RAMSEY-TYPE PROBLEMS FOR GENERALISED SIDON SETS

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ABSTRACT. We establish the existence of generalised Sidon sets enjoying additional Ramsey-type properties, which are motivated by questions of Erdős and Newman and of Alon and Erdős.

§1 INTRODUCTION

For a subset $X \subseteq \mathbb{N}$ and a fixed integer $k \geqslant 2$, we denote by $\varrho_{X,k} \colon \mathbb{N} \longrightarrow \mathbb{N}$ the function of the number of additive representations with k terms from X, i.e., for every $n \in \mathbb{N}$, we set

$$\varrho_{X,k}(n) = \left|\left\{(x_1, \dots, x_k) \in X^k \colon x_1 + \dots + x_k = n \text{ and } x_1 \leqslant \dots \leqslant x_k\right\}\right|.$$

Moreover, we define

$$\varrho_k(X) = \sup\{\varrho_{X,k}(n) \colon n \in \mathbb{N}\}$$

and we say X is a *$B_{k,\ell}$-set*, if $\varrho_k(X) = \ell$, i.e., if at least one integer has ℓ ordered k-term representations in X and no integer has more.

The study of $B_{k,\ell}$-sets can be traced back to the work of Sidon [14] (see, e.g., the textbook of Halberstam and Roth [6, Chapter II]) and $B_{2,1}$-sets are usually referred to as *Sidon sets* (or *Sidon sequences*). In general $B_{k,1}$-sets have the property that all k-term sums are distinct and for simplicity we refer to those sets as *B_k-sets*.

We are interested in the existence of infinite $B_{k,\ell}$-sets with special properties motivated by questions of Erdős and Newman (see, e.g., [3]) and by Alon and Erdős [1]. Below we briefly discuss these properties and our main result, Theorem 1.1, asserts the existence of $B_{k,\ell}$-sets enjoying all those qualities.

1.1. **Ramsey-type property.** Our starting point is a question independently proposed by Erdős and Newman. We shall employ the arrow notation from Ramsey theory and write

$$X \longrightarrow [k, \ell]_r \,,$$

to signify the statement that for every r-colouring $X = X_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} X_r$ there is some colour class X_q such that

$$\varrho_k(X_q) = \ell \,.$$

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In other words, for some colour class X_q there are ℓ distinct k-tuples $(x_i^{(j)})_{i=1}^k \in X_q^k$ with $x_1^{(j)} \leqslant \cdots \leqslant x_k^{(j)}$ for $j = 1, \ldots, \ell$ such that

$$\sum_{i=1}^{k} x_i^{(1)} = \sum_{i=1}^{k} x_i^{(2)} = \cdots = \sum_{i=1}^{k} x_i^{(\ell)} .$$

We note that, if $X \longrightarrow [k, \ell]_r$ holds for every $r \geqslant 2$, then for every finite colouring of X there still exists some integer which can be represented in ℓ different ways as a sum of k terms from the same colour class. It might seem plausible, that for such a partition relation to hold, the set X itself may have to represent some integers in more than ℓ ways. However, it turned out that this naïve idea is false. In fact, Erdős [3] established the existence of an infinite $B_{2,3}$-set X satisfying

$$X \longrightarrow [2, 3]_r$$

for every $r \geqslant 2$. This means, that even though no integer can be additively represented in four different ways by two elements of X, for every finite colouring of X there is still some integer enjoying three different 2-term representations with all terms from the same colour class. Formulated this way, this result has some resemblance to Folkman-type results in Ramsey theory for graphs (cf. the work of Folkman [4] and the textbook of Graham, Spencer, and Rothschild [5, §5.3]). Erdős conjectured, that such $B_{2,\ell}$-sets exist for every $\ell \geqslant 2$ (and he proved it for ℓ being a power of 2 and for ℓ of the form $\frac{1}{2}\binom{2s}{s}$ for some $s \geqslant 1$). This conjecture was addressed by Nešetřil and Rödl [8] and here we build on their work and extend their result for $B_{k,\ell}$-sets for arbitrary $k > 2$ (see property (i) in Theorem 1.1 below).

