\mathrm{d}z \, \|\mathrm{D}u\|^p_{p;\Omega} = \varepsilon^p \|\mathrm{D}u\|^p_{p;\Omega} \,.$$

This proves the claim for $p \in [1, \infty)$. The case $p = \infty$, finally, can be handled by similar and even slightly simpler estimates without integration in the x-variable. □

2.5 Weakly harmonic functions and regularity of harmonic functions

Consider an open set Ω in $\mathbb{R}^n$ and $h \in \mathrm{C}^2(\Omega)$. From the fundamental lemma of the calculus of variations (see the exercises) and integration by parts we obtain the following characterizations of harmonicity:

$$\begin{aligned} h \text{ harmonic on } \Omega \iff & \int_\Omega \Delta h \, \varphi \,\mathrm{d}x = 0 \text{ for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega) \\ \iff & \int_\Omega \nabla h \cdot \nabla \varphi \,\mathrm{d}x = 0 \text{ for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega) \\ \iff & \int_\Omega h \, \Delta \varphi \,\mathrm{d}x = 0 \text{ for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega) \,. \end{aligned}$$

Here, the conditions in the second and third line remain meaningful for non-C^2 functions and can thus be taken as generalized definitions of harmonicity. In the second line, it also common to replace ∇h by a weak gradient field G which may exist for non-C^1 functions, and on the precise technical level the definitions are then implemented as follows:

Definitions (weak gradient, (very) weak harmonicity). *Consider an open set Ω in $\mathbb{R}^n$ and $h \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$.*

(1) *We call $G \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ a **weak gradient** of h in Ω if we have*

$$\int_\Omega h \operatorname{div} \Phi \,\mathrm{d}x = -\int_\Omega G \cdot \Phi \,\mathrm{d}x \qquad \text{for all } \Phi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^n) \,.$$

(2) *We say that h is **weakly harmonic** in Ω if h has a weak gradient G in Ω such that there holds*

$$\int_\Omega G \cdot \nabla \varphi \,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega) \,.$$

(3) *We say that h is **very weakly harmonic** in Ω if we have*

$$\int_\Omega h \, \Delta \varphi \,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega) \,.$$

Remarks.

(1) It is shown in the exercise class (without usage of the terminology at hand, however) that the classical gradient of a C^1 function is also its (unique C^0) weak gradient.

(2) From the definitions we infer

$$h \text{ harmonic} \implies h \text{ weakly harmonic} \implies h \text{ very weakly harmonic}\,.$$

Here, the first implication is obvious by the above characterization (and by regarding the classical gradient as a weak one). The second implication follows by plugging $\Phi = \nabla\varphi$ into the definition of weak gradient and then combining it with the definition of weak harmonicity.

(3) For suitably regular functions, the fundamental lemma and integrations by parts yield that the implications in Remark (2) are indeed equivalences: So, every weakly harmonic C^2 function is harmonic, and every very weakly harmonic function which posseses a weak gradient (in particular, every very weakly harmonic C^1 function) is weakly harmonic.

Next we show that (very) weakly harmonic functions are automatically C^∞ and thus turn out to be classically harmonic *even without any additional regularity assumption.* This non-trivial fact may seem quite surprising at first, yet turns out to be typical in the theory of *elliptic* PDEs:

Theorem (Weyl lemma, $\mathbf{C^\infty}$ regularity of harmonic functions). *Consider an open set Ω in $\mathbb{R}^n$. Every very weakly harmonic function in Ω is (possibly after modification on an $\mathscr{L}^n$ null set) automatically in $\mathrm{C}^\infty(\Omega)$ and classically harmonic in Ω.*

Proof. We suppose that $h \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ is very weakly harmonic in Ω and rely on the following three facts, which will be verified in the exercises:

- The very weak harmonicity of h implies that the mollifications $h_\varepsilon \in \mathrm{C}^\infty(\Omega_\varepsilon)$ with $\varepsilon > 0$ are classically harmonic in Ω_ε.
- If — what is clearly possible and assumed in the remainder of this section — a rotationally symmetric mollification kernel is used, then the spherical mean value property of a classically harmonic function k in Ω implies $k_\varepsilon = k$ in Ω_ε for every $\varepsilon > 0$.

  Detailed justification: Clearly, the rotational symmetry of the mollification kernel η implies rotational symmetry of the scaled kernels η_ε, that is, $\eta_\varepsilon(x) = \chi_\varepsilon(|x|)$ for all $x \in \mathbb{R}^n$ and suitable functions χ_ε. Using, in turn, integration in spherical coordinates, the spherical mean value property of k, and radial integration, we infer

  $$\begin{aligned} k_\varepsilon(a) = \int_{\mathrm{B}_\varepsilon(a)} \eta_\varepsilon(a-x)k(x)\,\mathrm{d}x &= \int_0^\varepsilon \chi_\varepsilon(r) \int_{\mathrm{S}_r(a)} k\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}r \\ &= n\omega_n \int_0^\varepsilon \chi_\varepsilon(r) r^{n-1}\,\mathrm{d}r\,k(a) = \int_{\mathrm{B}_\varepsilon} \eta_\varepsilon(z)\,\mathrm{d}z\,k(a) = k(a) \end{aligned}$$

  for every $a \in \Omega_\varepsilon$. This shows the claim $k_\varepsilon = k$ in Ω_ε.
- For $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and arbitrary $\varepsilon, \delta > 0$, we have $(u_\varepsilon)_\delta = (u_\delta)_\varepsilon$ in $(\Omega_\varepsilon)_\delta = \Omega_{\varepsilon+\delta} = (\Omega_\delta)_\varepsilon$.

All in all, we infer

$$h_\varepsilon = (h_\varepsilon)_\delta = (h_\delta)_\varepsilon = h_\delta \qquad \text{in } \Omega_{\varepsilon+\delta}$$

for arbitrary $\varepsilon, \delta > 0$, and thus $h = \lim_{\delta \searrow 0} h_\delta = h_\varepsilon$ holds $\mathscr{L}^n$-a.e. in $\Omega_{2\varepsilon}$ for every $\varepsilon > 0$. Since h_ε is C^∞ and classically harmonic in the open set $\Omega_{2\varepsilon} \subset \Omega_\varepsilon$ and the union of all $\Omega_{2\varepsilon}$ with $\varepsilon > 0$ is Ω, this immediately yields the claim. □

Remarks (on the Weyl lemma and variants thereof)**.**

(1) In particular, the Weyl lemma **applies to classically harmonic functions**. Even in this case the **improvement from $\mathbf{C^2}$ to $\mathbf{C^\infty}$ regularity** may be considered as **surprising**.

The proof, however, simplifies in the classically harmonic case, since the mean value property of h directly leads to $h = h_\varepsilon$ in Ω_ε for every $\varepsilon > 0$. Thus, $h \in \mathrm{C}^\infty(\Omega)$ follows without any need of working with a second parameter δ.

(2) If $h \in \mathrm{C}^0(\Omega)$ has the mean value property, that is, either $\fint_{\mathrm{B}_r(a)} h \,\mathrm{d}x = h(a)$ for all balls $\overline{\mathrm{B}_r(a)} \subset \Omega$ or $\fint_{\mathrm{S}_r(a)} h \,\mathrm{d}\mathcal{H}^{n-1}(x) = h(a)$ for all spheres $\mathrm{S}_r(a) \subset \overline{\mathrm{B}_r(a)} \subset \Omega$, then, in some literature, h is called **generalized harmonic** in Ω. Occasionally this notion is even broadened to functions $h \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ which satisfy the mean value property for $\mathcal{L}^{n+1}$-almost all pairs (a,r) with $\overline{\mathrm{B}_r(a)} \subset \Omega$. In any case, also generalized harmonic functions h in Ω satisfy $h = h_\varepsilon$ in Ω_ε (justified earlier in case of $\mathrm{S}_r(a)$ mean values; for $\mathrm{B}_r(a)$ mean values see below) and thus **turn out to be $\mathbf{C}^\infty$ smooth and classically harmonic** (in the $\mathrm{L}^1_{\mathrm{loc}}$ setting after modification on an $\mathscr{L}^n$ null set).

Deducution of $h = h_\varepsilon$ from the $\mathrm{B}_r(a)$ *mean value property.* We choose a rotationally symmetric mollification kernel η on $\mathbb{R}^n$ such that the function χ with $\eta(x) = \chi(|x|)$ for $x \in \mathbb{R}^n$ strictly decreases on $[0,1]$. For $\varepsilon > 0$, this implies that χ_ε with $\eta_\varepsilon(x) = \chi_\varepsilon(|x|)$ for $x \in \mathbb{R}^n$ strictly decreases on $[0,\varepsilon]$. Now we fix $a \in \Omega_\varepsilon$. As a consequence of the previous observations, the superlevel sets $A_{t,\varepsilon}(a) := \{x \in \mathbb{R}^n : \eta_\varepsilon(a-x) > t\}$ are balls with center a and radius $\leq \varepsilon$. Now, for $h \in \mathrm{C}^0(\Omega)$, we argue with Fubini's theorem (applied twice), the mean value property on the balls $A_{t,\varepsilon}(a)$, and the normalization $\int_{\mathrm{B}_\varepsilon} \eta_\varepsilon \,\mathrm{d}x = 1$ of the scaled kernels. In this way, we infer

$$
\begin{aligned}
h_\varepsilon(a) &= \int_{\mathrm{B}_\varepsilon(a)} \eta_\varepsilon(a-x) h(x) \,\mathrm{d}x = \int_{\mathrm{B}_\varepsilon(a)} \int_0^{\eta_\varepsilon(a-x)} \mathrm{d}t \, h(x) \,\mathrm{d}x = \int_0^{\eta_\varepsilon(0)} \int_{A_{t,\varepsilon}(a)} h(x) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_0^{\eta_\varepsilon(0)} \int_{A_{t,\varepsilon}(a)} \mathrm{d}x \,\mathrm{d}t \, h(a) = \int_{\mathrm{B}_\varepsilon(a)} \int_0^{\eta_\varepsilon(a-x)} \mathrm{d}t \,\mathrm{d}x \, h(a) = \int_{\mathrm{B}_\varepsilon(a)} \eta_\varepsilon(a-x) \,\mathrm{d}x \, h(a) = h(a) \,.
\end{aligned}
$$

This shows the claim for $h \in \mathrm{C}^0(\Omega)$. For $h \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, the reasoning is, up to tracking of null sets, the same. ☐

(3) In conclusion, all concepts of harmonicity (classical, weak, very weak, generalized) coincide, and one may wonder why we have entered into the discussion of the different concepts at all. One answer is that the coincidence of the different definitions may and should indeed be seen an indication that harmonic functions are very natural and interesting objects. Another more practical answer is that both weakly harmonic functions and generalized harmonic functions are useful in obtaining existence results for harmonic functions in the sense of the original classic definition. In particular, existence of weakly harmonic functions (and weak solutions to more general PDEs) can be approached by powerful methods of functional analysis. Anyway, here we do not enter into such methods, but rather present a more specific existence proof, which involves generalized harmonic functions:

Theorem (solvability of the Dirichlet problem for harmonic functions *on balls*). *For $a \in \mathbb{R}^n$, $R \in (0,\infty)$, $\varphi \in \mathrm{C}^0(\mathrm{S}_R(a))$, the Dirichlet problem for harmonic functions*

$$
\begin{aligned}
\Delta h &\equiv 0 \qquad \text{in } \mathrm{B}_R(a)\,, \\
h &= \varphi \qquad \text{at } \mathrm{S}_R(a)
\end{aligned}
$$

has a solution $h \in \mathrm{C}^2(\mathrm{B}_R(a)) \cap \mathrm{C}^0(\overline{\mathrm{B}_R(a)})$ (which, by the maximum principle, is also unique).

Proof. W.l.o.g. we only treat the case $a = 0$, $R = 1$. By the Weierstraß approximation theorem[8], there exists a sequence $(p_k)_{k\in\mathbb{N}}$ of polynomials on $\mathbb{R}^n$ such that $\lim_{k\to\infty} p_k = \varphi$ uniformly at

[8] The Weierstraß approximation theorem (in n dimensions) asserts, for every continuous function on a compact subset K of $\mathbb{R}^n$, that there exists a sequence $(p_k)_{k\in\mathbb{N}}$ of polynomials on $\mathbb{R}^n$ which approximates φ uniformly in the sense of $\lim_{k\to\infty} p_k = \varphi$ uniformly on K. For $n = 1$ this is commonly proved by rather elementary means. For $n \geq 2$ the proof is often carried out in the setting of a general functional analysis principle, the Stone-Weierstraß theorem, which contains the Weierstraß approximation theorem as a special case.

S_1. In view of the solvability result in Section 2.2, for every $k \in \mathbb{N}$, we can find a (polynomial) solution h_k of the Dirichlet problem

$$\begin{aligned} \Delta h_k &\equiv 0 \qquad \text{in } \mathrm{B}_1\,, \\ h_k &= p_k \qquad \text{at } \mathrm{S}_1\,. \end{aligned}$$

By the maximum (and minimum) principle for the harmonic functions $h_k - h_\ell$, we obtain

$$\max_{\overline{\mathrm{B}_1}} |h_\ell - h_k| \le \max_{\mathrm{S}_1} |p_\ell - p_k| \qquad \text{for all } k, \ell \in \mathbb{N}\,.$$

Thus, the uniform Cauchy-property of $(p_k)_{k\in\mathbb{N}}$ at S_1 carries over to $(h_k)_{k\in\mathbb{N}}$ on $\overline{\mathrm{B}_1}$ and leads to the existence of a uniform limit $h := \lim_{k\to\infty} h_k$ on $\overline{\mathrm{B}_1}$. Since uniform limits preserve continuity, we obtain $h \in \mathrm{C}^0(\overline{\mathrm{B}_1})$ with $h = \varphi$ on S_1. In view of the uniform convergence it is easily verified that h inherits the mean value property from h_k, and thus h is generalized harmonic in B_1 in the sense of Remark (2) above. As pointed out there, h is then C^∞ in B_1 and turns out to be the classically harmonic solution of the Dirichlet problem on B_1. □

Remark. An alternative way of finalizing the proof above is worth pointing out: Instead of relying on the notion of generalized harmonic functions one may also employ the Weierstraß-type convergence theorem treated in the subsequent Section 2.6.

2.6 Liouville and convergence theorems, Harnack's inequality

For the remainder of Chapter 2 we generally assume Ω is an open set in $\mathbb{R}^n$. It has been observed in the Section 2.5 that the mean value property of a harmonic function h in Ω implies the crucial identity

$$h(a) = h_r(a) = \frac{1}{r^n}\int_{\mathrm{B}_r(a)} \eta\Big(\frac{a-x}{r}\Big) h(x)\,\mathrm{d}x = \frac{1}{r^n}\int_{\Omega} \eta\Big(\frac{a-x}{r}\Big) h(x)\,\mathrm{d}x$$

for $\overline{\mathrm{B}_r(a)} \subset \Omega$ provided that the mollification kernel η is rotationally symmetric. Differentiating with respect to a and exchanging the order of differentiation and integration (justified as usual), we infer

$$\partial^\alpha h(a) = \frac{1}{r^{n+|\alpha|}}\int_{\mathrm{B}_r(a)} \partial^\alpha\eta\Big(\frac{a-x}{r}\Big) h(x)\,\mathrm{d}x$$

for every multi-index $\alpha \in \mathbb{N}_0^n$. Once the kernel η is fixed, we have $\sup_{\mathbb{R}^n} |\partial^\alpha \eta| \le \mathrm{const}(n, |\alpha|)$ for a constant $\mathrm{const}(n,|\alpha|) \in [0,\infty)$ which depends solely on n and $|\alpha|$. Thus, from the previous expression for $\partial^\alpha h$ we deduce the important **interior estimates for harmonic functions**

$$\boxed{|\partial^\alpha h(a)| \le \frac{\mathrm{const}(n,|\alpha|)}{r^{n+|\alpha|}} \|h\|_{1;\mathrm{B}_r(a)}\,, \qquad \text{whenever } h \text{ is harmonic in } \Omega,\ \overline{\mathrm{B}_r(a)} \subset \Omega, \text{ and } \alpha \in \mathbb{N}_0^n.}$$

As a first application of these estimates, we establish the following result on entire (i.e. everywhere-defined) harmonic functions on $\mathbb{R}^n$.

Theorem (Liouville property of *entire* harmonic functions).

(I) *If h is harmonic and bounded in all of $\mathbb{R}^n$, then h is necessarily constant in $\mathbb{R}^n$.*

(II) *If h is harmonic in all of $\mathbb{R}^n$ with polynomial growth $\lim_{|x|\to\infty} \frac{h(x)}{|x|^{m+1}} = 0$ for some $m \in \mathbb{N}_0$, then h is necessarily a polynomial of degree $\le m$.*

Remark. Consider an entire harmonic function h. Then boundedness of h implies its constancy by the Liouville property in (I). This is sharpened by the case $m = 0$ in (II) which says that already sublinear growth of h implies its constancy.

A refined Liouville property will be established later in this section.

Proof. In order to prove (I), we use the interior estimates in the case $\alpha = \mathrm{e}_i$, $i \in \{1, 2, \ldots, n\}$ of a first-order partial derivative. We obtain

$$|\partial_i h(a)| \le \frac{\mathrm{const}(n)}{r^{n+1}} \|h\|_{1;\mathrm{B}_r(a)} \le \frac{\mathrm{const}(n)}{r} \sup_{\mathbb{R}^n} |h| \underset{r\to\infty}{\longrightarrow} 0 \qquad \text{for all } a \in \mathbb{R}^n \,,$$

where we have used the boundedness $\sup_{\mathbb{R}^n} |h| < \infty$ of h in the last step. In conclusion, ∇h vanishes in $\mathbb{R}^n$, and h is constant in $\mathbb{R}^n$.

For proving (II), we apply the interior estimates for $\alpha \in \mathbb{N}_0^n$, $|\alpha| = m+1$. In addition, we rely on the weak maximum principle and the assumed polynomial growth. In this way, we get

$$\left.\begin{aligned} |\partial^\alpha h(a)| &\le \frac{\mathrm{const}(n,m)}{r^{n+m+1}} \|h\|_{1;\mathrm{B}_r(a)} \le \frac{\mathrm{const}(n,m)}{r^{m+1}} \sup_{\mathrm{B}_r(a)} |h| \\ &\le \frac{\mathrm{const}(n,m)}{r^{m+1}} \sup_{\mathrm{S}_r(a)} |h| = \mathrm{const}(n,m) \sup_{x\in\mathrm{S}_r(a)} \frac{|h(x)|}{|x|^{m+1}} \underset{r\to\infty}{\longrightarrow} 0 \end{aligned}\right\} \quad \text{for all } a \in \mathbb{R}^n \,.$$

Thus, $\mathrm{D}^{m+1}h$ vanishes in $\mathbb{R}^n$, and h is a polynomial of degree $\le m$. □

As a second application of the interior estimates we deduce compactness and convergence results for sequences of harmonic functions:

Theorem. *Consider a sequence $(h_k)_{k\in\mathbb{N}}$ of harmonic functions in Ω.*

(I) *(Montel-type)* ***compactness theorem*****:** *If the sequence is locally uniformly bounded in Ω, that is, $\sup_{k\in\mathbb{N}} \sup_K |h_k| < \infty$ for every compact $K \subset \Omega$, then it has a subsequence which converges locally uniformly in Ω.*

(II) *(Weierstraß-type)* ***convergence theorem*****:** *If the sequence converges locally uniformly in Ω, then the limit function h is harmonic in Ω, and we have locally uniform convergence of derivatives $\lim_{k\to\infty} \partial^\alpha h_k = \partial^\alpha h$ in Ω for arbitrary $\alpha \in \mathbb{N}_0^n$.*

Remarks (on the compactness and convergence theorem)**.**

(1) The compactness theorem resembles a version of the Montel compactness theorem in complex analysis, which gives the same assertions for a sequence of holomorphic functions. Similarly the convergence theorem resembles the Weierstraß compactness theorem in complex analysis. In the case $n = 2$, where harmonic functions are nothing but real/imaginary parts of holomorphic functions, the above theorems are equivalent with their complex-analysis counterparts.

(2) For a sequence of harmonic functions in *bounded* Ω, continuous up to $\partial\Omega$, its uniform boundedness or uniform convergence merely at $\partial\Omega$ implies by the weak maximum principle its uniform boundedness or uniform convergence in all of Ω. In this sense, the hypotheses of the theorems follow from corresponding hypotheses at the boundary.

Proof of the compactness theorem. Given a convex[9] compact subset K of Ω, we can choose a larger compact subset $\widetilde{K}$ of Ω such that $r := \operatorname{dist}(K, \mathbb{R}^n \setminus \widetilde{K}) > 0$. Then the interior estimates for the harmonic functions h_k yield

$$|\nabla h_k(a)| \le \frac{\operatorname{const}(n)}{r^{n+1}} \|h_k\|_{1;\mathrm{B}_r(a)} \le \frac{\operatorname{const}(n)}{r} \sup_{\widetilde{K}} |h_k| \qquad \text{for all } a \in K \text{ and } k \in \mathbb{N}\,.$$

Thus, from uniform boundedness of $(h_k)_{k\in\mathbb{N}}$ on $\widetilde{K}$ (which we have by assumption) we infer uniform boundedness of $(\nabla h_k)_{k\in\mathbb{N}}$ on K, and this in turn implies that the h_k are equi-Lipschitz on K. The Arzelà-Ascoli theorem[10] then yields a subsequence $\big(h_{k_\ell}\big)_{\ell\in\mathbb{N}}$ which converges uniformly on K. Exhausting Ω with countably many suitable compact subsets K and using the diagonal sequence trick, one can then show the existence of *one* subsequence such that this convergence actually holds for *all* compact subsets K of Ω. □

Proof of the convergence theorem. Given a compact subset K of Ω, we choose a larger compact set $\widetilde{K} \subset \Omega$ and $r > 0$ as in the proof of the compactness theorem. Then, by the interior estimates, we obtain

$$|\partial^\alpha h_\ell(a) - \partial^\alpha h_k(a)| \le \frac{\operatorname{const}(n, |\alpha|)}{r^{|\alpha|}} \sup_{\widetilde{K}} |h_\ell - h_k| \qquad \text{for all } a \in K,\ \alpha \in \mathbb{N}_0^n, \text{ and } k, \ell \in \mathbb{N}\,.$$

Thus, from the uniform convergence of $(h_k)_{k\in\mathbb{N}}$ on $\widetilde{K}$ (which we have by assumption) we infer, for every $\alpha \in \mathbb{N}_0^n$, that $(\partial^\alpha h_k)_{k\in\mathbb{N}}$ is a uniform Cauchy sequence on K, thus uniformly convergent on K, and locally uniformly convergent in Ω. In this situation, a standard analysis result ensures that the limits are the 'correct' ones, that is, $\lim_{k\to\infty} \partial^\alpha h_k = \partial^\alpha h$ locally uniformly in Ω for all $\alpha \in \mathbb{N}_0^n$. In particular, we get $\Delta h = \lim_{k\to\infty} \Delta h_k$, and thus h inherits harmonicity from h_k. □

Theorem (Harnack inequality). *For every non-empty compact subset K of a connected Ω, there exists a constant $C = \operatorname{const}(K, \Omega) \in [1, \infty)$ such that*

$$\max_K h \le C \min_K h \qquad \textit{holds for all } \textbf{\textit{non-negative}} \textit{ harmonic functions } h \textit{ in } \Omega\,.$$

Proof. In a first step, we consider a ball $\mathrm{B}_r(a) \subset \Omega$ and arbitrary points $x, y \in \overline{\mathrm{B}_{r/4}(a)}$ in the smaller concentric ball $\overline{\mathrm{B}_{r/4}(a)}$. We observe $\mathrm{B}_{r/4}(x) \subset \mathrm{B}_{3r/4}(y)$ (and moreover that the closures of both these balls are contained in $\mathrm{B}_r(a)$ and Ω). Using the mean value property (twice) and non-negativity of h, we infer

$$h(x) = \frac{1}{\omega_n (r/4)^n} \int_{\mathrm{B}_{r/4}(x)} h \,\mathrm{d}z \le 3^n \frac{1}{\omega_n (3r/4)^n} \int_{\mathrm{B}_{3r/4}(y)} h \,\mathrm{d}z = 3^n h(y)\,.$$

[9] The convexity assumption is not restrictive. Indeed it suffices to verify the claim for all closed balls K in Ω.

[10] The Arzelà-Ascoli theorem can be stated as follows: If $(f_k)_{k\in\mathbb{N}}$ is a sequence of *equi-continuous* and pointwisely bounded (and then automatically uniformly bounded) functions on a *compact* metric space $\mathcal{X}$, then there exists a subsequence $\big(f_{k_\ell}\big)_{\ell\in\mathbb{N}}$ which converges uniformly on $\mathcal{X}$. As in the case at hand, this is often applied to a sequence of equi-Lipschitz functions f_k, i.e. to functions f_k which are all Lipschitz continuous with a fixed Lipschitz constant.

The resulting estimate is the Harnack inequality for $K = \overline{\mathrm{B}_{r/4}(a)}$.

In a second step, we carry over this estimate to the non-empty compact subset K from the statement of the theorem. In view of the connectedness of Ω we can assume that K is connected (for, if it is not, we can replace it with a larger compact subset which has this property). By compactness of K we can moreover find a finite cover $(B_i)_{i=1,2,\ldots,M}$ of K by balls $B_i = \mathrm{B}_{r_i/4}(a_i)$ such that $\mathrm{B}_{r_i}(a_i) \subset \Omega$ holds and the first step applies on B_i. For the moment, we now fix $x \in K$ and consider the auxiliary set S of points $y \in K$ which can be reached from x via a chain of balls from the cover in the sense that there exist distinct indices $i_1, i_2, \ldots, i_\ell \in \{1, 2, \ldots, M\}$ with $x \in B_{i_1}$, $B_{i_j} \cap B_{i_{j+1}} \neq \emptyset$ for $j = 1, 2, \ldots, \ell{-}1$, and $y \in B_{i_\ell}$. It turns out that S is both open and closed in K (since each $x \in S$ and $x \in K \setminus S$, respectively, are contained in a ball B_i, and then all points of this open ball belong to S and $K \setminus S$, respectively). Thus S equals the connected set K, and the connecting chain of balls B_{i_j} in the preceding sense exists for all $x, y \in K$. Once we know this, we can choose arbitrary points $x_j \in B_{i_j} \cap B_{i_{j+1}}$ and apply the estimate of the first step along the chain of balls as follows:

$$h(x) \le 3^n h(x_1) \le 3^{2n} h(x_2) \le 3^{3n} h(x_3) \le \ldots \le 3^{(\ell-1)n} h(x_{n-1}) \le 3^{\ell n} h(y)\,.$$

In view of $\ell \le M$ we infer $h(x) \le 3^{Mn} h(y)$ for arbitrary points $x, y \in K$. By taking the sup in $x \in K$ and the inf in $y \in K$ we then arrive at the claim with constant $C = 3^{Mn}$ (where M depends only on the initial choice of the cover and thus only on K and Ω). $\square$

Remark (on invariance of the Harnack constant)**.** The **Harnack constant**, that is, the optimal constant in the Harnack inequality, is **invariant under isometries and scaling**. More precisely, given (K, Ω) as in the theorem, $a \in \mathbb{R}^n$, $T \in \mathcal{O}(\mathbb{R}^n)$, and $r > 0$, the Harnack inequality holds for $(a{+}rT(K), a{+}rT(\Omega))$ with the same constant as for (K, Ω) itself.

The proof of this claim is based on the observation that *harmonic* functions h in Ω correspond to *harmonic* functions $\widetilde{h}$ in $a{+}rT(\Omega)$ through the transformation $\widetilde{h}(a{+}rTx) = h(x)$ for $x \in \Omega$.

Corollary (*one-sided* Liouville property for entire harmonic functions).

(I) *If h is harmonic in $\mathbb{R}^n$ and bounded from either above or below in $\mathbb{R}^n$, then h is necessarily constant in $\mathbb{R}^n$.*

(II) *If h is harmonic in $\mathbb{R}^n$ and either h_+ or h_- has polynomial growth $\lim_{|x|\to\infty} \frac{h_\pm(x)}{|x|^{m+1}} = 0$ with $m \in \mathbb{N}_0$, then h is a polynomial of degree $\le m$.*

(III) *If u is* sub*harmonic in $\mathbb{R}^2$ and u_+ grows sub-logarithmically, that is $\lim_{|x|\to\infty} \frac{u_+(x)}{\log |x|} = 0$, then u is necessariliy constant in $\mathbb{R}^2$.*

Remarks (on optimality of the growth conditions)**.**

(1) The growth condition in (II) is optimal in the following sense: There exist harmonic functions h on $\mathbb{R}^n$ (e.g. homogeneous harmonic polynomials of degree $m{+}1$) which satisfy $\limsup_{|x|\to\infty} \frac{|h(x)|}{|x|^{m+1}} < \infty$ (and thus $\lim_{|x|\to\infty} \frac{h(x)}{|x|^{m+1+\delta}} = 0$ for every $\delta > 0$), but are not polynomials of degree $\le m$. Clearly, this discussion also shows the optimality of the analogous growth condition in the both-sided Liouville property from the beginning of the section.

(2) Similarly the restriction to two dimensions and the growth in (III) are also optimal. This follows from the fact that $\max\{F, -1\}$ is a non-constant subharmonic harmonic function on $\mathbb{R}^n$ in the generalized sense of the subsequent Section 2.7, and a mollification u thereof is even

non-constant subharmonic in the classical sense. In dimensions $n \geq 3$ these functions are additionally bounded (from above *and* below), while in dimension $n = 2$ they are bounded from below and satisfy $\limsup_{|x|\to\infty} \frac{u_+(x)}{\log|x|} < \infty$.

Proof. For the proof of (I) it suffices to treat the case of boundedness from above, that is $M := \sup_{\mathbb{R}^n} h < \infty$. By the Harnack inequality for the non-negative harmonic function $M-h$ in $\mathbb{R}^n$, we then get

$$\max_{\overline{\mathrm{B}_r}} (M-h) \leq C \min_{\overline{\mathrm{B}_r}} (M-h) \qquad \text{for arbitrary } r > 0\,.$$

Sending $r \to \infty$, we conclude

$$\sup_{\mathbb{R}^n} (M-h) \leq C \min_{\mathbb{R}^n} (M-h) = 0$$

and read off $h \equiv M$ on $\mathbb{R}^n$.

The proofs of (II) and (III) are discussed in the exercise class. □

Corollary (**Harnack convergence theorem**). *Consider a domain Ω in $\mathbb{R}^n$ and a sequence $(h_k)_{k\in\mathbb{N}}$ of harmonic functions in Ω such that $h_1 \leq h_2 \leq h_3 \leq \ldots$ holds in Ω. Then, either we have $\lim_{k\to\infty} h_k(x) = \infty$ for all $x \in \Omega$, or the sequence $(h_k)_{k\in\mathbb{N}}$ converges locally uniformly in Ω to a harmonic limit function.*

Remark. In particular, if one knows $\lim_{k\to\infty} h_k(x_0) < \infty$ at a *single* point $x_0 \in \Omega$, the theorem can be applied (and is commonly used in this way) to deduce locally uniform convergence in Ω.

Proof. We assume $\lim_{k\to\infty} h_k(x_0) < \infty$ for some $x_0 \in \Omega$. For $k \leq \ell$ in $\mathbb{N}$ and a compact subset $K \subset \Omega$ with $x_0 \in K$, we then get

$$\max_K (h_\ell - h_k) \leq C \min_K (h_\ell - h_k) \leq C(h_\ell(x_0) - h_k(x_0))$$

from the Harnack inequality for the non-negative harmonic function $h_\ell - h_k$ in Ω. In view of this estimate, the Cauchy property of $(h_k(x_0))_{k\in\mathbb{N}}$ implies the uniform Cauchy property of $(h_k)_{k\in\mathbb{N}}$ on K. This property induces locally uniform convergence of $(h_k)_{k\in\mathbb{N}}$ in Ω. Harmonicity of the limit function then results from the previously treated Weierstraß-type convergence theorem. □

2.7 Generalized sub/superharmonic functions

In Section 2.5 the concept of harmonicity has been extended to non-C^2 functions in two rather different ways. On one hand, (very) weakly harmonic functions have been defined via integration-by-parts formulas. On the other hand, a notion of generalized harmonic functions, based on the mean value property, has been discussed. Both these approaches can be adapted in order to **explain *sub*harmonicity and *super*harmonicity for non-$\mathbf{C}^2$ functions**. In contrast to the harmonic case, however, one cannot expect that generalized sub/superharmonic functions exhibit additional regularity, and thus the resulting concepts are truly more general than the ones for C^2 functions.

Here we dispense with weak notions based on integration by parts (though also these are common in PDE theory). Rather, with applications in the later existence theory of the Section 2.10 in mind, we **turn directly to generalized notions based on mean value inequalities:**

Definition (**general sub/superharmonic functions**)**.** *An upper semicontinuous function* $u\colon \Omega \to [-\infty,\infty)$ *is* ***(generalized) subharmonic*** *in* Ω *if it satisfies the mean value inequality*[11]

$$u(a) \le \fint_{\mathrm{B}_r(a)} u \,\mathrm{d}x \qquad \text{for all } a \in \Omega,\ r>0 \text{ such that } \overline{\mathrm{B}_r(a)} \subset \Omega\,.$$

Similarly, a lower semicontinuous function $u\colon \Omega \to (-\infty,\infty]$ *is* ***(generalized) superharmonic*** *in* Ω *if it satisfies the mean value inequality*

$$u(a) \ge \fint_{\mathrm{B}_r(a)} u \,\mathrm{d}x \qquad \text{for all } a \in \Omega,\ r>0 \text{ such that } \overline{\mathrm{B}_r(a)} \subset \Omega\,.$$

Remarks (on general sub/superharmonic functions)**.**

(1) For $u \in \mathrm{C}^2(\Omega)$ it is clear from Section 2.4 that the generalized notions coincide with the classical requirements $\Delta u \ge 0$ and $\Delta u \le 0$, respectively.

(2) Most **previous results** on sub/superharmonic C^2 functions **extend verbatim** to generalized sub/superharmonic functions. Specifically, the next result shows that this is true for the basic maximum and comparison principles of Section 2.4.

(3) A subharmonic or superharmonic function u in Ω satisfies $\lim_{r\searrow 0} \fint_{\mathrm{B}_r(a)} |u-u(a)|\,\mathrm{d}x = 0$ for *all* $a \in \Omega$. (For $\mathbb{R}$-valued u this means that *all* points in Ω are Lebesgue points and that u itself is the Lebesgue representative).

Proof. We consider the subharmonic case. The mean value inequality and upper semicontinuity yield the chain of inequalities $u(a) \le \liminf_{r\searrow 0} \fint_{\mathrm{B}_r(a)} u\,\mathrm{d}x \le \limsup_{r\searrow 0} \fint_{\mathrm{B}_r(a)} u\,\mathrm{d}x \le \limsup_{\Omega\ni x\to a} u(x) \le u(a)$, and we infer $\lim_{r\searrow 0} \fint_{\mathrm{B}_r(a)} (u-u(a))\,\mathrm{d}x = 0$. In addition, we have $\lim_{r\searrow 0} \fint_{\mathrm{B}_r(a)} (u-u(a))_+\,\mathrm{d}x \le \limsup_{\Omega\ni x\to a} (u(x)-u(a))_+ = 0$ by upper semicontinuity. In view of $|f| = 2f_+ - f$ this is enough to conclude $\lim_{r\searrow 0} \fint_{\mathrm{B}_r(a)} |u-u(a)|\,\mathrm{d}x = 0$. $\square$

Lemma (**characterizations of subharmonic functions**)**.** *For an upper semicontinuous function* $u\colon \Omega \to [-\infty,\infty)$*, the following properties are* ***equivalent:***

(a) *u satisfies the* ***mean value inequality on balls****, that is, u is generalized subharmonic on Ω in the sense of the above definition.*

(b) *u satisfies the* ***mean value inequality on spheres***

$$u(a) \le \fint_{\mathrm{S}_r(a)} u \,\mathrm{d}\mathcal{H}^{n-1} \qquad \text{for all } a \in \Omega,\ r>0 \text{ such that } \overline{\mathrm{B}_r(a)} \subset \Omega\,.$$

(c) *u satisfies the* ***comparison principle*** *as follows:*

$$\left.\begin{array}{c} G \text{ bounded open set in } \mathbb{R}^n\,,\ \overline{G} \subset \Omega\,, \\ h \in \mathrm{C}^2(G) \cap \mathrm{C}^0(\overline{G}) \text{ harmonic in } G\,, \\ u \le h \text{ at } \partial G \end{array}\right\} \implies u \le h \text{ in } G\,.$$

(d) *u satisfies the* ***mean value inequality on*** **small** ***balls****, that is, for every $a \in \Omega$, there exists some (possibly small but positive) $r_a \in (0,\infty]$ such that*

$$u(a) \le \fint_{\mathrm{B}_r(a)} u \,\mathrm{d}x \qquad \text{for all } r \in (0,r_a) \text{ such that } \overline{\mathrm{B}_r(a)} \subset \Omega\,.$$

[11] Upper semicontinuity of u implies its Borel measurability and boundedness from above in $\mathrm{B}_r(a)$. Thus, the mean value integral exists in $[-\infty,\infty)$.

Specifically, the lemma shows that **generalized subharmonicity is**, in fact, **a local property** in the following sense: If $(O_i)_{i\in I}$ is a family of open sets (over an arbitrary index set I) with $\bigcup_{i\in I} O_i = \Omega$ and u is subharmonic on O_i for every $i \in I$, then u is also subharmonic on Ω itself. While this locality principle is obvious for subharmonicity in the classical $\Delta u \geq 0$ sense, for generalized subharmonicity it results only from property (d) in the lemma and is not obvious from (a), (b), or (c) alone. We foreshadow that the locality principle for generalized subharmonic functions will be very useful in establishing a basic existence result for *harmonic* functions in the later Section 2.10.

Proof of the lemma. We first record that (a) implies (d) trivially.

Moreover, (d) implies (c), by the reasoning from the proof of the strong maximum principle in Section 2.4. Indeed, this reasoning can be adapted to work on the connected components of G and derives the weak maximum principle for $u{-}h$ from the mean value property of $u{-}h$.

