CANONICAL COLOURINGS IN RANDOM GRAPHS

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Dedicated to Martin Aigner on the occasion of his 80th birthday

Abstract. Rödl and Ruciński [*Threshold functions for Ramsey properties*, J. Amer. Math. Soc. **8** (1995)] established Ramsey's theorem for random graphs. In particular, for fixed integers $r, \ell \geqslant 2$ they showed that $\hat{p}_{K_\ell,r}(n) = n^{-\frac{2}{\ell+1}}$ is a threshold for the Ramsey property that every r-colouring of the edges of the binomial random graph $G(n,p)$ yields a monochromatic copy of K_ℓ. We investigate how this result extends to arbitrary colourings of $G(n,p)$ with an unbounded number of colours. In this situation, Erdős and Rado [*A combinatorial theorem*, J. London Math. Soc. **25** (1950)] showed that *canonically coloured* copies of K_ℓ can be ensured in the deterministic setting. We transfer the Erdős–Rado theorem to the random environment and show that both thresholds coincide when $\ell \geqslant 4$. As a consequence, the proof yields $K_{\ell+1}$-free graphs G for which every edge colouring contains a canonically coloured K_ℓ.

The 0-statement of the threshold is a direct consequence of the corresponding statement of the Rödl–Ruciński theorem and the main contribution is the 1-statement. The proof of the 1-statement employs the transference principle of Conlon and Gowers [*Combinatorial theorems in sparse random sets*, Ann. of Math. (2) **184** (2016)].

§1. Introduction

In the last three decades, extremal and Ramsey-type properties of random graphs were considered, which led to several general approaches (see, e.g. [2, 3, 12, 15, 24–26] and the references therein). We consider Ramsey-type questions for the binomial random graph $G(n,p)$. For graphs G and H and an integer $r \geqslant 2$ we write

$$G \longrightarrow (H)_r$$

to signify the statement that every r-colouring of the edges of G yields a monochromatic copy of H. Ramsey's theorem [20] tells us that for fixed H and r the family of graphs G with $G \longrightarrow (H)_r$ is non-empty. Obviously, this family is monotone and, hence, there is a

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threshold function $\hat{p}_{H,r}\colon \mathbb{N} \longrightarrow [0,1]$ such that

$$\lim_{n\longrightarrow\infty} \mathbb{P}\big(G(n,p) \longrightarrow (H)_r\big) = \begin{cases} 0\,, & \text{if } p \ll \hat{p}_{H,r}, \\ 1\,, & \text{if } p \gg \hat{p}_{H,r}. \end{cases} \tag{1.1}$$

As usual we shall refer to any such function as *the* threshold of that property, even though it is not unique.

Rödl and Ruciński [23, 24] determined the threshold $\hat{p}_{H,r}$ for every graph H and every fixed number of colours r. We restrict ourselves to the situation when H is a clique K_ℓ and state their result for that case only.

Theorem 1.1 (Rödl & Ruciński)**.** *For every $r \geqslant 2$ and $\ell \geqslant 3$ we have $\hat{p}_{K_\ell,r}(n) = n^{-\frac{2}{\ell+1}}$.* □

In fact, Rödl and Ruciński established a *semi-sharp* threshold, i.e., the 0-statement in (1.1) holds as long as $p(n) \leqslant c_{\ell,r} n^{-\frac{2}{\ell+1}}$ for some sufficiently small constant $c_{\ell,r} > 0$ and, similarly, the 1-statement becomes true already if $p(n) \geqslant C_{\ell,r} n^{-\frac{2}{\ell+1}}$ for some $C_{\ell,r}$. This was sharpened recently in [9], where the gap between $c_{\ell,r}$ and $C_{\ell,r}$ was closed. Perhaps surprisingly, the asymptotic growth of the threshold function $\hat{p}_{K_\ell,r}(n)$ in Theorem 1.1 is independent of the number of colours r.

We are interested in arbitrary edge colourings of $G(n,p)$, i.e., colourings which are not restricted to a fixed number of colours. However, if the number of colours is unrestricted, then this allows injective edge colourings and, consequently, monochromatic (nontrivial) subgraphs might be prevented. Erdős and Rado [6], however, showed that certain *canonical* patterns are unavoidable in edge colourings of sufficiently large cliques. Obviously, the monochromatic and the injective pattern must be canonical. Two more canonical patterns arise by ordering the vertices of K_n and colouring every edge uv by $\min\{u,v\}$ or colouring every edge by its maximal vertex. More generally, for finite graphs G and H with ordered vertex sets, we write

$$G \xrightarrow{\;*\;} (H)$$

if for every edge colouring $\varphi\colon E(G) \longrightarrow \mathbb{N}$ there exists an order-preserving graph embedding $\varsigma\colon H \longrightarrow G$ such that one of the following holds:

(*a*) the copy $\varsigma(H)$ of H is monochromatic under φ,
(*b*) or φ restricted to $E(\varsigma(H))$ is injective,
(*c*) or for all edges e, $e' \in E(\varsigma(H))$ we have $\varphi(e) = \varphi(e') \iff \min(e) = \min(e')$,
(*d*) or for all edges e, $e' \in E(\varsigma(H))$ we have $\varphi(e) = \varphi(e') \iff \max(e) = \max(e')$.

We call an ordered copy of H in G *canonical* if it displays one of the four patterns described in (*a*)–(*d*).

Note that for the patterns described in (*a*) and (*b*) the orderings of the vertex sets have no bearing. Moreover, we shall refer to copies enjoying an injective colouring as *rainbow* copies of H (even if $|E(H)| \neq 7$). Similarly, we refer to the patterns appearing in *(c)* and *(d)* as *min-coloured* and *max-coloured*, respectively. In case only the backward implications in *(c)* or *(d)* are enforced, then we refer to those colourings as *non-strict*, e.g., if $\min(e) = \min(e')$ yields $\varphi(e) = \varphi(e')$ for all edges e, $e' \in E(\varsigma(H))$, then $\varsigma(H)$ is a non-strictly min-coloured copy of H. Obviously, monochromatic copies are also non-strictly min- and max-coloured.

From now on, the vertex sets of all graphs considered are ordered. In particular, for cliques and random graphs we simply assume

$$V(K_n) = [n] \qquad \text{and} \qquad V(G(n,p)) = [n]\,.$$

With this notation at hand, the aforementioned canonical Ramsey theorem of Erdős and Rado [6] restricted to the graph case asserts that canonical copies are unavoidable.