1.2. **Local-global structure.** The second property under consideration is motivated by a question of Alon and Erdős [1] for Sidon sets, inspired by Pisier's problem [10]. Alon and Erdős asked if the following property characterises sets X that are a finite union of Sidon sets:

For some $\delta > 0$ every finite $Y \subseteq X$ contains a Sidon set of size at least $\delta\,|Y|$. (1.1)

It is immediate that the local property (1.1) holds for every set X being a finite union of Sidon sets. In fact, this implication is true for every hereditary property and not specific to the property of Sidon sets. Hence, the question of Alon and Erdős asks if the local property (1.1) implies that X has a 'simple' global structure, rendered by being decomposable into a finite number of Sidon sets.

There are a few instances where indeed the global structure has such a local characterisation. Most notably, Horn [7] established such an assertion for independent sets in vector spaces and, more generally, Edmonds [2] proved such a statement for independent sets in matroids. However, in the context of Sidon sets Nešetřil, Rödl, and Sales [9] (partly based on earlier work with Erdős) showed that such a local characterisation fails (see also [12, §5]). We strengthen their result and show that the implication fails already for $\delta = 1/4$ (see properties *(i)* and *(ii)* for $k = 2$ and any $\ell \geqslant 2$ in Theorem 1.1 below).

1.3. **Main result.** Besides the properties discussed so far, our method yields $B_{k,\ell}$-sets X enjoying additional properties (see properties *(iii)*–*(vi)* in Theorem 1.1 below). Properties *(iii)* and *(iv)* imply that there is at most one additive representation for any integer with at most k terms from X. Consequently, assertions *(i)* and *(ii)* stay valid, even if we relax the definition of $B_{k,\ell}$-sets by considering all additive representations with *up to at most* k terms. Part *(v)* tells us that every integer has zero, one or ℓ additive representations from X and the last property asserts that all terms appearing in two different k-term representations are distinct.

Theorem 1.1. *For all integers $k \geqslant 2$, $\ell \geqslant 2$, there exists an infinite $B_{k,\ell}$-set $X \subseteq \mathbb{N}$ satisfying the following properties:*

(i) For every integer $r \geqslant 2$ we have $X \longrightarrow [k, \ell]_r$.

(ii) Every finite subset $Y \subseteq X$ contains a B_k-set of size at least $\frac{k-1}{2k}|Y|$.

In addition X also satisfies:

(iii) For every $h = 2, \dots, k-1$ the set X is a B_h-set.

(iv) For every $1 \leqslant i < j$ with $i + j < 2k$ we have $\varrho_{X,i}(n) + \varrho_{X,j}(n) \leqslant 1$.

(v) If $\varrho_{X,k}(n) \geqslant 2$ for some $n \in \mathbb{N}$, then $\varrho_{X,k}(n) = \ell$.

(vi) If $x_1 + \dots + x_k = y_1 + \dots + y_k$ for some $x_1 \leqslant \dots \leqslant x_k$, $y_1 \leqslant \dots \leqslant y_k$ from X, then either $x_i = y_i$ for all $i \in [k]$ or all $2k$ terms $x_1, \dots, x_k, y_1, \dots, y_k$ are distinct.

Following the idea from the work of Nešetřil and Rödl [8], for $\ell = 2$ the proof of Theorem 1.1 relies on the existence of Ramsey graphs for the cycle C_{2k} having girth $2k$ themselves. Such a result is a consequence of the work of Rödl and Ruciński [13] on thresholds for Ramsey properties in random graphs. For $\ell > 2$ the cycle C_{2k} is replaced by so-called generalised theta graphs $\Theta_{k,\ell}$ and the existence of a Ramsey graph of the same girth follows from recent work of the first two authors [11]. For the proof of Theorem 1.1 we shall employ a version of these results for ordered graphs. In the next section we introduce these results from Ramsey theory and we deduce Theorem 1.1 in Section 3.