Next we show that (c) implies (b). To this end, we fix a ball $\overline{\mathrm{B}_r(a)} \subset \Omega$, we assume $u \not\equiv -\infty$ in $\overline{\mathrm{B}_r(a)}$, and we rely, for $k \in \mathbb{N}$, on the often useful standard construction

$$\varphi_k(x) := \max_{y\in\overline{\mathrm{B}_r(a)}} \big(u(y){-}k|x{-}y|\big) \in \mathbb{R} \qquad \text{for } x \in \mathrm{S}_r(a)\,.$$

Here, compactness of $\overline{\mathrm{B}_r(a)}$ and upper semicontinuity imply that the maximum is attained. Furthermore, we record:

- It is clear from the definition that $\varphi_k \geq u$ and $\varphi_k \geq \varphi_{k+1}$ hold at $\mathrm{S}_r(a)$ for all $k \in \mathbb{N}$.
- The upper semicontinuity of u implies $\lim_{k\to\infty} \varphi_k(x) = u(x)$ for all $x \in \mathrm{S}_r(a)$.
  (Justification: In view of the above, $\lim_{k\to\infty} \varphi_k(x)$ exists in $[-\infty,\infty)$ and is $\geq u(x)$. From the definition of φ_k, we infer $\varphi_k(x) \leq \max\{\max_{\overline{\mathrm{B}_r(a)}\cap \mathrm{B}_\delta(x)} u, \max_{\overline{\mathrm{B}_r(a)}} u - k\delta\}$ for all $k \in \mathbb{N}$, $\delta > 0$. Sending first $k \to \infty$, then $\delta \searrow 0$, we infer first $\lim_{k\to\infty} \varphi_k(x) \leq \max_{\overline{\mathrm{B}_r(a)}\cap \mathrm{B}_\delta(x)} u$, then $\lim_{k\to\infty} \varphi_k(x) \leq u(x)$ by upper semicontinuity of u.)
- For every $k \in \mathbb{N}$, the function φ_k is the pointwise maximum of Lipschitz functions with Lipschitz constant k, and thus also φ_k itself is a Lipschitz function with Lipschitz constant $\leq k$, in particular $\varphi_k \in \mathrm{C}^0(\mathrm{S}_r(a))$.

By the existence theorem at the end of Section 2.5, we can then find, for every $k \in \mathbb{N}$, a solution h_k to the Dirichlet problem for harmonic functions

$$h_k \in \mathrm{C}^2(\mathrm{B}_r(a)) \cap \mathrm{C}^0(\overline{\mathrm{B}_r(a)})\,, \qquad \Delta h_k \equiv 0 \text{ in } \mathrm{B}_r(a)\,, \qquad h_k = \varphi_k \text{ at } \mathrm{S}_r(a)\,.$$

By the comparison principle, the inequality $u \leq \varphi_k = h_k$ at $\mathrm{S}_r(a)$ extends to $u \leq h_k$ in $\mathrm{B}_r(a)$ and specifically to $u(a) \leq h_k(a)$ for all $k \in \mathbb{N}$. Applying this observation together with the mean value property of h_k on spheres $\mathrm{S}_\varrho(a)$ with $0 < \varrho < r$ (then $\overline{\mathrm{B}_\varrho(a)}$ is contained in the domain $\mathrm{B}_r(a)$ where h_k are harmonic), we conclude

$$u(a) \leq h_k(a) \leq \lim_{\varrho\nearrow r} \fint_{\mathrm{S}_\varrho(a)} h_k \,\mathrm{d}\mathcal{H}^{n-1} = \fint_{\mathrm{S}_r(a)} h_k \,\mathrm{d}\mathcal{H}^{n-1} = \fint_{\mathrm{S}_r(a)} \varphi_k \,\mathrm{d}\mathcal{H}^{n-1}\,.$$

Finally, the monotone convergence theorem ensures $\lim_{k\to\infty} \fint_{\mathrm{S}_r(a)} \varphi_k \,\mathrm{d}\mathcal{H}^{n-1} = \fint_{\mathrm{S}_r(a)} u \,\mathrm{d}\mathcal{H}^{n-1}$, and thus the mean value inequality of (b) holds for $\mathrm{S}_r(a)$.

Finally, from (b) we get back to (a) by a spherical integration argument, which closely follows the last part of the proof of the mean value property in Section 2.4. □

The concept of generalized subharmonic functions turns out to be very convenient also in the treatment of the following (classes of) examples and basic principles:

Examples ('many' (generalized) subharmonic functions).

(1) If the **fundamental solution** F is extended by setting $F(0) := -\infty$, then — beside being harmonic in $\mathbb{R}^n \setminus \{0\}$ — it turns out to be **subharmonic in all of $\mathbb{R}^n$**. This can be verified with the help of the previous lemma: Indeed, the harmonicity of F in $\mathbb{R}^n \setminus \{0\}$ implies the validity of property (d) for $a \in \mathbb{R}^n \setminus \{0\}$ with $r_a := |a|$, while in view of $F(0) = -\infty$ this property holds trivially for $a = 0$ with $r_0 := \infty$.

(2) **Convex functions**[12] $u\colon \Omega \to [-\infty, \infty)$ in convex open sets $\Omega \subset \mathbb{R}^n$ **are subharmonic**.

Proof. It follows from the definition of convexity that u is either constant $\equiv -\infty$ or finite-valued. In any case, u is continuous on Ω (trivially in the former and by a basic result of convex analysis in the latter case). Moreover, Jensen's inequality gives

$$u(a) = u\bigg(\fint_{\mathrm{B}_r(a)} x \,\mathrm{d}x\bigg) \le \fint_{\mathrm{B}_r(a)} u(x) \,\mathrm{d}x \qquad \text{whenever } \overline{\mathrm{B}_r(a)} \subset \Omega\,.$$

By definition, this means that u is subharmonic on Ω. □

For example,y $u(x) := |x|$ and $u(x) := \mathrm{e}^{|x|}$ give subharmonic functions u in $\Omega = \mathbb{R}^n$.

(3) If u and v are two subharmonic functions in an open set $\Omega \subset \mathbb{R}^n$, also the **pointwise maximum** $\max\{u, v\}$ is **subharmonic** in Ω.

Proof. Whenever $\overline{\mathrm{B}_r(a)} \subset \Omega$, we clearly have $u(a) \le \fint_{\mathrm{B}_r(a)} u \,\mathrm{d}x \le \fint_{\mathrm{B}_r(a)} \max\{u, v\} \,\mathrm{d}x$ and $v(a) \le \fint_{\mathrm{B}_r(a)} v \,\mathrm{d}x \le \fint_{\mathrm{B}_r(a)} \max\{u, v\} \,\mathrm{d}x$, thus $\max\{u(a), v(a)\} \le \fint_{\mathrm{B}_r(a)} \max\{u, v\} \,\mathrm{d}x$. □

This implies, for instance, that $\max\{F, s\}$ with the fundamental solution F and $s \in \mathbb{R}$ and the positive part of any harmonic function are subharmonic on $\mathbb{R}^n$.

As an obvious consequence, also the pointwise maximum of any *finite* family of subharmonic functions in Ω remains subharmonic in Ω. Moreover, an analogous reasoning shows that even the pointwise supremum of an *arbitrary* family of subharmonic functions on Ω is subharmonic on Ω *provided that* this supremum is — what is not automatic in case of an infinite family — still upper semicontinuous.

(4) The following interconnected assertions hold for open $\Omega \subset \mathbb{R}^n$:

- If h is harmonic in Ω, then $|h|^s$ with any $s \in [1, \infty)$ is subharmonic in Ω.
- If h is harmonic on Ω and $\Phi\colon (a, b) \to [-\infty, \infty)$ is convex in an open interval $(a, b) \subset \mathbb{R}$ such that $h(\Omega) \subset (a, b)$, then $\Phi(h)$ is subharmonic in Ω.
- If $H\colon \Omega \to \mathbb{R}^N$ is a vector-valued harmonic function in Ω (that is, all its components H_i with $i \in \{1, 2, \ldots, N\}$ are harmonic in Ω) and $\Phi\colon C \to [-\infty, \infty)$ is convex in a convex open set $C \subset \mathbb{R}^N$ such that $H(\Omega) \subset C$, then $\Phi(H)$ is subharmonic in Ω.

[12] Convexity of $[-\infty, \infty)$-valued functions u on convex sets C is defined by the usual convexity inequality $u(\lambda x + (1{-}\lambda) y) \le \lambda u(x) + (1{-}\lambda) u(y)$ for all $x, y \in C$, $\lambda \in [0, 1]$.

Proof. We prove the last claim, which contains the previous ones as special cases. As in the proof of (2), Φ is either constant $\equiv -\infty$ or finite-valued, but in any case is continuous in C. Thus, $\Phi(H)$ is continuous in Ω, and via the mean value property of H and Jensen's inequality we infer

$$\Phi(H(a)) = \Phi\bigg(\fint_{\mathrm{B}_r(a)} H \,\mathrm{d}x\bigg) \le \fint_{\mathrm{B}_r(a)} \Phi(H)\,\mathrm{d}x \qquad \text{whenever } \overline{\mathrm{B}_r(a)} \subset \Omega\,.$$

By definition, this means that $\Phi(H)$ is subharmonic in Ω. □

For instance, this implies that $u(x) := |x_1x_2|^s$ defines, for every fixed $s \in [1,\infty)$, a subharmonic function u in $\Omega = \mathbb{R}^n$

(5) The following **convergence theorems for subharmonic functions** hold on open $\Omega \subset \mathbb{R}^n$:

(a) If u_k are subharmonic in Ω with $u_k \searrow u$ pointwisely in Ω (i.e. $u_1 \ge u_2 \ge u_3 \ge \ldots$ and $\lim_{k\to\infty} u_k = u$ on Ω), then u is subharmonic in Ω.

(b) If u_k are subharmonic in Ω with $u_k \to u$ pointwisely in Ω and $u_k \le u$ in Ω for $k \gg 1$ and if u is upper semicontinuous and finite-valued in Ω, then u is subharmonic in Ω.

(c) If u_k are subharmonic in Ω with $\lim_{k\to\infty} u_k = u$ locally uniformly in Ω for finite-valued u, then u is subharmonic in Ω.

Proof. In all cases, we first justify upper semicontinuity of u: Under the assumptions of (5a), it follows from $u = \inf_{k\in\mathbb{N}} u_k$. In (5b), it is assumed. In the situation of (5c), it results from the locally uniform convergence. Then we get

$$u(a) = \lim_{k\to\infty} u_k(a) \le \limsup_{k\to\infty} \fint_{\mathrm{B}_r(a)} u_k\,\mathrm{d}x \le \fint_{\mathrm{B}_r(a)} u\,\mathrm{d}x \qquad \text{whenever } \overline{\mathrm{B}_r(a)} \subset \Omega \qquad (*)$$

with different justifications for having $\limsup_{k\to\infty} \fint_{\mathrm{B}_r(a)} u_k\,\mathrm{d}x \le \fint_{\mathrm{B}_r(a)} u\,\mathrm{d}x$: In the (5a) case, it comes (even as equality) from the monotone convergence theorem (where $\sup_{\mathrm{B}_r(a)} u_1 < \infty$ by upper semicontinuity). In the situation of (5b), it results simply from $u_k \le u$ in Ω for $k \gg 1$. In case of (5c), it follows (even as equality) from uniform convergence in $\mathrm{B}_r(a)$. □

2.8 Green's representation formula and the Poisson integral

In this section we come back to the **Dirichlet problem**

$$\begin{aligned} \Delta u &= f &\qquad& \text{in } \Omega\,, \\ u &= \varphi && \text{at } \partial\Omega \end{aligned}$$

for the Poisson equation on a bounded open set $\Omega \subset \mathbb{R}^n$ (where $f \in \mathrm{C}^0(\Omega)$ and $\varphi \in \mathrm{C}^0(\partial\Omega)$ are given). While uniqueness of solutions u has been shown in Section 2.4 in large generality, so far we have solved the existence problem for solutions u only in specific cases, namely for $f \equiv 0$ and balls Ω; see the end of Section 2.5. We now strive for establishing **existence on general bounded domains** Ω on the basis of **explicit formulas for solutions**. Though in the end the taken approach will be fully successful only for specific domains Ω (most prominently again for balls), in this section we will lay some foundations for other approaches and gain insight on harmonic functions and the Dirichlet problem.

As a first natural step we deal with explicit formulas valid for *given solutions* u. Only eventually we are then in position for the second step, namely to *define solutions* u by these formulas. As a starting point in direction of the first step, we have:

Theorem (Green's representation formula). *Consider a Gauss-Green domain Ω in $\mathbb{R}^n$ and a function $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^1(\overline{\Omega})$. For each $x \in \Omega$ such that $\int_\Omega F(x-y)\Delta u(y)\,\mathrm{d}y$ exists in $[-\infty, \infty]$, we have*

$$u(x) = -\int_{\partial\Omega} F(x-y)\partial_\nu u(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) + \int_{\partial\Omega} u(y)(\partial_\nu)_y F(x-y)\,\mathrm{d}\mathcal{H}^{n-1}(y) + \int_\Omega F(x-y)\Delta u(y)\,\mathrm{d}y\,,$$

where F denotes the fundamental solution of the Laplace equation in $\mathbb{R}^n$.

Remarks (on the representation formula)**.**

(1) The single terms on the right-hand side have interpretations in potential theory: The term $\int_{\partial\Omega} F(x-y)\psi(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)$ is called a single-layer boundary potential, while the term $\int_{\partial\Omega} \varphi(y)(\partial_\nu)_y F(x-y)\,\mathrm{d}\mathcal{H}^{n-1}(y)$ is a double-layer dipole potential. Finally, the volume potential $\int_\Omega F(x-y)f(y)\,\mathrm{d}y$ is known as Newton potential and is studied in more detail in Section 2.11.

(2) The second and third terms on the right-hand side involve only the boundary values $u|_{\partial\Omega}$ and the Laplacian Δu of u, which correspond to the prescribed data f and φ in the Dirichlet problem. Thus, these terms are 'good' for defining u by a formula in terms of f and φ only. However, the first term on the right-hand side is unfortunately 'bad' in the sense that it contains the normal derivative $\partial_\nu u|_{\partial\Omega}$, which is not prescribed in the Dirichlet problem. We will return to this point below and will develop a strategy to circumvent it.

(As a side remark we record that the situation slightly changes in case of the Neumann problem: Then the first and third terms are 'good', while the second is 'bad'.)

Proof of Green's representation formula. We argue for fixed $x \in \Omega$ and for $\varepsilon > 0$ small enough that $\overline{\mathrm{B}_\varepsilon(x)} \subset \Omega$ holds. We start by applying Green's second identity on the domain[13] $\Omega \setminus \overline{\mathrm{B}_\varepsilon(x)}$ with boundary $(\partial\Omega) \cup \mathrm{S}_\varepsilon(x)$ to the functions u and $y \mapsto F(x-y)$. In this way we get

$$\begin{aligned}
&\int_{\Omega\setminus\mathrm{B}_\varepsilon(x)} F(x-y)\Delta u(y)\,\mathrm{d}y - \int_{\Omega\setminus\mathrm{B}_\varepsilon(x)} u(y)\Delta_y F(x-y)\,\mathrm{d}y\\
&\qquad = \int_{\partial\Omega} F(x-y)\partial_\nu u(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) - \int_{\partial\Omega} u(y)(\partial_\nu)_y F(x-y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\\
&\qquad\quad + \int_{\mathrm{S}_\varepsilon(x)} F(x-y)\,\tfrac{x-y}{\varepsilon}\cdot\nabla u(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) - \int_{\mathrm{S}_\varepsilon(x)} u(y)\,\tfrac{x-y}{\varepsilon}\cdot\nabla_y F(x-y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\,,
\end{aligned}$$

where at $y \in \mathrm{S}_\varepsilon(x)$ the outward unit normal ν takes the form $\nu(y) = \frac{x-y}{\varepsilon}$. Here the second integral on the left-hand side vanishes, since $y \mapsto F(x-y)$ is harmonic on $\mathbb{R}^n \setminus \{x\} \supset \Omega \setminus \mathrm{B}_\varepsilon(x)$.

[13] At this point, a marginal technical difficulty arises from the fact that only Ω has been assumed to be a Gauss-Green domain but not $\Omega \setminus \overline{\mathrm{B}_\varepsilon(x)}$. However, this can easily be overcome by cutting-off the singularity of F in the sense that one chooses some $F_\varepsilon \in \mathrm{C}^2(\mathbb{R}^n)$ which coincides with F on $\mathbb{R}^n \setminus \mathrm{B}_\varepsilon$. Then one may apply Green's identity to u and $y \mapsto F_\varepsilon(x-y)$ on both Ω and $\mathrm{B}_\varepsilon(x)$ and arrives at the above by subtracting the resulting formula on $\mathrm{B}_\varepsilon(x)$ from the one on Ω.

Moreover, taking into account the explicit form of F, the third integral on the right-hand side takes the form $-\frac{\varepsilon}{n-2}\fint_{\mathrm{S}_\varepsilon(x)}\frac{x-y}{\varepsilon}\cdot\nabla u(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)$ for $n\geq 3$ and $\varepsilon(\log\varepsilon)\fint_{\mathrm{S}_\varepsilon(x)}\frac{x-y}{\varepsilon}\cdot\nabla u(y)\,\mathrm{d}\mathcal{H}^{1}(y)$ for $n=2$, respectively. In view of the ε-prefactors, this integral will eventually disappear in the limit $\varepsilon\searrow 0$. For the treatment of the crucial fourth integral on the right-hand side, we compute $\frac{x-y}{\varepsilon}\cdot\nabla_y F(x-y)=-\frac{1}{n\omega_n\varepsilon^{n-1}}$ for $y\in\mathrm{S}_\varepsilon(x)$ and rewrite this integral as $-\fint_{\mathrm{S}_\varepsilon(x)}u\,\mathrm{d}\mathcal{H}^{n-1}$. Thus, in the limit $\varepsilon\searrow 0$ it turns simply into $-u(x)$. Incorporating all these remarks and sending $\varepsilon\searrow 0$ in the above identity, we arrive at

$$\begin{aligned}\int_\Omega F(x-y)\Delta u(y)\,\mathrm{d}y\\ &= \int_{\partial\Omega}F(x-y)\partial_\nu u(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\;-\int_{\partial\Omega}u(y)(\partial_\nu)_yF(x-y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\;+u(x)\,,\end{aligned}$$

where on the left-hand side we made use of $F\in\mathrm{L}^1(\mathrm{B}_\varepsilon(x))$ in order to guarantee the convergence of the integrals on $\Omega\setminus\mathrm{B}_\varepsilon(x)$ to the limit integral on all of Ω. By rearranging terms in the last equation we easily arrive at the claim. □

Next we aim at removing the ‘bad’ first term in Green’s representation formula. This will be approached by replacing the shifted fundamental solutions $y\mapsto F(x-y)$ in this formula with $y\mapsto F(x-y)-h_x(y)$, where the **corrector functions** h_x are chosen as a harmonic functions on Ω such that $y\mapsto F(x-y)-h_x(y)$ vanishes on $\partial\Omega$. With the properties of the resulting function $G(x,y)\coloneqq F(x-y)-h_x(y)$ in mind, we coin the following definition:

Definition (**Green function**)**.** *Consider an open set Ω in $\mathbb{R}^n$. We call a $2n$-variables function $G\colon\{(x,y)\in\Omega\times\Omega: y\neq x\}\to\mathbb{R}$ the **Green function of Ω** (or, at length, the Green function for the Dirichlet problem for harmonic functions in Ω) if it satisfies the following two conditions:*

(a) *For every $x\in\Omega$, the function $y\mapsto F(x-y)-G(x,y)$ on $\Omega\setminus\{x\}$ can be extended to a harmonic function in Ω.*

(b) *For every $x\in\Omega$, the function $y\mapsto G(x,y)$ vanishes at $\partial\Omega$ in the sense of $\lim_{\Omega\ni y\to y_0}G(x,y)=0$ for every $y_0\in\partial\Omega$ (and moreover $\lim_{\Omega\ni y\to\infty}G(x,y)=0$ in case of unbounded Ω).*

Remarks (on Green functions)**.**

(0) Clearly, the function $y\mapsto F(x-y)-G(x,y)$ in (a) plays the role of the harmonic corrector function h_x mentioned before.

(1) For fixed $x\in\Omega$ it follows from (a) and the harmonicity of F on $\mathbb{R}^n\setminus\{0\}$ that $G(x,\,\cdot\,)$ is harmonic in $\Omega\setminus\{x\}$. Moreover, the harmonic extension in (a) is continuous (and even smooth) near x. This means that $y\mapsto G(x,y)$ and $y\mapsto F(x-y)$ have a singularity of the same type at the point x and that the singularities cancel out when taking the difference. All in all, “$\Delta\big[G(x,\,\cdot\,)\big]=\delta_x$” holds in Ω (in the same heuristic sense in which we observed “$\Delta F=\delta_0$” in $\mathbb{R}^n$ in Section 2.1). In this light one may say that the **Green function consists of fundamental solutions with singularity at x and zero boundary values**.

(2) The **Green function is unique** if it exists. This follows from uniqueness in the Dirichlet problem solved by the corresponding harmonic corrector functions h_x on Ω (where uniqueness, in turn, results from the weak maximum principle and holds even in case of unbounded Ω, since $\infty_{\mathbb{R}^n}$ is suitably taken into account).

(3) Since the Green function has zero boundary values, the weak maximum principle implies that the **Green function is non-positive** if it exists. In fact, in order to apply the weak maximum principle here one may take two slightly different technical viewpoits. For arbitrary $x \in \Omega$, one may either extend $G(x, \cdot\,)$ by setting $G(x,x) := -\infty$ and view it as a subharmonic function in all of Ω in the sense of Section 2.7, or one may view $G(x, \cdot\,)$ as a harmonic function in $\Omega \setminus \{x\}$ only, but take into account $\lim_{y\to x} G(x,y) = -\infty \le 0$ at the additional boundary point x of $\Omega \setminus \{x\}$.

(4) If a Gauss-Green domain Ω has a Green function G such that $G(x, \cdot\,) \in \mathrm{C}^1(\overline{\Omega} \setminus \{x\})$ holds for all $x \in \Omega$, then — somewhat surprisingly — ***G* is symmetric** in the sense of $G(y,x) = G(x,y)$ for all $x, y \in \Omega$ with $y \neq x$.

Proof. Similar to the proof of Green's representation formula, we apply Green's second identity to the functions $v := G(x, \cdot\,)$ and $w := G(y, \cdot\,)$ on $\Omega \setminus (\overline{\mathrm{B}_\varepsilon(x)} \cup \overline{\mathrm{B}_\varepsilon(y)})$ (where $\varepsilon > 0$ is small enough that $\overline{\mathrm{B}_\varepsilon(x)}$ and $\overline{\mathrm{B}_\varepsilon(y)}$ are disjoint subsets of Ω). Since both v and w are harmonic in this domain and vanish at $\partial\Omega$, we deduce

$$0 = \int_{\mathrm{S}_\varepsilon(x)} \big(v\,\partial_\nu w - w\,\partial_\nu v\big)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\mathrm{S}_\varepsilon(y)} \big(v\,\partial_\nu w - w\,\partial_\nu v\big)\,\mathrm{d}\mathcal{H}^{n-1}\,,$$

where $\nu(z) = \frac{x-z}{\varepsilon}$ for $z \in \mathrm{S}_\varepsilon(x)$ and $\nu(z) = \frac{y-z}{\varepsilon}$ for $z \in \mathrm{S}_\varepsilon(y)$. Since v and ∇v blow up at x in the same way as $z \mapsto F(x-z)$ and $z \mapsto \nabla F(x-z)$ (the differences just being bounded functions), while w and $\partial_\nu w$ are continuous at x, we can follow the proof of Green's representation formula to infer $\lim_{\varepsilon\searrow 0} \int_{\mathrm{S}_\varepsilon(x)} v\,\partial_\nu w\,\mathrm{d}\mathcal{H}^{n-1} = 0$ and $\lim_{\varepsilon\searrow 0} \int_{\mathrm{S}_\varepsilon(x)} (-w\,\partial_\nu v)\,\mathrm{d}\mathcal{H}^{n-1} = -w(x)$. Clearly, the terms on $\mathrm{S}_\varepsilon(y)$ can be treated analogously, and thus we arrive at

$$0 = -w(x) + v(y) = G(x,y) - G(y,x)\,.$$

From this, the claim is immediate. □

If we assume that the Green function exists, then we indeed get the desired representation formula:

Theorem (Green function representation). *For a Gauss-Green domain Ω, assume that the Green function G exists and satisfies $G(x, \cdot\,) \in \mathrm{C}^1(\overline{\Omega} \setminus \{x\})$ for all $x \in \Omega$. For $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^1(\overline{\Omega})$, set $f := \Delta u$ and $\varphi := u|_{\partial\Omega}$. Then, for all $x \in \Omega$ such that $\int_\Omega G(x,y)f(y)\,\mathrm{d}y$ exists in $[-\infty, \infty]$, we have*

$$u(x) = \int_{\partial\Omega} \varphi(y)(\partial_\nu)_y G(x,y)\,\mathrm{d}\mathcal{H}^{n-1}(y) + \int_\Omega G(x,y)f(y)\,\mathrm{d}y\,.$$

Proof. We closely follow the proof of Green's representation formula, but decisively replace $F(x-y)$ with $G(x,y)$. Thus, we apply Green's second identity on $\Omega \setminus \overline{\mathrm{B}_\varepsilon(x)}$ for the functions u and $G(x, \cdot\,)$. On one hand we then use that $G(x, \cdot\,)$ is harmonic in $\Omega \setminus \{x\}$. On the other hand we exploit that $G(x,y) = F(x-y) + h_x(y)$ and $(\partial_\nu)_y G(x,y) = (\partial_\nu)_y F(x-y) + \partial_\nu h_x(y)$ hold for $y \in \mathrm{S}_\varepsilon(x)$ with h_x and ∇h_x bounded near x. Altogether this means that we can treat the integrals resulting from Green's second identity in the same way as in the proof of Green's representation formula, where the terms with h_x and $\partial_\nu h_x$ on $\mathrm{S}_\varepsilon(x)$ will vanish in the limit $\varepsilon \searrow 0$. The only essential difference in the computation is that the term $\int_{\partial\Omega} G(x,y)\partial_\nu u(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)$

now vanishes due to the zero boundary values of $G(x,\,\cdot\,)$ at $\partial\Omega$. Rearranging terms as before, we then arrive at the claim.

(In case that both $h_x\Delta u$ and $y\mapsto F(x-y)\Delta u(y)$ are in $\mathrm{L}^1(\Omega)$, which is ensured for $u\in\mathrm{C}^2(\overline{\Omega})$, for instance, one may alternatively proceed as follows. One may then apply Green's second identity on all of Ω for u and h_x to find

$$0=-\int_{\partial\Omega}h_x(y)\,\partial_\nu u(y)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\partial\Omega}u(y)\,\partial_\nu h_x(y)\,\mathrm{d}\mathcal{H}^{n-1}+\int_\Omega h_x(y)\,\Delta u(y)\,\mathrm{d}y\,.$$

One then subtracts this equality from Green's representation formula, takes into account $F(x-y)-h_x(y)=G(x,y)$ for $y\in\overline{\Omega}$, and observes that the terms with $\partial_\nu u$ cancel out in view of even $F(x-y)-h_x(y)=G(x,y)=0$ for $y\in\partial\Omega$. This confirms the claim once more.) ☐

Remarks (on the Green function representation)**.**

(1) As the most important special case, for a *harmonic* function $h\in\mathrm{C}^2(\Omega)\cap\mathrm{C}^1(\overline{\Omega})$ with $\varphi:=h|_{\partial\Omega}$, the Green function representation reads

$$h(x)=\int_{\partial\Omega}\varphi(y)(\partial_\nu)_yG(x,y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\qquad\text{for all }x\in\Omega\,.$$

(2) The Green function representation remains true for functions $u\in\mathrm{C}^2(\Omega)\cap\mathrm{C}^0(\overline{\Omega})$ which are *not necessarily* C^1 *up to the boundary*, and it also extends to suitable *unbounded* domains Ω provided that u and $f=\Delta u$ decay suitably at ∞.

(3) Though the Green function representation has been established for *given solutions* u, in principle it is suited for reversing the point of of view and making an attempt to establish the existence of a solution u to the Dirichlet problem with *given data* f and φ. Under strong assumptions on G, f, and φ, this is indeed possible, that is, it can be verified that the function u defined by the formula is a solution.

(4) However, it does not make sense to enter into the more technical details of (2) and the existence program (3), since we are left with the more **basic problem to secure the existence of the Green function** G at all. Indeed, proving existence of G is the same as proving the existence of the corrector functions h_x, which are themselves solutions of a Dirichlet problem for harmonic functions. Thus, we are still faced with the existence issue in a Dirichlet problem, and one may doubt that anything is won at all. Indeed, it will turn out that the Green function representation does not (easily) lead to an existence theory for general domains Ω. Rather, there is a straightforward gain **only for specific domains** Ω for which the Green function can be (more or less) explicitly determined. The most prominent such case follows:

Theorem (Green function of balls).

(I) *For $x\in\mathbb{R}^n\setminus\{0\}$, we denote by $x^*:=\frac{x}{|x|^2}$ the reflection of x at the unit sphere S_1, and as usual we write F for the fundamental solution of the Laplace equation in $\mathbb{R}^n$. With this terminology the Green function G_{B_1} of the unit ball B_1 in $\mathbb{R}^n$ is given by*

$$\begin{aligned}G_{\mathrm{B}_1}(x,y)&=F(y-x)-F(|x|(y-x^*))\\&=\begin{cases}-\frac{1}{n(n-2)\omega_n}\big[|y-x|^{2-n}-(|x|\,|y-x^*|)^{2-n}\big] & \text{if } n\ge 3\\ \frac{1}{2\pi}\big[\log|y-x|-\log(|x|\,|y-x^*|)\big] & \text{if } n=2\end{cases}\end{aligned}$$

for $y\neq x$ in B_1, where $|x|\,|y-x^*|=1$ is understood for $y\neq 0=x$.

(II) *The Green function $G_{\mathrm{B}_R(a)}$ of an arbitrary ball $\mathrm{B}_R(a)$ in $\mathbb{R}^n$ is given by*

$$G_{\mathrm{B}_R(a)}(x,y) = R^{2-n} G_{\mathrm{B}_1}\Big(\frac{x-a}{R}, \frac{y-a}{R}\Big) \qquad \textit{for } y \neq x \textit{ in } \mathrm{B}_R(a)\,.$$

Remarks (on Green functions of balls)**.**

(1) Since the fundamental solution yields the electric potential of a unit charge at the origin, $G_{\mathrm{B}_1}(x,\,\cdot\,)$ corresponds to the electric potential induced by two charges, a unit charge at the point x and an antipolar charge $|x|^{2-n}$ at the reflection point x^*.

(2) By the general Remark (4) on Green functions, $G_{\mathrm{B}_R(a)}$ is symmetric in its two arguments. However, symmetry of G_{B_1} and thus $G_{\mathrm{B}_R(a)}$ can also be deduced more elementarily: It is obvious that $|y-x|$ is symmetric in (x,y), and the computation $(|x|\,|y-x^*|)^2 = |x|^2(|y|^2-2y\cdot x^*+|x^*|^2) = |x|^2|y|^2-2y\cdot x+1$ reveals the same symmetry for $|x|\,|y-x^*|$.

(3) A related useful observation is

$$|x|\,|y-x^*| = |y-x| \qquad\qquad \textbf{for } y \neq x \text{ in } \mathbb{R}^n \text{ with } \boldsymbol{|y|=1}.$$

This results from $(|x|\,|y-x^*|)^2 = |x|^2|y|^2-2y\cdot x+1 \overset{|y|=1}{=} |x|^2-2y\cdot x+|y|^2 = |y-x|^2$.

Proof of the last theorem. We first check that the expression in (I) satisfies the defining properties (a), (b) of the Green function of B_1. Indeed, since F is harmonic on $\mathbb{R}^n \setminus \{0\}$, for every fixed $x \in \mathrm{B}_1$, the function $y \mapsto F(|x|(y-x^*))$ is harmonic in B_1 (which does not contain x^*), and (a) is immediate. Moreover, Remark (3) above yields $F(|x|(y-x^*)) = F(y-x)$ for $y \in \mathrm{S}_1$, and (b) follows.

Next we verify the definition for the expression in (II). We fix $x \in \mathrm{B}_R(a)$ and record $\frac{x-a}{R} \in \mathrm{B}_1$. Then the fact that $F\big(\frac{x-a}{R}-y\big)-G_{\mathrm{B}_1}\big(\frac{x-a}{R},y\big)$ is, when extended for $y=\frac{x-a}{R}$, harmonic in $y \in \mathrm{B}_1$ implies that $F(x-y)-R^{2-n}G_{\mathrm{B}_1}\big(\frac{x-a}{R},\frac{y-a}{R}\big) = R^{2-n}\big[F\big(\frac{x-a}{R}-\frac{y-a}{R}\big)-G_{\mathrm{B}_1}\big(\frac{x-a}{R},\frac{y-a}{R}\big)\big]+C(n,R)$ is, when extended for $y=x$, harmonic in $y \in \mathrm{B}_R(a)$ (where $C(n,R)$ actually equals $\frac{\log R}{2\pi}$ for $n=2$ and is zero for $n \geq 3$). This shows (a). Moreover, from $G_{\mathrm{B}_1}\big(\frac{x-a}{R},y\big)=0$ for $y \in \mathrm{S}_1$ we clearly get $R^{2-n}G_{\mathrm{B}_1}\big(\frac{x-a}{R},\frac{y-a}{R}\big)=0$ for $y \in \mathrm{S}_R(a)$, and thus also (b) is valid. $\square$

In order to make the Green function representation explicit — at least in the basic case of balls — we **need in fact the normal derivatives** of the Green function at the boundary. Thus, starting from the explicit formula for the Green function G_{B_1} of B_1 and using Remark (3), we compute for $y \in \mathrm{S}_1$:

$$\begin{aligned}
(\partial_\nu)_y G_{\mathrm{B}_1}(x,y) = y\cdot\nabla_y G_{\mathrm{B}_1}(x,y) &= y\cdot\frac{1}{n\omega_n}\bigg[\frac{y-x}{|y-x|^n}-\frac{|x|^2(y-x^*)}{|x|^n|y-x^*|^n}\bigg] \\
&= \frac{1}{n\omega_n}\bigg[\frac{|y|^2-x\cdot y}{|y-x|^n}-\frac{|x|^2(|y|^2-x^*\cdot y)}{|y-x|^n}\bigg] \\
&= \frac{1}{n\omega_n}\bigg[\frac{|y|^2-x\cdot y}{|y-x|^n}-\frac{|x|^2-x\cdot y}{|y-x|^n}\bigg] \\
&= \frac{1}{n\omega_n}\frac{|y|^2-|x|^2}{|y-x|^n}\,.
\end{aligned}$$

Moreover, from the formula for $G_{\mathrm{B}_R(a)}$ and the chain rule we infer

$$(\partial_\nu)_y G_{\mathrm{B}_R(a)}(x,y) = R^{2-n}(\partial_\nu)_y G_{\mathrm{B}_1}\Big(\frac{x-a}{R},\frac{y-a}{R}\Big)\frac{1}{R} = \frac{1}{n\omega_n R}\frac{|y-a|^2-|x-a|^2}{|y-x|^n} \qquad \text{for } y\in \mathrm{S}_R(a)\,,$$

which then leads to:

Definition (**Poisson kernel**). *For every $R\in(0,\infty)$, the function $\mathrm{P}_R\colon \mathrm{B}_R\times\mathrm{S}_R\to\mathbb{R}$, given by*

$$\mathrm{P}_R(x,y) := \frac{1}{n\omega_n R}\frac{|y|^2-|x|^2}{|y-x|^n} = \frac{1}{n\omega_n R}\frac{R^2-|x|^2}{|y-x|^n} \qquad \text{for } x\in \mathrm{B}_R\,,\ y\in\mathrm{S}_R\,,$$

*is called the n-dimensional **Poisson kernel** for radius R.*

With the normal derivative at hand we now restate the Green function representation on balls in a more explicit way and obtain the following **important formula for solutions of the Dirichlet problem for harmonic functions *on balls***:

Main Theorem (**Poisson integral formula (PIF)**). *Consider a ball $\mathrm{B}_R(a)$ in $\mathbb{R}^n$.*

(I) *Suppose that $h\in \mathrm{C}^2(\mathrm{B}_R(a))\cap \mathrm{C}^0(\overline{\mathrm{B}_R(a)})$ is harmonic in $\mathrm{B}_R(a)$, and set $\varphi := h|_{\mathrm{S}_R(a)}$. Then we have*

$$h(x) = \int_{\mathrm{S}_R(a)} \varphi(y)\,\mathrm{P}_R(x-a,y-a)\,\mathrm{d}\mathcal{H}^{n-1}(y) \qquad \text{for all } x\in\mathrm{B}_R(a)\,.$$

(II) *Consider a given $\varphi\in\mathrm{C}^0(\mathrm{S}_R(a))$. Then, by setting*

$$h(x) := \int_{\mathrm{S}_R(a)} \varphi(y)\,\mathrm{P}_R(x-a,y-a)\,\mathrm{d}\mathcal{H}^{n-1}(y) \qquad \text{for } x\in\mathrm{B}_R(a)$$

and $h(x):=\varphi(x)$ for $x\in\mathrm{S}_R(a)$ we obtain a solution $h\in\mathrm{C}^2(\mathrm{B}_R(a))\cap\mathrm{C}^0(\overline{\mathrm{B}_R(a)})$ to the Dirichlet problem

$$\Delta h\equiv 0 \text{ in } \mathrm{B}_R(a)\,, \qquad u=\varphi \text{ at } \mathrm{S}_R(a)\,.$$

Remark. In particular, (I) implies uniqueness of solutions to the Dirichlet problem in (II). However, we have established uniqueness already in the earlier Section 2.4.

Proof for Part (I) *of the theorem.* The definition of the Poisson kernel has been implemented such that $(\partial_\nu)_y G_{\mathrm{B}_R(a)}(x,y) = \mathrm{P}_R(x-a,y-a)$ holds for $x\in\mathrm{B}_R(a)$ and $y\in\mathrm{S}_R(a)$. If we use this and the harmonicity of h in the general Green function representation, we get the claim for $h\in\mathrm{C}^2(\mathrm{B}_R(a))\cap\mathrm{C}^1(\overline{\mathrm{B}_R(a)})$. If merely $h\in\mathrm{C}^2(\mathrm{B}_R(a))\cap\mathrm{C}^0(\overline{\mathrm{B}_R(a)})$ is assumed, we can apply the result of the previous consideration to the harmonic $C^2(\overline{\mathrm{B}_R(a)})$ functions $x\mapsto h(tx)$ with parameter $t\in(0,1)$. We infer $h(tx)=\int_{\mathrm{S}_R(a)} h(ty)\,\mathrm{P}_R(x-a,y-a)\,\mathrm{d}\mathcal{H}^{n-1}(y)$ for $x\in\mathrm{B}_R(a)$ and then pass to the limit $t\nearrow 1$ in this equality. Since the uniform continuity of h on $\overline{\mathrm{B}_R(a)}$ implies $\lim_{t\nearrow 1} h(ty)=h(y)=\varphi(y)$ uniformly in $y\in\mathrm{S}_R(a)$ and $\mathrm{P}_R(x-a,y-a)$ with fixed $x\in\mathrm{B}_R(a)$ is a bounded function of $y\in\mathrm{S}_R(a)$, the integrals converge suitably, and we arrive at the claim. □

Proof for Part (II) *of the theorem.* We know from Section 2.5 that a solution $h\in\mathrm{C}^2(\mathrm{B}_R(a))\cap\mathrm{C}^0(\overline{\mathrm{B}_R(a)})$ to the Dirichlet problem exists. Then, Part (I) applies to h, and thus h is indeed given by the integral formula on $\mathrm{B}_R(a)$ (and, as a solution to the Dirichlet problem, it also coincides with φ at $\mathrm{S}_R(a)$). □

Though the preceding proof for Part (II) is fully valid, we have not quite followed the strategy set out as principal aim of this section. Indeed, existence has not been achieved with the help of the explicit formula, but rather we have simply cited the existence result obtained by a different approach in a previous section. However, as we show next, it is also possible to follow the guiding idea of this section and establish the existence of the solution by a direct and quite illustrative analysis of the Poisson integral:

Alternative proof for Part (II) *of the theorem* (*without usage of the earlier existence result*). For simplicity of notation we assume $a = 0$ and $R = 1$. We first record that $\mathrm{P}_1(\,\cdot\,, y) \in \mathrm{C}^\infty(\mathrm{B}_1)$ is harmonic in B_1 for each fixed $y \in \mathrm{S}_1$. Indeed, this can be checked by explicit computation of the Laplacian or, alternatively by the following more abstract argument: It follows from the symmetry of G_{B_1} that $G_{\mathrm{B}_1}(x, y)$ is harmonic in $x \in \mathrm{B}_1$, and thus also $\mathrm{P}_1(x, y) = y \cdot \nabla_y G_{\mathrm{B}_1}(x, y)$ is harmonic in $x \in \mathrm{B}_1$. In any case, the smoothness and harmonicity of $\mathrm{P}_1(\,\cdot\,, y)$ then imply (by exchange of differentiation and integration with the usual justification) that the Poisson integral defines a harmonic $h \in \mathrm{C}^\infty(\mathrm{B}_1)$.