Theorem 1.2 (Erdős & Rado)**.** *For every $\ell \geqslant 3$, there exists n such that $K_n \xrightarrow{\ *\ } (K_\ell)$.* □

We are interested in a common generalisation of Theorems 1.1 and 1.2. Owing to Theorem 1.2, for any ordered graph H, the monotone family $\{G\colon G \xrightarrow{\ *\ } (H)\}$ is non-empty and it raises the problem of estimating the threshold $\hat{p}_H\colon \mathbb{N} \longrightarrow [0,1]$ such that

$$\lim_{n\longrightarrow\infty} \mathbb{P}\big(G(n,p) \xrightarrow{\ *\ } (H)\big) = \begin{cases} 0\,, & \text{if } p \ll \hat{p}_H,\\ 1\,, & \text{if } p \gg \hat{p}_H. \end{cases} \tag{1.2}$$

It follows from the definition, that for every graph H not admitting a vertex cover of size at most two, the only canonical colourings of H using at most two colours are monochromatic. Consequently,

$$\hat{p}_H \geqslant \hat{p}_{H,2}$$

for every such graph H. In particular, for cliques on at least four vertices this may suggest that the asymptotics of the thresholds for those Ramsey properties coincide and our main result verifies this.

Theorem 1.3. *For every $\ell \geqslant 4$, there exists $C > 0$ such that for $p = p(n) \geqslant Cn^{-\frac{2}{\ell+1}}$ we have*

$$\lim_{n\longrightarrow\infty} \mathbb{P}\big(G(n,p) \xrightarrow{\ *\ } (K_\ell)\big) = 1\,.$$

Combining Theorem 1.3 with the corresponding lower bound on $\hat{p}_{K_\ell,2}$ shows that the threshold for the canonical Ramsey property is semi-sharp for $\ell \geqslant 4$. For $\ell = 3$ we recall that the canonical Ramsey threshold is indeed smaller than the Ramsey threshold $n^{-1/2}$. In fact, one can check that every edge colouring of K_4 yields a canonical copy of the triangle and, hence, $\hat{p}_{K_3} \leqslant n^{-2/3}$.

Moreover, we note that for $p = O(n^{-\frac{2}{\ell+1}})$ the random graph $G(n,p)$ is likely to contain only $o(pn^2)$ cliques $K_{\ell+1}$. In the proof of Theorem 1.3 we can delete an edge from each such clique. Consequently, we obtain the following statement in structural Ramsey theory, which can be viewed as a Folkman-type extension of the Erdős–Rado theorem for graphs.

Corollary 1.4. *For every $\ell \geqslant 4$ there exists a $K_{\ell+1}$-free graph G such that $G \xrightarrow{*} (K_\ell)$. Moreover, G contains no two distinct copies of K_ℓ that share at least three vertices.*

In the context of Ramsey's theorem, the existence of such a graph G was asked for by Erdős and Hajnal [5]; for two colours this was established by Folkman [7], and for any fixed number of colours by Nešetřil and Rödl [18]. The graph G in Corollary 1.4 will be obtained by modifying the random graph and, hence, the proof is non-constructive. Reiher and Rödl [21] pointed out that the first part of Corollary 1.4 can also be proved in a constructive manner by means of the *partite construction method* of Nešetřil and Rödl [19]. While this approach falls short to exclude K_ℓ's intersecting in triangles, it has the advantage that it readily extends to k-uniform hypergraphs for every $k \geqslant 3$.

We conclude this introduction with a short overview of the main ideas of the proof of Theorem 1.3. Roughly speaking, the proof is inspired by the proof of the canonical graph Ramsey theorem laid out by Lefmann and Rödl [13] and Alon et al. [1]. This approach pivots on a case distinction of the edge colouring of the underlying graph K_n. The first case, when many different colours appear everywhere, which is captured by assuming that every vertex is incident to only $o(n)$ edges of the same colour, leads to rainbow copies of K_ℓ. In the other case, there is a vertex with a monochromatic neighbourhood of size $\Omega(n)$, which by iterated applications, as in the standard proof of Ramsey's theorem, leads to a non-strictly min- or max-coloured $K_{(\ell-2)^2+2}$. Such a non-strictly min/max-coloured clique contains a canonical K_ℓ by a straightforward application of Dirichlet's box principle.

Transferring such an approach from K_n to $G(n,p)$ for $p = O(n^{-\frac{2}{\ell+1}})$ faces several challenges. Firstly, we shall not use a $K_{(\ell-2)^2+2}$ in the coloured host graph, as such large cliques are not very likely to appear in $G(n,p)$ for that edge probability. Moreover, in the more challenging second case, when the colouring is unbounded, it is certainly not sufficient to consider one vertex with a large monochromatic neighbourhood (of size $\Omega(pn)$), as again, this neighbourhood is too sparse to contain any useful structure in $G(n,p)$. Thus we resort to a robust version of the above-mentioned argument, building a large non-strictly min- or max-coloured subgraph which contains $\Omega(n^2p)$ edges.

The bounded case, with at most λ edges of every colour incident to any given vertex of $G(n,p)$, is a problem of independent interest. For example, $\lambda = 1$ corresponds to studying proper edge colourings of $G(n,p)$ and anti-Ramsey properties (see, e.g. [10, 11, 16] and the references therein). In fact, for $\ell \geqslant 5$, there are proper colourings of $G(n,p)$ with $p = cn^{-\frac{2}{\ell+1}}$ which do not contain a rainbow copy of K_ℓ (see [11]), which is an alternative argument for $\hat{p}_{K_\ell} \geqslant cn^{-\frac{2}{\ell+1}}$ and another obstruction for Theorem 1.3. For the proof of Theorem 1.3 presented here, we will need to guarantee rainbow copies of K_ℓ under the weaker assumption that $\lambda = o(pn)$. This can be viewed as a partial extension of the work of Kohayakawa, Kostadinidis, and Mota [10]. In both cases (bounded and unbounded colourings) the transference principle for random discrete structures developed by Conlon and Gowers [3] is an integral part of the proof.

Organisation. In the next section, we present the two main lemmata rendering the case distinction sketched above, and deduce Theorem 1.3. The proofs of these lemmata are deferred to Sections 3 and 4, along with the corresponding preliminaries. We conclude with a discussion of possible generalisations of this work from cliques K_ℓ to general graphs H and of related open problems in Section 5.