§2 Local Ramsey theory

All graphs in this article have ordered vertex sets and all graph isomorphisms respect the orderings of the vertex sets. So for any two graphs there is at most one isomorphism between them. Given two integers $k, \ell \geqslant 2$ we fix a *generalised theta graph* $\Theta_{k,\ell}$ consisting of ℓ internally vertex disjoint, ascending paths of length k with common end points. Thus the graph $\Theta_{k,\ell}$ has

- $(k-1)\ell + 2$ vertices, say a_0, a_k, and $a_i^{(j)}$ with $(i, j) \in [k-1] \times [\ell]$,
- and $k\ell$ edges so that $a_0 a_1^{(j)} \dots a_{k-1}^{(j)} a_k$ is a path for every $j \in [\ell]$ (see Figure 2.1).

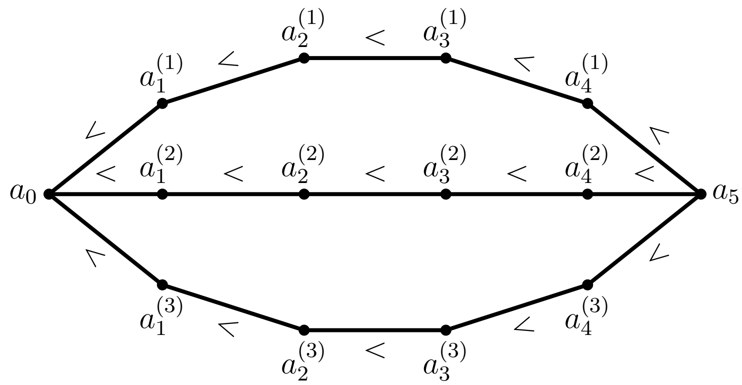


FIGURE 2.1. Ordered theta graph $\Theta_{5,3}$.

The demand that these path be ascending means that we require $a_0 < a_1^{(j)} < \dots < a_{k-1}^{(j)} < a_k$ for every $j \in [\ell]$. As it is immaterial to our argument how internal vertices of distinct paths compare to each other with respect to the vertex ordering of $\Theta_{k,\ell}$, we would like to leave this unspecified.

Roughly speaking, the reason why theta graphs help us in our investigation of $B_{k,\ell}$-sets is that when $V(\Theta_{k,\ell}) \subseteq \mathbb{N}$ and

$$x_i^{(j)} = a_i^{(j)} - a_{i-1}^{(j)},$$

where $a_0^{(j)} = a_0$ and $a_k^{(j)} = a_k$, then the ℓ sums $\sum_{i=1}^k x_i^{(j)}$ telescope to $a_k - a_0$ and, hence, they are equal. The set X promised by Theorem 1.1 will be derived from an appropriate (infinite) graph G with $V(G) \subseteq \mathbb{N}$ by setting

$$X = \{b - a : ab \in E(G) \text{ and } a < b\}.$$

Thus all occurrences of $\Theta_{k,\ell}$ in G correspond to numbers expressible in ℓ distinct ways as k-term sums with elements from X. Moreover, the partition relation $X \longrightarrow [k,\ell]_r$ can be enforced by letting G contain a Ramsey graph of $\Theta_{k,\ell}$ for r colours.

In view of the desired properties of X, we will also need G to have some further qualities, such as containing neither $\Theta_{k,\ell+1}$ nor cycles of length less than $2k$, but provided that the natural numbers in $V(G)$ are sufficiently 'far apart' it turns out that all demands on X translate to such 'local' properties of G. The local structure of Ramsey graphs has recently been analysed by the first two authors [11], who established the girth Ramsey theorem. Here is the precise statement for theta graphs, which we shall exploit.