It remains to prove the attainment of the boundary datum

$$\lim_{\mathrm{B}_1 \ni x \to x_0} h(x) = \varphi(x_0) \qquad \text{for every boundary point } x_0 \in \mathrm{S}_1\,, \tag{BC}$$

and this will be achieved by relying on the following crucial properties of the Poisson kernel:

- For every fixed $x \in \mathrm{B}_1$, we have $\mathrm{P}_1(x, \,\cdot\,) \geq 0$ on S_1 (clear from the explicit formula for P_1) and $\int_{\mathrm{S}_1} \mathrm{P}_1(x, \,\cdot\,)\,\mathrm{d}\mathcal{H}^{n-1} = 1$ (by Part (I) applied to the constant harmonic function $\equiv 1$). In view of these properties we may **understand the Poisson integral** $\int_{\mathrm{S}_1} h\,\mathrm{P}_1(x, \,\cdot\,)\,\mathrm{d}\mathcal{H}^{n-1}$ **as a weighted integral mean** of h with weight function $\mathrm{P}_1(x, \,\cdot\,)$.

- The **weight functions** $\mathrm{P}_1(x, \,\cdot\,)$ **concentrate at a boundary point** $x_0 \in \mathrm{S}_1$ **in the limit** $\mathrm{B}_1 \ni x \to x_0$ in the sense that we have locally uniform convergence $\lim_{\mathrm{B}_1 \ni x \to x_0} \mathrm{P}_1(x, \,\cdot\,) \equiv 0$ on $\mathrm{S}_1 \setminus \{x_0\}$ (which easy to check from the explicit formula for P_1).

On the basis of these observations, for arbitrary $x_0 \in \mathrm{S}_1$, $x \in \mathrm{B}_1$, $\delta > 0$, and for the function h defined by the Poisson integral, we estimate

$$\begin{aligned}
|h(x) - \varphi(x_0)| &= \left| \int_{\mathrm{S}_1} \varphi\,\mathrm{P}_1(x, \,\cdot\,)\,\mathrm{d}\mathcal{H}^{n-1} - \varphi(x_0) \int_{\mathrm{S}_1} \mathrm{P}_1(x, \,\cdot\,)\,\mathrm{d}\mathcal{H}^{n-1} \right| \\
&\leq \int_{\mathrm{S}_1} |\varphi - \varphi(x_0)|\,\mathrm{P}_1(x, \,\cdot\,)\,\mathrm{d}\mathcal{H}^{n-1} \\
&\leq 2 \max_{\mathrm{S}_1} |\varphi| \int_{\mathrm{S}_1 \setminus \mathrm{B}_\delta(x_0)} \mathrm{P}_1(x, \,\cdot\,)\,\mathrm{d}\mathcal{H}^{n-1} + \sup_{\mathrm{S}_1 \cap \mathrm{B}_\delta(x_0)} |\varphi - \varphi(x_0)| \int_{\mathrm{S}_1} \mathrm{P}_1(x, \,\cdot\,)\,\mathrm{d}\mathcal{H}^{n-1} \\
&= 2 \max_{\mathrm{S}_1} |\varphi| \int_{\mathrm{S}_1 \setminus \mathrm{B}_\delta(x_0)} \mathrm{P}_1(x, \,\cdot\,)\,\mathrm{d}\mathcal{H}^{n-1} + \sup_{\mathrm{S}_1 \cap \mathrm{B}_\delta(x_0)} |\varphi - \varphi(x_0)|\,.
\end{aligned}$$

Here, the first term on the right-hand side vanishes in the limit $\mathrm{B}_1 \ni x \to x_0$, and thus we have $\limsup_{\mathrm{B}_1 \ni x \to x_0} |h(x) - \varphi(x_0)| \leq \sup_{\mathrm{S}_1 \cap \mathrm{B}_\delta(x_0)} |\varphi - \varphi(x_0)|$. By continuity of φ at x_0, the remaining right-hand side vanishes when we send $\delta \searrow 0$. Thus we have $\lim_{\mathrm{B}_1 \ni x \to x_0} |h(x) - \varphi(x_0)| = 0$, which is nothing but the boundary condition (BC). $\square$

Remark. As a side benefit of (a slight variant of) this last proof we indeed obtain the following refined statement: For $\varphi \in \mathrm{L}^1(\mathrm{S}_1\,;\mathcal{H}^{n-1})$, the function h defined by the Poisson integral on B_1 is still smooth and harmonic, and it satisfies (BC) for all continuity points $x_0 \in \mathrm{S}_1$ of φ *even if φ is not continuous on the whole sphere* S_1.

Remarks (on the Poisson integral formula)**.**

(1) With the Poisson integral formula we have an explicit integral formula for solutions at hand. This is essentially the best possible situation for which one may reasonably hope in the theory of PDEs.

(2) The Poisson integral formula resembles the Cauchy integral formula from complex analysis, and indeed on discs in $\mathbb{R}^2$ these two formulas are essentially equivalent.

In more detail, the Cauchy integral formula — for simplicity of notation stated only in the case of center at 0 — asserts $H(z) = \frac{1}{2\pi\mathbf{i}}\int_{\kappa_R}\frac{H(\zeta)}{\zeta-z}\,\mathrm{d}\zeta = \frac{1}{2\pi R}\int_{\mathrm{S}_R}\frac{H(\zeta)}{\zeta-z}\zeta\,\mathrm{d}\mathcal{H}^1(\zeta)$ for $z \in \mathrm{B}_R$ and a holomorphic function H in a neighborhood of the disc $\overline{\mathrm{B}_R} \subset \mathbb{C}$, where $\kappa_R\colon [0,2\pi) \to \mathrm{S}_R\,,\ t \mapsto R\mathrm{e}^{\mathbf{i}t}$ is the standard parametrization of the circle S_R. Using the reflection $z^* = R^2z/|z|^2 = R^2/\overline{z} \notin \overline{\mathrm{B}_R}$ and subtracting the Cauchy integral for the holomorphic function $\zeta \mapsto \frac{H(\zeta)}{\zeta-z^*}\zeta$ in B_R, the formula can be rewritten as $H(z) = \frac{1}{2\pi R}\int_{\mathrm{S}_R}\Big[\frac{\zeta}{\zeta-z}-\frac{\zeta}{\zeta-z^*}\Big]H(\zeta)\,\mathrm{d}\mathcal{H}^1(\zeta)$ (where, for $z=0$ the term $\frac{\zeta}{\zeta-z^*}$ should be omitted). By computation one finds $\frac{1}{2\pi R}\Big[\frac{\zeta}{\zeta-z}-\frac{\zeta}{\zeta-z^*}\Big] = \frac{1}{2\pi R}\,\mathcal{Re}\,\frac{\zeta+z}{\zeta-z} = \mathrm{P}_R(z,\zeta)$ for $\zeta \in \mathrm{S}_R$, and thus the rewritten formula is nothing but the Poisson integral for the real and imaginary parts of H.

As a side benefit, we also record the representation $H(z)-\mathbf{i}\,\mathrm{const} = \frac{1}{2\pi R}\int_{\mathrm{S}_R}\frac{\zeta+z}{\zeta-z}\,\mathcal{Re}\,H(\zeta)\,\mathrm{d}\mathcal{H}^1(\zeta)$ for $z \in \mathrm{B}_R$ of the holomorphic function H, up to a purely imaginary constant, in terms of boundary values of the real part only. This representation follows directly from the Poisson integral formula $\mathcal{Re}\,H(z) = \frac{1}{2\pi R}\int_{\mathrm{S}_R}\big(\mathcal{Re}\,\frac{\zeta+z}{\zeta-z}\big)\,\mathcal{Re}\,H(z)\,\mathrm{d}\mathcal{H}^1(z)$ and the fact that the real part determines the imaginary part of H up to a constant.

(3) **Many properties of harmonic functions are closely connected to the Poisson integral formula.**

For instance, since $\mathrm{P}_R(0,y{-}a) = \frac{1}{\mathcal{H}^{n-1}(\mathrm{S}_R(a))}$ is constant in $y \in \mathrm{S}_R(a)$, the evaluation of the Poisson integral in the center $x=a$ of the relevant ball $\mathrm{B}_R(a)$ gives just the ordinary mean value. Thus, the Poisson integral formula for center points reduces to the spherical mean value property.

Moreover, from the Poisson integral formula one can also read off the following **special Harnack inequality** with sharp constants: If $h \in \mathrm{C}^2(\mathrm{B}_R(a))$ is a *non-negative* harmonic function in $\mathrm{B}_R(a) \subset \mathbb{R}^n$, then it holds

$$R^{n-2}\frac{R{-}|x{-}a|}{(R{+}|x{-}a|)^{n-1}}h(a) \le h(x) \le R^{n-2}\frac{R{+}|x{-}a|}{(R{-}|x{-}a|)^{n-1}}h(a) \qquad \text{for } x \in \mathrm{B}_R(a)$$

and

$$\sup_{\mathrm{B}_r(a)} h \le \left(\frac{R{+}r}{R{-}r}\right)^n \inf_{\mathrm{B}_r(a)} h \qquad \text{for } r \in (0,R)\,.$$

The deduction of these inequalities and the optimality of the constants involved will be discussed in the exercise class.

(4) By exchange of derivative and integral (which is easy to justify) one can deduce the **Poisson integral formula *for the derivatives***: If $h \in \mathrm{C}^2(\mathrm{B}_R(a))\cap\mathrm{C}^0(\overline{\mathrm{B}_R(a)})$ is harmonic in $\mathrm{B}_R(a)$ and $\alpha \in \mathbb{N}_0^n$ is an arbitrary multi-index, then there holds

$$\partial^\alpha h(x) = \int_{\mathrm{S}_R(a)} h(y)\left(\frac{\partial}{\partial x}\right)^\alpha \mathrm{P}_R(x{-}a,y{-}a)\,\mathrm{d}\mathcal{H}^{n-1}(y) \qquad \text{for all } x \in \mathrm{B}_R(a)\,.$$

Remarks (Green function and Green function representation *for other domains*). Here, by F we always denote the fundamental solution of the Laplace equation on $\mathbb{R}^n$.

(1) For $n \geq 3$, the **full space** $\mathbb{R}^n$ has the Green function $G_{\mathbb{R}^n}$ given by $G_{\mathbb{R}^n}(x,y) = F(y-x)$ for $x, y \in \mathbb{R}^n$. In contrast, the two-dimensional plane $\mathbb{R}^2$ does not have a Green function.

Proof. The first claim follows directly from the properties of F, where the decisive fact is $\lim_{|y|\to\infty} F(y) = 0$ in dimensions $n \geq 3$. To show the non-existence claim on $\mathbb{R}^2$, suppose the contrary. Then $F - G_{\mathbb{R}^2}(0,\,\cdot\,)$ would extend to a harmonic function h_0 on $\mathbb{R}^2$, and from $\lim_{|y|\to\infty} G_{\mathbb{R}^2}(0,y) = 0$ we would get $\lim_{|y|\to\infty} h_0(y) = \lim_{|y|\to\infty} F(y) = \infty$. Thus, h_0 would posses an interior minimum point and would necessarily be constant by the strong minimum principle. Clearly this contradicts $\lim_{|y|\to\infty} h_0(y) = \infty$, and thus we have indeed shown the non-existence claim. $\square$

(2) The **half-space** $\mathrm{H}_n := (0,\infty)\times\mathbb{R}^{n-1}$ in $\mathbb{R}^n$ has — this is easily verified from the definition — the Green function G_{H_n} given by

$$G_{\mathrm{H}_n}(x,y) = F(y-x) - F(y-x^-) \qquad \text{for } x, y \in \mathrm{H}_n$$

with the reflection $x^- := x - 2x_1\mathrm{e}_1$ of x at $\partial\mathrm{H}_n = \{0\}\times\mathbb{R}^{n-1}$. Similar to the case of balls, also $G_{\mathrm{H}_n}(x,\,\cdot\,)$ has an interpretation as the electric potential induced by a unit charge at x an an antipolar unit charge at x^-. Moreover, we remark that $F(y-x)$ and $F(y-x^-)$ individually tend to zero for $\mathrm{H}_n \ni y \to \infty$ in case $n \geq 3$, while in case $n = 2$ they tend to ∞ and only their difference satisfies, due to cancellation, the requirement $\lim_{\mathrm{H}_n \ni y\to\infty} G_{\mathrm{H}_n}(x,y) = 0$ on the Green function.

The corresponding Green function representation for solutions h of the Dirichlet problem

$$\Delta h \equiv 0 \text{ in } \mathrm{H}_n\,, \qquad\qquad h(0,\,\cdot\,) = \varphi \text{ in } \mathbb{R}^{n-1}$$

with $\varphi \in \mathrm{C}^0(\mathbb{R}^{n-1})$ takes the form

$$h(t,x') = \frac{2t}{n\omega_n}\int_{\mathbb{R}^{n-1}} \frac{\varphi(y)}{(t^2+|y-x'|^2)^{\frac{n}{2}}}\,\mathrm{d}y \qquad \text{for } (t,x') \in \mathrm{H}_n\,.$$

However, the representation only applies if the additional boundary point ∞ is suitably taken into account by imposing a decay condition on φ and h, respectively. Without such a condition the integral need not converge and uniqueness in the Dirichlet problem need not hold (where the basic non-uniqueness examples are given by $h_\alpha(t,x') := \alpha t$, while uniqueness holds under the assumptions of the Phragmén-Lindelöf principle on H_n).

The proof that the above formula indeed defines solutions of the Dirichlet problem can be carried out in analogy with the case of the Poisson integral formula. Alternatively, one can exploit a direct connection between the ball and the half-space case, which draws on the observation that reflection of the domain at S_1 in the sense of $\Omega \mapsto \Omega^* := \{x^* : x \in \Omega\}$ transforms $\big(\mathrm{B}_{\frac12}(\frac12\mathrm{e}_1)\big)$ into $\big(\mathrm{B}_{\frac12}(\frac12\mathrm{e}_1)\big)^* = \mathrm{e}_1 + \mathrm{H}_n$ (where e_1 is the first canonical basis vector in $\mathbb{R}^n$) and on the **Kelvin transformation**. Indeed, for $u\colon \Omega \to \mathbb{R}$ in $\Omega \subset \mathbb{R}^n \setminus \{0\}$, its Kelvin transform $u^*\colon \Omega^* \to \mathbb{R}$ in Ω^* is defined by $u^*(y) := |y|^{2-n}u(y^*)$ for $y \in \Omega^*$. It is easy to check that this is an involutory operation (i.e. $(u^*)^* = u$), and with some more effort one can also verify[14] $\Delta(u^*)(y) = |y|^{-4}(\Delta u)^*(y)$ for $u \in \mathrm{C}^2(\Omega)$ and $y \in \Omega^*$. In particular, this mean that the Kelvin transformation preserves harmonicity and carries solutions of Dirichlet problems on $\mathrm{B}_{\frac{1}{2}}(\frac{1}{2}\mathrm{e}_1)$ into solutions of Dirichlet problems on $\mathrm{e}_1{+}\mathrm{H}_n$. A deduction of the Green function representation on H_n from this correspondence will be treated in the exercises.

(3) For some simple domains with symmetries, the Green function G is a finite linear combination of shifted fundamental solutions and corresponds to the potential induced by finitely many charges placed at suitable reflection points. Indeed, the formulas for balls and half-spaces can be adapted to obtain fully explicit formulas also for the cases of **half-balls** (4 charges), **quarter-balls** (8 charges), **quarter-spaces** (4 charges), and **complements of balls** (2 charges), for instance. In these cases the charges and their positions can be guessed from intuition and pictures. As an example, the case of the unit half-ball will be treated in the exercises.

In a similar way, for **cubes** and **infinite strips**, one can think of infinitely many charges and obtain a series expansion of the Green function at least.

(4) **For general domains** Ω, in contrast, there is ***no*** **hope for an *explicit* representation** of the Green function G_Ω, and not even its existence is clear. However, even if existence of G_Ω were at hand, in order to obtain solutions via the Green function representation one would need that $(\partial_\nu)_y G_\Omega(x,y)$ exists for $y \in \partial\Omega$ and has properties similar to those of the Poisson kernel needed in the PIF proof. These are complicated matters which cannot be established without more background machinery. Thus, **the Green function approach is *not* well-suited for building an existence theory on general domains** Ω, and here we give up on that approach. However, by a different strategy we will eventually establish solvability of the Dirichlet problem for harmonic functions on quite general domains Ω in the later Section 2.10. We remark that, once this is achieved, we obtain existence of the Green function G_Ω and the Green function representation on Ω as side benefits (at least up to discussion of C^1 up-to-the-boundary regularity of $G_\Omega(x,\,\cdot\,)$).

Remarks (Green functions of the second kind; Neumann problem). Consider a Gauss-Green domain Ω in $\mathbb{R}^n$.

(1) For the **Neumann problem**

$$\Delta u = f \quad \text{in } \Omega\,, \qquad\qquad \partial_\nu u = \psi \quad \text{at } \partial\Omega$$

with prescribed $f \in \mathrm{C}^0(\Omega)$ and $\psi \in \mathrm{C}^0(\partial\Omega)$, the second term in Green's representation formula is 'bad' in the sense that it involves the non-prescribed boundary values $u|_{\partial\Omega}$. Thus, in connection with the Neumann problem one naturally attempts to eliminate this second term rather than the first one, which is 'bad' in the Dirichlet but 'good' in the Neumann case. In line with the previous approach, one may then hope to replace the normal derivative $(\partial_\nu)_y F(x{-}y)$ of the fundamental solution in this second term with $(\partial_\nu)_y F(x{-}y) - \partial_\nu h_x(y)$, where, for fixed $x \in \Omega$, the corrector function h_x is harmonic in Ω with $\partial_\nu h_x(y) = (\partial_\nu)_y F(x{-}y)$ for all $y \in \partial\Omega$. However, this cannot work out as stated, since we know from

[14] Indeed, the proof of this formula is elementary, but computations are more managable if one reduces to the case of polynomials/mononomials first. For further details, see [3, Lemma 4.4, Proposition 4.6], for instance.

Sections 2.1 and 2.3 that it holds

$$\int_{\partial\Omega} \partial_\nu h_x(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) = 0 \qquad \text{but} \qquad \int_{\partial\Omega} (\partial_\nu)_y F(x-y)\,\mathrm{d}\mathcal{H}^{n-1}(y) = 1\,.$$

Hence, we can hope at best that $(\partial_\nu)_y F(x-y)$ and $\partial_\nu h_x(y)$ differ by the constant $\mathcal{H}^{n-1}(\partial\Omega)^{-1}$, that is,

$$(\partial_\nu)_y F(x-y) - \partial_\nu h_x(y) = \mathcal{H}^{n-1}(\partial\Omega)^{-1} \qquad \text{for all } x \in \Omega\,,\ y \in \partial\Omega\,.$$

(2) On the basis of the considerations in (1) we introduce the following terminology:

Definition (Green function of the second kind, Neumann function). *We call a function $G^{\mathrm{N}}\colon \{(x,y) \in \Omega\times\Omega : y \neq x\} \to \mathbb{R}$ the* ***Green function of the second kind for Ω*** *or the* ***Neumann function of Ω*** *if it satisfies the following two conditions:*

(a) *For every $x \in \Omega$, the function $y \mapsto F(x-y)-G^{\mathrm{N}}(x,y)$ on $\Omega \setminus \{x\}$ can be extended to a harmonic function in Ω.*

(b) *For every $x \in \Omega$, the function $G^{\mathrm{N}}(x,\,\cdot\,) \in \mathrm{C}^1(\overline{\Omega} \setminus \{x\})$ has constant normal derivative $\partial_\nu(G^{\mathrm{N}}(x,\,\cdot\,)) \equiv \mathcal{H}^{n-1}(\partial\Omega)^{-1}$ at $\partial\Omega$.*

(3) The Green function of the second kind is **unique up to constants** in the following sense: If G^{N} and $\widetilde{G}^{\mathrm{N}}$ are Green functions of the second kind for Ω, then there exists constants $C_x \in \mathbb{R}$ such that $\widetilde{G}^{\mathrm{N}}(x,y) = G^{\mathrm{N}}(x,y)+C_x$ for all $x, y \in \Omega$. This follows from the observation that the harmonic extensions of $y \mapsto F(x-y)-G^{\mathrm{N}}(x,y)$ and $y \mapsto F(x-y)-\widetilde{G}^{\mathrm{N}}(x,y)$ solve the same Neumann problem, in which the solution is unique up to constants; see Section 2.3.

Furthermore, if a Green function G^{N} of the second kind for Ω is chosen in such a way that $x \mapsto \int_{\partial\Omega} G^{\mathrm{N}}(x,y)\,\mathrm{d}\mathcal{H}^{n-1}(y)$ is constant in Ω (where the constancy can always be ensured by a suitable choice of the previously mentioned C_x), then G^{N} is **symmetric** in the sense of $G^{\mathrm{N}}(y,x) = G^{\mathrm{N}}(x,y)$ for all $x, y \in \Omega$ with $y \neq x$. This follows by the same arguments previously used in proving the symmetry of the Green function for the Dirichlet problem.

(4) If G^{N} is a Green function of the second kind for Ω, we can follow the reasoning in the Dirichlet case once more (which essentially means that we apply Green's second identity on $\Omega \setminus \overline{\mathrm{B}_\varepsilon(x)}$ for the functions u and $G^{\mathrm{N}}(x,\,\cdot\,)$) to obtain the following **Green function representation**: For $u \in \mathrm{C}^2(\Omega)\cap\mathrm{C}^1(\overline{\Omega})$, set $f := \Delta u$ and $\psi := \partial_\nu u|_{\partial\Omega}$. Then, for all $x \in \Omega$ such that $\int_{\partial\Omega} G^{\mathrm{N}}(x,y)f(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)$ exists in $[-\infty,\infty]$, we have

$$u(x) = \fint_{\partial\Omega} u\,\mathrm{d}\mathcal{H}^{n-1} - \int_{\partial\Omega} G^{\mathrm{N}}(x,y)\psi(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) + \int_{\partial\Omega} G^{\mathrm{N}}(x,y)f(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\,.$$

Here, the term $\fint_{\partial\Omega} u\,\mathrm{d}\mathcal{H}^{n-1}$ is an x-independent constant whose occurrence is not at all surprising, since the solution of the Neumann problem is unique only up to constants.

(5) The Green function $G^{\mathrm{N}}_{\mathrm{B}_1}$ of the second kind **for the unit ball** $\mathrm{B}_1 \subset \mathbb{R}^n$ is given by fully **explicit formulas in dimension $n = 2$ and dimension $n = 3$**. Indeed, one instance of

such a Green function is given, for $x, y \in \mathrm{B}_1$ with $y \neq x$, by

$$G^{\mathrm{N}}_{\mathrm{B}_1}(x,y) = \frac{1}{2\pi}\big[\log|y{-}x| + \log(|x|\,|y{-}x^*|)\big] \qquad \text{if } n=2\,,$$
$$G^{\mathrm{N}}_{\mathrm{B}_1}(x,y) = -\frac{1}{4\pi}\left[\frac{1}{|y{-}x|} + \frac{1}{|x|\,|y{-}x^*|} - \log(1{-}x\cdot y{+}|x|\,|y{-}x^*|)\right] \qquad \text{if } n=3$$

(with $x^* := \frac{x}{|x|^2}$ and convention $|x|\,|y{-}x^*| = 1$ for $y \neq 0 = x$, as used earlier). For $n=2$, this means that $G^{\mathrm{N}}_{\mathrm{B}_1}(x,y) = F(y{-}x) + F(|x|(y{-}x^*))$ differs from the Green function in the Dirichlet case only in the sign between the two terms, and clearly $F(|x|(y{-}x^*))$ is harmonic in $y \in \mathrm{B}_1$ as required by (2a) above. For $n=3$, we have $G^{\mathrm{N}}_{\mathrm{B}_1}(x,y) = F(y{-}x) + F(|x|(y{-}x^*)) + \frac{1}{4\pi}\log(1{-}x{\cdot}y{+}|x||y{-}x^*|)$, where harmonicity of $F(|x|(y{-}x^*))$ in $y \in \mathrm{B}_1$ is evident, while harmonicity of $\log(1{-}x\cdot y{+}|x||y{-}x^*|)$ in $y \in \mathrm{B}_1$ can be checked by computation. Furthermore, in both cases one can verify the requirement (2b) for the normal derivative by explicit computation. The above instance of $G^{\mathrm{N}}_{\mathrm{B}_1}$ also has the property that $\int_{\mathrm{S}_1} G^{\mathrm{N}}_{\mathrm{B}_1}(x,y)\,\mathrm{d}\mathcal{H}^{n-1}(y)$ is constant in $x \in \mathrm{B}_1$ (with value 0 for $n=2$ and value $-2{+}\log 2$ for $n=3$; this follows from the coincidence $F(y{-}x) = F(|x|(y{-}x^*))$ for $y \in \mathrm{S}_1$ and the spherical mean value property for harmonic functions), and in accordance with (3) this choice of $G^{\mathrm{N}}_{\mathrm{B}_1}$ is symmetric.

For further background information, constructive derivations of the formulas in dimension $n \in \{2,3\}$, and series expansions of $G^{\mathrm{N}}_{\mathrm{B}_1}$ for $n \geq 4$, we refer to [7, Sections 2.10, 2.11, 2.12].

(6) In order to write out the **Green function representation and** deduce an **existence result for the Neumann problem on $\mathbf{B_1} \subset \mathbb{R}^{\boldsymbol{n}}$, $\boldsymbol{n} \in \{\mathbf{2},\mathbf{3}\}$,** we first observe that the formulas in (5) simplify for $y \in \mathrm{S}_1$ since $|x||y{-}x^*| = |y{-}x|$ holds in this case. Plugging the simplified formulas into the general Green function representation of (4), we obtain for the harmonic case $f = \Delta h \equiv 0$ with $⨍_{\mathrm{S}_1} h\,\mathrm{d}\mathcal{H}^{n-1} = 0$ or equivalently with $h(0)=0$:

Theorem (solvability of the Neumann problem on the 2d disc and the 3d ball). *Consider $n \in \{2,3\}$, the unit disc/ball $\mathrm{B}_1 \subset \mathbb{R}^n$, the unit circle/sphere $\mathrm{S}_1 = \partial\mathrm{B}_1 \subset \mathbb{R}^n$, and $\psi \in \mathrm{C}^0(\mathrm{S}_1)$. Setting, for $x \in \mathrm{B}_1$,*

$$h(x) := -\frac{1}{\pi}\int_{\mathrm{S}_1}\big(\log|y{-}x|\big)\psi(y)\,\mathrm{d}\mathcal{H}^1(y) \qquad \textit{if } n=2\,,$$
$$h(x) := \frac{1}{4\pi}\int_{\mathrm{S}_1}\left(\frac{2}{|y{-}x|} - \log(1{-}x{\cdot}y{+}|y{-}x|)\right)\psi(y)\,\mathrm{d}\mathcal{H}^2(y) \qquad \textit{if } n=3\,,$$

we obtain a harmonic function h in B_1 with $\lim_{\mathrm{B}_1\ni x\to x_0} x\cdot\nabla h(x) = \psi(x_0) - ⨍_{\mathrm{S}_1}\psi\,\mathrm{d}\mathcal{H}^{n-1}$ for all $x_0 \in \mathrm{S}_1$. In particular, if ψ has zero mean, i.e.

$$\int_{\mathrm{S}_1}\psi\,\mathrm{d}\mathcal{H}^{n-1} = 0\,, \qquad (*)$$

then $h \in \mathrm{C}^2(\mathrm{B}_1)$ with $h(0)=0$ solves the Neumann problem

$$\Delta h \equiv 0 \textit{ in } \mathrm{B}_1\,, \qquad \lim_{\mathrm{B}_1\ni x\to x_0} x\cdot\nabla h(x) = \psi(x_0) \textit{ for all } x_0 \in \mathrm{S}_1\,.$$

Sketch of proof. Harmonicity of h can be checked, as usual, by differentiation under the integral. In order to verify the boundary conditions, one computes, for $y \in \mathrm{S}_1$,

$$x \cdot \nabla_x \left(-\frac{1}{\pi} \log |y-x| \right) = \mathrm{P}_1(x,y) - \frac{1}{2\pi} \qquad \text{if } n = 2\,,$$
$$x \cdot \nabla_x \frac{1}{4\pi} \left(\frac{2}{|y-x|} - \log(1-x\cdot y+|y-x|) \right) = \mathrm{P}_1(x,y) - \frac{1}{4\pi} \qquad \text{if } n = 3$$

and thus establishes a connection with the Poisson kernel P_1. This readily gives

$$x \cdot \nabla h(x) = \int_{\mathrm{S}_1} \mathrm{P}_1(x,y)\psi(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) - \fint_{\mathrm{S}_1} \psi\,\mathrm{d}\mathcal{H}^{n-1} \underset{x\to x_0}{\longrightarrow} \psi(x_0) - \fint_{\mathrm{S}_1} \psi\,\mathrm{d}\mathcal{H}^{n-1}$$

for $x_0 \in \mathrm{S}_1$, where the convergence results from the known fact that the Poisson integral solves the Dirichlet problem. Finally, from the above formulas one also reads off $h(0) = 0$ in case $n = 2$ and $h(0) = (2-\log 2)\fint_{\mathrm{S}_1} \psi\,\mathrm{d}\mathcal{H}^2$ in case $n = 3$. This establishes all claims (with the indicated simplifications in the particular case $\int_{\mathrm{S}_1} \psi\,\mathrm{d}\mathcal{H}^{n-1} = 0$). □

We emphasize that the **zero-mean hypothesis (∗)** above and the corresponding requirement $\int_{\partial\Omega} \psi\,\mathrm{d}\mathcal{H}^{n-1} = 0$ for general Gauss-Green domains Ω are basic **necessary and sufficient conditions for solvability of the Neumann problem**. Indeed, the sufficiency of (∗) for solvability on the 2d disc and the 3d ball is demonstrated by the preceding theorem, while the necessity of the general condition for obtaining a harmonic $h \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^1(\overline{\Omega})$ with $\partial_\nu h = \psi$ at $\partial\Omega$ is clear from the observation $\int_{\partial\Omega} \psi\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial\Omega} \partial_\nu h\,\mathrm{d}\mathcal{H}^{n-1} = 0$.

Further Remarks (Green functions of the second kind on unbounded domains). Consider an unbounded open set Ω in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is C^1 in a neighborhood of $\mathcal{H}^{n-1}$-a.e. boundary point.

(7) For unbounded Ω, the situation differs from the one described in Remark (1) above insofar that $\int_{\partial\Omega} \partial_\nu h\,\mathrm{d}\mathcal{H}^{n-1}$ need no longer vanish for harmonic h but can take arbitrary values. Thus, one may reasonably hope to find Green functions of the second kind whose boundary normal derivative is not only constant but even zero, and indeed we coin the following definition: We call a function $G^{\mathrm{N}} \colon \{(x,y) \in \Omega\times\Omega : y \neq x\} \to \mathbb{R}$ the **Green function of the second kind for Ω** if, for every $x \in \Omega$, the function $y \mapsto F(x-y)-G^{\mathrm{N}}(x,y)$ extends harmonicly to Ω and it holds $\partial_\nu(G^{\mathrm{N}}(x,\,\cdot\,)) \equiv 0$ at $\partial\Omega$ together with $\lim_{\Omega\ni y\to\infty} G^{\mathrm{N}}(x,y) = 0$.

(8) For the **half-space** $\mathrm{H}_n := (0,\infty)\times\mathbb{R}^{n-1} \subset \mathbb{R}^n$ **in dimension $n \geq 3$**, the unique **Green function of the second kind** $G^{\mathrm{N}}_{\mathrm{H}_n}$ in the sense of (7) is given by the **explicit formula**

$$G^{\mathrm{N}}_{\mathrm{H}_n}(x,y) = F(y-x) + F(y-x^-) \qquad \text{for } x,y \in \mathrm{H}_n$$

(with fundamental solution F and $x^- := x-2x_1\mathrm{e}_1$, as used earlier). This function is easily seen to satisfy the above conditions and differs from its Dirichlet counterpart only in the sign between the two terms. The corresponding **Green function representation for** solutions h of **the Neumann problem on H_n, $n \geq 3$,**

$$\Delta h \equiv 0 \text{ in } \mathrm{H}_n\,, \qquad \partial_1 h(0,\,\cdot\,) = \psi \text{ in } \mathbb{R}^{n-1}\,, \qquad \lim_{\mathrm{H}_n\ni x\to\infty} h(x) = 0$$

with $\psi \in \mathrm{C}^0(\mathbb{R}^{n-1})$ reads

$$h(t,x') = \frac{2}{n(n-2)\omega_n} \int_{\mathbb{R}^{n-1}} \frac{\psi(y)}{(t^2+|y-x'|^2)^{\frac{n-2}{2}}}\,\mathrm{d}y \qquad \text{for } t \in (0,\infty)\,,\ x' \in \mathbb{R}^{n-1}$$

and is valid under suitable assumptions on the decay of h and ψ, respectively, at ∞.

The **half-plane** $\mathrm{H}_2 := (0,\infty)\times\mathbb{R} \subset \mathbb{R}^2$ **in dimension 2**, in contrast, **does not have a Green function of the second kind** in the sense of (7). This follows by a simple reflection argument from the earlier observation that the plane $\mathbb{R}^2$ does not posses a Green function (of the first kind). Still, the formula (which is more or less suggested by the higher-dimensional considerations)

$$h(t,x') = \frac{1}{2\pi}\int_{\mathbb{R}} \psi(y)\log(t^2+(y-x')^2)\,\mathrm{d}y \qquad \text{for } t\in(0,\infty)\,,\ x'\in\mathbb{R}$$

is a **sort-of Green function representation for the Neumann problem on $\mathbf{H_2}$** and is valid, once more, under suitable assumptions on h or ψ. The solutions h obtained from this formula cannot be expected, *in general*, to satisfy the decay $\lim_{\mathrm{H}_2\ni x\to\infty} h(x) = 0$ at ∞, since the integrand does not tend to 0 for $(t,x')\to\infty$. However, in the special case $\int_{\mathbb{R}}\psi\,\mathrm{d}y = 0$, the formula can be rewritten as $h(t,x') = \frac{1}{2\pi}\int_{\mathbb{R}}\psi(y)\big[\log(t^2+(y-x')^2) - \log(1+t^2+x'^2)\big]\,\mathrm{d}y$, and then in view of $\lim_{\mathrm{H}_2\ni(t,x')\to\infty}\big[\log(t^2+(y-x')^2) - \log(1+t^2+x'^2)\big] = 0$ one may expect the decay $\lim_{\mathrm{H}_2\ni x\to\infty} h(x) = 0$ again.

2.9 Isolated singularities, analyticity, and reflection principles

The first result of this section deals with removable singularities and singularities of fundamental-solution type.

Theorem (on **isolated singularities**). *Consider an open set Ω in $\mathbb{R}^n$, a point $a\in\mathbb{R}^n$, and a harmonic function h* ***in $\boldsymbol{\Omega\setminus\{a\}}$*** *(which is again open and has a as an isolated boundary point).*

(I) ***Removable-singularity theorem***: *If $\lim_{x\to a}\frac{h(x)}{|F(x-a)|} = 0$ holds, then h has an extension to a harmonic function in all of Ω.*

(II) *If either $\lim_{x\to a}|x-a|^{n-1}h_-(x) = 0$ or $\lim_{x\to a}|x-a|^{n-1}h_+(x) = 0$ is satisfied, then there exist a constant $c\in\mathbb{R}$ and a harmonic function h_0 in all of Ω such that there holds*

$$h(x) = cF(x-a) + h_0(x) \qquad \text{for all } x\in\Omega\setminus\{a\}\,.$$

Remarks (on the isolated-singularity theorem).

(1) Part (I) says: If a harmonic function h grows at isolated singularity a slower than the fundamental solution at 0, then the singularity is artificial and can be removed.

(2) Part (II) can be expressed as follows: If, for an harmonic function h, either h_+ or h_- grows at an isolated singularity less than one power faster than the fundamental solution at 0, then h exhibits at this singularity, quite precisely, the behavior of a multiple of the fundamental solution. Roughly speaking this means that a harmonic function h can only have an isolated singularity 'worse' than the one of the fundamental solution if both h_+ and h_- tend to ∞ (at least along some sequences) at least one power faster than the fundamental solution.

(3) Specifically and most crucially, Part (II) applies to *every* non-negative harmonic function on $\Omega\setminus\{a\}$ (then $c\le 0$) and *every* non-positive harmonic function on $\Omega\setminus\{a\}$ (then $c\ge 0$). This case of the result is known as **Bôcher's theorem**.

(4) The growth hypothesis in Part (II) is sharp in the sense that there exists a harmonic h with isolated singularity at a such that $\limsup_{x\to a}|x-a|^{n-1}|h(x)|<\infty$. Indeed, this happens for any first-order derivative $h(x)=\partial_i F(x-a)$ of the (shifted) fundamental solution.