§2. Proof of the main result

2.1. **Proof of the canonical Ramsey theorem for graphs.** The proof of Theorem 1.3 adopts some ideas of the canonical Ramsey theorem for graphs from the work of Lefmann and Rödl [13] and Alon et al. [1] and below we recall their argument. For $\ell \geqslant 3$ we fix

$$\delta = \frac{1}{4\ell^3} \quad \text{and} \quad n \geqslant 2^{6\ell^2(\log_2(\ell)+1)} \tag{2.1}$$

and first we consider bounded colourings $\varphi\colon E(K_n) \longrightarrow \mathbb{N}$. More precisely, we say such a colouring is δ-*bounded* if for every colour $c \in \mathbb{N}$ and every vertex $v \in V(K_n)$ we have

$$d_c(v) = \big|N_c(v)\big| = \big|\{w \in V(K_n)\colon \varphi(vw) = c\}\big| \leqslant \delta n\,.$$

Roughly speaking, bounded colourings have the property that many different colours are "present everywhere" and this yields rainbow copies of K_ℓ. In fact, a simple counting argument shows for δ-bounded colourings that at most $\delta n^3/2$ triples contain two edges of the same colour and at most $\delta n^4/8$ quadruples contain two disjoint edges of the same colour. Consequently, selecting every vertex of K_n independently with probability $2\ell/n$ and removing a vertex from every such triple and every such quadruple, establishes the existence of ℓ vertices inducing a rainbow K_ℓ.

The second part of the proof resembles the standard proof of Ramsey's theorem for graphs and iterates along large monochromatic neighbourhoods. Given the observation above for bounded colourings, we may assume that the edge colouring φ is unbounded in a hereditary way and this requires the exponential lower bound on n above.

More precisely, assuming that φ fails to induce a rainbow copy of K_ℓ gives rise to a vertex $v \in V(K_n)$, a colour c, and a comparability sign $\diamond \in \{<, >\}$ such that

$$d_c^\diamond(v) = \big|N_c^\diamond(v)\big| = \big|\{w \in V(K_n)\colon \varphi(vw) = c \text{ and } v \diamond w\}\big| > \frac{\delta n}{2}\,.$$

Restricting our attention to the colouring φ on the vecontained in $N_c^\diamond(v)$ and iterating this argument $L = 2(\ell-2)^2+2$ times leads to a sequence $(v_i, c_i, \diamond_i)_{i\in[L]}$ such that for every $i \in [L]$ we have

$$\left|\bigcap_{j=1}^{i} N_{c_j}^{\diamond_j}(v_j)\right| > \left(\frac{\delta}{2}\right)^i n\,. \tag{2.2}$$

In fact, owing to the choices in (2.1) we can iterate this step L times.

Furthermore, we may assume that there are indices $1 \leqslant i_0 < \cdots < i_{(\ell-2)^2} < L$ such that $\diamond_{i_j}$ is $<$ for all j. Consequently, the correspondingly indexed vertices $v_{i_0}, \dots, v_{i_{(\ell-2)^2}}$ together with v_L induce a non-strictly min-coloured clique on $(\ell-2)^2+2$ vertices. Finally, if one of the colours appears $\ell-1$ times among $c_{i_0}, \dots, c_{i_{(\ell-2)^2}}$, then this yields a monochromatic K_ℓ among $v_{i_0}, \dots, v_{i_{(\ell-2)^2}}$, and v_L. Otherwise, at least $\ell-1$ distinct colours appear and we are guaranteed to find a min-coloured K_ℓ instead.

2.2. **Bounded and unbounded colourings in random graphs.** For the proof of Theorem 1.3 we derive appropriate random versions of the facts above that analyse bounded and unbounded colourings of $G(n,p)$ (see Lemmata 2.1 and 2.2 below). We begin by defining a notion of boundedness central to our proof. Roughly speaking, an edge colouring of $G(n,p)$ is bounded if at most $o(pn)$ edges of the same colour are incident to any given vertex. However, similar to the proof in the deterministic setting, it will be useful to define this property for large subsets of vertices, which is made precise as follows.

Given a graph $G = (V,E)$ with an edge colouring $\varphi\colon E \longrightarrow \mathbb{N}$, a subset $U \subseteq V$, and reals $\delta > 0$, $p \in (0,1]$ we say φ is (δ,p)-*bounded on* U if for every colour $c \in \mathbb{N}$ and every vertex $u \in U$ we have

$$d_c(u,U) = \big|N_c(u,U)\big| = \big|\{w \in U : \varphi(uw) = c\}\big| \leqslant \delta p|U|\,.$$

The first lemma asserts that bounded edge colourings of $G(n,p)$ for $p \gg n^{-\frac{2}{\ell+1}}$ yield rainbow copies of K_ℓ asymptotically almost surely, i.e., with probability tending to 1 as $n \longrightarrow \infty$.

In view of Corollary 1.4, we define the ℓ-*clean* subgraph G_ℓ of a given graph G on $[n]$ as follows: Consider all edges of G in lexicographic order and remove an edge e in the current subgraph of G, if the edge e is contained in two distinct K_ℓ's intersecting in at least three vertices. Actually, the precise definition is not relevant for our argument, but it will be convenient that this way the ℓ-clean subgraph $G_\ell \subseteq G$ is unique. Note that G_ℓ contains no copy of $K_{\ell+1}$, since this would yield two K_ℓ's intersecting in $\ell-1$ vertices.

Lemma 2.1. *For all integers $\ell \geqslant 4$ and every $\nu > 0$ there is some constant $C > 0$ such that for $p = p(n) \geqslant Cn^{-\frac{2}{\ell+1}}$ asymptotically almost surely the following holds for $G \in G(n,p)$.*

If $\varphi\colon E(G) \longrightarrow \mathbb{N}$ is $(\ell^{-5}/4, p)$-bounded on some $U \subseteq V(G)$ with $|U| \geqslant \nu n$, then U induces a rainbow copy of K_ℓ in G.

Moreover, if in addition we have $p(n) \leqslant n^{-\frac{2\ell-2}{\ell^2+\ell-4}}/\omega(n)$ for some arbitrary function ω tending to infinity as $n \longrightarrow \infty$, then the ℓ-clean subgraph $G_\ell \subseteq G$ also contains a rainbow copy of K_ℓ.

Lemma 2.1 strengthens a result of Kohayakawa, Kostadinidis, and Mota [10], where a more restrictive boundedness assumption is required. The proof of Lemma 2.1 makes use of the transference principle of Conlon and Gowers [3], which allows us to transfer the bounded case in the deterministic setting to the random environment. We defer the proof of Lemma 2.1 to Section 3. The second lemma yields canonical copies in unbounded colourings.