Proposition 2.1. *For all integers $k, \ell \geqslant 2$ and $s \geqslant 2k$ there is an infinite graph G with vertex set $V(G) \subseteq \mathbb{N}$ possessing the following properties:*

(i) For every colouring of $E(G)$ with finitely many colours there is a monochromatic induced copy of $\Theta_{k,\ell}$.
(ii) Every induced cycle in G of length at most s has length exactly $2k$ and belongs to a unique copy of $\Theta_{k,\ell}$ in G.

For transparency we would like to point out that *(ii)* tells us, in particular, that G contains no cycles of length shorter than $2k$ and how all $2k$-cycles in G are ordered. All of them have two diametrically opposite vertices joined by two ascending paths of length k. In the remainder of this section we explain how Proposition 2.1 follows from known results [11], while the next section is reserved to the deduction of Theorem 1.1 from Proposition 2.1.

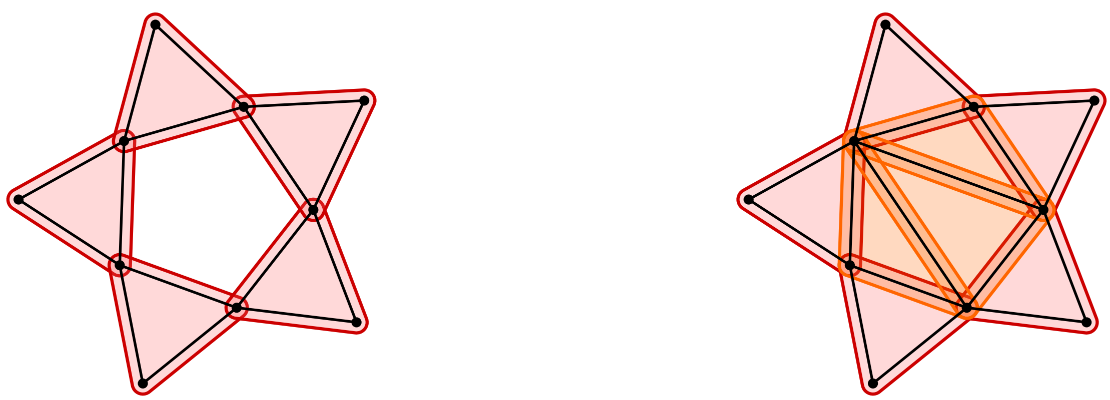
We shall utilise a strong version of the girth Ramsey theorem, whose formulation requires some preparations.

Definition 2.2 (Forests of copies)**.** A set of graphs $\mathscr{N}$ is called a *forest of copies* if there exists an enumeration $\mathscr{N} = \{F_1, \ldots, F_{|\mathscr{N}|}\}$ such that for every $j \in [2, |\mathscr{N}|]$ the set

$$V(F_j) \cap \bigcup_{i<j} V(F_i)$$

is either empty, a single vertex, or an edge belonging both to $E(F_j)$ and to $\bigcup_{i<j} E(F_i)$.

For instance, if all graphs in $\mathscr{N}$ are single edges, then $\mathscr{N}$ is a forest of copies if and only if these edges form a forest in the ordinary graph theoretical sense. However, this simple example might create the false impression that forests of copies would be closed under taking subsets. A counterexample is shown in Figure 2.2. On the left hand side we see a 'cycle of triangles', or more precisely a collection of five triangles not forming a forest of copies. By adding two further edges, however, we can hide this configuration inside a forest of eight triangles (see Figure 2.2b).



(a) A 'cycle' of triangles . . .

(b) . . . contained in a forest.

FIGURE 2.2. Some subforests fail to be forests.