The proof of the removable-singularity theorem in Part (I) combines solvability of the Dirichlet problem for harmonic functions on balls and an earlier Phragmén-Lindelöf principle as follows:

Proof of Part (I). We choose $\overline{\mathrm{B}_r(a)}\subset\Omega$ and consider the solution $h_0\in\mathrm{C}^2(\mathrm{B}_r(a))\cap\mathrm{C}^0(\overline{\mathrm{B}_r(a)})$ of the Dirichlet problem $\Delta h_0\equiv 0$ in $\mathrm{B}_r(a)$ and $h_0=h$ at $\mathrm{S}_r(a)$ (where existence of h_0 is known from either Section 2.5 or Section 2.8). Applying an earlier Phragmén-Lindelöf principle (see Remark (3) on such principles in Section 2.4) to both $h{-}h_0$ and $h_0{-}h$ on $\mathrm{B}_r(a)\setminus\{a\}$, we deduce from $h{-}h_0\equiv 0$ at $\mathrm{S}_r(a)$ that $h{-}h_0\equiv 0$ holds also in $\mathrm{B}_r(a)\setminus\{a\}$. We now extend h by setting $h(a):=h_0(a)$. Then $h=h_0$ is harmonic in $\mathrm{B}_r(a)$, and by assumption h is harmonic in $\Omega\setminus\{a\}$. Thus, the extended h is harmonic in all of Ω. □

The proof of Part (II) combines several earlier results in an interesting manner:

Proof of Part (II). We take $a=0$ for simplicity, and we treat the case of $\lim_{x\to a}|x|^{n-1}h_-(x)=0$ only. We fix $\overline{\mathrm{B}_r(a)}\subset\Omega$ and a cut-off function $\psi\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with $\psi\equiv 1$ in B_r. Moreover, for arbitrary $\varepsilon\in(0,r)$, we choose another cut-off function $\varphi_\varepsilon\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ such that we have $\varphi_\varepsilon\equiv 1$ near 0 and $\operatorname{spt}\varphi_\varepsilon\subset\mathrm{B}_\varepsilon$. Then the function $(\psi{-}\varphi_\varepsilon)h$ vanishes near $\partial\Omega\cup\{0\}$ and thus is (when extended with value 0 at 0) in $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$. Moreover, in view of $\psi{-}\varphi_\varepsilon=1{-}0=1$ in $\mathrm{B}_r\setminus\mathrm{B}_\varepsilon$, the function $(\psi{-}\varphi_\varepsilon)h$ equals h in $\mathrm{B}_r\setminus\overline{\mathrm{B}_\varepsilon}$ and in particular is harmonic there. All in all, these properties and Green's representation formula yield

$$h(x)=((\psi{-}\varphi_\varepsilon)h)(x)=\int_\Omega F(x{-}y)\Delta((\psi{-}\varphi_\varepsilon)h)(y)\,\mathrm{d}y=w_\varepsilon(x)+h_0(x)\qquad\text{for } x\in\mathrm{B}_r\setminus\overline{\mathrm{B}_\varepsilon}\,,$$

where we have set

$$w_\varepsilon(x):=\int_{\overline{\mathrm{B}_\varepsilon}}F(x{-}y)\Delta((1{-}\varphi_\varepsilon)h)(y)\,\mathrm{d}y\qquad\text{and}\qquad h_0(x):=\int_{\Omega\setminus\mathrm{B}_r}F(x{-}y)\Delta(\psi h)(y)\,\mathrm{d}y\,.$$

In fact, the latter definitions yield a harmonic function w_ε in all of $\mathbb{R}^n\setminus\overline{\mathrm{B}_\varepsilon}$ (since $y\mapsto F(x{-}y)$ is smooth and harmonic near $\overline{\mathrm{B}_\varepsilon}$ as soon as $x\notin\overline{\mathrm{B}_\varepsilon}$ and then Δw_ε can be computed as usual by exchange of differentiation and integration) and a harmonic function h_0 in B_r (for analogous reasons). Moreover, the behavior of F at infinity implies $\lim_{x\to\infty}w_\varepsilon(x)=0$ in case $n\ge 3$ and $\lim_{x\to\infty}(w_\varepsilon(x)-c_\varepsilon F(x))=0$ for $c_\varepsilon:=\int_{\mathrm{B}_\varepsilon}\Delta((1{-}\varphi_\varepsilon)h)\,\mathrm{d}y$ in case $n=2$. Applying the preceding for both $\widetilde{\varepsilon}$ and ε such that $0<\widetilde{\varepsilon}<\varepsilon<r$, we infer $w_\varepsilon+h_0=h=w_{\widetilde{\varepsilon}}+h_0$ in $\mathrm{B}_r\setminus\mathrm{B}_\varepsilon$. Thus we have $w_{\widetilde{\varepsilon}}=w_\varepsilon$ in $\mathrm{B}_r\setminus\mathrm{B}_\varepsilon$. This, however, leaves us with an ε-independent function w which is harmonic in all of $\mathbb{R}^n\setminus\{0\}$ and satisfies

$$h=w+h_0\qquad\text{in } \mathrm{B}_r\setminus\{0\}\,.$$

Indeed, one way of defining w is setting $w(x):=w_{r/2}(x)$ in case $|x|\ge r/2$ and $w(x):=w_\varepsilon(x)$ in case $\varepsilon\le|x|<r/2$, where in the latter case the choice of ε does not matter. In addition, we can also preserve the previously recorded limit behavior for $x\to\infty$ (in case $n=2$ with an ε-independent constant $c\in\mathbb{R}$).

We now continue the reasoning for the **case $n \geq 3$ only**. In that case, for w we have the properties

$$w \text{ harmonic in } \mathbb{R}^n \setminus \{0\}\,, \qquad \lim_{x\to 0} |x|^{n-1} w_-(x) = 0\,, \qquad \lim_{x\to\infty} w(x) = 0\,,$$

where the behavior for $x \to 0$ follows from the hypothesis on h and the fact that h_0 stays bounded near 0. For the Kelvin transform w^*, given by $w^*(y) = |y|^{2-n} w(y^*)$ for $y \in \mathbb{R}^n \setminus \{0\}$, one may check that the preceding properties imply

$$w^* \text{ harmonic in } \mathbb{R}^n \setminus \{0\}\,, \qquad \lim_{y\to\infty} \frac{w^*_-(y)}{|y|} = 0\,, \qquad \lim_{y\to 0} |y|^{n-2} w^*(y) = 0$$

(where one should notice that $y \to \infty$ means $y^* \to 0$ and vice versa). At this stage, recalling $|F(y)| = \frac{1}{n(n-2)\omega_n} |y|^{2-n}$, we may apply Part (I) of the theorem to deduce that the singularity of w^* at 0 is removable. Then, however, a one-sided Liouville property from Section 2.6 applies and shows that w^* is in fact constant in $\mathbb{R}^n \setminus \{0\}$. By definition of the Kelvin transform, this means $w = cF$ in $\mathbb{R}^n \setminus \{0\}$ with a constant $c \in \mathbb{R}$, and thus we arrive at $h = cF + h_0$ at first in $\mathrm{B}_r \setminus \{0\}$. But clearly we can set $h_0 := h - cF$ outside B_r to achieve $h = cF + h_0$ in all of $\Omega \setminus \{0\}$ with h_0 harmonic in all of Ω. This is the claim.

Finally, in the **case $n = 2$** the latter part of the reasoning works similarly, but requires a number of adaptations due to the different limit behavior of w at ∞ and the logarithmic growth of the fundamental solution F. Details on these issues will be treated in the exercises. □

Remark. If one strengthens the growth assumption in Part (II) of the theorem to either $\limsup_{x\to a} |x-a|^{n-2} h_-(x) < \infty$ or $\limsup_{x\to a} |x-a|^{n-2} h_+(x) < \infty$ (that means in other words $\limsup_{x\to a} \frac{h_-(x)}{|F(x-a)|} < \infty$ or $\limsup_{x\to a} \frac{h_+(x)}{|F(x-a)|} < \infty$), one can also give a completely different proof. This alternative argument is essentially based on comparison with the fundamental solution F (which is implemented via several different applications of maximum and comparison principles) and on Harnack's inequality. Anyway, since the outcome is only a suboptimal version of the statement, we do not go into the details of the alternative approach.

The next result shows that harmonic function are, in a sense, even better than C^∞.

Theorem (analyticity of harmonic functions). *If h is harmonic in an open set Ω in $\mathbb{R}^n$, then h is indeed real-analytic in Ω.*

Remark (on analyticity in multiple variables)**.** A function $f\colon \Omega \to \mathbb{R}$ is called **(real-)analytic** in Ω or function of **class C^ω** in Ω if every point $a \in \Omega$ has a neighborhood U in which f can be expanded as a (uniformly) convergent power series $\sum_{\alpha\in\mathbb{N}_0^n} c_\alpha (x-a)^\alpha$ with coefficients $c_\alpha \in \mathbb{R}$ and center a in the precise meaning that $\lim_{m\to\infty} \sum_{|\alpha|\leq m} c_\alpha (x-a)^\alpha = f(x)$ converges (uniformly) for $x \in U$. If this is the case, then f is of class C^∞ in Ω, and the power series is necessarily the Taylor series of f at a, that is, its coefficients are the Taylor coefficients $c_\alpha = \frac{1}{\alpha_1!\alpha_2!\cdot\ldots\cdot\alpha_n!} \partial^\alpha f(a)$.

We now record some consequences of analyticity:

Corollary (identity theorem). *If h is harmonic in a connected open set Ω in $\mathbb{R}^n$ with either $h \equiv 0$ in a non-empty open subset of Ω or $\partial^\alpha h(x_0) = 0$ for all $\alpha \in \mathbb{N}_0^n$ at one point $x_0 \in \Omega$, then $h \equiv 0$ holds in all of Ω.*

Proof. The set $S := \{x \in \Omega : \partial^\alpha h(x) = 0 \text{ for all } \alpha \in \mathbb{N}_0^n\}$ is non-empty, since it contains either the open subset or the point x_0 from the assumption. It is also open, since the Taylor series at $a \in S$ is the null series, and thus, by analyticity, h vanishes in a neighborhood of a and that neighborhood is contained in S. Finally, since all $\partial^\alpha h$ are continuous, S is also closed in Ω. In conclusion, S is non-empty, open, and closed in the connected set Ω. This implies $S = \Omega$. □

Corollary (refined strong maximum/minimum principle). *If h is harmonic in a connected open set Ω in $\mathbb{R}^n$ and if there exists a* ***local*** *maximum or minimum point of h in Ω, then h is constant in Ω.*

Proof. Clearly, the local maximum/minimum point is a global maximum/minimum point for h restricted to an open neighborhood of this point. By the strong maximum/minimum principle, h is constant in the neighborhood, and by the identity theorem, h is constant even in Ω. □

Finally, we turn to the proof of the analyticity result which is crucially based on estimates for the derivatives of harmonic functions. Indeed suitable estimates can be obtained in (at least) two ways: One can either rely on the Poisson integral formula for the derivatives and estimate the derivatives of the Poisson kernel, or one can use a comparably elementary induction argument. Here, we follow the latter approach whose outcome is summarized in the following lemma:

Lemma (refined interior estimates for harmonic functions). *Consider a harmonic function h in an open set Ω in $\mathbb{R}^n$ and a ball $\overline{\mathrm{B}_r(a)} \subset \Omega$. Then, for every $m \in \mathbb{N}_0$, we have*

$$|\mathrm{D}^m h(a)| \le \frac{2^n}{\omega_n r^n}\left(\frac{2nm}{r}\right)^m \|h\|_{1;\mathrm{B}_r(a)}\,,$$

where $|\mathrm{D}^m h(a)| := \sup_{v_1,v_2,\ldots,v_m \in \mathrm{S}_1^{n-1}} |\mathrm{D}^m h(a)(v_1,v_2,\ldots,v_m)|$ denotes the operator norm of the symmetric m-linear mapping $\mathrm{D}^m h(a) \in \mathcal{L}^m_{\mathrm{sym}}(\mathbb{R}^n)$.

Remark. The estimates in the lemma resemble those obtained in Section 2.6 by taking derivatives of a mollification kernel η. However, as the decisive advantage, the present lemma provides better and more explicit constants. The underlying reason for this lies in the fact that mollification kernels (with compact support) are not analytic themselves and thus cannot be expected to yield constants which are suitable for a proof of analyticity. (In contrast, the Poisson kernel is analytic, which is the basis for the alternative approach mentioned above.)

Proof of the lemma. We first show by induction on $m \in \mathbb{N}_0$ that, for $\overline{\mathrm{B}_{m\varrho}(a)} \subset \Omega$, we have

$$|\mathrm{D}^m h(a)| \le \left(\frac{n}{\varrho}\right)^m \sup_{\mathrm{B}_{m\varrho}(a)} |h|\,.$$

Since this claim trivially holds for $m = 0$, we can directly proceed to the induction step, in which we assume the estimate for $\mathrm{D}^m h(a)$ with $m \in \mathbb{N}_0$ and establish it for $\mathrm{D}^{m+1}h(a)$ in case $\overline{\mathrm{B}_{(m+1)\varrho}(a)} \subset \Omega$. To this end, we consider $v_1, v_2, \ldots, v_m, w \in \mathrm{S}_1^{n-1}$ and $g := \partial_{v_m} \ldots \partial_{v_2}\partial_{v_1} h$. Using the mean value property and the divergence theorem for $\partial_w g = \operatorname{div}(gw)$, we find $|\partial_w g(a)| = \frac{1}{\omega_n \varrho^n}\big|\int_{\mathrm{B}_\varrho(a)} \partial_w g \,\mathrm{d}x\big| \le \frac{1}{\omega_n \varrho^n}\int_{\mathrm{S}_\varrho(a)} |g|\,\mathrm{d}\mathcal{H}^{n-1} \le \frac{n}{\varrho}\sup_{\mathrm{S}_\varrho(a)}|g|$, and in view of the above choices we infer $|\mathrm{D}^{m+1}h(a)| \le \frac{n}{\varrho}\sup_{\mathrm{S}_\varrho(a)}|\mathrm{D}^m h|$. In view of the induction hypothesis, we can control the right-hand side of the last estimate by $|\mathrm{D}^m h(b)| \le \left(\frac{n}{\varrho}\right)^m \sup_{\mathrm{B}_{m\varrho}(b)}|h| \le \left(\frac{n}{\varrho}\right)^m \sup_{\mathrm{B}_{(m+1)\varrho}(a)}|h|$ at each point $b \in \mathrm{S}_\varrho(a)$, and thus we arrive at $|\mathrm{D}^{m+1}h(a)| \le \left(\frac{n}{\varrho}\right)^{m+1}\sup_{\mathrm{B}_{(m+1)\varrho}(a)}|h|$. This completes the induction. Combining the outcome of the induction argument with $\varrho = \frac{r}{2m}$ and the mean value estimate $|h(b)| \le \frac{1}{\omega_n(r/2)^n}\|h\|_{1;\mathrm{B}_r(a)}$ for $b \in \mathrm{B}_{r/2}(a)$, we finally end up with $|\mathrm{D}^m h(a)| \le \big(\frac{2nm}{r}\big)^m \sup_{\mathrm{B}_{r/2}(a)}|h| \le \frac{2^n}{\omega_n r^n}\big(\frac{2nm}{r}\big)^m\|h\|_{1;\mathrm{B}_r(a)}$. $\square$

Proof of the analyticity theorem. Fix a ball $\overline{\mathrm{B}_{2r}(a)} \subset \Omega$. We estimate, for $m \in \mathbb{N}$, the remainder $\mathrm{R}_a^{m-1}h(x) := h(x) - \sum_{k=0}^{m-1}\frac{1}{k!}\mathrm{D}^k h(a)(x{-}a, x{-}a, \ldots, x{-}a) = h(x) - \sum_{|\alpha|\le m-1}\frac{\partial^\alpha h(a)}{\alpha!}(x{-}a)^\alpha$ in the Taylor formula. To this end, we use the well-known Lagrange estimate for the remainder $\mathrm{R}_a^{m-1}h$, the estimate $|\mathrm{D}^m h(x)| \le \frac{2^n}{\omega_n r^n}\big(\frac{2nm}{r}\big)^m\|h\|_{1;\mathrm{B}_{2r}(a)}$ for $x \in \mathrm{B}_r(a)$ from the preceding lemma and the observation[15] that $m! \ge (\frac{m}{\mathrm{e}})^m$ for $m \in \mathbb{N}$. We infer

$$|\mathrm{R}_a^{m-1}h(x)| \le \frac{|x{-}a|^m}{m!}\sup_{\mathrm{B}_r(a)}|\mathrm{D}^m h| \le \frac{2^n}{\omega_n r^n}\left(\frac{2\mathrm{e}n|x{-}a|}{r}\right)^m\|h\|_{1;\mathrm{B}_{2r}(a)}$$

for $x \in \mathrm{B}_r(a)$, and thus we have shown $\lim_{m\to\infty}\mathrm{R}_a^{m-1}h(x) = 0$ in case $|x{-}a| < r/(2\mathrm{e}n)$ (and in fact we can also read off that the convergence is uniform in $x \in \mathrm{B}_{r/(3\mathrm{e}n)}(a)$). This means that the Taylor series of h at an arbitrary point $a \in \Omega$ converges (uniformly) in a neighborhood of a and coincides with h itself there. So, we have established analyticity of h. $\square$

As the final topic in this section, we discuss the extension of harmonic functions by reflection:

Theorem (**reflection principles** for harmonic functions)**.** *Consider an open set Ω in $\mathbb{R}^n$, and decompose it into $\Omega_+ := \{x \in \Omega : x_1 > 0\}$, $\Omega_- := \{x \in \Omega : x_1 < 0\}$, and $\Omega_0 := \{x \in \Omega : x_1 = 0\}$. Moreover, assume that Ω is symmetric with respect to reflection at the hyperplane $\{0\}{\times}\mathbb{R}^{n-1}$, that is $\{x^- : x \in \Omega\} = \Omega$ with the notation $x^- := x{-}2x_1\mathrm{e}_1 = (-x_1, x_2, x_3, \ldots, x_n)$ for $x \in \mathbb{R}^n$.*

(I) ***Odd reflection principle**: If $h \in \mathrm{C}^2(\Omega_+) \cap \mathrm{C}^0(\Omega_+ \cup \Omega_0)$ is harmonic in Ω_+ with **zero boundary values** $h \equiv 0$ at Ω_0, then odd reflection*

$$\overline{h}(x) := \begin{cases} h(x) & \text{for } x \in \Omega_+ \cup \Omega_0 \\ -h(x^-) & \text{for } x \in \Omega_- \end{cases}$$

defines a harmonic function $\overline{h}$ in Ω.

(II) ***Even reflection principle**: If $h \in \mathrm{C}^2(\Omega_+) \cap \mathrm{C}^1(\Omega_+ \cup \Omega_0)$ is harmonic in Ω_+ with **zero boundary normal derivative** $\partial_1 h \equiv 0$ at Ω_0, then even reflection*

$$\overline{h}(x) := \begin{cases} h(x) & \text{for } x \in \Omega_+ \cup \Omega_0 \\ h(x^-) & \text{for } x \in \Omega_- \end{cases}$$

defines a harmonic function $\overline{h}$ in Ω.

Remarks (on the reflection principles)**.**

(1) In both reflection principles it is clear that $\overline{h}$ is harmonic in $\Omega_+ \cup \Omega_-$ (simply by taking into account $\Delta_x[h(x^-)] = (\Delta h)(x^-)$). The essential and non-trivial claim, however, is C^2 regularity of $\overline{h}$ **across Ω_0** (i.e. in an open neighborhood of Ω_0 in Ω). With this regularity is at hand it becomes evident that $\overline{h}$ is harmonic across Ω_0 (by continuity of $\Delta\overline{h}$) and then also smooth and analytic across Ω_0 (by earlier theorems).

[15] This observation can be proved by an induction argument (based on the estimate $\big(1{+}\frac{1}{m}\big)^m \le \mathrm{e}$). Alternatively it can be viewed — at least for $m \gg 1$ which suffices for our purposes — as a consequence of the famous Stirling formula $\lim_{m\to\infty}\frac{m!}{(\frac{m}{\mathrm{e}})^m\sqrt{m}} = \sqrt{2\pi}$, which describes the growth of the factorials.

(2) In particular, the **reflection principles yield C^ω boundary regularity of harmonic functions** in the cases considered: If $h \in \mathrm{C}^2(\Omega_+) \cap \mathrm{C}^0(\Omega_+ \cup \Omega_0)$ is harmonic in Ω_+, then we already know analyticity $h \in \mathrm{C}^\omega(\Omega_+)$ in the interior. Beyond that, in case of zero Dirichlet boundary values on the boundary portion Ω_0, the odd reflection principle implies even up-to-the-boundary analyticity $h \in \mathrm{C}^\omega(\Omega_+ \cup \Omega_0)$. Similarly, the even reflection principle yields up-to-the-boundary analyticity in case of zero Neumann boundary values on the boundary portion Ω_0.

(3) For the even reflection principle, it suffices to impose only the slightly weaker hypotheses $h \in \mathrm{C}^2(\Omega_+) \cap \mathrm{C}^0(\Omega_+ \cup \Omega_0)$, $\partial_1 h \in \mathrm{C}^0(\Omega_+ \cup \Omega_0)$ instead of $h \in \mathrm{C}^2(\Omega_+) \cap \mathrm{C}^1(\Omega_+ \cup \Omega_0)$. This will be clear from one of the proofs at least.

Next we provide two alternative proofs for the odd reflection principle.

1st proof for Part (I) *of the theorem.* Since h vanishes at Ω_0, we clearly have $\overline{h} \in \mathrm{C}^0(\Omega)$. For $a \in \Omega$, we set $r_a := |a_1| > 0$ in case $a \in \Omega_+ \cup \Omega_-$ and $r_a := \infty$ in case $a \in \Omega_0$. With this choice of r_a we claim $⨍_{\mathrm{B}_r(a)} \overline{h}\,\mathrm{d}x = \overline{h}(a)$ for every ball $\overline{\mathrm{B}_r(a)} \subset \Omega$ with $r \in (0, r_a)$. Indeed, in case $a \in \Omega_\pm$ this is immediate by the mean value property of the harmonic function $\overline{h}$ in $\Omega_\pm$, since the choice of r_a ensures $\overline{\mathrm{B}_r(a)} \subset \Omega_\pm$. In case $a \in \Omega_0$, we first find $\int_{\mathrm{B}_r(a)\cap\Omega_+} h(x)\,\mathrm{d}x = \int_{\mathrm{B}_r(a)\cap\Omega_-} h(x^-)\,\mathrm{d}x$, by reflection of the x_1 variable, and then end up with $⨍_{\mathrm{B}_r(a)} \overline{h}\,\mathrm{d}x = 0 = \overline{h}(a)$. All in all, we have shown that $h \in \mathrm{C}^0(\Omega)$ satisfies the mean value property on balls with radii $r \in (0, r_a)$, and then harmonicity of $\overline{h}$ in Ω follows by the characterization lemma in Section 2.7. □

2nd proof for Part (I) *of the theorem.* Since h vanishes at Ω_0, we clearly have $\overline{h} \in \mathrm{C}^0(\Omega)$. For $a \in \Omega_0$ and $\overline{\mathrm{B}_r(a)} \subset \Omega$, the Poisson integral formula

$$\begin{aligned} h_0(x) &:= \int_{\mathrm{S}_R(a)} \overline{h}(y) \mathrm{P}_r(x,y)\,\mathrm{d}\mathcal{H}^{n-1}(y) \\ &= \int_{\mathrm{S}_r(a)\cap\Omega_+} h(y) \mathrm{P}_r(x,y)\,\mathrm{d}\mathcal{H}^{n-1}(y) - \int_{\mathrm{S}_r(a)\cap\Omega_-} h(y^-) \mathrm{P}_r(x,y)\,\mathrm{d}\mathcal{H}^{n-1}(y) \end{aligned}$$

for $x \in \mathrm{B}_r(a)$ provides a harmonic function $h_0 \in \mathrm{C}^2(\mathrm{B}_r(a)) \cap \mathrm{C}^0(\overline{\mathrm{B}_r(a)})$ with $h_0 = \overline{h}$ at $\mathrm{S}_r(a)$. Specifically, for $x \in \mathrm{B}_r(a) \cap \Omega_0$, by reflection of the y-variable and the fact that $\mathrm{P}_r(x, y^-) = \mathrm{P}_r(x, y)$ for $x_1 = 0$ we infer that the integral on $\mathrm{S}_r(a) \cap \Omega_+$ equals the one on $\mathrm{S}_r(a) \cap \Omega_-$. Thus, these integrals cancel out, and we read off that h_0 vanishes at $\mathrm{B}_r(a) \cap \Omega_0$. Consequently, $\overline{h} - h_0 \in \mathrm{C}^2(\mathrm{B}_r(a) \cap \Omega_\pm) \cap \mathrm{C}^0(\overline{\mathrm{B}_r(a) \cap \Omega_\pm})$ is harmonic in $\mathrm{B}_r(a) \cap \Omega_\pm$ with zero boundary values. Applying earlier maximum/uniqueness principles on $\mathrm{B}_r(a) \cap \Omega_\pm$, we conclude that $\overline{h} = h_0$ is harmonic in all of $\mathrm{B}_r(a)$. Since every $a \in \Omega_0$ is contained in a suitable ball $\mathrm{B}_r(a)$, this proves that $\overline{h}$ is harmonic across Ω_0. □

Here, the second proof of (I) can be adapted to establish the even reflection principle in Part (II) of theorem. This adaptation and also an alternative proof of (II) by reduction to (I) will be treated in the exercises.

Further Remarks (on the reflection principles)**.**

(4) Similar principles for the reflection at the unit sphere S_1 are based on the Kelvin transform (defined by $u^*(y) := |y|^{2-n} u(y^*)$; see Remark (2) on Green functions in Section 2.8).

In detail, these principles apply on an open set Ω in $\mathbb{R}^n \setminus \{0\}$ which is symmetric with respect to reflection at S_1, and on such Ω they can be stated as follows:

Odd reflection principle: If $h \in \mathrm{C}^2(\Omega \cap \mathrm{B}_1) \cap \mathrm{C}^0(\Omega \cap \overline{\mathrm{B}_1})$ is harmonic in $\Omega \cap \mathrm{B}_1$ with $h \equiv 0$ at $\Omega \cap \mathrm{S}_1$, then

$$\overline{h}(x) := \begin{cases} h(x) & \text{for } x \in \Omega \cap \overline{\mathrm{B}_1} \\ -h^*(x) & \text{for } x \in \Omega \setminus \overline{\mathrm{B}_1} \end{cases}$$

defines a harmonic function $\overline{h}$ in Ω.

Even reflection principle: If $h \in \mathrm{C}^2(\Omega \cap \mathrm{B}_1) \cap \mathrm{C}^1(\Omega \cap \overline{\mathrm{B}_1})$ is harmonic in $\Omega \cap \mathrm{B}_1$ with $\partial_\nu h \equiv 0$ at $\Omega \cap \mathrm{S}_1$ (where ν denotes the outward unit normal to B_1), then

$$\overline{h}(x) := \begin{cases} h(x) & \text{for } x \in \Omega \cap \overline{\mathrm{B}_1} \\ h^*(x) & \text{for } x \in \Omega \setminus \overline{\mathrm{B}_1} \end{cases}$$

defines a harmonic function $\overline{h}$ in Ω.

Sketch of proof. We first argue that we have the boundary regularity $h \in \mathrm{C}^1(\Omega \cap \overline{\mathrm{B}_1})$ also in case of the odd reflection principle. Indeed, this follows from the boundary regularity recorded in the above Remark (2) for flat boundaries, since the Kelvin transformation connects the Dirichlet problem on (parts of) B_1 to the one on (parts of) the half-space H_n; compare Remark (2) in Section 2.8 once more. With the regularity $h \in \mathrm{C}^2(\Omega \cap \mathrm{B}_1) \cap \mathrm{C}^1(\Omega \cap \overline{\mathrm{B}_1})$ at hand, it is then straightforward to check by computations of derivatives of h^* that $\overline{h} \in \mathrm{C}^2(\Omega \setminus \mathrm{S}_1) \cap \mathrm{C}^1(\Omega)$ holds in the situation of both principles. Moreover, since the Kelvin transformation preserves harmonicity, $\overline{h}$ is harmonic in $\Omega \setminus \mathrm{S}_1$. From Green's first identity we next obtain $\int_{\Omega \cap \mathrm{B}_1} \nabla \overline{h} \cdot \nabla \varphi \,\mathrm{d}x = \int_{\Omega \cap \mathrm{S}_1} \partial_\nu \overline{h}\, \varphi \,\mathrm{d}\mathcal{H}^{n-1}$ and $\int_{\Omega \setminus \overline{\mathrm{B}_1}} \nabla \overline{h} \cdot \nabla \varphi = -\int_{\Omega \cap \mathrm{S}_1} \partial_\nu \overline{h}\, \varphi \,\mathrm{d}\mathcal{H}^{n-1}$ for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ (even with zero right-hand sides in the situation of the even reflection principle). Adding up these two equations, we conclude that $\overline{h}$ is weakly harmonic and thus harmonic in Ω. □

(5) There are reflection principles (with applications to boundary regularity) also for other PDEs.

2.10 Perron's method for the Dirichlet problem on general domains

In this section we return to the existence issue in the Dirichlet problem for harmonic functions

$$\Delta h \equiv 0 \text{ in } \Omega\,, \qquad h = \varphi \text{ at } \partial\Omega\,, \tag{DP}$$

on (quite) general domains Ω. Indeed, in the sequel we will generally assume that Ω is a bounded open set in $\mathbb{R}^n$. We will see, however, that this alone does suffice for solutions h to exist and that more assumptions on Ω will come into play.

We now describe an elegant method, known as the Perron method, for solving the Dirichlet problem on general domains. This method, which decisively draws on Section 2.7, has the advantage that it produces a candidate (in fact the only candidate) for a solution very quickly:

Definition (**subfunction** for the Dirichlet problem)**.** *Consider a bounded open set Ω in $\mathbb{R}^n$ and a bounded function $\varphi\colon \partial\Omega \to \mathbb{R}$. A **subfunction** u for the boundary datum φ is a subharmonic function $u \in \mathrm{C}^0(\Omega)$ (in the generalized sense of Section 2.7) such that $\limsup_{\Omega \ni x \to a} u(x) \le \varphi(a)$ holds for all $a \in \partial\Omega$*

Definition (**Perron function**)**.** *Consider a bounded open set Ω in $\mathbb{R}^n$ and a bounded function $\varphi\colon \partial\Omega \to \mathbb{R}$. The **Perron function** h for φ is the function $h\colon \Omega \to \mathbb{R}$ obtained as the pointwise supremum of all subfunctions for φ, that is,*

$$\boxed{h(x) := \sup\{u(x) : u \text{ is a subfunction for } \varphi\} \qquad \text{for all } x \in \Omega\,.}$$

Remarks (on the definition of the Perron function)**.** Consider a bounded open set Ω in $\mathbb{R}^n$ and a bounded function $\varphi\colon \partial\Omega \to \mathbb{R}$.

(1) The supremum in the definition of the Perron function h is always finite, and indeed we have

$$\inf_{\partial\Omega} \varphi \le h \le \sup_{\partial\Omega} \varphi \qquad \text{in } \Omega\,.$$

Proof. The left-hand inequality results from the fact that the constant function with value $\inf_{\partial\Omega} \varphi$ is a subfunction for φ. The right-hand inequality follows from the observation that every subfunction u for φ satisfies $u \le \sup_{\partial\Omega} \varphi$ on Ω by the weak maximum principle (in a form recorded towards the end of Section 2.4). □

(2) *If* a solution $h_0 \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ to the Dirichlet problem (DP) exists, then this solution h_0 necessarily equals the Perron function h. Hence, the Perron function is a **perfectly reasonable candidate solution** and indeed yields a solution whenever one exists at all.

Proof. Since h_0 is a subfunction for φ, the Perron function h satisfies, by its very definition, $h \ge h_0$ in Ω. Moreover, by a comparison principle, every subfunction u for φ satisfies $u \le h_0$ in Ω, and from this we infer, again by definition, $h \le h_0$ in Ω. □

Regardless of the preceding remark it is not evident and still needs to be proved under suitable assumptions that the Perron function for φ actually solves the Dirichlet problem (DP). One out of two major steps in this direction is addressed in the following theorem.

Theorem (**harmonicity of the Perron function**)**.** *Consider a bounded open set Ω in $\mathbb{R}^n$ and a bounded function $\varphi\colon \partial\Omega \to \mathbb{R}$. Then the Perron function h for φ is harmonic in Ω.*

In order to approach the proof of this theorem we first establish an auxiliary lemma:

Lemma (**harmonic replacement**)**.** *Consider an open set Ω in $\mathbb{R}^n$, a ball $\overline{\mathrm{B}_R(a)} \subset \Omega$, and a subharmonic function u in Ω which is continuous on $\mathrm{S}_R(a)$. Then there exists a function $h \in \mathrm{C}^2(\mathrm{B}_R(a)) \cap \mathrm{C}^0(\overline{\mathrm{B}_R(a)})$ which is harmonic in $\mathrm{B}_R(a)$ and satisfies $h = u$ at $\mathrm{S}_R(a)$. Moreover, by setting*

$$\overline{u}(x) := \begin{cases} h(x) & \text{for } x \in \mathrm{B}_R(a) \\ u(x) & \text{for } x \in \Omega \setminus \mathrm{B}_R(a) \end{cases},$$

we obtain another subharmonic function $\overline{u}$ in Ω which satisfies $\overline{u} \ge u$ in Ω.

The function $\overline{u}$ in the lemma is sometimes called the harmonic replacement of u (with regard to the ball $\mathrm{B}_R(a)$).

Proof of the lemma. The existence of h is known from Section 2.5 and Section 2.8, respectively. Moreover, by the comparison principle of Section 2.7 we infer $h \ge u$ in $\mathrm{B}_R(a)$ and thus $\overline{u} \ge u$ in Ω. To complete the proof we show that $\overline{u}$ is subharmonic. To this end, as seen in Section 2.7, it suffices to verify $\overline{u}(x) \le \fint_{\mathrm{B}_r(x)} \overline{u}\,\mathrm{d}y$ whenever $\overline{\mathrm{B}_r(x)} \subset \Omega$ with $r \in (0, r_x)$, where r_x are arbitrary positive numbers. Indeed, in case $x \in \mathrm{B}_R(a)$, for $r < r_x := R - |x-a| > 0$, we have $\mathrm{B}_r(x) \subset \mathrm{B}_R(a)$ and $\overline{u}(x) = h(x) = \fint_{\mathrm{B}_r(x)} h\,\mathrm{d}y = \fint_{\mathrm{B}_r(x)} \overline{u}\,\mathrm{d}y$ by the mean value property of h. In case $x \in \Omega \setminus \mathrm{B}_R(a)$, the subharmonicity of u implies $\overline{u}(x) = u(x) \le \fint_{\mathrm{B}_r(x)} u\,\mathrm{d}y \le \fint_{\mathrm{B}_r(x)} \overline{u}\,\mathrm{d}y$ even for arbitrary $r < r_x := \infty$. □

Proof of the theorem. We fix an arbitrary ball $\overline{\mathrm{B}_R(a)} \subset \Omega$. The definition of the Perron function h yields a sequence $(u_k)_{k\in\mathbb{N}}$ of subfunctions for φ such that $\lim_{k\to\infty} u_k(a) = h(a)$. Possibly replacing u_k with the pointwise maximum $\max\{u_1, u_2, \ldots, u_k\}$ (which is still subharmonic; see Section 2.7), we can assume that the sequence is non-decreasing, that is, $u_{k+1} \ge u_k$ in Ω for all $k \in \mathbb{N}$. Since subfunctions are continuous by definition, we can apply the lemma and consider the harmonic replacements $\overline{u_k}$ of u_k, here all taken with regard to the fixed ball $\mathrm{B}_R(a)$. By the lemma, the $\overline{u_k}$ are still subharmonic in Ω and hence are subfunctions for φ. In particular, the maximum principle yields the uniform bound $\overline{u_k} \le \sup_{\partial\Omega} \varphi$ in Ω for all $k \in \mathbb{N}$. Moreover, since the comparison principle implies $h_1 \le h_2 \le h_3 \le \ldots$ for the harmonic functions h_k in the definition of $\overline{u_k}$, we also get $\overline{u_{k+1}} \ge \overline{u_k}$ in Ω for all $k \in \mathbb{N}$. Now, the Harnack convergence theorem from Section 2.6 applies to the non-decreasing sequence $(\overline{u_k})_{k\in\mathbb{N}}$ and guarantees that the in-$\mathrm{B}_R(a)$-harmonic functions $\overline{u_k}$ converge for $k \to \infty$ to a harmonic limit function $\overline{h}$ in $\mathrm{B}_R(a)$. Since the Perron function h lies above all the subfunctions $\overline{u_k}$, it also lies above $\overline{h}$ in $\mathrm{B}_R(a)$, that is, $\overline{h} \le h$ in $\mathrm{B}_R(a)$. However, in view of $\overline{h}(a) = \lim_{k\to\infty} \overline{u_k}(a) \ge \lim_{k\to\infty} u_k(a) = h(a)$, we have equality $\overline{h}(a) = h(a)$ at the center point a. Now we use a contradiction argument to show $\overline{h} = h$ even in the full ball $\mathrm{B}_R(a)$. Indeed, assume that this is false and there exists some $x \in \mathrm{B}_R(a)$ with $\overline{h}(x) < h(x)$. Then, by definition of the Perron function, $\overline{h}(x) < u^*(x)$ holds for some subfunction u^* for φ. Furthermore, the functions $u_k^* := \max\{u_k, u^*\}$ and their harmonic replacements $\overline{u_k^*}$, still taken with regard to $\mathrm{B}_R(a)$, are subfunctions for φ with $\overline{u_{k+1}^*} \ge \overline{u_k^*}$ on Ω for all $k \in \mathbb{N}$. As before, the Harnack convergence theorem then yields a harmonic limit function $\overline{h^*} := \lim_{k\to\infty} \overline{u_k^*}$ in $\mathrm{B}_R(a)$ which satisifies $\overline{h^*}(a) = h(a)$. In view of $u_k^* \ge u_k$ we have $\overline{u_k^*} \ge \overline{u_k}$ and $\overline{h^*} \ge \overline{h}$ in $\mathrm{B}_R(a)$. Thus, $\overline{h} - \overline{h^*}$ is a non-positive harmonic function in $\mathrm{B}_R(a)$ and satisfies $\overline{h}(a) - \overline{h^*}(a) = h(a) - h(a) = 0$. Hence, the strong maximum principle implies the coincidence $\overline{h} = \overline{h^*}$ in all of $\mathrm{B}_R(a)$. With this, we finally arrive at $\overline{h}(x) = \overline{h^*}(x) \ge \overline{u_1^*}(x) \ge u_1^*(x) \ge u^*(x)$ which contradicts the choice of u^* above. In conclusion, we have proved that $h = \overline{h}$ is harmonic in $\mathrm{B}_R(a)$. Since this applies for an arbitrary ball $\overline{\mathrm{B}_R(a)} \subset \Omega$, the claimed harmonicity of the Perron function h in Ω is established. □

The second step in the existence program consists in proving that the Perron function attains the prescribed boundary values. In fact, the next theorem characterizes situations in which this is the case with the help of some more terminology:

Definition (barriers and regular boundary points). *Consider an open set Ω in $\mathbb{R}^n$.*

(I) *A function $b \colon \overline{\Omega} \to \mathbb{R}$ is called an* ***(upper) barrier*** *for a boundary point $a \in \partial\Omega$ on Ω if b is continuous on $\Omega \cup \{a\}$, superharmonic in Ω, and lower semicontinuous on $\overline{\Omega}$ with $b > 0$ on $\overline{\Omega} \setminus \{a\}$ and $b(a) = 0$.*

(II) *We say that there exists a* ***local barrier*** *for a boundary point $a \in \partial\Omega$ if there exist some $r > 0$ and a barrier for a on $\Omega \cap \mathrm{B}_r(a)$. A boundary point $a \in \partial\Omega$ is called (a)* ***regular (boundary point)*** *for Ω if a local barrier for a exists.*

Theorem (regular boundary points and attainment of boundary values). *For a bounded open set Ω in $\mathbb{R}^n$ and a boundary point $a \in \partial\Omega$, the following assertions are* ***equivalent****:*

(1) *The point a is a regular boundary point for Ω, that is, there exists a local barrier for a.*

(1′) *There exists a barrier for a on Ω.*

(2) *For every $\varphi \in \mathrm{C}^0(\partial\Omega)$, the Perron function h for φ satisfies $\lim_{\Omega\ni x\to a} h(x) = \varphi(a)$.*

(2′) *For every bounded* $\varphi\colon \partial\Omega \to \mathbb{R}$ *which is continuous at* a*, the Perron function* h *for* φ *satisfies* $\lim_{\Omega\ni x\to a} h(x) = \varphi(a)$.