Lemma 2.2. *For all integers $\ell \geqslant 3$ and every $\delta > 0$ there is some constant $C > 0$ such that for $p = p(n) \geqslant Cn^{-\frac{2}{\ell+1}}$ asymptotically almost surely the following holds for $G \in G(n,p)$.*

If $\varphi\colon E(G) \longrightarrow \mathbb{N}$ has the property that every $U \subseteq V(G)$ with size $|U| \geqslant \delta^{5\ell^2} n$ satisfies

$$\big|\{u \in U\colon d_c(u,U) \geqslant 8\delta p|U| \text{ for some colour } c\}\big| \geqslant \frac{|U|}{2}\,, \tag{2.3}$$

then G contains a canonical copy of K_ℓ.

Moreover, if in addition we have $p(n) \leqslant n^{-\frac{2\ell-2}{\ell^2+\ell-4}}/\omega(n)$ for some arbitrary function ω tending to infinity as $n \longrightarrow \infty$, then the ℓ-clean subgraph $G_\ell \subseteq G$ also contains a rainbow copy of K_ℓ.

As in the unbounded case in the deterministic setting, the proof of Lemma 2.2 yields either a monochromatic, or a min-coloured, or a max-coloured copy of K_ℓ. The proof of Lemma 2.2 is more involved and we give a detailed outline in Section 4.1. We conclude this section with the short proof of Theorem 1.3 and Corollary 1.4 based on Lemmata 2.1 and 2.2.

Proof of Theorem 1.3 and Corollary 1.4. Given $\ell \geqslant 4$ we set $\delta = \ell^{-5}/64$ and $\nu = \delta^{5\ell^2}/2$ and let C be sufficiently large so that we can appeal to Lemma 2.1 with ℓ and ν and to Lemma 2.2 with ℓ and δ. Owing to the monotonicity of the canonical Ramsey property, for the proof of Theorem 1.3 we may assume $p = p(n) = Cn^{-\frac{2}{\ell+1}}$. Since $\ell \geqslant 4$ this implies $p(n) \leqslant n^{-\frac{2\ell-2}{\ell^2+\ell-4}}/\omega(n)$ for some function ω tending to infinity with n.

Let $G \in G(n,p)$ satisfy the conclusion of both lemmata and consider an arbitrary edge colouring $\varphi\colon E(G) \longrightarrow \mathbb{N}$ of G.

For every $U \subseteq V(G)$ we consider its subset of unbounded vertices in U

$$B(U) = \{w \in U\colon d_c(w,U) \geqslant 8\delta p|U| \text{ for some colour } c\}\,.$$

Owing to Lemma 2.2 we may assume that there is a set $U \subseteq V(G)$ satisfying $|U| \geqslant \delta^{5\ell^2} n$ and $|B(U)| < |U|/2$. Removing the unbounded vertices from U we arrive at a set

$$U' = U \smallsetminus B(U) \quad \text{with} \quad |U'| > \frac{|U|}{2} \geqslant \nu n\,.$$

For every vertex $u \in U'$ and every colour c we have

$$d_c(u,U') \leqslant d_c(u,U) < 8\delta p|U| < 16\delta p|U'|\,.$$

In other words, φ is $(16\delta, p)$-bounded on U' and Lemma 2.1 yields asymptotically almost surely a rainbow copy of K_ℓ in the ℓ-clean subgraph $G_\ell \subseteq G$. □

§3. Rainbow cliques in bounded colourings of random graphs

We shall use the following notation. For a graph $G = (V, E)$, a vertex v, and a set $U \subseteq V$ we write $d_G(v, U)$ for the size of the neighbourhood of v in U. For subsets $X, Y \subseteq V$ we denote by $e_G(X, Y)$ the number of edges with one vertex in X and one vertex in Y, where edges in $X \cap Y$ are counted twice, i.e.,

$$e_G(X,Y) = \big|\{(x,y) \in X \times Y : xy \in E\}\big| = \sum_{x \in X} d_G(x, Y)\,.$$

Moreover, for some integer $\ell \geqslant 2$ we denote by $\kappa_\ell(G)$ the number of (labeled) copies of K_ℓ in G. For a family $\mathcal{U} = (U_1, \dots, U_\ell)$ of mutually disjoint vertex subsets of V we write $G[\mathcal{U}]$ for the ℓ-partite subgraph induced by the sets $U_1, \dots, U_\ell$.

The proof of Lemma 2.1 is based on the transference principle of Conlon and Gowers [3], which we use in the following form [4, Theorem 3.2].

Theorem 3.1 (Conlon & Gowers)**.** *For all integers $\ell \geqslant 3$ and every $\varepsilon > 0$ there is some constant $C > 0$ such that for $p = p(n)$ with $Cn^{-\frac{2}{\ell+1}} \leqslant p \leqslant 1/C$ and every $\zeta > 0$ asymptotically almost surely the following holds for $G \in G(n,p)$.*

For every family $\mathcal{U} = (U_1, \dots, U_\ell)$ of mutually disjoint vertex subsets of $V(G)$ and every ℓ-partite subgraph S of $G[\mathcal{U}]$, there exists an ℓ-partite subgraph D of $K_n[\mathcal{U}]$ such that

- *(i) for all subsets $X, Y \subseteq V(G)$ we have $|e_S(X,Y) - p \cdot e_D(X,Y)| \leqslant \varepsilon p n^2$*
- *(ii) and $\big|\kappa_\ell(S) - p^{\binom{\ell}{2}} \cdot \kappa_\ell(D)\big| \leqslant \varepsilon p^{\binom{\ell}{2}} n^\ell$.*

Moreover, we have

- *(iii) for every $X \subseteq V(G)$ all but at most εn vertices $v \in V(G)$ satisfy*

$$|d_S(v,X) - p \cdot d_D(v,X)| \leqslant \varepsilon p n$$

- *(iv) and if G' is obtained from G by removing at most $\zeta p n^2$ edges, then*

$$\big|\kappa_\ell(G'[\mathcal{U}]) - p^{\binom{\ell}{2}}|U_1| \cdots |U_\ell|\big| \leqslant (\varepsilon + \zeta) p^{\binom{\ell}{2}} n^\ell\,.$$ □