Given two ordered graphs F and H we write $\binom{H}{F}$ for the set of all ordered copies of F in H, i.e., for the set of all induced, ordered subgraphs of H isomorphic to F. For a subsystem $\mathscr{H} \subseteq \binom{H}{F}$ and a number of colours r the partition relation $\mathscr{H} \longrightarrow (F)_r$ indicates that for every colouring $f\colon E(H) \longrightarrow [r]$ one of the copies in $\mathscr{H}$ is monochromatic. The girth Ramsey theorem confirms the intuition that some such Ramsey systems are locally just forests of copies. The rôle of $\mathscr{X}$ in the following theorem is due to the fact that, for the above reason, $\mathscr{N}$ can only be demanded to be a subset of a forest copies rather than an actual forest of copies.

Theorem 2.3 (Girth Ramsey theorem)**.** *Given a graph F and $r, n \in \mathbb{N}$ there exists a graph H together with a system of copies $\mathscr{H} \subseteq \binom{H}{F}$ satisfying not only $\mathscr{H} \longrightarrow (F)_r$, but also the following statement: For every $\mathscr{N} \subseteq \mathscr{H}$ with $|\mathscr{N}| \in [2, n]$ there exists a set $\mathscr{X} \subseteq \mathscr{H}$ such that $|\mathscr{X}| \leqslant |\mathscr{N}| - 2$ and $\mathscr{N} \cup \mathscr{X}$ is a forest of copies.* □

For unordered graphs Theorem 2.3 follows from the work of the first two authors [11, Theorems 13.12]. For ordered graphs Theorem 2.3 can be obtained by the same proof of Theorem 13.12, where for the Ramsey construction Φ we appeal to Ramsey's theorem for ordered graphs (see, e.g., [11, §3.5] for details on the partite construction method for ordered graphs).

Proof of Proposition 2.1. Given integers k, ℓ, and s we first observe that the desired infinite graph G can be taken as the disjoint union of finite graphs $(H_r)_{r \geqslant 2}$ satisfying:

(*a*) $H_r \longrightarrow (\Theta_{k,\ell})_r$,
(*b*) every induced cycle in H_r of length at most s has length $2k$,
(*c*) and every such cycle is contained in a unique copy of $\Theta_{k,\ell}$ in H_r.

Now given r we appeal to Theorem 2.3 with $F = \Theta_{k,\ell}$ and $n = \max\{s, k(\ell+1)\}$, thereby obtaining a graph H and a system of copies $\mathscr{H} \subseteq \binom{H}{\Theta_{k,\ell}}$. We may assume that $E(H)$ is the union of the edge sets of the copies of $\Theta_{k,\ell}$ in $\mathscr{H}$, because throwing away further edges of H would not destroy either of the relevant properties of H and $\mathscr{H}$. We shall prove that the graph $H_r = H$ displays properties (*a*)–(*c*).

Part (*a*) immediately follows from $\mathscr{H} \longrightarrow (\Theta_{k,\ell})_r$. For the proof of assertion (*b*) we consider an induced cycle C in H whose length is at most s and we show that there is a copy of $\Theta_{k,\ell}$ in $\mathscr{H}$ containing it. To see this we take a set $\mathscr{N} \subseteq \mathscr{H}$ of size $|\mathscr{N}| \leqslant s \leqslant n$ such that $\bigcup_{\Theta^\star \in \mathscr{N}} E(\Theta^\star)$ contains all edges of C. By Theorem 2.3 there exists a set $\mathscr{X} \subseteq \mathscr{H}$ such that $\mathscr{N} \cup \mathscr{X}$ is a forest of copies. Let the enumeration $\mathscr{N} \cup \mathscr{X} = \{\Theta^\star_1, \dots, \Theta^\star_m\}$ exemplify this state of affairs and let $j \leqslant m$ be minimal such that $E(C) \subseteq \bigcup_{i \leqslant j} E(\Theta^\star_i)$.

If $j = 1$ then C is contained in $\Theta^\star_1$ and we are done. So suppose $j \in [2, m]$ from now on. Recall that the set

$$z_j = V(\Theta^\star_j) \cap \bigcup_{i<j} V(\Theta^\star_i)$$

satisfies either $|z_j| \leqslant 1$ or $z_j \in E(\Theta^\star_j) \cap \bigcup_{i<j} E(\Theta^\star_i)$.