Proof of the implication (1) $\Longrightarrow$ (1′). By assumption there exist some $r > 0$ and a barrier b for a on $\overline{\Omega \cap \mathrm{B}_{2r}(a)}$. It follows from the lower semicontinuity of b on $\overline{\Omega \cap \mathrm{B}_{2r}(a)}$ and $b > 0$ on $\overline{\Omega \cap \mathrm{B}_{2r}(a)} \setminus \{a\}$ that $b_0 := \min_{\overline{\Omega\cap \mathrm{B}_{2r}(a)}\setminus \mathrm{B}_r(a)} b$ exists and is positive. We now claim that a barrier $\widetilde{b}$ for a on Ω is obtained by setting $\widetilde{b} := \min\{b, b_0\}$ on $\overline{\Omega} \cap \mathrm{B}_{2r}(a)$ and $\widetilde{b} := b_0$ on $\overline{\Omega} \setminus \overline{\mathrm{B}_r(a)}$ (where both these sets are open in $\overline{\Omega}$ and the definitions coincide on their overlap). Indeed, continuity, superharmonicity, and lower semicontinuity carry over from b to $\min\{b, b_0\}$ and then, by locality, to $\widetilde{b}$; compare Section 2.7 for the operations with superharmonicity. Moreover, we clearly have $\widetilde{b} > 0$ on $\overline{\Omega} \setminus \{a\}$ and $\widetilde{b}(a) = 0$. So, $\widetilde{b}$ is a barrier for a on Ω as claimed. □

Next we establish the implication which is crucial for the existence theory.

Proof of the implication (1′) $\Longrightarrow$ (2′). By assumption there exists a barrier b for a on Ω. We consider a bounded $\varphi\colon \partial\Omega \to \mathbb{R}$ which is continuous at a and an arbitrary $\varepsilon > 0$. Thanks to the continuity of φ at a we can fix $\delta > 0$ such that

$$|\varphi - \varphi(a)| \le \varepsilon \qquad \text{in } \mathrm{B}_\delta(a) \cap \Omega\,.$$

Since $\sup_{\partial\Omega} |\varphi|$ is finite and $\min_{\partial\Omega\setminus \mathrm{B}_\delta(a)} b$ is positive (the latter by lower semicontinuity of b and $b > 0$ at $\partial\Omega \setminus \{a\}$), we can next fix $C \in [0, \infty)$ such that

$$2 \sup_{\partial\Omega} |\varphi| \le C \min_{\partial\Omega\setminus \mathrm{B}_\delta(a)} b\,.$$

Finally, since b is continuous on $\Omega \cup \{a\}$ with $b(a) = 0$, we can find some $\widetilde{\delta} \in (0, \delta]$ such that

$$Cb \le \varepsilon \qquad \text{in } \mathrm{B}_{\widetilde{\delta}}(a) \cap \Omega\,.$$

Now, the superharmonicity of b implies that $\varphi(a) - \varepsilon - Cb$ is subharmonic in Ω. In addition, $\varphi(a) - \varepsilon - Cb$ is upper semicontinuous on $\overline{\Omega}$ and $\le \varphi$ at $\partial\Omega$ by the choices of δ and C above (and the fact that $Cb \ge 0$ on $\overline{\Omega}$). All in all, $\varphi(a) - \varepsilon - Cb$ is a subfunction for φ, and analogously $\varphi(a) + \varepsilon + Cb$ is a *super*function for φ (which can be defined by saying that $-(\varphi(a) + \varepsilon + Cb)$ is a subfunction for $-\varphi$). For the Perron function h for φ, these considerations yield

$$\varphi(a) - \varepsilon - Cb \le h \le \varphi(a) + \varepsilon + Cb \qquad \text{in } \Omega\,,$$

where the left-hand inequality follows directly from the definition of the Perron function, while the right-hand inequality combines the definition with the observation that every subfunction for φ stays below the superfunction $\varphi(a)+\varepsilon+Cb$. In other words, we have shown $|h-\varphi(a)| \le \varepsilon+Cb$ in Ω, and taking into account the choice of $\widetilde{\delta}$ we end up with

$$|h - \varphi(a)| \le 2\varepsilon \qquad \text{in } \mathrm{B}_{\widetilde{\delta}}(a) \cap \Omega\,.$$

Since we started with an arbitrary $\varepsilon > 0$, we have indeed shown $\lim_{\Omega\ni x\to a} h(x) = \varphi(a)$. □

At this stage, we record that the implication (2′) $\Longrightarrow$ (2) is trivial. Thus, we can complete the proof of the theorem by providing the following reasoning.

Proof of the implication (2) $\Longrightarrow$ (1). Setting $u(x) := |x-a|$ for $x \in \mathbb{R}^n$, we obtain a continuous convex function u on $\mathbb{R}^n$. In particular, u is subharmonic in $\mathbb{R}^n$ and on Ω (compare Section 2.7) and is a subfunction for its boundary values $\varphi := u|_{\partial\Omega} \in \mathrm{C}^0(\partial\Omega)$. Consequently, the Perron function h for φ satisfies $h \geq u$ on Ω. Moreover, from the previous theorem we infer that h is harmonic, thus superharmonic and continuous, in Ω, and from the assumption (2) we get $\lim_{\Omega \ni x \to a} h(x) = u(a) = 0$. Once we extend h by setting $h(y) := \liminf_{\Omega \ni x \to y} h(x)$ for $y \in \partial\Omega$, then h is also lower semicontinuous on $\overline{\Omega}$ with $h \geq u > 0$ on $\overline{\Omega}$ and $h(a) = 0$. All in all, h is a barrier for a (even) on (all of) Ω. □

By combination of the last two theorems we indeed **solve the Dirichlet problem** for harmonic functions on a bounded open set Ω in $\mathbb{R}^n$ **under a necessary and sufficient condition**, namely regularity, **at the boundary $\partial\Omega$**. We summarize this situation and moreover provide a first sufficient geometric criterion for regularity in the next theorem.

Main Theorem (Dirichlet problem for harmonic functions *on general domains*). *Consider a bounded open set Ω in $\mathbb{R}^n$.*

(I) *The Dirichlet problem*

$$\Delta h \equiv 0 \ \text{in } \Omega\,, \qquad\qquad h = \varphi \ \text{at } \partial\Omega$$

has, for every $\varphi \in \mathrm{C}^0(\partial\Omega)$, a (unique) solution $h \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ ***if and only if*** *all boundary points of Ω are regular for Ω.*

(II) *If Ω satisfies an* ***exterior ball condition*** *at a boundary point $a \in \partial\Omega$, that is, there exist $y \in \mathbb{R}^n$ and $r > 0$ such that $\overline{\mathrm{B}_r(y)} \cap \overline{\Omega} = \{a\}$, then the boundary point a is regular for Ω.*

In particular, if Ω satisfies an exterior ball condition at every *boundary point in $\partial\Omega$, then the Dirichlet problem with an arbitrary continuous boundary datum has a (unique) solution.*

Proof. We first prove (I). On one hand, if the Dirichlet problem is generally solvable, then condition (2) in the previous theorem is satisfied for every $a \in \partial\Omega$, and that theorem yields that all $a \in \partial\Omega$ are regular for Ω. On the other hand, if all points $a \in \partial\Omega$ are regular for Ω, then the last two theorem imply that the Perron function h for given $\varphi \in \mathrm{C}^0(\partial\Omega)$ is harmonic in Ω with $\lim_{\Omega \ni x \to a} h(x) = \varphi(a)$, that is, h extends continuously to $\overline{\Omega}$ and solves the Dirichlet problem.

In order to prove (II) it suffices to provide a local barrier for a. Indeed, using the ball $\mathrm{B}_r(y)$ of the exterior ball condition and setting

$$b(x) := \begin{cases} r^{2-n} - |x-y|^{2-n} & \text{if } n \geq 3 \\ -\log r + \log|x-y| & \text{if } n = 2 \end{cases}$$

for $x \in \overline{\mathbb{R}}^n \setminus \{y\}$, we obtain a harmonic function b in $\mathbb{R}^n \setminus \{y\}$, which is strictly positive in $\mathbb{R}^n \setminus \overline{\mathrm{B}_r(y)}$ and vanishes at $\mathrm{S}_r(y)$. In particular, in view of $\Omega \setminus \{a\} \subset \mathbb{R}^n \setminus \overline{\mathrm{B}_r(y)}$ and $a \in \mathrm{S}_r(y)$, this means that b is barrier for a (even) on (all of) Ω. □

With the preceding theorem at hand, the existence issue for the Dirichlet problem is reduced to the question if the domain under consideration has only regular boundary points. Thus, we now discuss the latter and still non-trivial question in some detail — without detailed proofs and full background explanations, however.

Remarks (on **regular and irregular boundary points**).

(0) Regularity (and thus also irregularity) of a boundary point $a \in \partial\Omega$ for an open set Ω is a **local property** of Ω near a, i.e. it depends only on $\Omega \cap \mathrm{B}_r(a)$ with arbitrarily small $r > 0$.

Moreover, regularity of a boundary point $a \in \partial\Omega$ for Ω is preserved when Ω is made smaller. More precisely, if a is regular for Ω, and $\widetilde{\Omega}$ is an open subset of Ω such that $a \in \partial\widetilde{\Omega}$, then a is regular for $\widetilde{\Omega}$ as well.

Both these properties are immediate from the definition of regularity, and the first property may be seen as an advantage of using *local* barriers in the definition of regularity.

(1) **Convex domains** $\Omega \subset \mathbb{R}^n$ satisfy an exterior ball condition at every point $a \in \partial\Omega$. In fact, in this case there exists even a half-space $H = \{y \in \mathbb{R}^n : \nu \cdot (y-a) > 0\}$ with $\nu \in \mathrm{S}_1^{n-1}$ such that $\overline{\Omega} \cap H = \emptyset$, and as a consequence we have $\overline{\Omega} \cap \overline{\mathrm{B}_r(a+r\nu)} = \{a\}$ for every $r > 0$.

An open set $\Omega \subset \mathbb{R}^n$ has $\mathbf{C^2}$ **boundary** if, for every $a \in \partial\Omega$, there exist $\delta > 0$, $T \in \mathcal{O}(\mathbb{R}^n)$, and $f \in \mathrm{C}^2(\mathbb{R}^{n-1})$ such that $\Omega \cap \mathrm{B}_r(a) = T(\{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y < f(x)\}) \cap \mathrm{B}_r(a)$, that is, Ω coincides locally near a with the rotated subgraph of a C^2 function. Starting from the observation that C^2 functions are locally majorized by parabolas, one can prove that C^2 subgraphs and then also open sets in $\mathbb{R}^n$ with C^2 boundary satisfy an exterior ball condition at every boundary point.

An open set $\Omega \subset \mathbb{R}^n$ is called $\mathbf{C^2}$**-quasiconvex** if, for every $a \in \partial\Omega$, there exist $r > 0$, a convex open set C in $\mathbb{R}^n$, and a C^2 diffeomorphism Φ from $\mathbb{R}^n$ onto an open subset of $\mathbb{R}^n$ such that $\Omega \cap \mathrm{B}_r(a) = \Phi(C) \cap \mathrm{B}_r(a)$, that is, Ω coincides locally near a with the image of a convex set under a C^2 diffeomorphism. This notion includes both convex domains and open sets with C^2 boundary, and still C^2-quasiconvex open sets in $\mathbb{R}^n$ can be shown to satisfy an exterior ball condition at every boundary point.

In conclusion, the preceding theorem therefore guarantees the solvability of the Dirichlet problem for harmonic functions on all these domains Ω and thus on a quite rich class of Ωs.

(2) However, the exterior ball condition is only sufficient, not necessary for regularity and can, in fact, be weakened as follows:

If an open set $\Omega \subset \mathbb{R}^n$ satisfies an **exterior cone condition** at $a \in \partial\Omega$, that is, there exists a non-empty open cone[16] $C \subset \mathbb{R}^n$ with vertex at the origin such that $\overline{\Omega} \cap (a+\overline{C}) = \{a\}$, then a is still regular for Ω. This can be proved by a refined reasoning based on the previously treated cases, maximum principles, and scaling invariance of cones.

Specifically **in two dimensions**, a point $a \in \partial\Omega$ is even regular for an open set $\Omega \subset \mathbb{R}^2$, if Ω merely satisfies an **exterior segment condition** at a, that is, there exists $y \in \mathbb{R}^2 \setminus \{a\}$ such that the line segment from a to y does not intersect Ω. In order to prove this, it suffices to consider the basic case of the disc-with-a-cut $D_* := \mathrm{B}_1^2 \setminus ((-\infty,0] \times \{0\})$. However, the solvability of the Dirichlet problem on D_* (and as a consequence the regularity of all boundary points of D_*) follows from the solvability of the Dirichlet problem on the half-disc $D_+ := \mathrm{B}_1^2 \cap ((0,\infty) \times \mathbb{R})$, since harmonic functions h in D_* correspond via $\widetilde{h}(x) := h(x_1^2 - x_2^2, 2x_1x_2)$ to harmonic functions $\widetilde{h}$ in D_+. Here, the background reason for this correspondency is that $\mathbb{R}^2 \to \mathbb{R}^2, x \mapsto (x_1^2 - x_2^2, 2x_1x_2)$ can be identified with the holomorphic map $\mathbb{C} \to \mathbb{C}, z \mapsto z^2$, but clearly one can also compute $\Delta\widetilde{h}(x) = 4|x|^2 \Delta h(x_1^2 - x_2^2, 2x_1x_2)$ and in this way check the correspondency 'by hands'.

[16] Here a set $C \subset \mathbb{R}^n$ is a cone with vertex at the origin if $x \in C$ implies $\mathbb{R}_{>0}x \subset C$.

(3) Examples of **irregular boundary points** are isolated boundary points, points near which the boundary is σ-finite for $\mathcal{H}^{n-2}$, and in case $n \geq 3$ tip points of certain sharp inward cusps (with a classical example of such a cusp in dimension $n = 3$ known as Lebesgue's thorn).

(4) In principle, there is even a **necessary and sufficient** (though not always easy-to-check) criterion for regularity, the **Wiener criterion** or **Wiener test**: A boundary point $a \in \partial\Omega$ is regular for an open set Ω in $\mathbb{R}^n$, $n \geq 3$, if and only if there holds

$$\sum_{k=0}^{\infty} 2^{(n-2)k} \mathrm{Cap}_2\big(\overline{\mathrm{B}_{2^{-k}}(a)} \setminus \Omega\big) = \infty \,,$$

where the harmonic capacity or 2-capacity Cap_2 is the set function defined in case $n \geq 3$ by

$$\mathrm{Cap}_2(K) := \inf\left\{ \int_{\mathbb{R}^n} |\nabla u|^2 \,\mathrm{d}x \;:\; u \in \mathrm{C}^1_{\mathrm{cpt}}(\mathbb{R}^n)\,,\, u \geq 1 \text{ on } K \right\} \qquad \text{for compact } K \subset \mathbb{R}^n\,.$$

The theory of (this) capacity and the proof of the Wiener criterion are topics in (advanced) potential theory and go beyond the scope of this lecture.

However, to provide a bit of background explanation, we mention that the capacity can be estimated by

$$\mathrm{const}(n)\mathscr{L}^n(K)^{\frac{n-2}{n}} \leq \mathrm{Cap}_2(K) \leq \mathrm{const}(n)\mathcal{H}^{n-2}(K) \qquad \text{for compact } K \subset \mathbb{R}^n\,.$$

Thus, the 2-capacity of K is related to the measures $\mathscr{L}^n(K)$ and $\mathcal{H}^{n-2}(K)$ and may — though it does not truly behave like a measure itself — be regarded as an indicator value for a sort-of size of K. In this light, the Wiener criterion indeed expresses that the complement $\mathrm{B}_r(a) \setminus \Omega$ of Ω in $\mathrm{B}_r(a)$ does not decrease, in way quantified via Cap_2, too fast for $r \searrow 0$.

In dimension $n = 2$, Wiener's test applies with slight technical adaptations. For instance, the above statement remains valid if one uses *relative* capacity with respect to a fixed bounded open set $G_0 \subset \mathbb{R}^2$ such that $\overline{\mathrm{B}_1(a)} \subset G_0$, that is, one understands $\mathrm{Cap}_2(K) := \mathrm{Cap}_2(K, G_0) := \inf\big\{ \int_{G_0} |\nabla u|^2 \,\mathrm{d}x \,:\, u \in \mathrm{C}^1_{\mathrm{cpt}}(G_0)\,,\, u \geq 1 \text{ on } K \big\}$ for compact $K \subset G_0$.

For more details on the criteria mentioned above and on regularity and irregularity of boundary points in general, we refer to [2, Chapters 6.6 and 7.7], for instance.

Finally, we briefly mention that Perron's method for solving the Dirichlet problem has a closely related predecessor in **Poincaré's balayage method** (where the French word 'balayage' signals that non-harmonicity is successively swept out in this approach). Also this method work on bounded open $\Omega \subset \mathbb{R}^n$ and for a given a bounded datum $\varphi\colon \partial\Omega \to \mathbb{R}$. One writes $\Omega = \bigcup_{i=1}^{\infty} B_i$ as a countable union of balls B_i such that $\overline{B_i} \subset \Omega$ and such that the sequence $(B_i)_{i\in\mathbb{N}}$ contains each of its balls B_i not only once, but rather the same ball occurs for infinitely many indices i. Starting from an arbitrary subfunction u for φ, one then defines u_1 as the harmonic replacement of u with respect to B_1 and inductively defines u_{i+1} as the harmonic replacement of u_i with respect to B_{i+1}. While this clearly yields an increasing sequence of subfunctions, one should observe that the harmonic replacement at a later step may evidently wreck harmonicity on a ball occurring earlier if that ball intersects the currently relevant one. However, in view of the re-occurrence of each ball, one may hope that non-harmonicity is successively swept out (as indicated by the name of the method), and indeed by a reasoning not too much different from the one earlier in this section it can be shown that u_i converge for $i \to \infty$ to a harmonic limit function h in Ω. Moreover, if suitable barriers are available, the subfunctions u_i can be bounded from above by upper barriers and may be modified to lie above certain lower barriers. Therefore, when suitably elaborating on the details one can reach an existence theorem very much similar to the one in this section, and all in all it becomes clear that the main advantage of Perron's method over the balayage method lies mainly in the very convenient formula for directly writing down the candidate solution, while otherwise the methods have quite a lot in common.

2.11 The Newton potential as a solution of the Poisson equation

We first introduce a class of function spaces, which serve as a decisive tool in this section.

Definition (**Hölder spaces**)**.**

(I) *Consider a function $g\colon \mathcal{X} \to \mathbb{R}^N$ on a metric space $\mathcal{X}$ and $\alpha \in (0,1]$. We say that g is* ***α-Hölder continuous*** *or Hölder continuous with exponent α on $\mathcal{X}$ if there exists a constant $C \in [0,\infty)$ such that*

$$|g(y) - g(x)| \le C\mathrm{d}_{\mathcal{X}}(y,x)^\alpha \qquad \text{holds for all } x,y \in \mathcal{X}\,.$$

The smallest possible constant

$$[g]_{\alpha;\mathcal{X}} := \sup_{\substack{x,y\in\mathcal{X}\\ y\neq x}} \frac{|g(y)-g(x)|}{\mathrm{d}_{\mathcal{X}}(y,x)^\alpha} \in [0,\infty)$$

in the inequality is called Hölder constant or ***(α-)Hölder seminorm*** *of g on $\mathcal{X}$. We complement this definition for the case $\alpha = 0$ with the convention that g is 0-Hölder-continuous on $\mathcal{X}$ if it is continuous and bounded on $\mathcal{X}$, with corresponding seminorm $[g]_{0;X} := \operatorname{osc}_{\mathcal{X}} g := \sup_{x,y\in\mathcal{X}} |g(y)-g(x)| \in [0,\infty)$.*

(II) *Consider an open set Ω in $\mathbb{R}^n$, a non-negative integer $k \in \mathbb{N}_0$, and an exponent $\alpha \in [0,1]$. The* ***Hölder space*** *$\mathrm{C}^{k,\alpha}(\Omega)$ consists of all functions $u \in \mathrm{C}^k(\Omega)$ such that $\partial^\beta u$ is bounded on Ω for all $\beta \in \mathbb{N}_0^n$ with $|\beta| \le k$ and additionally α-Hölder-continuous in Ω in case $|\beta| = k$. This space is equipped with the* ***$\mathrm{C}^{k,\alpha}$-norm***

$$\|u\|_{\mathrm{C}^{k,\alpha}(\Omega)} := \max_{|\beta|\le k} \Big(\sup_\Omega |\partial^\beta u|\Big) + \max_{|\beta|=k} [\partial^\beta u]_{\alpha;\Omega}\,.$$

The Hölder space $\mathrm{C}^{k,\alpha}(\Omega,\mathbb{R}^N)$ of $\mathbb{R}^N$-valued functions u is defined analogously.

It is straightforward to verify that the Hölder spaces are indeed Banach spaces with the given norms, and thus we refrain from explicating any details on this.

Remarks (on Hölder spaces)**.**

(1) Hölder continuous functions with Hölder exponent 1 are also called Lipschitz continuous. Correspondingly, in this case the Hölder constant is also known as Lipschitz constant.

(2) The local Hölder space $\mathrm{C}^{k,\alpha}_{\mathrm{loc}}(\Omega)$ consists of all $u \in \mathrm{C}^k(\Omega)$ such that, for every $x \in \Omega$, there exists an open neighborhood O of x such that $u|_O \in \mathrm{C}^{k,\alpha}(O)$. The space $C^{k,\alpha}_{\mathrm{cpt}}(\Omega)$ consists of all $u \in \mathrm{C}^{k,\alpha}(\Omega)$ with compact support in Ω.

(3) In case of convex Ω there holds[17] $C^{k+1,\alpha}(\Omega) \subset \mathrm{C}^{k,1}(\Omega)$, and an inductive application of this fact shows that, for $u \in \mathrm{C}^{k,\alpha}(\Omega)$, the lower-order partial derivatives $\partial^\beta u$ with $|\beta| \le k-1$ are all Lipschitz continuous on Ω. Specifically, $C^{k+1,\alpha}_{\mathrm{loc}}(\Omega) \subset \mathrm{C}^{k,1}_{\mathrm{loc}}(\Omega)$ holds even on arbitrary Ω.

[17] To prove this assertion one reasons that boundedness of $\nabla\partial^\beta u$ on Ω implies Lipschitz continuity of $\partial^\beta u$ on Ω by the standard estimate $|\partial^\beta u(y) - \partial^\beta u(x)| = \big|\int_0^1 \nabla\partial^\beta u(x+t(y-x))\cdot(y-x)\,\mathrm{d}t\big| \le \big(\sup_\Omega |\nabla\partial^\beta u|\big)|y-x|$.

(4) The space $\mathrm{C}^{k,0}_{\mathrm{loc}}(\Omega)$ is nothing but the standard space $\mathrm{C}^k(\Omega)$ of k-times continuously differentiable functions on Ω.

Now we turn to a systematic treatment of the **Poisson equation**

$$\Delta u = f \qquad \text{in } \Omega$$

with given right-hand side $f\colon \Omega \to \mathbb{R}$ and unknown $u\colon \Omega \to \mathbb{R}$. In the case $\Omega = \mathbb{R}^n$, the last term in Green's representation formula from Section 2.8 provides a candidate solution, for which we introduce a specific name:

Definition (**Newton potential**). *For $f \in \mathrm{L}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$, the **Newton potential** $\mathrm{N}_f\colon \mathbb{R}^n \to \mathbb{R}$ of f is defined as the convolution of f with the fundamental solution F, that is,*

$$\mathrm{N}_f(x) := (F{*}f)(x) = \int_{\mathbb{R}^n} F(x{-}y)f(y)\,\mathrm{d}y \qquad \textit{for } x \in \mathbb{R}^n\,.$$

In this definition, the integral exists with finite value, since we have $F \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $f \in \mathrm{L}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$. In particular, thanks to the compact support assumption on f, the fact that $F \notin \mathrm{L}^1(\mathbb{R}^n)$ does not cause trouble.

Remarks (on **basic solution properties of the Newton potential**).

(1) The heuristic equation "$\Delta F = \delta_0$" from Section 2.1 suggests the **heuristic computation**

$$\Delta\mathrm{N}_f(x) = \int_{\mathbb{R}^n} \Delta F(x{-}y)f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} f(x{-}z)\Delta F(z)\,\mathrm{d}z = \int_{\mathbb{R}^n} f(x{-}z)\,\mathrm{d}\delta_0(z) = f(x)$$

for $x \in \mathbb{R}^n$. Thus, one may conjecture (though not yet on a very solid basis) that N_f solves the Poisson equation with right-hand side f, that is,

$$\boxed{\Delta\mathrm{N}_f = f \qquad \text{in } \mathbb{R}^n}\,.$$

(2) The prediction of (1) is actually correct in many situations, and for

$$f \in \mathrm{C}^2_{\mathrm{cpt}}(\Omega)$$

we now provide a fully valid quick proof: We first rewrite the definition of N_f as $\mathrm{N}_f(x) = \int_{\mathbb{R}^n} f(x{-}z)F(z)\,\mathrm{d}z$ for $x \in \mathbb{R}^n$. Then, since the pure second derivatives $\partial_i^2 f$ are bounded on $\mathbb{R}^n$ with compact support and $F \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ holds, the differentiation

$$\Delta\mathrm{N}_f(x) = \int_{\mathbb{R}^n} \Delta f(x{-}z)F(z)\,\mathrm{d}z = \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^n\setminus\mathrm{B}_\varepsilon} \big[\Delta_z f(x{-}z)\big]F(z)\,\mathrm{d}z$$

is justified. For fixed $x \in \mathbb{R}^n$ we choose a radius $R \in (0,\infty)$ such that $x{-}\operatorname{spt} f \subset \mathrm{B}_R$. Then the domain of the last integral can be replaced by $\mathrm{B}_R \setminus \overline{\mathrm{B}_\varepsilon}$, and from Green's second identity (with vanishing boundary term on S_R) and the fact that $\Delta F \equiv 0$ in $\mathrm{B}_R \setminus \overline{\mathrm{B}_\varepsilon}$ we infer

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus\mathrm{B}_\varepsilon} \big[\Delta_z f(x{-}z)\big]F(z)\,\mathrm{d}z\\ &\qquad= \bigg[\int_{\mathrm{S}_\varepsilon} \tfrac{z}{|z|}\cdot\nabla f(x{-}z)\,F(z)\,\mathrm{d}\mathcal{H}^{n-1}(z) + \int_{\mathrm{S}_\varepsilon} \tfrac{z}{|z|}\cdot\nabla F(z)\,f(x{-}z)\,\mathrm{d}\mathcal{H}^{n-1}(z)\bigg]\,.\end{aligned}$$

Using the explicit form of F — as in the proof of Green's representation formula in Section 2.8 — we see that the first integral on the right-hand side vanishes in the limit $\varepsilon \searrow 0$, while the second integral turns out to be the mean value integral $\fint_{\mathrm{S}_\varepsilon} f(x-z)\,\mathrm{d}\mathcal{H}^{n-1}(z)$. Summarizing and using the continuity of f, we thus conclude

$$\Delta \mathrm{N}_f(x) = \lim_{\varepsilon\searrow 0} \fint_{\mathrm{S}_\varepsilon} f(x-z)\,\mathrm{d}\mathcal{H}^{n-1}(z) = f(x) \qquad \text{for all } x \in \mathbb{R}^n\,.$$

(3) The C^2 assumption on the right-hand side f in (2) is artificial, since the Poisson equation $\Delta u = f$ does not at all involve derivatives of f. However, if we merely assume

$$f \in \mathrm{L}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)\,,$$

by similar arguments we can show at least that $\mathrm{N}_f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ is generally a **very weak solution** to the Poisson equation in the sense of

$$\int_{\mathbb{R}^n} \mathrm{N}_f \Delta\varphi\,\mathrm{d}x = \int_{\mathbb{R}^n} f\varphi\,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)\,.$$

Indeed, in order to prove this, we fix $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$ and rewrite with Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{N}_f \Delta\varphi\,\mathrm{d}x &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(x-y)f(y)\,\mathrm{d}y\,\Delta\varphi(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^n} F(z)\Delta\varphi(y+z)\,\mathrm{d}z\,\mathrm{d}y\,.\end{aligned}$$

For arbitrary $y \in \mathbb{R}^n$ we choose $L \in (0,\infty)$ such that $\operatorname{spt}\varphi \subset \mathrm{B}_L(y)$. Then, based on Green's second identity, the harmonicity of F on $\mathbb{R}^n \setminus \{0\}$, and the form of ∇F we compute, once more in the spirit of Section 2.8,

$$\begin{aligned}&\int_{\mathbb{R}^n} F(z)\Delta\varphi(y+z)\,\mathrm{d}z\\ &\qquad= \lim_{\varepsilon\searrow 0}\int_{\mathrm{B}_L\setminus\mathrm{B}_\varepsilon} F(z)\Delta\varphi(y+z)\,\mathrm{d}z\\ &\qquad= \lim_{\varepsilon\searrow 0}\left[-\int_{\mathrm{S}_\varepsilon} F(z)\tfrac{z}{|z|}\cdot\nabla\varphi(y+z)\,\mathrm{d}\mathcal{H}^{n-1}(z) + \int_{\mathrm{S}_\varepsilon} \tfrac{z}{|z|}\cdot\nabla F(z)\varphi(y+z)\,\mathrm{d}\mathcal{H}^{n-1}(z)\right]\\ &\qquad= \lim_{\varepsilon\searrow 0}\fint_{\mathrm{S}_\varepsilon}\varphi(y+z)\,\mathrm{d}\mathcal{H}^{n-1}(z) = \varphi(y)\,.\end{aligned}$$

Inserting the result of the last computation into the preceding equality, we end up with

$$\int_{\mathbb{R}^n} \mathrm{N}_f \Delta\varphi\,\mathrm{d}x = \int_{\mathbb{R}^n} f\varphi\,\mathrm{d}y \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)\,,$$

that is, we arrive at the claimed weak solution property.

(4) In the sequel we further refine these observations and reach natural results which do not require the C^2 assumption for the right-hand side f, but still yield solutions in a better sense than just the very weak one. These refinements, however, require more elaborate regularity estimates in Hölder spaces:

Theorem (regularity and solution properties of the Newton potential).

(I) *Suppose*

$$f \in \mathrm{L}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)\,.$$

Then the Newton potential $u := \mathrm{N}_f$ *satisfies*

$$u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n) \qquad \textit{for all } \alpha \in [0,1)$$

with corresponding estimate

$$\|u\|_{\mathrm{C}^{1,\alpha}(\Omega)} \le \mathrm{const}(n,\alpha,R)\|f\|_{\infty;\mathbb{R}^n}$$

for all bounded open subsets Ω *of* $\mathbb{R}^n$ *with bound* $\mathrm{diam}(\Omega \cup \mathrm{spt}\, f) \le R < \infty$*. Moreover,* u *is a* ***weak solution*** *to the Poisson equation* $\Delta u = f$ *in* $\mathbb{R}^n$ *in the sense of*

$$-\int_{\mathbb{R}^n} \nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_{\mathbb{R}^n} f\varphi \,\mathrm{d}x \qquad \textit{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)\,.$$

(II) *Suppose*

$$f \in \mathrm{C}^{0,\alpha}_{\mathrm{cpt}}(\mathbb{R}^n) \qquad \textit{for some } \alpha \in (0,1)\,.$$

Then the Newton potential $u := \mathrm{N}_f$ *satisfies*

$$u \in \mathrm{C}^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$$

with corresponding estimate

$$\|u\|_{\mathrm{C}^{2,\alpha}(\Omega)} \le \mathrm{const}(n,\alpha,R)\|f\|_{C^{0,\alpha}(\mathbb{R}^n)}$$

for all bounded open subsets Ω *of* $\mathbb{R}^n$ *with bound* $\mathrm{diam}(\Omega \cup \mathrm{spt}\, f) \le R < \infty$*. Moreover,* u *is a* ***classical solution*** *to the Poisson equation* $\Delta u = f$ *in* $\mathbb{R}^n$*.*

Proof of Statement (I). We first show that we have $u \in \mathrm{C}^1(\mathbb{R}^n)$ with gradient given by

$$\nabla u(x) = \int_{\mathbb{R}^n} \nabla F(x-y) f(y)\,\mathrm{d}y \qquad \text{for } x \in \mathbb{R}^n\,. \tag{$*$}$$

Here, the integral on the right-hand side exists with finite value in $\mathbb{R}^n$, since we have $\nabla F \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n)$ and $f \in \mathrm{L}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$. We emphasize, however, that the identity ($*$) cannot be derived directly from the definition of u as Newton potential of f by the usual exchange of differentiation and integration, since there exists no x-independent majorant for the mappings $y \mapsto \nabla F(x-y)$ with singularity at x. However, the problem can be circumvented by "cutting out" the singularity as follows. We choose a C^∞ function $\eta\colon \mathbb{R} \to [0,1]$ such that $\eta \equiv 0$ on $(-\infty,1]$ and $\eta \equiv 1$ on $[2,\infty)$. Then, for $\varepsilon > 0$, we set

$$u_\varepsilon(x) := \int_{\mathbb{R}^n} F(x-y)\eta\Big(\tfrac{|x-y|}{\varepsilon}\Big) f(y)\,\mathrm{d}y \in \mathbb{R} \qquad \text{for } x \in \mathbb{R}^n\,,$$

and we infer pointwise convergence $\lim_{\varepsilon\searrow} u_\varepsilon(x) = u(x)$ for $x \in \mathbb{R}^n$ from the dominated convergence theorem with majorant $y \mapsto |F(x-y)f(y)|$. For u_ε, exchange of differentiation and integration is justified (since $z \mapsto F(z)\eta(|z|/\varepsilon)$ is smooth in $\mathbb{R}^n$ and thus $y \mapsto \nabla_x\big[F(x-y)\eta(|x-y|/\varepsilon)\big]$ is bounded on $\operatorname{spt} f$). Therefore, we get

$$\begin{aligned}\nabla u_\varepsilon(x) &= \int_{\mathbb{R}^n} \nabla_x\Big[F(x-y)\eta\Big(\tfrac{|x-y|}{\varepsilon}\Big)\Big] f(y)\,\mathrm{d}y\\ &= \int_{\mathbb{R}^n} \nabla F(x-y)\eta\Big(\tfrac{|x-y|}{\varepsilon}\Big) f(y)\,\mathrm{d}y + \frac{1}{\varepsilon}\int_{\mathbb{R}^n} F(x-y)\eta'\Big(\tfrac{|x-y|}{\varepsilon}\Big)\tfrac{x-y}{|x-y|} f(y)\,\mathrm{d}y\,.\end{aligned}$$

Using this formula and taking into account $\eta' \equiv 0$ on $\mathbb{R}^n \setminus \mathrm{B}_{2\varepsilon}$, it is not difficult to estimate

$$\begin{aligned}&\left|\nabla u_\varepsilon(x) - \int_{\mathbb{R}^n} \nabla F(x-y) f(y)\,\mathrm{d}y\right|\\ &\qquad\le \bigg[\int_{x-\operatorname{spt} f} |\nabla F(z)|\Big|\eta\Big(\tfrac{|z|}{\varepsilon}\Big)-1\Big|\,\mathrm{d}z + \frac{1}{\varepsilon}\int_{\mathrm{B}_{2\varepsilon}} |F|\,\mathrm{d}z\Big(\sup_{\mathbb{R}^n}|\eta'|\Big)\bigg]\|f\|_{\infty;\mathbb{R}^n} \xrightarrow[\varepsilon\searrow 0]{} 0\,,\end{aligned}$$

where the former integral vanishes in the limit by dominated convergence with majorant $|\nabla F|$, while the latter term is controlled through the explicit form of the fundamental solution F. Indeed, with this we have shown

$$\lim_{\varepsilon\searrow 0} \nabla u_\varepsilon(x) = \int_{\mathbb{R}^n} \nabla F(x-y) f(y)\,\mathrm{d}y \qquad \text{locally uniformly in } x \in \mathbb{R}^n$$

(where "locally" stems from the fact that one occurrence of x in the domain of integration in the previous estimate could not be eliminated). Combining the locally uniform convergence of the gradients with the previously observed pointwise convergence $\lim_{\varepsilon\searrow 0} u_\varepsilon = u$, we conclude that we have $u \in \mathrm{C}^1(\mathbb{R}^n)$ and that ∇u is indeed given by $(*)$.

With the knowledge that $u \in \mathrm{C}^1(\mathbb{R}^n)$ holds, the claimed weak solution property follows via a simple integration by parts from the very weak solution property obtained in Remark (3). Alternatively, at this stage, one may deduce the weak solution property from Remark (2) by a mollification argument, which is discussed in the exercises.