We say the graph D provided by Theorem 3.1 is a *dense model* for the subgraph S of the sparse random graph. Furthermore, we remark that the moreover-part is not stated in [4, Theorem 3.2]. However, it easily follows from *(i)* and *(ii)* applied with an appropriately chosen $\varepsilon' \ll \varepsilon$. In fact, part *(iii)* follows from *(i)* applied to X and Y^+ being the set of vertices having too high degree in D, and a second application to X and a similarly defined set Y^- (see, e.g., proof of Lemma 4.4 in Section 4.2). Part *(iv)* can be deduced by applying *(i)* and *(ii)* for $S' = G'[\mathcal{U}]$. In fact, for this choice, part *(i)* combined with Chernoff's inequality implies that all $\binom{\ell}{2}$ bipartite subgraphs $D'[U_i, U_j]$ have density close to 1. More precisely, in this case $D'[\mathcal{U}]$ and $K_n[\mathcal{U}]$ differ by at most $(2\varepsilon' + \zeta)n^2$ edges. Consequently, $\big|\kappa_\ell(D') - |U_1|\cdots|U_\ell|\big| \leqslant (2\varepsilon' + \zeta)n^\ell$, which can be transferred to $S' = G'[\mathcal{U}]$ by *(ii)*. The two conclusions of the following lemma further strengthen the upper bound of part *(iv)* and can be viewed as a customised version of part *(ii)* for our proof of Lemma 2.1.

Lemma 3.2. *For all integers $\ell \geqslant 4$ and every $\varepsilon > 0$ there is some constant $C > 0$ such that for $p = p(n)$ with $Cn^{-1/m_2(H)} \leqslant p \leqslant 1/C$ asymptotically almost surely the following holds for $G \in G(n,p)$.*

For every family $\mathcal{U} = (U_1, \ldots, U_\ell)$ of mutually disjoint vertex subsets of $V(G)$ and every ℓ-partite subgraph S of $G[\mathcal{U}]$ the following holds:

(a) For $d_{12} = \frac{e_S(U_1,U_2)}{p|U_1||U_2|}$ and $d_{34} = \frac{e_S(U_3,U_4)}{p|U_3||U_4|}$ we have

$$\kappa_\ell(S) \leqslant d_{12}d_{34} \cdot p^{\binom{\ell}{2}}|U_1|\cdots|U_\ell| + \varepsilon p^{\binom{\ell}{2}}n^\ell\,.$$

(b) For $c_{123} = \frac{\sum_{u\in U_1} d_S(u,U_2)d_S(u,U_3)}{p^2|U_1||U_2||U_3|}$ we have

$$\kappa_\ell(S) \leqslant c_{123} \cdot p^{\binom{\ell}{2}}|U_1|\cdots|U_\ell| + \varepsilon p^{\binom{\ell}{2}}n^\ell\,.$$

Proof. We only prove part *(b)*, since the proof of *(a)* is very similar. Given ℓ and ε, let C be sufficiently large, so that Theorem 3.1 applies for ℓ and $\varepsilon/16$.

For the given ℓ-partite subgraph S on partition classes $U_1, \ldots, U_\ell$ we may assume, without loss of generality, that $|U_i| \geqslant \varepsilon n/2$, since otherwise the bound easily follows from $\kappa_\ell(S) \leqslant \kappa_\ell(G)$ and part *(iv)* of Theorem 3.1. Moreover, due to the properties of the random graph $G \in G(n,p)$, we can bound the degrees by

$$d_S(v) \leqslant d_G(v) \leqslant 2pn \tag{3.1}$$

for every vertex $v \in V(G)$.

Let D be the dense model of S provided by Theorem 3.1. In view of part *(ii)* of Theorem 3.1, it suffices to show that

$$\kappa_\ell(D) \leqslant c_{123} \cdot |U_1|\cdots|U_\ell| + \frac{\varepsilon}{2}n^\ell\,. \tag{3.2}$$

Let Y^+ be the set of vertices $u \in U_1$ for which $p \cdot d_D(u,U_2) > d_S(u,U_2) + \varepsilon pn/16$ or $p \cdot d_D(u,U_3) > d_S(u,U_3) + \varepsilon pn/16$. Part *(iii)* tells us $|Y^+| \leqslant \varepsilon n/8$ and combined with (3.1) it follows

$$\begin{aligned} p^2 \cdot \sum_{u\in U_1} d_D(u,U_2)d_D(u,U_3) &\leqslant \sum_{u\in U_1} d_S(u,U_2)d_S(u,U_3) + \Big(\frac{\varepsilon}{4} + \frac{\varepsilon^2}{16}\Big)p^2n^3 + |Y^+|p^2n^2 \\ &\leqslant c_{123} \cdot p^2|U_1||U_2||U_3| + \frac{\varepsilon}{2}p^2n^3\,. \end{aligned}$$

Consequently, the number of $K_{1,2}$ in D with center vertex in U_1 and leaves in U_2 and U_3 is bounded from above by

$$c_{123}\cdot|U_1||U_2||U_3|+\frac{\varepsilon}{2}n^3$$

and (3.2) by bounding the extension of each of these $K_{1,2}$ trivially by $|U_4|\cdots|U_\ell|$. □

We conclude this section with the proof of Lemma 2.1, which yields rainbow cliques in bounded colourings of the random graph.

Proof of Lemma 2.1. Given $\ell\geqslant 4$ and $\nu>0$ we define the auxiliary constant $\delta=\ell^{-4}/2$ and let C be sufficiently large so that the Theorem 3.1 and Lemma 3.2 apply for

$$\varepsilon=\Big(\frac{\nu}{2\ell}\Big)^{\ell}. \tag{3.3}$$

Suppose $G\in G(n,p)$ satisfies the conclusions of Theorem 3.1 and Lemma 3.2. We may also assume that for every subset $X\subseteq V(G)$ of size at least $|X|\geqslant\frac{\nu n}{2\ell}$ we have $d_G(v,X)\leqslant 1.1p|X|$ for all but at most $n/\log(n)$ vertices v.

Below we only prove the moreover-part of the lemma for the ℓ-clean subgraph $G_\ell\subseteq G$, since the proof for G without the upper bound on p is identical. Hence, we assume $p=p(n)\leqslant n^{-\frac{2\ell-2}{\ell^2+\ell-4}}/\omega(n)$ for some function ω tending to infinity. From this upper bound on p it follows by Markov's inequality that asymptotically almost surely the number of distinct pairs of K_ℓ sharing more than two vertices is at most $o(pn^2)$ and, therefore, we may assume

$$\big|E(G)\smallsetminus E(G_\ell)\big|\leqslant\varepsilon pn^2\,. \tag{3.4}$$

Let $\varphi\colon E(G)\longrightarrow\mathbb{N}$ be an edge colouring, which is $(\ell^{-5}/4,p)$-bounded on $U\subseteq V(G)$ of size at least νn.