In the former case the graph $\bigcup_{i \leqslant j} \Theta^\star_i$ is the disjoint union or a one-point amalgamation of $\bigcup_{i<j} \Theta^\star_i$ and $\Theta^\star_j$. As these graph operations never create new cycles, the minimality of j shows that C is entirely contained in $\Theta^\star_j$ and we are done again.

If z_j is an edge, it is still true that C is contained in either $\bigcup_{i<j} \Theta^\star_i$ or in $\Theta^\star_j$, since otherwise the edge z_j contradicts the fact that C is induced in H. This concludes the proof that C is contained in a copy $\Theta^\star$ of $\Theta_{k,\ell}$ from $\mathscr{H}$ and establishes part (*b*).

Before addressing the uniqueness in part *(c)*, we would like to argue that H is $\Theta_{k,\ell+1}$-free. Due to $e(\Theta_{k,\ell+1}) = k(\ell+1) \leqslant n$ we could otherwise find a forest of copies $\{\Theta_1^\star, \dots, \Theta_m^\star\} \subseteq \mathscr{H}$ whose union contains a copy $\Theta^{\star\star}$ of $\Theta_{k,\ell+1}$. Again we consider the least integer j with $E(\Theta^{\star\star}) \subseteq \bigcup_{i \leqslant j} E(\Theta_i^\star)$, observe $j \in [2, m]$, and look at the set $z_j = V(\Theta_j^\star) \cap \bigcup_{i<j} V(\Theta_i^\star)$. Since $\Theta_{k,\ell+1}$ is 2-connected, the size of z_j needs to be at least two, wherefore z_j is an edge which $\bigcup_{i<j} \Theta_i^\star$ and $\Theta_j^\star$ have in common.

Since the deletion of any two adjacent vertices from $\Theta_{k,\ell+1}$ yields a connected graph, z_j cannot belong to $E(\Theta^{\star\star})$, since this would force the copy $\Theta^{\star\star}$ of $\Theta_{k,\ell+1}$ to be contained in the copy $\Theta_j^\star$ of $\Theta_{k,\ell}$, which is impossible. Consequently, the edge z_j witnesses that $\Theta^{\star\star}$ is a non-induced subgraph of H. But this causes H to contain a cycle whose length is at most $k+1$, contrary to the already established part *(b)*.

Let us finally address part *(c)*. Given an induced cycle C in H of length at most s we already know that C has length $2k$ and is contained in some copy $\Theta^\star$ of $\Theta_{k,\ell}$ in H. Thanks to our vertex orderings, the vertices a_0 and a_k of this copy need to be determined by $a_0 = \min V(C)$ and $a_k = \max V(C)$. Now $\Theta^\star$ consists of ℓ internally vertex disjoint ascending path from a_0 to a_k, two of which form C. It suffices to show that there is no further ascending path from a_0 to a_k of length k. If such a path P existed, then none of its inner vertices could be in $V(\Theta^\star)$, because H has girth $2k$. But this means that P and $\Theta^\star$ form a copy of $\Theta_{k,\ell+1}$ in H, which is absurd. □

§3 Proof of the main theorem

When proving Theorem 1.1 we sometimes need to know that if two short sums of elements of our set X agree, then under certain conditions their summands agree. In all cases this will ultimately boil down to the following basic arithmetic principle.

Fact 3.1. *If $a_0 + a_1 m + \dots + a_q m^q = 0$ holds for some integers $m \geqslant 2$, $q \geqslant 1$, and $a_0, \dots, a_q \in (-m, m)$, then $a_0 = \dots = a_q = 0$.*

Proof. Otherwise there exists a least index i such that $a_i \neq 0$. Since $a_i m^i + \dots + a_q m^q = 0$ is divisible by m^{i+1}, the coefficient a_i needs to be divisible by m. Together with $|a_i| < m$ this yields the contradiction $a_i = 0$. □

Proof of Theorem 1.1. For two given numbers $k, \ell \geqslant 2$ we apply Proposition 2.1 to $s = 2k$, thereby obtaining an infinite graph G with $V(G) \subseteq \mathbb{N}$ such that

- *(1)* $G \longrightarrow (\Theta_{k,\ell})_r$ holds for every positive integers r,
- *(2)* all cycles in G have length at least $2k$,
- *(3)* and every $2k$-cycle in G is contained in a unique copy of $\Theta_{k,\ell}$.