Finally, we turn to the $\mathrm{C}^{1,\alpha}$ estimate. For bounded open $\Omega \subset \mathbb{R}^n$ and $x, \widetilde{x} \in \Omega$ such that $d := |\widetilde{x}-x| > 0$, we obtain from $(*)$ the initial estimate

$$|\nabla u(\widetilde{x}) - \nabla u(x)| \le \int_{\operatorname{spt} f} |\nabla F(\widetilde{x}-y) - \nabla F(x-y)|\,\mathrm{d}y\,\|f\|_{\infty;\mathbb{R}^n}\,.$$

For $y \notin \mathrm{B}_{2d}(x)$, the line segment $[x-y, \widetilde{x}-y]$ from $x-y \notin \mathrm{B}_{2d}$ to $\widetilde{x}-y$ has length d and thus stays outside $\mathrm{B}_{|x-y|/2}$. Taking into account that $\nabla^2 F$ is homogeneous of degree $-n$, we then gain the control $|\nabla F(\widetilde{x}-y) - \nabla F(x-y)| \le d\sup_{[x-y,\widetilde{x}-y]} |\nabla^2 F| \le \mathrm{const}(n)d|x-y|^{-n}$ for $y \notin \mathrm{B}_{2d}(x)$. For $y \in \mathrm{B}_{2d}(x) \subset \mathrm{B}_{3d}(\widetilde{x})$, in contrast, since ∇F is homogeneous of degree $1-n$, we directly get $|\nabla F(\widetilde{x}-y) - \nabla F(x-y)| \le \mathrm{const}(n)\big[|\widetilde{x}-y|^{1-n} + |x-y|^{1-n}\big]$. Using these two estimates and observing also $d \le R$ and $\operatorname{spt} f \subset \mathrm{B}_R(x)$ (by the choice of R in the statement of the theorem), we end up with

$$\begin{aligned}&|\nabla u(\widetilde{x}) - \nabla u(x)|\\ &\le \mathrm{const}(n)\bigg[d\int_{\mathrm{B}_{2R}(x)\setminus\mathrm{B}_{2d}(x)} |x-y|^{-n}\,\mathrm{d}y + \int_{\mathrm{B}_{3d}(\widetilde{x})} |\widetilde{x}-y|^{1-n}\,\mathrm{d}y + \int_{\mathrm{B}_{2d}(x)} |x-y|^{1-n}\,\mathrm{d}y\bigg]\|f\|_{\infty;\mathbb{R}^n}\,.\end{aligned}$$

Explicit computation of the integrals on the right-hand side in spherical coordinates then gives

$$|\nabla u(\widetilde{x}) - \nabla u(x)| \le \mathrm{const}(n)\Big[d\log\tfrac{R}{d} + d\Big]\|f\|_{\infty;\mathbb{R}^n} \le \mathrm{const}(n,\alpha)R^{1-\alpha}d^\alpha\|f\|_{\infty;\mathbb{R}^n}$$

for all exponents $\alpha \in (0,1)$. Thus, we have shown α-Hölder continuity of all partial derivatives $\partial_i u$, $i \in \{1,2,\ldots,n\}$ for all $\alpha \in (0,1)$ with corresponding gradient Hölder estimate

$$[\partial_i u]_{\alpha;\Omega} \le \mathrm{const}(n,\alpha)R^{1-\alpha}\|f\|_{\infty;\mathbb{R}^n}\,.$$

We combine this with the (much) simpler estimates

$$|u(x)| \le \int_{\mathrm{B}_R(x)} |F(x-y)|\,\mathrm{d}y\,\|f\|_{\infty;\mathbb{R}^n} \le \mathrm{const}(n,R)\|f\|_{\infty;\mathbb{R}^n} \qquad \text{for } x\in\Omega\,,$$

$$|\partial_i u(x)| \le \int_{\mathrm{B}_R(x)} |\nabla F(x-y)|\,\mathrm{d}y\,\|f\|_{\infty;\mathbb{R}^n} \le \mathrm{const}(n,R)\|f\|_{\infty;\mathbb{R}^n} \qquad \text{for } x\in\Omega$$

and arrive at the claim

$$\|u\|_{\mathrm{C}^{1,\alpha}(\Omega)} \le \mathrm{const}(n,\alpha,R)\|f\|_{\infty;\mathbb{R}^n}\,.$$

This completes the proof of Statement (I). □

Proof of Statement (II). Here, we are first concerned with the claim that $u\in\mathrm{C}^2(\mathbb{R}^n)$ holds and the second derivatives $\partial_i\partial_j u$, $i,j\in\{1,2,\ldots,n\}$, of u are given by

$$\partial_i\partial_j u(x) = \int_{\mathrm{B}_{2R}(x)} \partial_i\partial_j F(x-y)\big(f(y)-f(x)\big)\,\mathrm{d}y + \frac{\delta_{ij}}{n}f(x) \qquad \text{for } x\in\Omega\,, \tag{$**$}$$

where Ω and R satisfy the requirements from the statement, specifically $\mathrm{spt}\,f\subset\mathrm{B}_R(x)$. We emphasize that one may not hope — though it is a tempting conjecture in view of the analogy with $(*)$ — for $\partial_i\partial_j u(x) = \int_{\mathrm{B}_{2R}(x)}\partial_i\partial_j F(x-y)f(y)\,\mathrm{d}y$, since $\partial_i\partial_j F$ is homogeneous of degree $-n$ and thus the latter integral does not even converge whenever $f(x)\neq 0$. In contrast, the integral on the right-hand side of $(**)$ does exist with finite real value, since we assumed $f\in\mathrm{C}^{0,\alpha}_{\mathrm{cpt}}(\mathbb{R}^n)$ and thus the integrand is majorized, up to a multiplicative constant, by the L^1 function $y\mapsto|y-x|^{\alpha-n}$ on $\mathrm{B}_R(x)$. In order to establish $(**)$ we proceed similar to the proof of Statement (I) and choose once more a C^∞ function $\eta\colon\mathbb{R}\to[0,1]$ with $\eta\equiv 0$ on $(-\infty,1]$ and $\eta\equiv 1$ on $[2,\infty)$. Then, for $\varepsilon>0$, we set

$$g^j_\varepsilon(x) := \int_{\mathrm{B}_{2R}(x)} \partial_j F(x-y)\eta\Big(\tfrac{|x-y|}{\varepsilon}\Big)f(y)\,\mathrm{d}y \qquad \text{for } x\in\Omega\,.$$

The dominated convergence theorem, the inclusion $\mathrm{spt}\,f\subset\mathrm{B}_R(x)$, and the identity $(*)$ from the previous proof then yield pointwise convergence $\lim_{\varepsilon\searrow 0} g^j_\varepsilon(x) = \int_{\mathbb{R}^n}\partial_j F(x-y)f(y)\,\mathrm{d}y = \partial_j u(x)$ for $x\in\Omega$. Moreover, on the level of g^j_ε we may differentiate once more to obtain

$$\begin{aligned}
\partial_i g^j_\varepsilon(x) &= \int_{\mathrm{B}_{2R}(x)} \frac{\partial}{\partial x_i}\Big[\partial_j F(x-y)\eta\Big(\tfrac{|x-y|}{\varepsilon}\Big)\Big]f(y)\,\mathrm{d}y \\
&= \int_{\mathrm{B}_{2R}(x)} \frac{\partial}{\partial x_i}\Big[\partial_j F(x-y)\eta\Big(\tfrac{|x-y|}{\varepsilon}\Big)\Big]\big(f(y)-f(x)\big)\,\mathrm{d}y \\
&\qquad + f(x)\int_{\mathrm{B}_{2R}(x)} \frac{\partial}{\partial x_i}\Big[\partial_j F(x-y)\eta\Big(\tfrac{|x-y|}{\varepsilon}\Big)\Big]\,\mathrm{d}y\,.
\end{aligned}$$

In order to simplify the last term for $0<\varepsilon\ll 1$, we first apply the divergence theorem and then take into account that in case $\varepsilon\le R$ it holds $\eta(|x-y|/\varepsilon)=1$ for $y\in \mathrm{S}_{2R}(x)$. In this way, we find

$$\begin{aligned}\int_{\mathrm{B}_{2R}(x)}\frac{\partial}{\partial x_i}\Big[\partial_j F(x-y)\eta\Big(\tfrac{|x-y|}{\varepsilon}\Big)\Big]\,\mathrm{d}y &= -\int_{\mathrm{B}_{2R}(x)}\frac{\partial}{\partial y_i}\Big[\partial_j F(x-y)\eta\Big(\tfrac{|x-y|}{\varepsilon}\Big)\Big]\,\mathrm{d}y\\ &= -\int_{\mathrm{S}_{2R}(x)}\partial_j F(x-y)\tfrac{y_i-x_i}{|y_i-x_i|}\,\mathrm{d}y = \fint_{\mathrm{S}_1} z_j z_i\,\mathrm{d}z = \frac{\delta_{ij}}{n}\,,\end{aligned}$$

and in summary we get

$$\partial_i g^j_\varepsilon(x) = \int_{\mathrm{B}_{2R}(x)}\frac{\partial}{\partial x_i}\Big[\partial_j F(x-y)\eta\Big(\tfrac{|x-y|}{\varepsilon}\Big)\Big]\big(f(y)-f(x)\big)\,\mathrm{d}y+\frac{\delta_{ij}}{n}f(x)\,.$$

Using this formula, computing the x_i-derivative with the product rule, and taking into account $\eta'\equiv 0$ on $\mathbb{R}^n\setminus \mathrm{B}_{2\varepsilon}$, it is not difficult to estimate

$$\begin{aligned}&\bigg|\partial_i g^j_\varepsilon(x)-\int_{\mathrm{B}_{2R}(x)}\partial_i\partial_j F(x-y)\big(f(y)-f(x)\big)\,\mathrm{d}y-\frac{\delta_{ij}}{n}f(x)\bigg|\\ &\qquad\le \int_{\mathrm{B}_{2R}(x)}|\partial_i\partial_j F(x-y)|\,\Big|\eta\Big(\tfrac{|x-y|}{\varepsilon}\Big)-1\Big|\,\big|f(y)-f(x)\big|\,\mathrm{d}y\\ &\qquad\qquad+\frac{1}{\varepsilon}\int_{\mathrm{B}_{2R}(x)}|F(x-y)|\,\Big|\eta'\Big(\tfrac{|x-y|}{\varepsilon}\Big)\Big|\,\big|f(y)-f(x)\big|\,\mathrm{d}y\\ &\qquad\le \mathrm{const}(n)\bigg[\int_{\mathrm{B}_{2\varepsilon}(x)}|x-y|^{\alpha-n}\,\mathrm{d}y+\frac{1}{\varepsilon}\int_{\mathrm{B}_{2\varepsilon}(x)}|x-y|^{1+\alpha-n}\,\mathrm{d}y\bigg][f]_{\alpha;\mathbb{R}^n}\\ &\qquad\le \mathrm{const}(n,\alpha)\varepsilon^\alpha[f]_{\alpha;\mathbb{R}^n}\underset{\varepsilon\searrow 0}{\longrightarrow}0\,.\end{aligned}$$

Hence we obtain

$$\lim_{\varepsilon\searrow 0}\partial_i g^j_\varepsilon(x)=\int_{\mathrm{B}_{2R}(x)}\partial_i\partial_j F(x-y)\big(f(y)-f(x)\big)\,\mathrm{d}y+\frac{\delta_{ij}}{n}f(x)\qquad\text{uniformly in } x\in\Omega\,.$$

Recalling that we already know $u\in \mathrm{C}^1(\mathbb{R}^n)$ and $\lim_{\varepsilon\searrow 0} g^j_\varepsilon=\partial_j u$ in Ω, we thus conclude that we have $u\in\mathrm{C}^2(\mathbb{R}^n)$ and that the second derivatives $\partial_i\partial_j u$ are indeed given by ($**$).

With the knowledge that $u\in\mathrm{C}^2(\mathbb{R}^n)$ holds, the very weak solution property of Remark (3) implies in a standard way that u is a classical solution of the Poisson equation $\Delta u=f$ on $\mathbb{R}^n$. Alternatively, at this stage, one may deduce the same from Remark (3) via a mollification argument.

Finally, we turn to the $\mathrm{C}^{2,\alpha}$ estimate. For bounded open $\Omega\subset\mathbb{R}^n$ and $x,\widetilde{x}\in\Omega$ with $d:=|x-\widetilde{x}|>0$, we first obtain from ($**$) the initial estimate

$$|\partial_i\partial_j u(x)-\partial_i\partial_j u(\widetilde{x})|\le T_1+T_2+T_3$$

with the three right-hand side terms

$$\begin{aligned}T_1 &:= \bigg|\int_{\mathrm{B}_{2R}(x)}\big[\partial_i\partial_j F(x-y)\big(f(y)-f(x)\big)-\partial_i\partial_j F(\widetilde{x}-y)\big(f(y)-f(\widetilde{x})\big)\big]\,\mathrm{d}y\bigg|\,,\\ T_2 &:= \int_{\mathrm{B}_{2R}(x)\Delta \mathrm{B}_{2R}(\widetilde{x})}|\partial_i\partial_j F(\widetilde{x}-y)|\,\big|f(y)-f(\widetilde{x})\big|\,\mathrm{d}y\,,\\ T_3 &:= \frac{1}{n}\big|f(x)-f(\widetilde{x})\big|\end{aligned}$$

(where we used the notation $A\Delta B := (A\setminus B)\cup(B\setminus A)$ for the symmetric difference of sets A and B). Clearly, for T_3 we have the simple estimate

$$T_3 \leq \mathrm{const}(n)d^\alpha[f]_{\alpha;\mathbb{R}^n}\,.$$

Moreover, observing first $\mathrm{B}_{2R}(x)\Delta\mathrm{B}_{2R}(\widetilde{x}) \subset \mathrm{B}_{2R+d}(\widetilde{x})\setminus\mathrm{B}_{2R-d}(\widetilde{x})$ and $d \leq R$, we get

$$\mathscr{L}^n\big(\mathrm{B}_{2R}(x)\Delta\mathrm{B}_{2R}(\widetilde{x})\big) \leq \omega_n(2R{+}d)^n - \omega_n(2R{-}d)^n \leq \mathrm{const}(n)dR^{n-1}\,.$$

Thus, with the homogeneity of $\partial_i\partial_j F$ we can estimate

$$\begin{aligned}
T_2 &\leq \mathrm{const}(n)\int_{\mathrm{B}_{2R}(x)\Delta\mathrm{B}_{2R}(\widetilde{x})} |\widetilde{x}{-}y|^{\alpha-n}\,\mathrm{d}y\,[f]_{\alpha;\mathbb{R}^n}\\
&\leq \mathrm{const}(n)\mathscr{L}^n\big(\mathrm{B}_{2R}(x)\Delta\mathrm{B}_{2R}(\widetilde{x})\big)(2R{-}d)^{\alpha-n}[f]_{\alpha;\mathbb{R}^n}\\
&\leq \mathrm{const}(n)dR^{\alpha-1}[f]_{\alpha;\mathbb{R}^n} \leq \mathrm{const}(n)d^\alpha[f]_{\alpha;\mathbb{R}^n}\,.
\end{aligned}$$

For the main term T_1, we first split the domain of integration into $\mathrm{B}_{2R}(x)\setminus\mathrm{B}_{2d}(x)$ and $\mathrm{B}_{2d}(x)$ and then decompose it further as

$$T_1 \leq T_1^1 + T_1^2 + T_1^3$$

with

$$\begin{aligned}
T_1^1 &:= \int_{\mathrm{B}_{2R}(x)\setminus\mathrm{B}_{2d}(x)} \big|\partial_i\partial_j F(x{-}y) - \partial_i\partial_j F(\widetilde{x}{-}y)\big|\,\big|f(y){-}f(x)\big|\,\mathrm{d}y\,,\\
T_1^2 &:= \bigg|\int_{\mathrm{B}_{2R}(x)\setminus\mathrm{B}_{2d}(x)} \partial_i\partial_j F(\widetilde{x}{-}y)\,\mathrm{d}y\bigg|\,\big|f(x){-}f(\widetilde{x})\big|\,,\\
T_1^3 &:= \int_{\mathrm{B}_{2d}(x)} \big[|\partial_i\partial_j F(x{-}y)|\,\big|f(y){-}f(x)\big| - |\partial_i\partial_j F(\widetilde{x}{-}y)|\big|f(y){-}f(\widetilde{x})\big|\big]\,\mathrm{d}y\,.
\end{aligned}$$

In order to bound T_1^1, we consider $y \in \mathbb{R}^n\setminus\mathrm{B}_{2d}(x)$, and, by the same reasoning as in the proof of Statement (I), we infer $|\partial_i\partial_j F(x{-}y) - \partial_i\partial_j F(\widetilde{x}{-}y)| \leq d\sup_{[\widetilde{x}-y,x-y]}|\nabla\partial_i\partial_j F| \leq \mathrm{const}(n)d|x{-}y|^{1-n}$ for such y. Thus, we obtain

$$T_1^1 \leq \mathrm{const}(n)d\int_{\mathbb{R}^n\setminus\mathrm{B}_{2d}(x)} |x{-}y|^{\alpha-1-n}\,\mathrm{d}y\,[f]_{\alpha;\mathbb{R}^n} \leq \mathrm{const}(n,\alpha)d^\alpha[f]_{\alpha;\mathbb{R}^n}\,.$$

For T_1^2, an application of the divergence theorem on the annulus $\mathrm{B}_{2R}(x)\setminus\mathrm{B}_{2d}(x)$ yields

$$\begin{aligned}
T_1^2 &\leq d^\alpha\int_{\mathrm{S}_{2d}(x)\cup\mathrm{S}_{2R}(x)} |\partial_j F(\widetilde{x}{-}y)|\,\mathrm{d}y\,[f]_{\alpha;\mathbb{R}^n}\\
&\leq \frac{d^\alpha}{n\omega_n}\int_{\mathrm{S}_{2d}(x)\cup\mathrm{S}_{2R}(x)} |\widetilde{x}{-}y|^{1-n}\,\mathrm{d}y\,[f]_{\alpha;\mathbb{R}^n} \leq 2^n d^\alpha[f]_{\alpha;\mathbb{R}^n}\,,
\end{aligned}$$

where in the last step we have taken into account $|\widetilde{x}{-}y|^{1-n} \leq d^{1-n}$ for $y \in \mathrm{S}_{2d}(x) \subset \mathbb{R}^n\setminus\mathrm{B}_d(\widetilde{x})$ and $|\widetilde{x}{-}y|^{1-n} \leq (2R{-}d)^{1-n} \leq R^{1-n}$ for $y \in \mathrm{S}_{2R}(x) \subset \mathbb{R}^n\setminus\mathrm{B}_{2R-d}(\widetilde{x})$. Finally, an estimation in the spirit of the proof of Statement (I) leaves us with

$$\begin{aligned}
T_1^3 &\leq \mathrm{const}(n)\int_{\mathrm{B}_{2d}(x)} \big[|x{-}y|^{\alpha-n} + |\widetilde{x}{-}y|^{\alpha-n}\big]\,\mathrm{d}y\,[f]_{\alpha;\mathbb{R}^n}\\
&\leq \mathrm{const}(n)\bigg[\int_{\mathrm{B}_{2d}(x)} |x{-}y|^{\alpha-n}\,\mathrm{d}y + \int_{\mathrm{B}_{3d}(\widetilde{x})} |\widetilde{x}{-}y|^{\alpha-n}\,\mathrm{d}y\bigg][f]_{\alpha;\mathbb{R}^n}\\
&\leq \mathrm{const}(n,\alpha)d^\alpha[f]_{\alpha;\mathbb{R}^n}\,.
\end{aligned}$$

Collecting the estimates for T_1, T_2, T_3, T_1^1, T_1^2, T_1^3, we end up with

$$|\partial_i\partial_j u(x)-\partial_i\partial_j u(\widetilde{x})| \le \mathrm{const}(n,\alpha)d^\alpha[f]_{\alpha;\mathbb{R}^n} \qquad \text{for all } x,\widetilde{x}\in\Omega\,,$$

that is, with α-Hölder continuity of all second-order derivatives $\partial_i\partial_j u$ of u and with the bound

$$[\partial_i\partial_j u]_{\alpha;\Omega} \le \mathrm{const}(n,\alpha)[f]_{\alpha;\mathbb{R}^n}\,.$$

In order to reach an estimate for the full $\mathrm{C}^{2,\alpha}$-norm we additionally record simple sup-estimates for u, $\partial_i u$, and $\partial_i\partial_j u$. Indeed, the estimates

$$|u(x)| \le \mathrm{const}(n,R)\sup_{\mathbb{R}^n}|f|\,, \qquad |\partial_i u(x)| \le \mathrm{const}(n,R)\sup_{\mathbb{R}^n}|f| \qquad \text{for } x\in\Omega$$

have already been recorded at the end of the proof of Statement (I). Furthermore, rewriting $\partial_i\partial_j u$ via (∗∗), we find

$$\begin{aligned}
|\partial_i\partial_j u(x)| &\le \int_{\mathrm{B}_{2R}(x)} |\partial_i\partial_j F(x-y)|\,\big|f(y)-f(x)\big|\,\mathrm{d}y + \tfrac{1}{n}|f(x)|\\
&\le \mathrm{const}(n)\int_{\mathrm{B}_{2R}(x)} |x-y|^{\alpha-n}\,\mathrm{d}y\,[f]_{\alpha;\mathbb{R}^n} + \sup_{\mathbb{R}^n}|f|\\
&\le \mathrm{const}(n,\alpha,R)[f]_{\alpha;\mathbb{R}^n} + \sup_{\mathbb{R}^n}|f|
\end{aligned}$$

for $x\in\Omega$. All in all, we have estimated the Hölder seminorm $[\partial_i\partial_j u]_{\alpha;\Omega}$ of $\partial_i\partial_j u$ and the suprema of u, $\partial_i u$, and $\partial_i\partial_j u$ on Ω, and we get

$$\|u\|_{\mathrm{C}^{2,\alpha}(\Omega)} \le \mathrm{const}(n,\alpha,R)\|f\|_{\mathrm{C}^{0,\alpha}(\mathbb{R}^n)}\,.$$

This is the last claim, and thus the proof of the theorem is complete. □

Corollary ($\mathbf{C}^{k+2,\alpha}$ estimates for the Newton potential). *Consider $k\in\mathbb{N}_0$ and $\alpha\in(0,1)$. Then, for $f\in\mathrm{C}^{k,\alpha}_{\mathrm{cpt}}(\mathbb{R}^n)$, we have $\mathrm{N}_f\in\mathrm{C}^{k+2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ with*

$$\|\mathrm{N}_f\|_{\mathrm{C}^{k+2,\alpha}(\Omega)} \le \mathrm{const}(n,k,\alpha,R)\|f\|_{\mathrm{C}^{k,\alpha}(\mathbb{R}^n)}$$

for all bounded open subsets Ω of $\mathbb{R}^n$ such that $\mathrm{diam}(\Omega\cup\mathrm{spt}\,f)\le R<\infty$.

Proof. For every $\beta\in\mathbb{N}_0^n$ such that $|\beta|\le k$, since $\partial^\beta f\in\mathrm{C}^0_{\mathrm{cpt}}(\mathbb{R}^n)$ is bounded with compact support and $F\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ holds, we can justify the necessary exchange of differentiation and integration to get

$$\partial^\beta(\mathrm{N}_f)=\mathrm{N}_{\partial^\beta f} \qquad \text{in } \mathbb{R}^n\,.$$

Combining this with the $\mathrm{C}^{2,\alpha}$ estimate from Statement (II) of the previous theorem, we find (for Ω as in the statements)

$$\|\mathrm{N}_f\|_{\mathrm{C}^{k+2,\alpha}(\Omega)} \le \sum_{|\beta|\le k}\|\partial^\beta\mathrm{N}_f\|_{\mathrm{C}^{2,\alpha}(\Omega)} = \sum_{|\beta|\le k}\|\mathrm{N}_{\partial^\beta f}\|_{\mathrm{C}^{2,\alpha}(\Omega)} \le \mathrm{const}(n,\alpha,R)\sum_{|\beta|\le k}\|\partial^\beta f\|_{\mathrm{C}^{0,\alpha}(\mathbb{R}^n)}\,.$$

In addition, by distinguishing the cases $|x-y|<1$ and $|x-y|\ge 1$ in the definition of the Hölder seminorm we see $[g]_{\mathrm{C}^{0,\alpha}(\mathbb{R}^n)}\le 2\sup_{\mathbb{R}^n}|g|+\sup_{\mathbb{R}^n}|\nabla g|$ for arbitrary $g\in\mathrm{C}^{0,\alpha}(\mathbb{R}^n)$. As a consequence we have

$$\|\partial^\beta f\|_{\mathrm{C}^{0,\alpha}(\mathbb{R}^n)} \le 3\sup_{\mathbb{R}^n}|\partial^\beta f| + \sum_{i=1}^n\sup_{\mathbb{R}^n}|\partial^{\beta+\mathrm{e}_i}f| \le (n{+}3)\|f\|_{\mathrm{C}^{k,\alpha}(\mathbb{R}^n)} \qquad \text{in case } |\beta|\le k{-}1\,,$$

while we trivially have

$$\|\partial^\beta f\|_{\mathrm{C}^{0,\alpha}(\mathbb{R}^n)} \le \|f\|_{\mathrm{C}^{k,\alpha}(\mathbb{R}^n)} \qquad \text{in case } |\beta| = k\,.$$

Using the last two estimates on the right-hand side of the estimate for $\|\mathrm{N}_f\|_{\mathrm{C}^{k+2,\alpha}(\Omega)}$, we arrive at the claimed estimate. Specifically, we read off $\mathrm{N}_f \in \mathrm{C}^{k+2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, and the proof is complete. □

Corollary (interior $\mathbf{C}^{k+2,\alpha}$ regularity for the Poisson equation). *Consider an open set Ω in $\mathbb{R}^n$, $k \in \mathbb{N}_0$, and $\alpha \in (0,1)$. Then, every $u \in \mathrm{C}^2(\Omega)$ such that $\Delta u \in \mathrm{C}^{k,\alpha}_{\mathrm{loc}}(\Omega)$ satisfies in fact $u \in \mathrm{C}^{k+2,\alpha}_{\mathrm{loc}}(\Omega)$. Specifically, every $u \in \mathrm{C}^2(\Omega)$ such that $\Delta u \in \mathrm{C}^\infty(\Omega)$ satisfies $u \in \mathrm{C}^\infty(\Omega)$.*

Proof. For $u \in \mathrm{C}^2(\Omega)$ such that $\Delta u \in \mathrm{C}^{k,\alpha}_{\mathrm{loc}}(\Omega)$ and arbitrary $x \in \Omega$, we choose $\varepsilon > 0$ and $f \in \mathrm{C}^{k,\alpha}_{\mathrm{cpt}}(\mathbb{R}^n)$ such that $\Delta u = f$ in $\mathrm{B}_\varepsilon(x) \subset \Omega$. By Statement (II) in the last theorem, we have $\mathrm{N}_f \in \mathrm{C}^2(\mathbb{R}^n)$ and $\Delta \mathrm{N}_f = f$ in $\mathbb{R}^n$. Therefore, $u - \mathrm{N}_f$ is harmonic and thus C^∞ in $\mathrm{B}_\varepsilon(x)$, while the previous corollary gives that N_f is $\mathrm{C}^{k+2,\alpha}$ in $\mathrm{B}_\varepsilon(x)$. So, we have $u = u - \mathrm{N}_f + \mathrm{N}_f \in \mathrm{C}^{k+2,\alpha}_{\mathrm{loc}}(\mathrm{B}_\varepsilon(x))$ and, all in all, also $u \in \mathrm{C}^{k+2,\alpha}_{\mathrm{loc}}(\Omega)$.

For $u \in \mathrm{C}^2(\Omega)$ such that $\Delta u \in \mathrm{C}^\infty(\Omega)$, the statement just proven applies for arbitrary $k \in \mathbb{N}_0$, $\alpha \in (0,1)$ and gives $u \in \mathrm{C}^{k+2}(\Omega)$ for arbitrarily large $k \in \mathbb{N}$. □

Remark. In rough summary the preceding corollary asserts that **solutions u to the Poisson equation $\Delta u = f$ in Ω are "two degrees better" than the right-hand side f**. However, some care is needed, since this applies only in Hölder spaces with exponents $\alpha \in (0,1)$, but *not* in the limit cases $\alpha = 0$ and $\alpha = 1$. For instance, on the unit disc $\mathrm{B}_1 \subset \mathbb{R}^2$, a function $u \in \bigcap_{\alpha \in [0,1)} \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\mathrm{B}_1)$ with $\Delta u \in \mathrm{C}^0(\mathrm{B}_1)$, but $\partial_1^2 u$ unbounded near 0 and thus $u \notin \mathrm{C}^{1,1}_{\mathrm{loc}}(\mathrm{B}_1)$ and $u \notin \mathrm{C}^2(\mathrm{B}_1)$ is given by $u(x) := (x_1^2 - x_2^2)\sqrt{-\log|x|}$ for $x \in \mathrm{B}_1 \setminus \{0\}$ and $u(0) := 0$.

The *special* solution N_f to the Poisson equation $\Delta \mathrm{N}_f = f$ in $\mathbb{R}^n$ provides a starting point for solving the *general* Dirichlet problem

$$\Delta u = f \text{ in } \Omega\,, \qquad\qquad u = \varphi \text{ at } \partial\Omega\,.$$

Indeed, solutions u can be obtained as sums $u = \mathrm{N}_f + h$ of N_f and solutions h to the half-homogeneous Dirichlet problem

$$\Delta h = 0 \text{ in } \Omega\,, \qquad\qquad h = \varphi - \mathrm{N}_f \text{ at } \partial\Omega\,.$$

However, the latter is just a Dirichlet problem for a harmonic function h (which corrects the boundary values of N_f) and has been solved in a large generality in Section 2.10. So, all the tools for solving the general problem are at hand, and the preceding idea can be worked out to obtain the following statement on **existence, regularity, and a-priori estimates for solutions** u.

Main Theorem (on the **Dirichlet problem for the Poisson equation**). *Consider a bounded open set Ω in $\mathbb{R}^n$ such that all points in $\partial\Omega$ are regular for Ω in the sense of Section 2.10, $k \in \mathbb{N}_0$, and $\alpha \in (0,1)$. Then, for every $\varphi \in \mathrm{C}^0(\partial\Omega)$ and every $f \in \mathrm{C}^{k,\alpha}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, there exists a unique solution $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ to the Dirichlet problem*

$$\Delta u = f \text{ in } \Omega \qquad\qquad u = \varphi \text{ at } \partial\Omega\,, \tag{DP}$$

and indeed this solution satisfies $u \in \mathrm{C}^{k+2,\alpha}_{\mathrm{loc}}(\Omega)$ with ***interior a-priori estimate***

$$\|u\|_{\mathrm{C}^{k+2,\alpha}(\Omega')} \le \frac{\mathrm{const}(n,k,\alpha,R)}{d^{k+2+\alpha}}\Big(\|f\|_{\mathrm{C}^{k,\alpha}(\Omega'')} + \sup_\Omega |f| + \sup_{\partial\Omega} |\varphi|\Big) \quad \text{whenever } \overline{\Omega'} \subset \Omega'',\ \overline{\Omega''} \subset \Omega$$

for open sets Ω, Ω', Ω'' in $\mathbb{R}^n$ with $d := \min\{1, \mathrm{dist}(\Omega', \mathbb{R}^n \setminus \Omega'')\}$ and $\operatorname{diam}\Omega \le R < \infty$.

Before spelling out the proof, we put on record a basic observation which will be useful at a couple of points: If, for $d > 0$, we denote by $\mathcal{U}_{d/2}(\Omega)$ the $(d/2)$-neighborhood of an open set $\Omega \subset \mathbb{R}^n$, then we have the auxiliary estimate

$$[g]_{\alpha;\Omega} \le 2d^{-\alpha} \sup_{\Omega} |g| + d^{1-\alpha} \sup_{\mathcal{U}_{d/2}(\Omega)} |\nabla g| \qquad \text{for } g \in \mathrm{C}^1(\mathcal{U}_{d/2}(\Omega))\,. \tag{2.1}$$

Indeed, (2.1) is easily obtained from an elementary estimate for $\frac{|g(y)-g(x)|}{|y-x|^\alpha}$ in case $|y-x| \ge d$ and a straightforward estimate via the gradient in case $|y-x| < d$.

Proof. For a given solution $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ of (DP), the regularity $u \in \mathrm{C}^{k+2,\alpha}_{\mathrm{loc}}(\Omega)$ has already been obtained in the preceding corollary. Next, we quantify the reasoning used there in order to establish the interior a-priori estimate for such a given solution u. To this end, we first choose a cut-off function $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$ such that $\eta \equiv 1$ in $\mathcal{U}_{d/2}(\Omega')$, such that $\operatorname{spt}\eta \subset \Omega''$, and such that $\|\mathrm{D}^\ell\eta\|_{\infty;\mathbb{R}^n} \le \mathrm{const}(n,\ell)d^{-\ell}$ for all $\ell \in \mathbb{N}_0$. Such a function can be obtained, for instance as mollification of the characteristic function of the $3d/4$-neighborhood $\mathcal{U}_{3d/4}(\Omega')$ with mollification radius $d/5$. With η at hand, we introduce $f_0 := \eta f$ and observe $\|f_0\|_{\mathrm{C}^{k,\alpha}(\mathbb{R}^n)} \le \mathrm{const}(n,k)d^{-k-\alpha}\|f\|_{\mathrm{C}^{k,\alpha}(\Omega'')}$. We now set $h := u - \mathrm{N}_{f_0}$ and rely on the auxiliary estimate (2.1) (with $d/2$ in place of d) in order to derive

$$\begin{aligned}\|u\|_{\mathrm{C}^{k+2,\alpha}(\Omega')} &\le \|\mathrm{N}_{f_0}\|_{\mathrm{C}^{k+2,\alpha}(\Omega')} + \|h\|_{\mathrm{C}^{k+2,\alpha}(\Omega')}\\ &\le \mathrm{const}(n,k,\alpha)\Big(\|\mathrm{N}_{f_0}\|_{\mathrm{C}^{k+2,\alpha}(\Omega')} + d^{-\alpha} \sup_{\ell \le k+2} \sup_{\Omega'} |\mathrm{D}^\ell h| + d^{1-\alpha} \sup_{\mathcal{U}_{d/4}(\Omega')} |\mathrm{D}^{k+3}h|\Big)\,.\end{aligned}$$

We further estimate the right-hand side via the estimates for the Newton potential in an earlier corollary, the interior estimates for the harmonic function h on $\mathcal{U}_{d/2}(\Omega')$ (see Section 2.6), and the previously observed control for the $\mathrm{C}^{k,\alpha}$-norm of f_0. This leaves us with

$$\begin{aligned}\|u\|_{\mathrm{C}^{k+2,\alpha}(\Omega')} &\le \mathrm{const}(n,k,\alpha,R)\Big(\|f_0\|_{\mathrm{C}^{k,\alpha}(\mathbb{R}^n)} + d^{-k-2-\alpha} \sup_{\Omega''} |h|\Big)\\ &\le \frac{\mathrm{const}(n,k,\alpha,R)}{d^{k+2+\alpha}}\Big(\|f\|_{\mathrm{C}^{k,\alpha}(\Omega'')} + \sup_{\Omega} |u| + \sup_{\Omega} |\mathrm{N}_{f_0}|\Big)\,.\end{aligned}$$

In addition, we have

$$\sup_{\Omega} |u| \le \mathrm{const}(R)\big(\sup_{\Omega} |f| + \sup_{\Omega} |\varphi|\big)\,, \qquad \sup_{\Omega} |\mathrm{N}_{f_0}| \le \mathrm{const}(n,R) \sup_{\mathbb{R}^n} |f_0| \le \mathrm{const}(n,R) \sup_{\Omega} |f|$$

by the corollary on continuous dependence in Section 2.4 and an easy estimate for the Newton potential (which is also contained in the first theorem of this section). All in all, we thus arrive at

$$\|u\|_{\mathrm{C}^{k+2,\alpha}(\Omega')} \le \frac{\mathrm{const}(n,k,\alpha,R)}{d^{k+2+\alpha}}\Big(\|f\|_{\mathrm{C}^{k,\alpha}(\Omega'')} + \sup_{\Omega} |f| + \sup_{\Omega} |\varphi|\Big)\,,$$

which is the claimed interior a-priori estimate.

It is left to prove, for $f \in \mathrm{C}^{0,\alpha}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ with $\alpha \in (0,1)$, the existence of a solution $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ to the Dirichlet problem. To this end, for each $i \in \mathbb{N}$, we consider a cut-off function $\eta_i \in \mathrm{C}^\infty(\mathbb{R}^n)$ such that $\mathbb{1}_{\Omega_{2/i}} \le \eta \le \mathbb{1}_{\Omega_{1/i}}$ in $\mathbb{R}^n$ (where $\Omega_\delta = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) > \delta\}$). We then introduce $f_i := \eta_i f \in \mathrm{C}^{0,\alpha}_{\mathrm{cpt}}(\mathbb{R}^n)$ and obtain from the theorem on the Newton potential that N_{f_i} is a C^2 solution to

$$\Delta \mathrm{N}_{f_i} = f_i \text{ in } \mathbb{R}^n$$

and that N_{f_i} satisfies

$$\|\mathrm{N}_{f_i}\|_{\mathrm{C}^{1,\alpha}(\Omega)} \le \mathrm{const}(n,\alpha,R)\|f_i\|_{\infty;\mathbb{R}^n} \le \mathrm{const}(n,\alpha,R)\|f\|_{\infty;\mathbb{R}^n}\,.$$

In particular, $(\mathrm{N}_{f_i})_{i\in\mathbb{N}}$ is a sequence of equi-Lipschitz and pointwisely bounded functions, and the Arzelà-Ascoli theorem implies that a subsequence $(\mathrm{N}_{f_{i_\ell}})_{\ell\in\mathbb{N}}$ converges uniformly on Ω. Taking into account the regularity of all boundary points of Ω, the main theorem of Section 2.10 provides, for each $i\in\mathbb{N}$, a solution $h_i\in\mathrm{C}^2(\Omega)\cap\mathrm{C}^0(\overline{\Omega})$ to the Dirichlet problem

$$\Delta h_i \equiv 0 \text{ in } \Omega\,, \qquad\qquad h_i = \varphi - \mathrm{N}_{f_i} \text{ at } \partial\Omega\,.$$

Consequently, $u_i := \mathrm{N}_{f_i} + h_i \in \mathrm{C}^2(\Omega)\cap\mathrm{C}^0(\overline{\Omega})$ solves the Dirichlet problem

$$\Delta u_i = f_i \text{ in } \Omega\,, \qquad\qquad u_i = \varphi \text{ at } \partial\Omega\,,$$

and we are led to discuss the convergence of these problems for $i\to\infty$. Since the maximum principle gives $\|h_i-h_j\|_{\infty;\Omega} \le \|\mathrm{N}_{f_i}-\mathrm{N}_{f_j}\|_{\infty;\Omega}$ for all $i,j\in\mathbb{N}$, as a first step, the uniform Cauchy property carries over from $(\mathrm{N}_{f_{i_\ell}})_{\ell\in\mathbb{N}}$ to $(h_{i_\ell})_{\ell\in\mathbb{N}}$. As a consequence, the solutions $(u_{i_\ell})_{\ell\in\mathbb{N}}$ converge uniformly on Ω. Since all u_i are continuous up $\partial\Omega$ and coincide with φ there, the convergence is in fact uniform on $\overline{\Omega}$ with limit function $u\in\mathrm{C}^0(\overline{\Omega})$ such that $u=\varphi$ at $\partial\Omega$. Finally, we apply the case $k=0$ of the by-now-established interior a-priori estimate to the solutions u_i to obtain

$$\|u_i\|_{\mathrm{C}^{2,\alpha}(\Omega')} \le \frac{\mathrm{const}(n,\alpha,R)}{d^{2+\alpha}}\Big(\|f_i\|_{\mathrm{C}^{0,\alpha}(\Omega'')} + \sup_\Omega |f_i| + \sup_{\partial\Omega}|\varphi|\Big)$$

for all $i\in\mathbb{N}$ and all open sets Ω', Ω'' which meet the requirements of the theorem. On the right-hand side of this estimate, we uniformly bound $\sup_\Omega|f_i|\le\sup_\Omega|f|$ for all i and $\|f_i\|_{\mathrm{C}^{0,\alpha}(\Omega'')} = \|f\|_{\mathrm{C}^{0,\alpha}(\Omega'')}$ at least for $i\ge 2/\mathrm{dist}(\Omega'',\mathbb{R}^n\setminus\Omega)$. Thus, for every open Ω' such that $\overline{\Omega'}\subset\Omega$, the Hölder norms $\|u_i\|_{\mathrm{C}^{2,\alpha}(\Omega')}$ stay bounded for $i\to\infty$, and the Arzelà-Ascoli theorem ensures that a further subsequence of the Hessians $(\nabla^2 u_{i_\ell})_{\ell\in\mathbb{N}}$ converges uniformly on Ω'. By a basic analysis result, u is then C^2 in Ω', and the uniform limit is $\nabla^2 u$. In particular, we may pass to the limit $i\to\infty$ (along the subsequence) in the solution property $\Delta u_i = f_i$ to infer $\Delta u = f$ in Ω'. Since each $x\in\Omega$ is contained in a suitable Ω', all in all we have shown that $u\in\mathrm{C}^2(\Omega)\cap\mathrm{C}^0(\overline{\Omega})$ satisfies $\Delta u = f$ in Ω and $u=\varphi$ at $\partial\Omega$, that is, u is the searched-for solution to (DP). □

2.12 On the eigenvalue problem for the Laplace operator

In close analogy to the notions of eigenvalues and eigenvectors in linear algebra, one introduces eigenvalues and eigenfunctions related to (the Dirichlet problem for) the Laplace operator Δ, and in the sequel we provide a basic discussion of these objects. For a reason that will explained below we prefer, in fact, to coin the notions for the operator $-\Delta$ rather than for Δ itself.