Let $U_1'\,\dot\cup\,\dots\,\dot\cup\,U_\ell'=U$ be a balanced partition of U. After removing a few vertices of too high degree, i.e., vertices $u\in U_i'$ for which $d_G(u,U_j')>1.1\,p|U_j'|$ for some $j\neq i$, we arrive at a collection $\mathcal{U}=(U_1,\dots,U_\ell)$ of mutually disjoint sets of size m such that

$$|U_1|=\dots=|U_\ell|=m\geqslant\frac{|U|}{2\ell}\geqslant\frac{\nu}{2\ell}n$$

and for every every vertex $u_i\in U_i$ and $j\in[\ell]$ we have

$$d_G(u_i,U_j)\leqslant 2p|U_j|=2pm\,. \tag{3.5}$$

In addition, the boundedness of φ and the choice of δ ensures for every colour $c\in\mathbb{N}$

$$d_c(u_i,U_j)\leqslant d_c(u_i,U)\leqslant\frac{1}{4\ell^5}p|U|\leqslant\frac{1}{2\ell^4}p|U_j|=\delta pm\,. \tag{3.6}$$

In view of (3.4), part *(iv)* of Theorem 3.1 yields

$$\kappa_\ell(G_\ell[\mathcal{U}])\geqslant(1-2\varepsilon)p^{\binom{\ell}{2}}m^\ell$$

and below we shall bound the number of non-rainbow copies of K_ℓ in $G[\mathcal{U}]\supseteq G_\ell[\mathcal{U}]$.

For that it will be useful to classify the non-rainbow copies according to where the repeated colour occurs. We consider two cases depending on whether the two edges of the same colour share a vertex or form a matching. Hence, we consider the number $\kappa_\ell^{\wedge}(G[\mathcal{U}],\varphi)$ of copies of K_ℓ in $G[\mathcal{U}]$ containing edges $e \in E_G(U_1,U_2)$ and $e' \in E_G(U_1,U_3)$ such that $\varphi(e)=\varphi(e')$. Similarly, we define $\kappa_\ell^{::}(G[\mathcal{U}],\varphi)$ to be those copies with the two edges of the same colour being from $E_G(U_1,U_2)$ and $E_G(U_3,U_4)$. We will exploit the boundedness of φ to deduce the following claim from Lemma 3.2.

Claim 3.3. *We have* $\kappa_\ell^{::}(G[\mathcal{U}],\varphi) \leqslant 5\delta p^{\binom{\ell}{2}} m^\ell$ *and* $\kappa_\ell^{\wedge}(G[\mathcal{U}],\varphi) \leqslant 5\delta p^{\binom{\ell}{2}} m^\ell$.

Applying the claim to cover all $\binom{\binom{\ell}{2}}{2}$ possibilities where the two identically coloured edges may appear within the pairs of classes of $\mathcal{U}$ yields at least

$$(1-2\varepsilon)p^{\binom{\ell}{2}}m^\ell - \frac{\ell^4}{8}\cdot 5\delta p^{\binom{\ell}{2}}m^\ell > \frac{1}{2}p^{\binom{\ell}{2}}m^\ell$$

rainbow copies of K_ℓ in $G_\ell[\mathcal{U}]$. Hence, for the proof of Lemma 2.1 it only remains to verify Claim 3.3. □

Proof of Claim 3.3. We first bound $\kappa_\ell^{::}(G[\mathcal{U}],\varphi)$. Note that if each colour occupies $\delta p m^2$ edges of $G[U_1,U_2]$, then the claimed upper bound follows easily from Lemma 3.2 *(a)*. We shall reduce the problem to this case.

Fix a partition $C_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} C_r = \mathbb{N}$ of the set of colours such that

$$r \leqslant \frac{2}{\delta} \qquad \text{and} \qquad \left|\varphi^{-1}(C_\varrho) \cap E_G(U_1,U_2)\right| \leqslant 2\delta p m^2 \text{ for every } \varrho \in [r].$$

Note that due to (3.5) and (3.6) such a partition of the colours can be found greedily, by adding colours to a class C_ϱ as long as the bound on the number of edges in the preimage holds.

For $\varrho \in [r]$ let S_ϱ be the subgraph obtained from $G[\mathcal{U}]$ by restricting the edges from $G[U_1,U_2]$ and $G[U_3,U_4]$ to $\varphi^{-1}(C_\varrho) \cap E_G(U_1,U_2)$ and $\varphi^{-1}(C_\varrho) \cap E_G(U_3,U_4)$, i.e., to those edges having a colour from C_ϱ. Note that every copy counted by $\kappa_\ell^{::}(G[\mathcal{U}],\varphi)$ is contained in some S_ϱ.

Applying Lemma 3.2 *(a)*, we obtain

$$\kappa_\ell(S_\varrho) \leqslant 2\delta p m^2 \cdot \left|E_{S_\varrho}(U_3,U_4)\right| \cdot p^{\binom{\ell}{2}-2} m^{\ell-4} + \varepsilon p^{\binom{\ell}{2}} n^\ell,$$

and summing over all $\varrho \in [r]$ and recalling (3.5) yields the desired bound

$$\kappa_\ell^{::}(G[\mathcal{U}],\varphi) \leqslant 4\delta p^{\binom{\ell}{2}} m^\ell + \varepsilon p^{\binom{\ell}{2}} n^\ell \overset{(3.3)}{\leqslant} 5\delta p^{\binom{\ell}{2}} m^\ell .$$

For the bound on $\kappa_\ell^{\wedge}(G[\mathcal{U}],\varphi)$, we will again partition the colours, to reduce it to $2/\delta$ applications of Lemma 3.2 *(b)*. However, every vertex in U_1 will define its own partition.

For every vertex $u \in U_1$ we fix a partition $C_1^u \cup \ldots \cup C_{r_u}^u = \mathbb{N}$ such that for every $\varrho \in [r_u]$ we have

$$\sum_{c \in C_\varrho^u} d_c(u, U_2) \leqslant 2\delta p m^2 .$$

Again it follows from (3.5) and (3.6) that such a partition exists for some $r_u \leqslant 2/\delta$. For simplicity we allow empty partition classes and, hence, we may assume that $r_u = r = \lfloor 2/\delta \rfloor$ for every vertex $u \in U_1$.