By relabelling the vertices of G we may assume that they are powers of $m = 2k+1$, i.e.,

$$V(G) = \{m^n : n \in \mathbb{N}\}.$$

We shall prove that the set

$$X = \{b - a \colon ab \in E(G) \text{ and } a < b\}$$

has all desired properties *(i)*–*(vi)* of Theorem 1.1. Most claims on short sums of elements of X will follow from the following statement.

Claim 3.2. *If for two positive integers s, t with $s + t \leqslant 2k$ there are elements $x_1 \leqslant \dots \leqslant x_s$ and $y_1 \leqslant \dots \leqslant y_t$ of X with $x_1 + \dots + x_s = y_1 + \dots + y_t$, then one of the following is true.*

- *(a) Either $s = t$ and $(x_1, \dots, x_s) = (y_1, \dots, y_t)$*
- *(b) or $s = t = k$ and there is a unique copy of $\Theta_{k,\ell}$ in G, say with vertices a_0, a_k, and $a_i^{(j)}$, and there are unique distinct indices $j, j' \in [\ell]$ such that $x_i = a_i^{(j)} - a_{i-1}^{(j)}$ and $y_i = a_i^{(j')} - a_{i-1}^{(j')}$ holds for all $i \in [k]$ (where, as usual, $a_0^{(j)} = a_0^{(j')} = a_0$ and $a_k^{(j)} = a_k^{(j')} = a_k$) and the $2k$ terms $x_1, \dots, x_k, y_1 \dots, y_k$ are all distinct.*

Proof. We may assume

$$\{x_1, \dots, x_s\} \cap \{y_1, \dots, y_t\} = \varnothing\,, \tag{3.1}$$

because otherwise we can delete two identical terms from the given equation and apply induction on $s + t$, thus ending up in case *(a)*.

Since every integer can be expressed in at most one way as a difference of two powers of m, for every $x \in X$ there exists a unique edge $e(x)$ of G with

$$x = \max e(x) - \min e(x)\,.$$

Setting

$$f(n) = \big|\{i \in [s] \colon \max e(x_i) = m^n\}\big| - \big|\{i \in [s] \colon \min e(x_i) = m^n\}\big|$$

for every positive integer n, we have $|f(n)| \leqslant s$ and

$$x_1 + \dots + x_s = \sum_{n=1}^{\infty} f(n) m^n\,,$$

where the sum on the right side has only finitely many nonzero terms. Similarly, we write

$$y_1 + \dots + y_t = \sum_{n=1}^{\infty} g(n) m^n\,,$$

where the function g is defined analogously by

$$g(n) = \big|\{i \in [t] \colon \max e(y_i) = m^n\}\big| - \big|\{i \in [t] \colon \min e(y_i) = m^n\}\big|$$

and satisfies $|g(n)| \leqslant t$. Now the given equality $x_1 + \dots + x_s = y_1 + \dots + y_t$ entails

$$\sum_{n=1}^{\infty} \big(f(n) - g(n)\big) m^n = 0\,.$$

Since for every $n \in \mathbb{N}$ we have $|f(n) - g(n)| \leqslant |f(n)| + |g(n)| \leqslant s + t \leqslant 2k < m$, Fact 3.1 tells us that $f = g$.