Definitions (eigenvalue equation, eigenvalues, eigenfunctions). *Consider a bounded open set Ω in $\mathbb{R}^n$.*

(I) *The partial differential equation*

$$-\Delta u = \lambda u \qquad \text{in } \Omega$$

with parameter $\lambda\in\mathbb{R}$ is called eigenvalue equation for the operator $-\Delta$ on Ω or Helmholtz equation on Ω. If this equation holds for $\lambda\in\mathbb{R}$ and $u\in\mathrm{C}^2(\Omega)$ which is not constant zero, one calls λ an eigenvalue and u an eigenfunction for the operator $-\Delta$ on Ω.

(II) *If both the eigenvalue equation and the zero Dirichlet boundary condition*

$$\begin{aligned} -\Delta u &= \lambda u && \textit{in } \Omega \\ u &\equiv 0 && \textit{at } \partial\Omega \end{aligned}$$

are satisfied for $\lambda \in \mathbb{R}$ *and* $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ *which is not constant zero, one calls* λ *an eigenvalue and* u *an eigenfunction to the Dirichlet problem for the operator* $-\Delta$ *on* Ω*.*

Remarks (on the eigenvalue problem)**.** Consider a bounded open subset Ω of $\mathbb{R}^n$.

(1) If u is an eigenfunction to the Dirichlet problem for $-\Delta$ on Ω with corresponding eigenvalue λ, then we have

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x = -\int_\Omega u\,\Delta u \,\mathrm{d}x = \lambda \int_\Omega u^2 \,\mathrm{d}x < \infty\,,$$

where the first equality is based[18] on an integration by parts, the second equality results from the eigenvalue equation, and the finiteness of the last integral follows from the requirement $u \in \mathrm{C}^0(\overline{\Omega})$. In particular, this ensures $\nabla u \in \mathrm{L}^2(\Omega, \mathbb{R}^n)$ for eigenfunctions u and $\lambda \geq 0$ for eigenvalues λ. Moreover, by uniqueness in the Dirichlet problem for harmonic functions, $\lambda = 0$ is *not* an eigenvalue, and we conclude that **eigenvalues** to the Dirichlet problem for $-\Delta$ on Ω **are always positive**. In this sense of having only positive eigenvalues, $\boldsymbol{-\Delta}$ **is a positive operator**, and the intention of the initial sign convention is to work with this operator rather than its "negative" counterpart Δ.

(2) In principle, one may admit complex eigenvalues and $\mathbb{C}$-valued eigenfunctions in the above definitions. However, the reasoning in Remark (1) can be adapted to show that, still, all eigenvalues to the Dirichlet problem for $-\Delta$ are positive *real* numbers and thus the $\mathbb{C}$-valued eigenfunctions have $\mathbb{R}$-valued eigenfunctions as real and imaginary parts. Thus, working over $\mathbb{C}$ does not bring any truly new information, and we prefer to keep working over $\mathbb{R}$.

(3) The eigenfunctions (to the Dirichlet problem) for $-\Delta$ on Ω *for a fixed eigenvalue* λ, together with the zero function, form a real vector space. This vector space is called the eigenspace for the eigenvalue λ.

(4) One can reasonably combine the eigenvalue equation with other homogeneous boundary conditions than just the zero Dirichlet one. The most prominent such alternative is the zero Neumann boundary condition.

[18] For a Gauss-Green domain Ω and $u \in \mathrm{C}^2(\overline{\Omega})$ such that $u \equiv 0$ at $\partial\Omega$, the divergence theorem applies to the vector field $u\nabla u \in \mathrm{C}^1(\overline{\Omega})$ and shows directly $\int_\Omega |\nabla u|^2 \,\mathrm{d}x = -\int_\Omega u\,\Delta u\,\mathrm{d}x$. The following reasoning yields the same identity even in the generality of the remark: We set $\eta(t) := 0$ for $t \in [0,1]$, $\eta(t) := (t{-}1)^2/4$ for $t \in [1,3]$, $\eta(t) := t{-}2$ for $t \in [3,\infty)$, and then extend to an odd function $\eta \in \mathrm{C}^1(\mathbb{R})$. Abbreviating $\eta_\varepsilon(t) := \varepsilon\eta(t/\varepsilon)$ for $t \in \mathbb{R}$, we infer $\eta_\varepsilon(u) \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega)$ and $\nabla[\eta_\varepsilon(u)] = \eta_\varepsilon'(u)\nabla u$ in Ω for every $\varepsilon > 0$. Thus, integration by parts for the compactly supported test function $\eta_\varepsilon(u)$ gives

$$\int_\Omega \eta_\varepsilon'(u)|\nabla u|^2 \,\mathrm{d}x = \int_\Omega \nabla[\eta_\varepsilon(u)] \cdot \nabla u \,\mathrm{d}x = -\int_\Omega \eta_\varepsilon(u)\,\Delta u \,\mathrm{d}x = \lambda \int_\Omega \eta_\varepsilon(u) u \,\mathrm{d}x \qquad \text{for every } \varepsilon > 0\,.$$

At this point, the monotone convergence theorem and the observation that $\nabla u \equiv 0$ holds $\mathscr{L}^n$-a.e. on $\{u{=}0\}$ (which in turn follows since $\{u{=}0, \nabla u{\neq}0\}$ is an $(n{-}1)$-dimensional submanifold of $\mathbb{R}^n$ and hence has zero $\mathscr{L}^n$ measure) yield $\lim_{\varepsilon\searrow 0} \int_\Omega \eta_\varepsilon'(u)|\nabla u|^2 \,\mathrm{d}x = \int_\Omega |\nabla u|^2 \,\mathrm{d}x$, while the dominated convergence theorem guarantees $\lim_{\varepsilon\searrow 0} \int_\Omega \eta_\varepsilon(u) u \,\mathrm{d}x = \int_\Omega u^2 \,\mathrm{d}x$. In view of these two convergences we now pass to the limit $\varepsilon \searrow 0$ in the preceding integral equality and arrive at the above claim.

In the next theorem we summarize basic results on the Dirichlet eigenvalue problem for $-\Delta$.

Theorem (on the **Dirichlet eigenvalue problem for the Laplace operator**). *Fix a bounded open set Ω in $\mathbb{R}^n$. Then the eigenvalues, eigenfunctions, and eigenspaces in the Dirichlet problem for $-\Delta$ on Ω have the following properties:*

(I) *The **eigenvalues are positive real numbers**, the **set of eigenvalues is at most countable**, and the **set of eigenvalues has no cluster point in** $\mathbb{R}$ (which, however, leaves ∞ as a possible cluster point).*

(II) *The **eigenspaces are finite-dimensional**.*

(III) *The **eigenspaces are pairwise orthogonal** to each other in the sense that*

$$\int_\Omega uv \,\mathrm{d}x = 0 = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x$$

holds whenever u and v are eigenfunctions to different eigenvalues.

(IV) *The **eigenfunctions are C^∞** functions on Ω.*

We directly proceed to the proofs of all four parts of the theorem:

Proof of Part (IV). Fix an arbitrary $\alpha \in (0,1)$, for instance $\alpha = \frac{1}{2}$. Since an eigenfunction u satisfies $u \in \mathrm{C}^2(\Omega)$ by definition, we trivially get $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega)$ as well. The interior regularity theory of Section 2.11 applies to the Poisson equation $\Delta u = -\lambda u$ in Ω and yields that u is two degrees better than $-\lambda u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega)$, that is, $u \in \mathrm{C}^{3,\alpha}_{\mathrm{loc}}(\Omega)$. However, then interior regularity also gives $u \in \mathrm{C}^{5,\alpha}_{\mathrm{loc}}(\Omega)$, and then even $u \in \mathrm{C}^{7,\alpha}_{\mathrm{loc}}(\Omega)$. Inductively one concludes $u \in \mathrm{C}^\infty(\Omega)$. □

Proof of Part (III). If u is an eigenfunction to an eigenvalue λ and v is an eigenfunction to an eigenvalue ν, the eigenvalue equations and suitable[19] integrations by parts give

$$\lambda \int_\Omega u\,v \,\mathrm{d}x = -\int_\Omega (\Delta u)\,v = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = -\int_\Omega u\,\Delta v \,\mathrm{d}x = \nu \int_\Omega u\,v \,\mathrm{d}x\,.$$

Thus, in the case $\lambda \neq \nu$ of different eigenvalues, we infer first $\int_\Omega u\,v\,\mathrm{d}x = 0$ and then also $\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = 0$. □

[19] Similar to the computation in Remark (1) above, the integrations by parts work easily for a Gauss-Green domain Ω and $u,v \in \mathrm{C}^2(\overline{\Omega})$ with zero boundary values. However, they can also be justified without such extra assumptions: Using η_ε from Footnote 18 and the test function $\eta_\varepsilon(v) \in \mathrm{C}^2_{\mathrm{cpt}}(\Omega)$, we infer

$$\lambda \int_\Omega u\,v \,\mathrm{d}x = \lim_{\varepsilon \searrow 0} \lambda \int_\Omega u\,\eta_\varepsilon(v) \,\mathrm{d}x = \lim_{\varepsilon \searrow 0} \int_\Omega (-\Delta u)\,\eta_\varepsilon(v) = \lim_{\varepsilon \searrow 0} \int_\Omega \eta_\varepsilon'(v)\,\nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\,,$$

where the convergences are ensured by dominated convergence. Specifically, the second convergence also draws on $\mathscr{L}^n$-a.e. vanishing of ∇v on $\{v{=}0\}$ once more and on the fact that Remark (1) ensures first $\nabla u, \nabla v \in \mathrm{L}^2(\Omega,\mathbb{R}^n)$ and then also $\nabla u \cdot \nabla v \in \mathrm{L}^1(\Omega)$. Moreover, testing with $\eta_\varepsilon(u) \in \mathrm{C}^2_{\mathrm{cpt}}(\Omega)$, we analogously get

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \nu \int_\Omega u\,v \,\mathrm{d}x\,,$$

and all in all we end up with the claims made in the above proof.

Proof of Part (I). We already know from Remark (1) that the eigenvalues are positive real numbers.

Moreover, once we show that they have no cluster point in $\mathbb{R}$, the countability claim follows. Indeed, assume that the set of eigenvalues were uncountable. Then, for some sufficiently large $n \in \mathbb{N}$, there would be infinitely many eigenvalues in $(0,n]$. However, by the Bolzano-Weierstraß theorem, these eigenvalues would necessarily have a cluster point in $(0,n]$.

So, we are left to rule out the existence of a cluster point in $\mathbb{R}$. In order to reach a contradiction, suppose the converse, that is, the existence of eigenfunctions u_k to eigenvalues λ_k for all $k \in \mathbb{N}$ such that $\lambda := \lim_{k\to\infty} \lambda_k \in \mathbb{R}$ exists, but $\lambda_k \neq \lambda$ holds for all $k \in \mathbb{N}$. In this situation, since the eigenfunctions u_k are not constantly zero, we have $\sup_\Omega |u_k| > 0$ for all $k \in \mathbb{N}$. Then, possibly passing from u_k to $u_k/\sup_\Omega |u_k|$, which is still and eigenfunction to the eigenvalue λ_k, we can indeed assume $\sup_\Omega |u_k| = 1$ for all $k \in \mathbb{N}$. We now introduce $f_k \in \mathrm{L}^\infty(\mathbb{R}^n) \cap \mathrm{C}^2(\Omega)$ by setting $f_k := -\lambda_k u_k$ in Ω and $f_k :\equiv 0$ on $\mathbb{R}^n \setminus \Omega$ and proceed in analogy with the last reasoning in Section 2.11. Fixing an arbitrary $\alpha \in (0,1)$, by $\mathrm{C}^{1,\alpha}$ estimates for the Newton potential, we first bound

$$\|\mathrm{N}_{f_k}\|_{\mathrm{C}^{1,\alpha}(U)} \le \mathrm{const}(n,\alpha,U)\|f_k\|_{\infty;\mathbb{R}^n} \le \mathrm{const}(n,\alpha,U)|\lambda_k| \underset{k\to\infty}{\longrightarrow} \mathrm{const}(n,\alpha,U)|\lambda|$$

for any bounded open neighborhood U of $\overline{\Omega}$. The Arzelà-Ascoli theorem then yields a subsequence such that $(\mathrm{N}_{f_{k_i}})_{i\in\mathbb{N}}$ and $(\nabla \mathrm{N}_{f_{k_i}})_{i\in\mathbb{N}}$ converge uniformly on $\overline{\Omega}$. Taking into account $\Delta u_k = f_k = \Delta \mathrm{N}_{f_k}$ in Ω and $u_k \equiv 0$ at $\partial\Omega$, the functions $h_k := u_k - \mathrm{N}_{f_k}$ are harmonic in Ω with $h_k = -\mathrm{N}_{f_k}$ at $\partial\Omega$. The maximum principle then ensures the Cauchy property $\sup_{\overline{\Omega}} |h_{k_j} - h_{k_i}| \le \sup_{\partial\Omega} |\mathrm{N}_{f_{k_j}} - \mathrm{N}_{f_{k_i}}| \to 0$ for $i,j \to \infty$, and we deduce uniform convergence of $(h_{k_i})_{i\in\mathbb{N}}$ on $\overline{\Omega}$. In conclusion, the $\mathrm{C}^0(\overline{\Omega})$ functions $u_{k_i} = h_{k_i} + \mathrm{N}_{f_{k_i}}$ with $u_{k_i} \equiv 0$ at $\partial\Omega$ converge for $i \to \infty$ uniformly on $\overline{\Omega}$ to a limit $u \in \mathrm{C}^0(\overline{\Omega})$ such that $u \equiv 0$ at $\partial\Omega$. Moreover, since we assume $\sup_\Omega |u_k| = 1$, we also get $\sup_\Omega |u| = 1$ and in particular $u \not\equiv 0$ on Ω. In addition, we infer from the Weierstraß type convergence theorem in Section 2.6 that the gradients ∇h_{k_i} and thus also ∇u_{k_i} converge for $i \to \infty$ locally uniformly on Ω, and this in turn ensures $\sup_{i\in\mathbb{N}} \|u_{k_i}\|_{\mathrm{C}^{0,\alpha}(\Omega'')} < \infty$ for all open sets Ω'' with $\overline{\Omega''} \subset \Omega$. Now we employ the interior a-priori estimate[20] from Section 2.11. We find

$$\begin{aligned}\|u_k\|_{\mathrm{C}^{2,\alpha}(\Omega')} &\le \mathrm{const}(n,\alpha,\Omega,\Omega',\Omega'')\Big(\|f_k\|_{\mathrm{C}^{0,\alpha}(\Omega'')} + \sup_\Omega |f_k|\Big)\\ &\le \mathrm{const}(n,\alpha,\Omega,\Omega',\Omega'')|\lambda_k|\Big(\|u_k\|_{\mathrm{C}^{0,\alpha}(\Omega'')} + 1\Big)\end{aligned}$$

whenever Ω', Ω'' are open sets with $\overline{\Omega'} \subset \Omega''$ and $\overline{\Omega''} \subset \Omega$. Since we have bounded the right-hand side (at least along a subsequence), another application of the Arzelà-Ascoli theorem yields uniform convergence of (a further subsequence of) the Hessians $\nabla^2 u_{k_i}$ to $\nabla^2 u$ and the Laplacians Δu_{k_i} to Δu in every open Ω' such that $\overline{\Omega'} \subset \Omega$. This is finally enough to conclude $u \in \mathrm{C}^2(\Omega)$ and pass to the limit $k \to \infty$ along subsequences in the eigenvalue equations $-\Delta u_k = \lambda_k u_k$ in Ω. All in all, we end up with an eigenfunction u to a new eigenvalue λ which differs from all λ_k. But then, involving the conclusion of Part (III), we arrive at

$$\int_\Omega u^2 \,\mathrm{d}x = \lim_{i\to\infty} \int_\Omega u_{k_i} u \,\mathrm{d}x = 0\,.$$

[20] The a-priori estimate has been stated in Section 2.11 under the assumption that Ω has only regular boundary points, but still, here we do not need this assumption. This can be checked by inspection of the earlier proof (where the assumption was needed for existence only). Alternatively, it can be justified by applying the a-priori estimate on a smooth domain slightly smaller than Ω.

This contradicts the earlier observation that $u \not\equiv 0$ on Ω and thus completes the proof. □

Proof of Part (II). We argue once more by contradiction and start by assuming the converse of the claim, that is, the existence of an eigenvalue λ and infinitely many linearly independent eigenfunctions $u_1, u_2, u_3, \ldots$ to this eigenvalue. By the Gram-Schmidt process for the L^2 inner product, we can assume $\int_\Omega u_k\, u_\ell \,\mathrm{d}x = 0$ for $k \neq \ell$ in $\mathbb{N}$. As in the proof of Part (I), we can further arrange for $\sup_\Omega |u_k| = 1$ and establish uniform convergence of a subsequence u_{k_i} on $\overline{\Omega}$ to a limit $u \in \mathrm{C}^0(\overline{\Omega})$ such that $u \not\equiv 0$ on Ω. Then we observe

$$\int_\Omega u^2 \,\mathrm{d}x = \lim_{i\to\infty} \int_\Omega u_{k_i} u \,\mathrm{d}x = \lim_{i\to\infty} \lim_{j\to\infty} \underbrace{\int_\Omega u_{k_i} u_{k_j} \,\mathrm{d}x}_{=0 \text{ for } j>i} = 0$$

and reach a contradiction to $u \not\equiv 0$ on Ω. This completes the proof. □

Before closing the section, we add — without proofs or full details — some more comments on the eigenvalue problem.

Further Remarks (on the eigenvalue problem)**.** Consider a bounded open subset Ω of $\mathbb{R}^n$

(1) One can show that there are always infinitely many eigenvalues to the Dirichlet problem for $-\Delta$ on Ω and that there exists an orthonormal Hilbert space basis out of eigenfunctions in $\mathrm{L}^2(\Omega)$. Since the set of eigenvalues is countable $\subset (0,\infty)$ and has no finite cluster point, the first assertion implies that the eigenvalues form a strictly increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $(0,\infty)$ with infinite limit $\lim_{k\to\infty} \lambda_k = \infty$. In particular, we may speak of the first eigenvalue λ_1. If Ω is connected, it can be further shown that this first eigenvalue λ_1 is always simple (that is, the corresponding eigenspace has dimensions 1) with eigenfunctions of constant sign (that is, every eigenfunction to λ_1 is either positive in all of Ω or negative in all of Ω).

    The proofs of these facts are typically carried out in a functional analysis framework and are not addressed here.

(2) For special domains Ω such as balls or cuboids, the eigenvalues and eigenfunctions to the Dirichlet problem for $-\Delta$ on Ω can be computed quite explicitly. This is partially explicated in the exercises.

(3) A famous question asks whether the domain Ω is uniquely determined by the sequence of eigenvalues to the Dirichlet problem for $-\Delta$ on Ω. Since the eigenvalues have an interpretation as resonant oscillation frequencies of an Ω-shaped elastic membrane with clamped boundary, the question may be roughly rephrased as **'Can one hear the shape of a drum?'**. Indeed, answering this question has been a famous open problem for a while. Nowadays it is known that the answer is 'No!' in general, but 'Yes!' under the considerable extra assumptions on Ω (e.g. if Ω is a 2-dimensional convex domain with analytic boundary).

Chapter 3

The heat equation

In this chapter, we investigate (scalar solutions of) the (scalar) **heat equation** or **diffusion equation**

$$\Delta u - u_t \equiv 0 \qquad \text{in } \Omega_T$$

and its inhomogeneous variant

$$\Delta u - u_t = f \qquad \text{in } \Omega_T\,.$$

Here, Ω is an open (and often also bounded) set in $\mathbb{R}^n$, and $T \in (0,\infty]$ is a given time horizon. In addition, the right-hand side of the inhomogeneous heat equation is a given function $f\colon \Omega_T \to \mathbb{R}$, which is called **inhomogeneity**. Moreover, we will generally rely on the following notations and conventions (in parts already used in stating the equations above):

- For arbitrary $A \subset \mathbb{R}^n$, we set $A_T := (0,T) \times A \subset \mathbb{R}^{n+1}$ and call A_T the **space-time cylinder** over A with final time T (and initial time 0). For $(t,x) \in \Omega_T \subset \mathbb{R}^{n+1}$ we regard $t \in (0,T) \subset \mathbb{R}$ as time variable and $x \in \Omega \subset \mathbb{R}^n$ as space variable.

- Correspondingly, we understand the unknown $u\colon \Omega_T \to \mathbb{R}$ as **function of n+1 variables**, 1 time variable and n space variables, where $n \in \mathbb{N}$ is arbitrary (and also the case $n = 1$ of a single space variable, but two variables in total is included). This differs from the convention used when first mentioning the equation in Chapter 1 (n there is now n+1).

- We use the abbreviations

$$\begin{aligned}
u_t &:= \frac{\partial u}{\partial t} && \text{for the \textbf{time derivative}},\\
\Delta u &:= \Delta_x u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} && \text{for the \textbf{spatial Laplacian}},\\
\nabla u &:= \nabla_x u = \Big(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\Big) && \text{for the \textbf{spatial gradient}},\\
\partial_\nu u &:= \big(\partial_\nu\big)_x u = \nu \cdot \nabla_x u \text{ where } \nu(t,x) := \nu_\Omega(x) && \text{for the normal derivative at } (\partial\Omega)_T\,.
\end{aligned}$$

  In particular, even if not explicitly indicated, the operators Δ, ∇, and ∂_ν act with respect to the space variable x only.

- The heat equation is usually coupled with boundary conditions posed on the **parabolic boundary**
$$\partial_{\mathrm{p}}\Omega_T := \big(\{0\}\times\overline{\Omega}\big)\cup(\partial\Omega)_T\,.$$
In fact, $\partial_{\mathrm{p}}\Omega_T$ us the disjoint union of bottom $\{0\}\times\overline{\Omega}$ and **lateral boundary** $(\partial\Omega)_T$ of the space-time cylinder Ω_T and in case $T<\infty$ differs from the full boundary $\partial\big(\Omega_T\big)$ by omission of the top $\{T\}\times\overline{\Omega}$ of the cylinder.

- The most basic **boundary condition** which prescribes the values of the solution itself at $\partial_{\mathrm{p}}\Omega_T$ is usually considered as combination of an **initial condition (IC)**
$$u(0,\,\cdot\,)=u_0 \qquad \text{in } \Omega \qquad\qquad \text{with given } u_0\colon\Omega\to\mathbb{R}$$
and a **Dirichlet boundary condition (BC)**
$$u=\varphi \qquad \text{at } (\partial\Omega)_T \qquad\qquad \text{with given } \varphi\colon(\partial\Omega)_T\to\mathbb{R}\,.$$
The combination of heat equation, initial condition and Dirichlet boundary condition is then called an **initial-boundary value problem (IBVP)** and more specifically a **Cauchy-Dirichlet problem** (where 'Cauchy' indicates the initial condition and 'Dirichlet' the precise boundary condition). When replacing the Dirichlet boundary condition with the Neumann boundary condition $\partial_\nu u=\psi$ at $(\partial\Omega)_T$, one speaks of a Cauchy-Neumann problem. In case $\Omega=\mathbb{R}^n$, the condition at the lateral boundary $(\partial\Omega)_T$ drops out (or is replaced with conditions on the behavior for $x\to\infty$). Correspondingly, in this case one calls the combination of heat equation and the initial condition an **initial value problem (IVP)** or a **Cauchy problem** on full space $\mathbb{R}^n$.

- The heat equation makes sense and is typically posed for solutions in the space
$$\mathrm{C}^2_1\big(\Omega_T\big) := \big\{w\in\mathrm{C}^1\big(\Omega_T\big)\,:\,\nabla_x^2 w \text{ exists and is continuous in } \Omega_T\big\}\,,$$
that is, solutions are required to be C^2 in the space variable(s) x, but only C^1 in the time variable t. Correspondingly, the initial-boundary value problem makes sense for solutions in $\mathrm{C}^2_1\big(\Omega_T\big)\cap\mathrm{C}^0\big(\Omega_T\cup\partial_{\mathrm{p}}\Omega_T\big)$.

We now enter into the **theory of the heat equation** and will see **many parallels to** the previously discussed theory of the **Laplace and Poisson equations**:

3.1 Consequences of the divergence theorem

Applications (of the divergence theorem to the heat equation)**.** In what follows we suppose that Ω is a Gauss-Green domain and
$$u\in\mathrm{C}^2_1\big(\Omega_T\big)\cap\mathrm{C}^1\big(\big(\overline{\Omega}\,\big)_T\big)\cap\mathrm{C}^0\big(\Omega_T\cup\partial_{\mathrm{p}}\Omega_T\big)$$
is a **solution to the heat equation** $\Delta u-u_t\equiv 0$ in Ω_T.

(1) For arbitrary fixed $t\in(0,T)$, we compute
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t,x)\,\mathrm{d}x=\int_\Omega u_t(t,x)\,\mathrm{d}x=\int_\Omega \Delta u(t,x)\,\mathrm{d}x=\int_{\partial\Omega}\partial_\nu u(t,x)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,,$$
where the first step is based on exchange of differentiation and integration (justified, since u and u_t are continuous and thus bounded on the compact set $\{t\}\times\overline{\Omega}$), the second step uses the solution property, and the third step works by the divergence theorem for the vector

field $\nabla u(t,\,\cdot\,)$. In conclusion, we have

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t,x)\,\mathrm{d}x = \int_{\partial\Omega} \partial_\nu u(t,x)\,\mathrm{d}\mathcal{H}^{n-1}(x)} \qquad \text{for all } t\in(0,T)\,.$$

Specifically, in case of the homogeneous Neumann condition $\partial_\nu u\equiv 0$ at $(\partial\Omega)_T$ this shows that $\int_\Omega u(t,x)\,\mathrm{d}x$ and $⨍_\Omega u(t,x)\,\mathrm{d}x$ are constant functions of $t\in[0,T)$, and this goes well along with physical intuition: In the heat equation interpretation, u models temperature, and the constancy conclusion expresses that in case of perfect isolation the mean of temperature is constant in time. In the diffusion equation interpretation, u models density of particles, and the same conclusion now expresses that in a closed system of particles the total mass in constant in time.

(2) Again, for arbitrary fixed $t\in(0,T)$, we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega u(t,x)^2\,\mathrm{d}x &= \int_\Omega u(t,x)u_t(t,x)\,\mathrm{d}x = \int_\Omega u(t,x)\Delta u(t,x)\,\mathrm{d}x\\ &= -\int_\Omega |\nabla u(t,x)|^2\,\mathrm{d}x + \int_{\partial\Omega} u(t,x)\partial_\nu u(t,x)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,,\end{aligned}$$

where we proceed analogous to the previous computation, this time with divergence theorem for $u(t,\,\cdot\,)\nabla u(t,\,\cdot\,)$. In conclusion, if either the homogeneous Dirichlet condition $u\equiv 0$ at $(\partial\Omega)_T$ or the homogeneous Neumann condition $\partial_\nu u\equiv 0$ at $(\partial\Omega)_T$ is satisfied, then we have

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega u(t,x)^2\,\mathrm{d}x = -\int_\Omega |\nabla u(t,x)|^2\,\mathrm{d}x} \qquad \text{for all } t\in(0,T)\,,$$

and consequently $\int_\Omega u(t,x)^2\,\mathrm{d}x$ is a non-increasing function of $t\in[0,T)$. If additionally the homogeneous Cauchy condition $u(0,\,\cdot\,)\equiv 0$ in Ω is in force, this observation necessarily implies $u\equiv 0$ in all of Ω_T.

(3) The preceding point (2) directly implies a **uniqueness result for solutions** $u\in \mathrm{C}^2_1(\Omega_T)\cap \mathrm{C}^1\big(\big(\overline{\Omega}\big)_T\big)\cap \mathrm{C}^0\big(\Omega_T\cup\partial_{\mathrm{p}}\Omega_T\big)$ to the **Cauchy-Dirichlet problem for the heat equation**

$$\Delta u - u_t \equiv 0 \text{ in } \Omega_T\,, \qquad u(0,\,\cdot\,)=u_0 \text{ in } \Omega\,, \qquad u=\varphi \text{ at } (\partial\Omega)_T$$

and an analogous uniqueness conclusion for the Cauchy-Neumann problem. Indeed, to see this, it suffices to apply the conclusion of (2) to the difference of two solutions. (In the Dirichlet case the comparably strong regularity requirement for the solutions up to $(\partial\Omega)_T$ will be weakened in the later Section 3.3.)

(4) Once more, for arbitrary fixed $t\in(0,T)$, we compute in a similar way (exchange of differentiation and integration, divergence theorem for $u_t(t,\,\cdot\,)\nabla u(t,\,\cdot\,)$, solution property)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\nabla u(t,x)|^2\,\mathrm{d}x &= \int_\Omega \nabla u_t(t,x)\cdot\nabla u(t,x)\,\mathrm{d}x\\ &= -\int_\Omega u_t(t,x)\Delta u(t,x)\,\mathrm{d}x + \int_{\partial\Omega} u_t(t,x)\partial_\nu u(t,x)\,\mathrm{d}\mathcal{H}^{n-1}(x)\\ &= -\int_\Omega \Delta u(t,x)^2\,\mathrm{d}x + \int_{\partial\Omega} u_t(t,x)\partial_\nu u(t,x)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,.\end{aligned}$$

Specifically, if either $u_t \equiv 0$ at $(\partial\Omega)_T$ or $\partial_\nu u \equiv 0$ at $(\partial\Omega)_T$ is valid (that is, if either the boundary values are time-independent or the Neumann condition is in force), then the **"energy" $\frac{1}{2}\int_\Omega |\nabla u(t,x)|^2\,\mathrm{d}x$ is a non-increasing function of $t \in (0,T)$**. This puts the heat equation into the realm of energy-dissipating equations or in brief **dissipative equations**.

On a level of technical detail we point out that the computation just made is not directly valid under the above-made regularity assumption for u, essentially $u \in \mathrm{C}^2_1(\Omega_T) \cap \mathrm{C}^1\big((\overline{\Omega})_T\big)$, since this assumption does not even ensure existence of the mixed second-order derivative ∇u_t. However, if not just u, but also u_t is in this space and in fact also if one localizes the assumption in time by replacing $(0,T)$ with $(t-\delta, t+\delta)$ for some small $\delta > 0$, then the preceding computation is fully justified and shows (with marginal reformulation due to $u_t = \Delta u$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\nabla u(t,x)|^2\,\mathrm{d}x = -\int_\Omega u_t(t,x)^2\,\mathrm{d}x + \int_{\partial\Omega} u_t(t,x)\partial_\nu u(t,x)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,.$$

With the help of mollifications u_ε *in the time-variable only* one can then show that this equality stays valid for solutions u such that merely $u \in \mathrm{C}^2_1(\Omega_T) \cap \mathrm{C}^1\big((\overline{\Omega})_T\big)$. Indeed, to this end one first observes that u_ε and $(u_\varepsilon)_t$ (the latter equally good as u_ε itself due to the mollification in t) satisfy the necessary assumptions as soon as ε and δ are small enough to ensure $[t-\delta, t+\delta] \subset (\varepsilon, T-\varepsilon)$ and that therefore the preceding equality is valid for u_ε in place of u. In addition, for $\varepsilon \searrow 0$ one gets uniform convergence $\nabla u_\varepsilon \to \nabla u$ and $(u_\varepsilon)_t \to u_t$ on $(t-\delta, t+\delta) \times \overline{\Omega}$ as well as $\partial_\nu u_\varepsilon \to \partial_\nu u$ at $(t-\delta, t+\delta) \times \partial\Omega$. This induces corresponding convergences for $\varepsilon \searrow 0$ for the three relevant integral terms in a locally-uniform-in-t manner. Then, however, the analysis result on exchange of differentiation and limit under locally uniform convergence of the derivatives can be applied to extend the claim to a solution $u \in \mathrm{C}^2_1(\Omega_T) \cap \mathrm{C}^1\big((\overline{\Omega})_T\big)$.

(5) In case of either $u_t \equiv 0$ at $(\partial\Omega)_T$ or $\partial_\nu u \equiv 0$ at $(\partial\Omega)_T$, by another similar computation, which is treated in the exercises, one checks

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{1}{2}\int_\Omega |\nabla u(t,x)|^2\,\mathrm{d}x \geq 0 \qquad \text{for all } t \in (0,T)$$

This confirms that the **"energy" $\frac{1}{2}\int_\Omega |\nabla u(t,x)|^2\,\mathrm{d}x$ is a convex function of $t \in (0,T)$**.

In the **case $T = \infty$** of infinite time horizon, the convexity observation in fact opens a way for analyzing the long-time asymptotic behavior of solutions to the heat equation as follows. The monotonicity (from point (4)) and convexity of the energy $\mathrm{E}(t) := \frac{1}{2}\int_\Omega |\nabla u(t,x)|^2\,\mathrm{d}x$ mean that E' is a non-positive and non-decreasing function and therefore $\lim_{t\to\infty}\mathrm{E}'(t)$ exists in $(-\infty, 0]$. However, if the limit were negative, this would give the contradiction $E(t) < 0$ for $t \gg 1$. Thus, one deduces that necessarily $\lim_{t\to\infty}\mathrm{E}'(t) = 0$ holds, and in view of point (4) this gives

$$\lim_{t\to\infty}\int_\Omega \Delta u(t,x)^2\,\mathrm{d}x = -\lim_{t\to\infty}\mathrm{E}'(t) = 0\,.$$

In other words, one finds $\lim_{t\to\infty}\Delta u(t,\,\cdot\,) = 0$ in $\mathrm{L}^2(\Omega)$. This may raise hopes for convergence $\lim_{t\to\infty} u(t,\,\cdot\,) = h$ to a harmonic limit function, and indeed by using some L^2 theory for the Poisson equation (which is not covered in this lecture) one can prove the following: If Ω has only regular boundary points and if the solution u fulfills a time-independent boundary condition $u(t,x) = \varphi(x)$ for all $(t,x) \in (0,\infty) \times \partial\Omega$ with $\varphi \in \mathrm{C}^0(\partial\Omega)$, then $u(t,\,\cdot\,)$ converges for $t \to \infty$ in $\mathrm{L}^2(\Omega)$ to the harmonic function h with $h = \varphi$ at $\partial\Omega$. We omit further details. Rather we confine ourselves to adding the final remark that the last-mentioned convergence principle and its variants for other PDEs directly connect elliptic and parabolic theory and thus in some cases allow for studying the one by means of the other.

3.2 The fundamental solution and the mean value property

As mentioned earlier, harmonicity (i.e. the solution property for the Laplace equation) is invariant under isometries such as translations, rotations, reflections. In the same way also the

solution property for the heat equation is invariant under isometries of the x variable, that is, if u solves the heat equation in $(0,\infty)\times\mathbb{R}^n$, then, for every $x_0\in\mathbb{R}^n$ and every $T\in\mathcal{O}(\mathbb{R}^n)$ also $(t,x)\mapsto u(t,x_0+Tx)$, solves the heat equation in $(0,\infty)\times\mathbb{R}^n$. Additionally, there is a scaling invariance, known as **parabolic scaling**:

If u solves the heat equation in $(0,\infty)\times\mathbb{R}^n$, then, for every $\lambda>0$, also $(t,x)\mapsto u(\lambda^2 t,\lambda x)$ solves the heat equation in $(0,\infty)\times\mathbb{R}^n$.

Proof for the parabolic-scaling property. If u solves and $\lambda>0$ is given, one straightforwardly computes

$$\Big(\Delta_x-\frac{\mathrm{d}}{\mathrm{d}t}\Big)u(\lambda^2t,\lambda x)=\lambda^2\Delta u(\lambda^2t,\lambda x)-\lambda^2u_t(\lambda^2t,\lambda x)=0$$

for all $(t,x)\in(0,\infty)\times\mathbb{R}^n$. $\square$

At this point, when trying to **adapt the idea of a fundamental solution** from the case of the Laplace equation to the case of the heat equation, it seems reasonable to look for a function $K\in\mathrm{C}^2_1((0,\infty)\times\mathbb{R}^n)$ which solves the heat equation, is rotationally invariant, and changes under parabolic scaling only by a λ-dependent constant. In terms of formulas this means that we require

$$\Delta K(t,x)-K_t(t,x)=0\,,\qquad K(t,x)=\widetilde{K}(t,|x|)\,,\qquad C(\lambda)K(\lambda^2t,\lambda x)=K(t,x)$$

for all $(t,x)\in(0,\infty)\times\mathbb{R}^n$ and each $\lambda>0$ with an auxiliary function $\widetilde{K}\colon(0,\infty)\times[0,\infty)\to\mathbb{R}$ and with λ-dependent constants $C(\lambda)\in\mathbb{R}$. Choosing $\lambda=1/\sqrt{t}$, the latter two properties imply

$$K(t,x)=C\Big(\frac{1}{\sqrt{t}}\Big)K\Big(1,\frac{x}{\sqrt{t}}\Big)=C\Big(\frac{1}{\sqrt{t}}\Big)\widetilde{K}\Big(1,\frac{|x|}{\sqrt{t}}\Big)$$

for all $(t,x)\in(0,\infty)\times\mathbb{R}^n$. In addition, we would prefer having a fast decay of $K(t,x)$ and it spatial derivatives towards 0 for $|x|\to\infty$. If this requirement is met, then in spirit of point (1) in Section 3.1 it should imply constancy of $\int_{\mathbb{R}^n}K(t,x)\,\mathrm{d}x$ as a function of $t\in(0,\infty)$, and as a normalization we may even take $\int_{\mathbb{R}^n}K(t,x)\,\mathrm{d}x=1$ for all $t\in(0,\infty)$. By bringing in the previously found form of K and by the change of variables $y=\frac{x}{\sqrt{t}}$, we then find

$$1=C\Big(\frac{1}{\sqrt{t}}\Big)\int_{\mathbb{R}^n}\widetilde{K}\Big(1,\frac{|x|}{\sqrt{t}}\Big)\,\mathrm{d}x=C\Big(\frac{1}{\sqrt{t}}\Big)\sqrt{t^n}\int_{\mathbb{R}^n}\widetilde{K}\big(1,|y|\big)\,\mathrm{d}y\,.$$

Since the last integral is independent of t, we infer that the constants $C\big(1/\sqrt{t}\,\big)$ necessarily depend on t in form of

$$C\Big(\frac{1}{\sqrt{t}}\Big)=\frac{C(1)}{\sqrt{t^n}}\,.$$

So, abbreviating $h:=C(1)\widetilde{K}(1,\,\cdot\,)$, we come out with

$$K(t,x)=\frac{1}{\sqrt{t^n}}h\Big(\frac{|x|}{\sqrt{t}}\Big)$$

as a reasonable **ansatz for the fundamental solution** of the heat equation. As will be discussed in the exercises, one may plug this ansatz into the heat equation to derive a second-order ODE for h and may then solve this ODE explicitly to find

$$K(t,x)=\frac{1}{\sqrt{t^n}}\exp\Big(-\frac{|x|^2}{4t}\Big)\Big[c+d\int_1^{\frac{|x|}{\sqrt{t}}}\mathrm{e}^{\sigma^2/4}\sigma^{1-n}\,\mathrm{d}\sigma\Big]\qquad\text{for all }(t,x)\in(0,\infty)\times\mathbb{R}^n$$

with two constants $c, d \in \mathbb{R}$. Finally, for simplification and for ensuring the decay for $|x| \to \infty$ one takes $d = 0$ and may then determine c from the normalization condition[1]. In this way we arrive at the following definition.