For $\varrho \in [r]$ this time we let S_ϱ be the subgraph obtained from $G[\mathcal{U}]$ by restricting the edges in $G[U_1, U_2]$ and $G[U_1, U_3]$ incident to a vertex $u \in U_1$ to those, which received a colour from C_ϱ^u, i.e.,

$$E_{S_\varrho}(U_1, U_2) = \bigcup_{u \in U_1} \{uv \in E(G) \colon v \in U_2 \text{ and } \varphi(uv) \in C_\varrho^u\}$$

and $E_{S_\varrho}(U_1, U_3)$ is defined in an analogous way. This definition guarantees that every K_ℓ in $G[\mathcal{U}]$ containing a monochromatic $K_{1,2}$ with center in U_1 and leaves in U_2 and U_3 is contained in S_ϱ for some $\varrho \in [r]$.

In view of Lemma 3.2 (*b*), we have

$$\begin{aligned}\kappa_\ell(S_\varrho) &\leqslant \sum_{u \in U_1} 2\delta p m \cdot \sum_{c \in C_\varrho^u} d_c(u, U_3) \cdot p^{\binom{\ell}{2}-2} m^{\ell-3} + \varepsilon p^{\binom{\ell}{2}} n^\ell \\ &= \left|E_{S_\varrho}(U_1, U_3)\right| \cdot 2\delta p^{\binom{\ell}{2}-1} m^{\ell-2} + \varepsilon p^{\binom{\ell}{2}} n^\ell .\end{aligned}$$

In view of (3.5), summing over all $\varrho \in [r]$ yields the desired bound

$$\kappa_\ell^{\wedge}(G[\mathcal{U}], \varphi) \leqslant 4\delta p^{\binom{\ell}{2}} m^\ell + \varepsilon p^{\binom{\ell}{2}} n^\ell \overset{(3.3)}{\leqslant} 5\delta p^{\binom{\ell}{2}} m^\ell ,$$

which concludes the proof of Claim 3.3 and, hence, Lemma 2.1 is established. □

Remark 3.4. Theorem 3.1 from [4] is more general and applies not only to cliques K_ℓ, but to all *strictly balanced* graphs H for $p \geqslant Cn^{-\frac{|V(H)|-2}{|E(H)|-1}}$, i.e., for graphs H satisfying

$$\frac{|E(H')| - 1}{|V(H')| - 2} < \frac{|E(H)| - 1}{|V(H)| - 2}$$

for all proper subgraphs $H' \subsetneq H$ on at least three vertices. Starting with this general version of Theorem 3.1, the arguments tfrom this section can be carried out verbatim for such graphs H. This yields a version of Lemma 2.1 guaranteeing rainbow copies of strictly balanced graphs H for (δ, p)-bounded edge colourings of $G(n, p)$ for $p \geqslant Cn^{-\frac{|V(H)|-2}{|E(H)|-1}}$ as long as δ is sufficiently small and C is sufficiently large depending on H.

§4. Canonical cliques in unbounded colourings of random graphs

This section contains the proof of Lemma 2.2 along with the required prerequisites. We begin with an overview of the proof.

4.1. **Outline of the proof.** Again, the main tool in the proof of Lemma 2.2 is the transference principle developed by Conlon and Gowers. This result asserts that asymptotically almost surely every subgraph F of $G(n,p)$ has a dense model (see, e.g., Theorem 3.1 *(i)*). Moreover, if $p \gg n^{-\frac{2}{\ell+1}}$, then the dense model D and the subgraph F, have closely related distributions of the copies of K_ℓ, which is made precise in part *(ii)* of Theorem 3.1. Roughly speaking, we may think of F being *close to* a random subgraph of D whose edges are sampled independently with probability p.

For mimicking the argument from Section 2 for colourings of K_n, the main obstacle is that the neighbourhoods in $G(n,p)$ are of order pn, not allowing the iteration rendered in (2.2). We circumvent this by considering *suitable* subgraphs $F_1, F_2, \dots, F_L$ of $G(n,p)$ for $L = 2(\ell-1)(\ell-2)+2$ and obtain their dense models $D_1, \dots, D_L$, which yield linear-sized sets as neighbourhoods in the dense models. Another challenge is that, for transference to be useful, the dense model has to contain $\Omega(n^\ell)$ copies of K_ℓ, so we also need a robust, counting version of the argument for the case of unbounded colourings of K_n.

In the proof of Lemma 2.2 the ordering of the underlying vertex set will be important. For that, we refine the definition of $d_c(u,U)$ for a given edge colouring φ and for $\diamond \in \{<\,,>\}$ we set

$$d_c^\diamond(u,U) = \left|N_c^\diamond(u,U)\right| = \left|\{w \in U\colon \varphi(uw) = c \text{ and } u \diamond w\}\right|.$$

In Lemma 2.2 we assume that every sufficiently large set U of vertices of the random graph has the property that half of its vertices u have a large monochromatic neighbourhood. For simplicity, in the outline below we shall always assume that, in fact, these vertices u always lie before their monochromatic neighbourhood, i.e., $d_c^<(u,U) \geqslant 4\delta p|U|$ for at least $|U|/4$ vertices.

The subgraphs $F_i \subseteq G(n,p)$ for $i \in [L]$ will be selected in an iterative manner. For the definition of F_1, let V_1 be those $n/4$ vertices v such that $d_{c(v)}^<(v) \geqslant 4\delta pn$ for some associated colour $c(v)$. The graph F_1 is then defined as the union of those edges, i.e.,

$$E(F_1) = \{vw \in E(G(n,p))\colon v \in V_1, v < w, \text{ and } \varphi(vw) = c(v)\}.$$

In other words, the colour of each edge $e \in E(F_1)$ is defined by its *starting point* $\min(e)$. The transference principle yields a dense model D_1 of F_1, and by part *(iii)* of Theorem 3.1 most $v \in V_1$ satisfy $p \cdot d_{D_1}^<(v)$ is well approximated by $d_{F_1}^<(v)$. Consequently, typically the neighbourhood of v in D_1 defines a subset $S_1(v)$ of size at least $3\delta n$.