We consider the graph $M \subseteq G$ with edge set $E(M) = \{e(x_1), \dots, e(x_s), e(y_1), \dots, e(y_t)\}$. If M had a vertex m^n of degree one, then $f(n) = g(n)$ would yield a contradiction to (3.1). Thus M contains a cycle C and by *(2)* the length of C is at least $2k$. On the other hand, we know $e(C) \leqslant e(M) = s + t \leqslant 2k$, and thus we must have equality throughout. In particular, the $2k$ terms $x_1, \dots, x_s, y_1, \dots, y_t$ are distinct.

By *(3)* there is a unique copy of $\Theta_{k,\ell}$ in G containing C. Denote the vertices of this copy in the usual manner by a_0, a_k, and $a_i^{(j)}$, where $(i, j) \in [k-1] \times [\ell]$. Invoking $f = g$ once more, one easily sees $s = t = k$ and that there exist distinct indices $j, j' \in [\ell]$ such that $\{e(x_1), \dots, e(x_k)\}$ and $\{e(y_1), \dots, e(y_k)\}$ are the edge sets of the paths

$$a_0 a_1^{(j)} \cdots a_{k-1}^{(j)} a_k \quad \text{and} \quad a_0 a_1^{(j')} \cdots a_{k-1}^{(j')} a_k\,,$$

respectively. Since the numbers $a_0 < a_1^{(j)} < \dots < a_{k-1}^{(j)} < a_k$ are powers of m, their consecutive differences $a_1^{(j)} - a_0, \dots, a_k - a_{k-1}^{(j)}$ form a strictly increasing sequence and thus we have $x_i = a_i^{(j)} - a_{i-1}^{(j)}$ for every $i \in [k]$. The same argument also yields $y_i = a_i^{(j')} - a_{i-1}^{(j')}$.

We have thereby found a copy of $\Theta_{k,\ell}$ such that the summands $x_1, \dots, x_k$ and $y_1, \dots, y_k$ can be expressed as in case *(b)* of the claim. It remains to observe that these summands determine the cycle C and, therefore, the copy of $\Theta_{k,\ell}$ in a unique manner. □

Claim 3.2 clearly implies the clauses *(iii)*–*(vi)* of Theorem 1.1. Moreover, since G needs to contain at least one copy of $\Theta_{k,\ell}$, there is at least one natural number n satisfying $\varrho_{X,k}(n) = \ell$ and thus X is indeed a $B_{k,\ell}$-set. The reason why the partition property *(1)* leads to clause *(i)* was already alluded to in the previous section and altogether it only remains to address *(ii)*.

Let us recall to this end that the map $ab \longmapsto |a - b|$ establishes a bijection between $E(G)$ and X. Combined with Claim 3.2 this reduces our task to the verification of the following statement:

For every finite set $E' \subseteq E(G)$ there is a subset $E'' \subseteq E'$ of size $|E''| \geqslant \frac{k-1}{2k}|E'|$ such that the graph $(\mathbb{N}, E'')$ contains no ascending path of length k.

Given E' a standard averaging argument yields a partition of $\mathbb{N}$ into k classes $V_1, \dots, V_k$ such that the subset $P \subseteq E'$ consisting of all edges connecting vertices from distinct classes satisfies $|P| \geqslant \frac{k-1}{k}|E'|$. We subdivide P further into the two sets

$$P' = \{xy \in P \colon x < y \text{ and there are } i < j \text{ with } x \in V_i, y \in V_j\}$$

and

$$P'' = \{xy \in P \colon x < y \text{ and there are } i > j \text{ with } x \in V_i, y \in V_j\}\,.$$

Neither of the graphs $(\mathbb{N}, P')$ or $(\mathbb{N}, P'')$ can contain an ascending path of length k. Moreover, due to $|P'| + |P''| = |P| \geqslant \frac{k-1}{k}|E'|$ we have $\max\{|P'|, |P''|\} \geqslant \frac{k-1}{2k}|E'|$ and, hence, at least one of the choices $E'' = P'$ or $E'' = P''$ is permissible. Thereby Theorem 1.1 is proved. □

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