Definition (fundamental solution). *The function $K\colon (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, given by*

$$K(t,x) := \frac{1}{\sqrt{(4\pi t)^n}} \exp\left(-\frac{|x|^2}{4t} \right) \qquad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^n$$

is called the ***fundamental solution of the heat equation*** *in $\mathbb{R}^n$ or the* ***heat kernel*** *on $\mathbb{R}^n$.*

Remarks (on the fundamental solution)**.**

(1) From the construction and the explicit formula one reads off that the fundamental solution K is a positive solution to the heat equation $\Delta K - K_t \equiv 0$ in $(0,\infty) \times \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} K(t,x)\,\mathrm{d}x = 1 \quad \text{for all } t > 0 \qquad \text{and} \qquad \lim_{t \searrow 0} K(t,x) = \begin{cases} 0 & \text{for } x \in \mathbb{R}^n \setminus \{0\} \\ \infty & \text{for } x = 0 \end{cases}.$$

In a sense this means that K solves the initial value problem for the heat equation on full space $\mathbb{R}^n$ with the **heuristic initial condition "$\boldsymbol{K(0,\,\cdot\,) = \delta_0}$ in $\mathbb{R}^n$"**, where δ_0 denotes once more the Dirac measure at the origin. In terms of physical interpretation, this initial condition for time $t = 0$ corresponds to zero temperature in $\mathbb{R}^n \setminus \{0\}$, but a "unit amount of heat" concentrated at the origin. However, since $K(t,\,\cdot\,)$ is positive in all of $\mathbb{R}^n$ already for arbitrarily small $t > 0$, the temperature immediately starts to raise even at points arbitrarily far from the origin — though there extremely slowly. In this sense the heat equation models heat propagation with an **infinite speed of propagation**.

(2) The fundamental solution K can be extended to a C^∞ solution K of the heat equation in $\big((-\infty,\infty) \times \mathbb{R}^n\big) \setminus \{(0,0)\}$ simply by setting $K :\equiv 0$ on $\big((-\infty,0] \times \mathbb{R}^n\big) \setminus \{(0,0)\}$. Indeed, the C^∞ property of this extension follows essentially from the well-known fact that the single-variable function $s \mapsto \begin{cases} \mathrm{e}^{-1/s} & \text{for } s > 0 \\ 0 & \text{for } s \le 0 \end{cases}$ is C^∞ in $\mathbb{R}$, and the solution property is then immediate. Moreover, in the same way as this single-variable function is not analytic near 0, the extended fundamental solution is not analytic in t near $\{0\} \times \mathbb{R}^n$. Therefore, we see that **in general $\boldsymbol{C^\infty}$ solutions** to the heat equation **need not be analytic in the $\boldsymbol{t}$ variable**.

With the fundamental solution at hand, we now turn to a heat-equation version of the mean value property which in fact works by taking mean values on the superlevel sets of the fundamental solution.

Definition (heat balls). *For $(t,x) \in (-\infty,\infty) \times \mathbb{R}^n$, $r > 0$, one defines the* ***heat ball*** *$\mathrm{W}_r(t,x) \subset (-\infty,t) \times \mathbb{R}^n$ as*

$$\mathrm{W}_r(t,x) := \{(s,y) \in (-\infty,t) \times \mathbb{R}^n \,:\, K(t-s, x-y) > r^{-n}\}$$

[1]Indeed, observe that $\frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x|^2}{4t}\right) \mathrm{d}x = \frac{1}{\sqrt{\pi^n}} \int_{\mathbb{R}^n} \mathrm{e}^{-|y|^2}\,\mathrm{d}y = 1$ holds by the change of variables $y = \frac{x}{\sqrt{4t}}$ and the well-known formula for the value of the Gaussian integral (in arbitrary dimension n).

Remark. It is decisive that by definition $\boldsymbol{W_r(t,x)}$ **lies in the past of** $\boldsymbol{t}$, that is, $\mathrm{W}_r(t,x) \subset (-\infty,t)\times\mathbb{R}^n$. Moreover, bringing in the explicit formula for K and rearranging the defining inequality of $\mathrm{W}_r(t,x)$ one can also show for $(s,y)\in(-\infty,t)\times\mathbb{R}^n$ that

$$(s,y)\in\mathrm{W}_r(t,x) \iff |x{-}y|^2 < -2n(t{-}s)\log\left(\frac{4\pi(t{-}s)}{r^2}\right).$$

Then one sees by standard calculus that the right-hand side of the rearranged inequality is positive precisely for $s > t{-}\frac{1}{4\pi}r^2$ and reaches for $s = t{-}\frac{1}{4\pi\mathrm{e}}r^2$ the maximum value $\frac{n}{2\pi\mathrm{e}}r^2$. These considerations yield in particular the inclusion

$$\mathrm{W}_r(t,x) \subset \left(t, t{-}\frac{1}{4\pi}r^2\right) \times \mathrm{B}_{\sqrt{\frac{n}{2\pi\mathrm{e}}}\,r}(x)$$

with the usual Euclidean ball $\mathrm{B}_{\sqrt{\frac{n}{2\pi\mathrm{e}}}\,r}(x)\subset\mathbb{R}^n$.

Theorem (mean value property). *Consider an open set $\Omega\subset\mathbb{R}^n$ and a solution $u\in\mathrm{C}^2_1(\Omega_T)$ to the heat equation $\Delta u - u_t \equiv 0$ in Ω_T. Then, for every heat ball $\overline{\mathrm{W}_r(t,x)}\subset\Omega_T$, there holds*

$$u(t,x) = \frac{1}{4r^n}\int_{\mathrm{W}_r(t,x)} u(s,y)\frac{|y{-}x|^2}{(t{-}s)^2}\,\mathrm{d}(s,y)\,,$$

where $\mathrm{d}(s,y)$ means integration with respect to the $(n{+}1)$-dimensional Lebesgue measure.

Remark. There holds

$$\frac{1}{4r^n}\int_{\mathrm{W}_r(t,x)} \frac{|y{-}x|^2}{(t{-}s)^2}\,\mathrm{d}(s,y) = 1\,,$$

and thus the integral in the theorem can and should be understood as a **weighted mean value integral** with respect to the weight $(s,y)\mapsto \frac{1}{4r^n}\frac{|y-x|^2}{(t-s)^2}$.

Sketch of proof for the normalization claimed in the remark. By a change of variables one can reduce to the case $(t,x)=(0,0)$ and $r=1$. For treating this case, one first computes by Fubini's theorem (with sections of $\mathrm{W}_1(0,0)$ determined by the previous remark) and radial integration

$$\begin{aligned}\int_{\mathrm{W}_1(0,0)}\frac{|y|^2}{s^2}\,\mathrm{d}(s,y) &= \int_{-\frac{1}{4\pi}}^0 \frac{1}{s^2}\int_{\mathrm{B}_{\sqrt{2ns\log(-4\pi s)}}} |y|^2\,\mathrm{d}y\,\mathrm{d}s\\ &= \frac{(2n)^{\frac{n+2}{2}}n\omega_n}{n{+}2}\int_{-\frac{1}{4\pi}}^0 (-s)^{\frac{n-2}{2}}\big({-}\log(-4\pi s)\big)^{\frac{n+2}{2}}\,\mathrm{d}s\,.\end{aligned}$$

One way of computing the resulting integral is then using the change of variables $\eta = -\frac{n}{2}\log(-4\pi s)$ to transform the last expression into

$$\frac{8\omega_n}{(n{+}2)\pi^{n/2}}\int_0^\infty \mathrm{e}^{-\eta}\eta^{\frac{n+2}{2}}\,\mathrm{d}\eta\,,$$

where by definition and properties of the Γ function it is

$$\int_0^\infty \mathrm{e}^{-\eta}\eta^{\frac{n+2}{2}}\,\mathrm{d}\eta = \Gamma\!\left(\frac{n{+}2}{2}+1\right) = \frac{n{+}2}{2}\,\Gamma\!\left(\frac{n{+}2}{2}\right) = \frac{n{+}2}{2}\cdot\frac{\pi^{n/2}}{\omega_n}\,.$$

All in all this gives

$$\int_{\mathrm{W}_1(0,0)}\frac{|y|^2}{s^2}\,\mathrm{d}(s,y) = 4$$

and thus confirms the claim. □

Proof of the theorem. By a translation we can reduce to the case $(t,x)=(0,0)$, and by mollification we can reduce to considering solutions $u \in \mathrm{C}^2\big(\overline{\mathrm{W}_r}\big)$, where we abbreviate $\mathrm{W}_r := \mathrm{W}_r(0,0)$. Then, partially similar to the proof of the mean value property in Section 2.4, we set

$$\Phi(r) := \frac{1}{4r^n}\int_{\mathrm{W}_r} u(s,y)\frac{|y|^2}{s^2}\,\mathrm{d}(s,y)$$

and aim at computing the derivative of Φ. To this end we first change variables $(s,y)=(r^2\sigma, r\upsilon)$ (where the Jacobian determinant can be expressed by formally writing $\mathrm{d}(s,y)=r^{n+2}\mathrm{d}(\sigma,\upsilon)$), then compute the derivative via exchange of differentiation and integration and the chain rule, and afterwards change variables back. In this way we find

$$\begin{aligned}
\Phi'(r) &= \frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{4}\int_{\mathrm{W}_1} u(r^2\sigma, r\upsilon)\,\frac{|\upsilon|^2}{\sigma^2}\,\mathrm{d}(\sigma,\upsilon)\\
&= \frac{1}{4}\int_{\mathrm{W}_1}\big[\nabla u(r^2\sigma,r\upsilon)\cdot\upsilon + u_t(r^2\sigma,r\upsilon)\,2r\sigma\big]\frac{|\upsilon|^2}{\sigma^2}\,\mathrm{d}(\sigma,\upsilon)\\
&= \frac{1}{4r^{n+1}}\int_{\mathrm{W}_r}\left[\nabla u(s,y)\cdot\frac{|y|^2y}{s^2} + u_t(s,y)\,\frac{2|y|^2}{s}\right]\mathrm{d}(s,y)
\end{aligned}$$

At this point we conveniently rewrite $\frac{2|y|^2}{s} = 4y\cdot\nabla\psi(s,y)$ in terms of an auxiliary function given by $\psi(s,y) := \frac{|y|^2}{4s} - n\log(\sqrt{-s}) - n\log\frac{\sqrt{4\pi}}{r}$ for $(s,y)\in(-\infty,0)\times\mathbb{R}^n$. By rearranging the defining inequality of the heat ball, we see in fact $\{\psi>0\}=\mathrm{W}_r$ and $\{\psi=0\}=\partial\,\mathrm{W}_r\setminus\{(0,0)\}$, in particular ψ has zero boundary values at $\partial\,\mathrm{W}_r\setminus\{(0,0)\}$. We now exploit this fact several times. In fact, in a first step we integrate by parts w.r.t. y in the just-rewritten term. In a second step we integrate by parts w.r.t. s in the then-second term and at the same time exploit $\operatorname{div}_y y = n$ and the solution property for the heat equation. In a third step, we again integrate by parts w.r.t. y in the then-third term, and in a fourth step we factor out ∇u. This procedure yields

$$\begin{aligned}
\Phi'(r) &= \frac{1}{4r^{n+1}}\int_{\mathrm{W}_r}\left[\nabla u(s,y)\cdot\frac{|y|^2y}{s^2} - \nabla u_t(s,y)\cdot 4y\,\psi(s,y) - 4(\operatorname{div}_y y)\,u_t(s,y)\,\psi(s,y)\right]\mathrm{d}(s,y)\\
&= \frac{1}{4r^{n+1}}\int_{\mathrm{W}_r}\left[\nabla u(s,y)\cdot\frac{|y|^2y}{s^2} + \nabla u(s,y)\cdot 4y\,\psi_t(s,y) - 4n\,\Delta u(s,y)\,\psi(s,y)\right]\mathrm{d}(s,y)\\
&= \frac{1}{4r^{n+1}}\int_{\mathrm{W}_r}\left[\nabla u(s,y)\cdot\frac{|y|^2y}{s^2} + \nabla u(s,y)\cdot 4y\,\psi_t(s,y) + 4n\,\nabla u(s,y)\cdot\nabla\psi(s,y)\right]\mathrm{d}(s,y)\\
&= \frac{1}{4r^{n+1}}\int_{\mathrm{W}_r}\nabla u(s,y)\cdot\left[\frac{|y|^2y}{s^2} + 4\psi_t(s,y)y + 4n\nabla\psi(s,y)\right]\mathrm{d}(s,y)\,.
\end{aligned}$$

On a level of technical detail, at this point we add the remark that the auxiliary function ψ is not defined at $(0,0)$ and in fact becomes singular there, specifically $\lim_{s\nearrow 0}\psi(s,0)=\infty$. Still, it can be checked that all singular factors in the above integrals remain in $\mathrm{L}^1(\mathrm{W}_r)$ (seen for $|y|^2/s^2$ in the preceding remark; $|y|/s$ and $|y|\log(\sqrt{-s})$ then controlled as well) and also that the singular behavior at the single point $(0,0)$ is negligible for all relevant integrations by parts.

Coming back to the main argument, we next observe that the the terms inside the brackets in the last line of the integral computation cancel out to zero by straightforward computation of the derivatives of the explicit function ψ. Thus, we have verified $\Phi'(r)=0$, and Φ is in fact a constant function of r. This, however, leads to

$$\Phi(r) = \lim_{\varrho\searrow 0}\Phi(\varrho) = u(0,0)\,,$$

where the first equality results from the previous reasoning, the second one from the continuity of u and the previous remarks which uncovered the nature of $\Phi(\varrho)$ as a weighted mean value on the heat ball $\mathrm{W}_\varrho(0,0)$ which shrinks towards $(0,0)$ for $\varrho \searrow 0$. All in all, we have shown $u(0,0)=\Phi(r)$, which is nothing but the claim for $(t,x)=(0,0)$. □

Remarks (on the mean value property).

(1) For subsolutions $u \in \mathrm{C}^2_1(\Omega_T)$ to the heat equation in the sense that $\Delta u - u_t \geq 0$ in Ω_T, an inspection of the above proof shows the mean value inequality

$$u(t,x) \leq \frac{1}{4r^n}\int_{\mathrm{W}_r(t,x)} u(s,y)\frac{|y-x|^2}{(t-s)^2}\,\mathrm{d}(s,y)$$

whenever $\overline{\mathrm{W}_r(t,x)} \subset \Omega_T$, and in case of additionally $\Delta u(t,x) - u_t(t,x) > 0$ the inequality is even strict. Clearly, for supersolutions one has the reverse inequality.

We remark that in establishing strict inequality it is in fact technically advantageous to set up the quantitative claim that $\Delta u - u_t \geq \delta > 0$ in $\mathrm{W}_{r_0}(t,x)$ for some $\mathrm{W}_{r_0}(t,x) \subset \Omega_T$ implies $u(t,x) + C_\delta \leq \frac{1}{4r^n}\int_{\mathrm{W}_r(t,x)} u(s,y)\frac{|y-x|^2}{(t-s)^2}\,\mathrm{d}(s,y)$ for all $r>0$ such that $\overline{\mathrm{W}_r(t,x)} \subset \Omega_T$, where $C_\delta > 0$ is a constant which depends only on n, δ, r_0, and r. This can be verified for C^2 solutions along the lines of the preceding argument and in view of the quantification with C_δ independent of u also the initial reduction from C^2_1 solutions to C^2 solutions (or equivalently an eventual extension from C^2 solutions to C^2_1 solutions) by mollification still works.

(2) By the same basic argument discussed earlier for the Laplace equation, one now finds that the respective mean value property **characterizes** subsolutions, supersolutions and most importantly **solutions to the heat equation**.

3.3 Maximum principles

We next turn to maximum principles for the heat equation. On basic level these principles resemble the ones for the Laplace equation, but additionally the "one-sidedness" of the time variable is taken into account and the true significance of the parabolic boundary is uncovered.

Theorem (maximum principles). *For an open set Ω in $\mathbb{R}^n$, a time horizon $T \in (0,\infty]$, and a function $u \in \mathrm{C}^2_1(\Omega_T)$, the following principles are valid:*

(I) *If $\Delta u - u_t > 0$ holds in Ω_T, then u has no local maximum point in Ω_T (and, what is slightly more, even u restricted to $(0,\tau]\times\Omega$ with arbitrary $\tau \in (0,T)$ has no local maximum point).*

(II) ***Weak maximum principle:*** *If Ω is bounded, if u is a subsolution in the sense of $\Delta u - u_t \geq 0$ in Ω_T, and if additionally $u \in \mathrm{C}^0(\Omega_T \cup \partial_{\mathrm{p}}\Omega_T)$ is continuous up to $\partial_{\mathrm{p}}\Omega_T$, then there holds*

$$u \leq \sup_{\partial_{\mathrm{p}}\Omega_T} u \qquad \text{in } \Omega_T\,.$$

(III) ***Strong maximum principle:*** *Assume that Ω is connected, that u is a subsolution in the sense of $\Delta u - u_t \geq 0$ in Ω_T, and that $M := \sup_{\Omega_T} u < \infty$ holds. Whenever $u(\tau,\chi) = M$ happens for a point $(\tau,\chi) \in \Omega_T$, then necessarily $u \equiv M$ holds on $(0,\tau]\times\Omega$. (Similarly, in case $T<\infty$, if $\lim_{\ell\to\infty} u(\tau_\ell,\chi_\ell) = M$ happens for points $(\tau_\ell,\chi_\ell) \in \Omega_T$ such that $\lim_{\ell\to\infty}(\tau_\ell,\chi_\ell) = (T,\chi)$ for $\chi \in \Omega$, this still implies that necessarily $u \equiv M$ holds in Ω_T.)*

Remark. In general the conclusion of the strong maximum principle can*not* be improved to having $u \equiv M$ in all of Ω_T, since there exist solutions u to the heat equation in Ω_T such that $u \equiv 0$ in $(0,\tau] \times \Omega$ and $u > 0$ in $(\tau,T) \times \Omega$ for $\tau \in (0,T)$ (whence $-u \equiv \sup_{\Omega_T}(-u) = 0$ in $(0,\tau]\times\Omega$ and $-u < 0$ in $(\tau,T)\times\Omega$ for the solution $-u$). Indeed, in case $\Omega \neq \mathbb{R}^n$, such a solution u can be constructed from the extended fundamental solution $K \in \mathrm{C}^\infty\big(((-\infty,\infty)\times\mathbb{R}^n)\setminus\{(0,0)\}\big)$ by choosing an arbitrary $x_0 \in \mathbb{R}^n \setminus \Omega$ and setting $u(t,x) := K(t-\tau, x-x_0)$ for $(t,x) \in \Omega_T \subset ((-\infty,\infty)\times\mathbb{R}^n)\setminus\{(\tau,x_0)\}$. In case $\Omega = \mathbb{R}^n$, the construction of an example will be addressed at the end of this section. In any case, a basic interpretation of such examples is that a physical system stays at zero temperature for a while, but after time τ begins to heat up by external influence, in simple but striking words at time τ "a heating at $\partial\Omega$ is switched on".

In principle, all three proofs of the maximum principle given in Section 2.4 can be adapted to the heat-equation case: The 1^{st} of these proofs based on a test function method gives the weak maximum principle. The 2^{nd} proof based on calculus criteria for maximum points yields first the assertion (I) above and can then be refined to reach the weak maximum principle once more. Finally, the 3^{rd} proof based on the mean value property establishes the strong maximum principle and as a corollary[2] also the weak maximum principle. In the sequel we only derive the strong maximum principle by the mentioned adaptation of the 3^{rd} proof. The details of the other approaches will be addressed in the exercises.

Proof of the strong maximum principle. We first consider the case that $u(\tau,\chi) = M = \sup_{\Omega_T} u$ happens for $(\tau,\chi) \in \Omega_T$ and aim at proving $u(t_*,x_*) = M$ for arbitrary $(t_*,x_*) \in (0,\tau)\times\Omega$ (which then implies the claim by continuity). To this aim, we first exploit connectedness and openness of Ω, which imply that the two points $\chi, x_* \in \Omega$ can be connected by a polygonal curve in Ω, that is, there exist $k \in \mathbb{N}$ and points $\chi = x_0,\ x_1,\ x_2,\ \ldots,\ x_{k-1},\ x_k = x_*$ all in Ω such that also the line segment from x_{i-1} to x_i is fully contained in Ω for all $i \in \{1,2\ldots,k\}$. In addition, we choose arbitrary times $\tau = t_0 > t_1 > t_2 > \ldots > t_k = t_*$ and observe that by construction the closed line segment S_i from (t_{i-1},x_{i-1}) to (t_i,x_i) is fully contained in Ω_T for all $i \in \{1,2,\ldots,k\}$. With these choices at hand we now establish $u(t_1,x_1) = M$. Indeed, since u is continuous and takes the value M at the endpoint $(t_0,x_0) = (\tau,\chi)$ of S_1, we may consider the point $(t,x) \in S_1$ with minimal t such that still $u(t,x) = M$. For $r > 0$ small enough that $\overline{\mathrm{W}_r(t,x)} \subset \Omega_T$, the mean value inequality then gives

$$M = u(t,x) \le \frac{1}{4r^n}\int_{\mathrm{W}_r(t,x)} u(s,y)\frac{|y-x|^2}{(t-s)^2}\,\mathrm{d}(s,y)\,,$$

but at the same time we also know $u \le M$ in $\mathrm{W}_r(t,x) \subset \Omega_T$. Since the integral is indeed a mean value (weight has integral 1), this is only possible if $u \equiv M$ holds on the whole heat ball $\mathrm{W}_r(t,x)$. Furthermore, the boundary of $\mathrm{W}_r(t,x)$ intersects the time axis orthogonally at (t,x), and therefore in case $t > t_1$ there exist points in $S_1 \cap \mathrm{W}_r(t,x)$ with time coordinates slightly smaller than t, which contradicts the assumed minimality of t. Hence, the only option is $t = t_1$, and we have shown $u(t_1,x_1) = M$. We then proceed by an entirely analogous reasoning along

[2]To see that the strong maximum principle implies the weak maximum principle, one first observes that there is no loss of generality in assuming connectedness of Ω, since one may deal with the connected components of Ω separately. Then, for each $\widetilde{T} \in (0,T)$, one claims $u \le \sup_{\partial_{\mathrm{p}}\Omega_T} u$ in $\Omega_{\widetilde{T}}$. Indeed, the maximum of $u \in \mathrm{C}^0\big(\overline{\Omega_{\widetilde{T}}}\big)$ on the compact set $\overline{\Omega_{\widetilde{T}}}$ is attained at a point $(\tau,\chi) \in \overline{\Omega_{\widetilde{T}}} \subset \Omega_T \cup \partial_{\mathrm{p}}\Omega_T$. In case $(\tau,\chi) \in \Omega_T$ the strong maximum principle enforces $u(\tau,\chi) \le \sup_{\partial_{\mathrm{p}}\Omega_T} u$, while in case $(\tau,\chi) \in \partial_{\mathrm{p}}\Omega_T$ one directly has $u(\tau,\chi) \le \sup_{\partial_{\mathrm{p}}\Omega_T} u$. Thus, the preceding claim is valid in all cases, and by arbitrariness of $\widetilde{T} \in (0,T)$ this means $u \le \sup_{\partial_{\mathrm{p}}\Omega_T} u$ in all of Ω_T.

the line segment S_2 and conclude next that also $u(t_2,x_2)=M$ holds. Continuing iteratively, we ultimately arrive at $u(t_*,x_*)=u(t_k,x_k)=M$ as required.

Finally, we turn to the case that $T<\infty$ holds and $\lim_{\ell\to\infty}u(\tau_\ell,\chi_\ell)=M$ happens for $(\tau_\ell,\chi_\ell)\in\Omega_T$ such that $\lim_{\ell\to\infty}(\tau_\ell,\chi_\ell)=(T,\chi)$ for $\chi\in\Omega$. In this case, we can fix a small enough $r>0$ such that $\overline{\mathrm{W}_r(\tau_\ell,\chi_\ell)}\subset\Omega_T$ for $\ell\gg 1$. Then, from the mean value inequality and Fatou's lemma we obtain

$$\begin{aligned} M=\lim_{\ell\to\infty}u(\tau_\ell,\chi_\ell) &\le \limsup_{\ell\to\infty}\frac{1}{4r^n}\int_{\mathrm{W}_r(\tau_\ell,\chi_\ell)}u(s,y)\frac{|y-\chi_\ell|^2}{(\tau_\ell-s)^2}\,\mathrm{d}(s,y)\\ &\le \frac{1}{4r^n}\int_{\mathrm{W}_r(T,\chi)}u(s,y)\frac{|y-\chi|^2}{(T-s)^2}\,\mathrm{d}(s,y)\,, \end{aligned}$$

where in controlling the lim sup via Fatou's lemma we exploited that is bounded above by M and the constant M is L^1 on the bounded subset $\bigcup_{\ell\gg 1}\mathrm{W}_r(\tau_\ell,\chi_\ell)$ of Ω_T. The resulting inequality can then replace the mean value inequality on the specific heat ball $\mathrm{W}_r(T,\chi)$ — even though $(T,\chi)\notin\Omega_T$ and $u(T,\chi)$ is not defined — and with this inequality (plus of course the usual mean value inequality for all other relevant heat balls) the previous reasoning via the polygonal curve and the line segments S_i can be rerun with (T,χ) in place of (τ,χ). This reasoning yields $u(t_*,x_*)=M$ for arbitrary $(t_*,x_*)\in\Omega_T$, and the proof is complete. □

Remarks (on the maximum principles)**.** Consider an open set Ω in $\mathbb{R}^n$ and $T\in(0,\infty]$.

(0) Clearly, there are also analogous **minimum principles** for supersolutions u in the sense of $\Delta u-u_t\le 0$ in Ω_T. These follow by applying the maximum principles to $-u$.

(1) The **comparison principle** asserts that, for bounded Ω and $u,v\in\mathrm{C}^2_1(\Omega_T)\cap\mathrm{C}^0(\Omega_T\cup\partial_\mathrm{p}\Omega_T)$, that one has the implication

$$\left.\begin{array}{ll}\Delta u-u_t\ge\Delta v-v_t & \text{in }\Omega_T\,,\\ u\le v & \text{at }\partial_\mathrm{p}\Omega_T\end{array}\right\}\quad\Longrightarrow\quad u\le v\quad\text{in }\Omega_T\,.$$

It follows by applying the maximum principle to the subsolution $u-v$.

(2) For bounded Ω, one infers **uniqueness of solutions** $u\in\mathrm{C}^2_1(\Omega_T)\cap\mathrm{C}^0(\Omega_T\cup\partial_\mathrm{p}\Omega_T)$ **to the Cauchy-Dirichlet problem**

$$\Delta u-u_t=f\text{ in }\Omega_T\,,\qquad u(0,\,\cdot\,)=u_0\text{ in }\Omega\,,\qquad u=\varphi\text{ at }\partial\Omega_T$$

with arbitrary prescribed data $f\in\mathrm{C}^0(\Omega_T)$, $u_0\in\mathrm{C}^0(\Omega)$, and $\varphi\in\mathrm{C}^0((\partial\Omega)_T)$. This is a direct consequence of the comparison principle and in comparison with the uniqueness assertion in the previous Section 3.1 comes out under a refined natural assumption for the solutions.

(3) For bounded Ω and solutions $u\in\mathrm{C}^2_1(\Omega_T)\cap\mathrm{C}^0(\Omega_T\cup\partial_\mathrm{p}\Omega_T)$ to the inhomogeneous heat equation $\Delta u-u_t=f$, one has **a-priori sup estimates** of the two types

$$\sup_{\Omega_T}|u|\le\sup_{\partial_\mathrm{p}\Omega_T}|u|+T\sup_{\Omega_T}|f|$$

and

$$\sup_{\Omega_T}|u|\le\sup_{\partial_\mathrm{p}\Omega_T}|u|+\frac{(R_\Omega)^2}{2n}\sup_{\Omega_T}|f|\,,$$

with $R_\Omega := \min\{R \in [0,\infty) : \Omega \subset \mathrm{B}_R(x_0)$ for some $x_0 \in \mathbb{R}^n\}$. Here, the former estimate has the better constant if T is small, while the latter one has the better constant if Ω is small and covers even for $T = \infty$. Moreover, we stress that, for boundary data specified in the same way as in the previous point, $\sup_{\partial_\mathrm{p}\Omega_T} |u|$ may be written out as $\max\{\sup_\Omega |u_0|, \sup_{(\partial\Omega)_T} \varphi\}$ and thus is an a-priori given quantity.

Proof of the sup *estimates.* Setting $M := \sup_{\Omega_T} |f|$ and $w(t,x) := u(t,x) - tM$, we have $\Delta w - w_t = \Delta u - u_t + M = f + M \geq 0$. Then, with the help of the maximum principle for w we find

$$\sup_{\Omega_T} u \leq \sup_{\Omega_T} w + \sup_{(t,x)\in\Omega_T} tM \leq \sup_{\partial_\mathrm{p}\Omega_T} w + TM \leq \sup_{\partial_\mathrm{p}\Omega_T} u + TM.$$

In combination with the same estimate for $-u$ this gives the first claimed estimate. The proof of the second estimate is based on the choice $w(t,x) := u(t,x) + \frac{|x-x_0|^2}{2n} M$ (which again satisfies $\Delta w - w_t \geq 0$) and on an analogous estimation. □

(4) By applying the preceding sup estimates to the difference of two solutions (with possibly different inhomogeneity and boundary data) one straightforwardly reads off estimates expressing the **continuous dependence** of solutions to the Cauchy-Dirichlet problem for the possibly inhomogeneous heat equation on the data f, u_0, φ.

(5) There exist also **maximum principles** for the heat equation over ***unbounded*** Ω, that is, analogs to the Phragmén-Lindelöf principles for the Laplace equation. One such principle (compare [9, Chapter 7.1(b)]) asserts, for arbitrary $M \in \mathbb{R}$,

$$\left.\begin{array}{r} u \in \mathrm{C}^2_1(\Omega_T)\,,\ \Delta u - u_t \geq 0 \text{ in } \Omega_T\,, \\ \displaystyle\limsup_{\Omega_T\ni(t,x)\to(\tau,\chi)} u(t,x) \leq M \text{ for all } (\tau,\chi) \in \partial_\mathrm{p}\Omega_T\,, \\ \text{for each } \tau \in (0,T)\colon\ u(t,x) \leq \mathrm{e}^{\mathrm{const}(\tau)(|x|^2+1)} \text{ for all } (t,x) \in \Omega_\tau \end{array}\right\} \implies \quad u \leq M \text{ in } \Omega_T\,,$$

where $\mathrm{const}(\tau) \in [0,\infty)$ denotes an arbitrary τ-dependent constant (which in the worst case may satisfy $\lim_{\tau\nearrow T} \mathrm{const}(\tau) = \infty$) and where the last assumption bounds the growth of u for $|x| \to \infty$ by exponential rates. In particular, whenever u solves the Cauchy problem on full $\mathbb{R}^n$, that is, u satisfies $\Delta u - u_t \equiv 0$ in $(0,T)\times\mathbb{R}^n$ and $u(0,\,\cdot\,) = u_0$ in $\mathbb{R}^n$, and additionally the growth assumption is valid, then there holds $|u| \leq \sup_{\mathbb{R}^n} |u_0|$ in $(0,T)\times\mathbb{R}^n$.

Sketch of proof. For the moment, we fix $\tau \in (0,T)$ and choose some $\sigma \in \left(0, \frac{1}{4\mathrm{const}(\tau)}\right)$ such that $m\sigma = \tau$ for some $m \in \mathbb{N}$. Then, it can be shown that the variant K^*_σ of the fundamental solution given by

$$K^*_\sigma(t,x) := \frac{1}{\sqrt{(\sigma-t)^n}} \exp\left(\frac{|x|^2}{4(\sigma-t)}\right)$$

solves the heat equation in Ω_σ. For arbitrary $\varepsilon > 0$, we now consider the subsolution $u-\varepsilon K^*_\sigma$ in Ω_σ. By a reasoning with maximizing sequences, the strong maximum principle, and the growth assumption (similar to the derivation of the basic Phragmén-Lindelöf principle in Section 2.4) one can then verify $\sup_{\Omega_\sigma}(u - \varepsilon K^*_\sigma) \leq M$, and in the limit $\varepsilon \searrow 0$ this implies $\sup_{\Omega_\sigma} u \leq M$. An analogous reasoning can be used iteratively to bound u next in $(\sigma,2\sigma)\times\Omega$, then in $(2\sigma,3\sigma)\times\Omega$, and finally in $((m-1)\sigma,m\sigma)\times\Omega$. Altogether this means $\sup_{\Omega_\tau} u \leq M$. Finally, in the limit $\tau \nearrow T$ we infer $\sup_{\Omega_T} u \leq M$. □

Remark. Without the growth condition from point (5) of the previous remark, the (weak) maximum principle and uniqueness of solutions do *not* hold true for the heat equation over *unbounded* $\Omega \subset \mathbb{R}^n$. This will now be shown already for $n = 1$ and $\Omega = \mathbb{R}$ by looking at exemplary solutions u given in form of

$$u(t,x) := \sum_{k=0}^{\infty} g^{(k)}(t) \frac{x^{2k}}{(2k)!} \qquad \text{for } (t,x) \in (-\infty,\infty) \times \mathbb{R}$$

with a fixed $g \in \mathrm{C}^\infty((-\infty,\infty))$. Indeed, if $\lim_{k\to\infty} \frac{g^{(k)}(t)}{(2k)!} = 0$ holds and this convergence is faster than exponential in a way[3] which is locally uniform in $t \in (-\infty,\infty)$, then one may differentiate the preceding series term by term with result

$$u_{xx}(t,x) - u_t(t,x) = \sum_{k=1}^{\infty} g^{(k)}(t) \frac{x^{2(k-1)}}{(2(k-1))!} - \sum_{k=0}^{\infty} g^{(k+1)}(t) \frac{x^{2k}}{(2k)!} = 0\,,$$

and thus u is a C^∞ solution to the heat equation in $(-\infty,\infty) \times \mathbb{R}$.

Specifically, if one chooses g above as $g(t) := \mathrm{e}^t$, one gets an explicit solution u given by

$$u(t,x) = \mathrm{e}^t \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = \mathrm{e}^t \cosh x \qquad \text{for } (t,x) \in (-\infty,\infty) \times \mathbb{R}\,.$$

More interestingly, however, one can verify that also g given by $g(t) := \mathrm{e}^{-1/t^2}$ for $t > 0$ and $g(t) := 0$ for $t \leq 0$ is admissible. This latter choice in fact produces a C^∞ solution u to the heat equation in $(-\infty,\infty) \times \mathbb{R}$ such that u is constant zero in $(-\infty,0] \times \mathbb{R}$, but non-constant in all of $(-\infty,\infty) \times \mathbb{R}$. This solution is known as **Tychonoff's exotic solution** and can serve as a counterexample in various regards.

To complete the construction of Tychonoff's exotic solution it remains to confirm the earlier-mentioned convergence hypothesis on $g^{(k)}(t)/(2k)!$ for the last-mentioned choice of g such that $g(t) := \mathrm{e}^{-1/t^2}$ for $t > 0$ and $g(t) := 0$ for $t \leq 0$. Here, a difficulty is that there is no straightforward explicit formula for $\frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathrm{e}^{-1/t^2}$ with *arbitrary* $k \in \mathbb{N}$. However, by an induction argument (which is not spelled out here) one can indeed check that, for every $k \in \mathbb{N}$, there exist $\ell_1, \ell_2, \ldots, \ell_{2^{k-1}} \in \{0,1,2,\ldots,k-1\}$ and $a_1, a_2, \ldots, a_{2^{k-1}} \in \mathbb{Z}$ such that $|a_i| \leq 3^k k^{\ell_i}$ for $i = 1,2,\ldots,2^{k-1}$ and

$$g^{(k)}(t) = \sum_{i=1}^{2^{k-1}} \frac{a_i}{t^{3k-2\ell_i}} \mathrm{e}^{-1/t^2} \qquad \text{for all } t > 0\,.$$

From this formula we read off the estimate

$$\frac{|g^{(k)}(t)|}{(2k)!} \leq \sum_{i=1}^{2^{k-1}} \frac{3^k k^{\ell_i}}{(2k)!\, t^{3k-2\ell_i}} \mathrm{e}^{-1/t^2} \qquad \text{for all } k \in \mathbb{N} \text{ and } t > 0\,.$$

Now we fix $\tau \geq 1$ and consider $t \in (0,\tau)$. In order to further bound the summands on the right-hand side of the previous estimate, we use on one hand the standard bound $\frac{1}{m!}\big(\frac{1}{t^2}\big)^m \leq \mathrm{e}^{1/t^2}$ for the specific choice $m := \lceil \frac{3}{2}k - \ell_i \rceil \leq \frac{3}{2}k - \ell_i + \frac{1}{2} \leq 2k$ (where $\lceil x \rceil$ denotes the smallest integer $\geq x$) to get

$$\mathrm{e}^{-1/t^2} \leq m!\, t^{2m} \leq m^m t^{2m} \leq (2k)^{\frac{3}{2}k-\ell_i+\frac{1}{2}} t^{3k-2\ell_i} \tau \leq 2^{2k} k^{\frac{3}{2}k-\ell_i+\frac{1}{2}} t^{3k-2\ell_i} \tau\,,$$

on the other hand we use the well-known inequality

$$(2k)! \geq \left(\frac{2k}{\mathrm{e}}\right)^{2k}.$$

[3] The precise requirement meant here is that, for each $\tau \in [0,\infty)$, one may bound $\frac{|g^{(k)}(t)|}{(2k)!} \leq M_\tau(k)$ for all $t \in (-\tau,\tau)$ and all $k \in \mathbb{N}_0$ by some $M_\tau(k) \in [0,\infty)$ such that $\lim_{k\to\infty} M_\tau(k) b^k = 0$ holds for every $b \in \mathbb{R}$.

With the help of both these facts we then bound the single summands above by

$$\frac{3^k k^{\ell_i}}{(2k)!\,t^{3k-2\ell_i}}\mathrm{e}^{-1/t^2} \le \frac{(3\mathrm{e}^2)^k k^{1/2}}{k^{k/2}}\tau$$

(which the right-hand side no longer depends on i) and altogether arrive at the targeted estimate

$$\frac{|g^{(k)}(t)|}{(2k)!} \le \frac{(6\mathrm{e}^2)^k k^{1/2}}{k^{k/2}}\tau =: M_\tau(k) \qquad \text{for all } k \in \mathbb{N} \text{ and } t \in (0,\tau)\,.$$

Here, $(6\mathrm{e}^2)^k k^{1/2}$ grows at an exponential rate for $k \to \infty$, but decisively $k^{k/2}$ grows even faster than exponential for $k \to \infty$, and thus $M_\tau(k)$ tends to 0 faster than exponential for $k \to \infty$. Since moreover $g^{(k)}(t) = 0$ holds for all $k \in \mathbb{N}$ in case $t \le 0$, the preceding suffices to verify the required (speed of) convergence.

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