We continue and define F_2 based on the sets $S_1(v_1)$ for $v_1 \in V_1$. Again appealing to the assumption (2.3) of Lemma 2.2 as described above for $U = S_1(v_1)$, we obtain $|S_1(v_1)|/4$ pairs $(v_2, c) \in S_1(v_1) \times \mathbb{N}$ with

$$d_c^<(v_2, S_1(v_1)) \geqslant 4\delta p|S_1(v_1)| = \Omega(\delta^2 pn)\,. \tag{4.1}$$

However, the colour c depends on v_1 and, similarly, as in the definition of F_1 we shall find a “large” monochromatic neighbourhood of v_2 for the definition of F_2. On the other hand, since the degree of v_2 is close to pn in $G(n,p)$ at most $1/\delta^2$ different colours might occur for different choices of v_1. We let $c(v_2)$ be the *majority* colour and restrict to it for the definition of F_2. Moreover, for an appropriately chosen subset $V_2 \subseteq \bigcup\{S_1(v_1) \colon v_1 \in V_1\}$ and we shall define F_2 in such a way that

$$E(F_2) \subseteq \{v_2 w \colon v_2 \in V_2, v_2 < w, \text{ and } \varphi(v_2 w) = c(v_2)\}\,.$$

In particular, the colours of the edges in F_2 are determined again by their starting point. For some technical reasons the definition of F_2 is a bit more involved and, for example, we will impose F_2 to be bipartite (see Claim 4.5 below for the formal statement). Similar as before, we obtain a dense model D_2 for F_2. Owing to (4.1), the vertices in V_2 have $\Omega(\delta^2 pn)$ neighbours in F_2. Consequently, most vertices in V_2 have $\Omega(\delta^2 n)$ neighbours in D_2. Those neighbourhoods yield the linear sized sets $S_2(v_2)$, which allows us to iterate the argument to obtain a subgraph $F_3 \subseteq G(n,p)$ with all edges colours determined by its starting vertex and a dense model D_3 of F_3 with linear sized neighbourhoods.

This way we obtain graphs $F = F_1 \cup \dots \cup F_L$, and $D = D_1 \cup \dots \cup D_L$. We shall show that the dense graph $D =$ contains $\Omega(n^\ell)$ copies of K_ℓ by its construction through “nested neighbourhoods.” Consequently, the transference principle in form of part *(ii)* of Theorem 3.1 tells us that the sparse subgraph $F \subseteq G(n,p)$ contains $\Omega(p^{\binom{\ell}{2}} n^\ell)$ copies of K_ℓ and by the choice of the F_i these copies will be non-strictly min-coloured.

It remains to analyse the induced vertex colourings of these non-strictly min-coloured copies of K_ℓ in $G(n,p)$. However, those induced vertex colours are not synchronised within each of the sets $V_1, \dots, V_L$, i.e., the edges edges of F_i are not monochromatic and may induce different colours for the vertices in V_i. We address this issue by a more careful selection of the graphs $F_1, \dots, F_\ell$ (see *(b)* of Claim 4.5 below).

The proof of Lemma 2.2 is based on a more involved application of the transference principle compared to the proof of Lemma 2.1 in Section 3. In Section 4.2 we review the required results from [3] and some helpful consequences for the proof of Lemma 2.2 presented in Section 4.3.

4.2. **The transference principle.** In this section, we present the prerequisites for the proof of Lemma 2.2. The statements are written for arbitrary strictly balanced graphs H, although we shall only employ them for $H = K_\ell$. For a strictly balanced graph H on at least three vertices, we can define its 2-*density* by

$$m_2(H) = \frac{|E(H)|-1}{|V(H)|-2}\,,$$

and we note that $m_2(K_\ell) = (\ell+1)/2$ appears in the exponents of $p = p(n)$ in the in the assumptions of the earlier statements in Sections 1–3. Similarly, below we shall impose $p = p(n) \geqslant Cn^{-1/m_2(H)}$ for some sufficiently large constant C for some strictly balanced graph H on the vertex set $[\ell]$ for some $\ell \geqslant 3$.

For some (large) integer n, we shall work within the set of functions from $[n]^{(2)}$ to $\mathbb{R}$, which naturally corresponds to the set of *weighted graphs* on the vertex set $[n]$. Therefore, we often identify a graph F on $[n]$ with the indicator function $\mathbb{1}_F$ of its edge set and a dense model $\mathfrak{d}$ for F is a function $\mathfrak{d}\colon [n]^{(2)} \longrightarrow [0,1]$, which is "close" to $p^{-1}\mathbb{1}_F$ in terms of its distribution of weighted edges and copies of H. The definitions of edge counts and vertex degrees extend straightforwardly to functions $\mathfrak{f}\colon [n]^{(2)} \longrightarrow \mathbb{R}$ and for that we set

$$e(\mathfrak{f}) = \sum_{u<w} \mathfrak{f}(uw)\,, \quad e_{\mathfrak{f}}(U,W) = \sum_{u\in U, w\in W} \mathfrak{f}(uw)\,, \quad \text{and} \quad d_{\mathfrak{f}}(v,U) = \sum_{u\in U\smallsetminus\{v\}} \mathfrak{f}(uv).$$

With this notation at hand we can define the *cut-norm* by

$$\|\mathfrak{f}\|_{\square} = \frac{1}{n^2} \max_{U,W\subseteq[n]} e_{\mathfrak{f}}(U,W)\,.$$

This norm allows us to compare the edge distributions of two weighted graphs on $[n]$. In fact, we will consider weighted graphs $\mathfrak{f}$ and $\mathfrak{d}$ to be "close," if $\|\mathfrak{f}-\mathfrak{d}\|_{\square}$ is "small" (see, e.g., Theorem 3.1 *(i)* and Theorem 4.1 *(i)* below). Similarly, for a graph H with vertex set $[\ell]$ we define its homomorphism density in $\mathfrak{f}$ by

$$\Lambda_H(\mathfrak{f}) = \frac{1}{n^\ell} \sum_{v_1,\dots,v_\ell\in[n]} \prod_{ij\in e(H)} \mathfrak{f}(v_iv_j)\,,$$

where we use the convention $\mathfrak{f}(v_iv_j) = 0$ in case $v_i = v_j$. Consequently, for cliques the quantity $\Lambda_{K_\ell}(\mathfrak{f})$ corresponds to the weighted K_ℓ-density in $\mathfrak{f}$ and for $\mathfrak{f}$ being an unweighted graph we can recover the notation $\kappa_\ell(\mathfrak{f}) = n^\ell \cdot \Lambda_{K_\ell}(\mathfrak{f})$ from Section 3.

As discussed in the outline for the proof of Lemma 2.2, Theorem 3.1 will be applied in stages to the subgraphs $F_1,\dots,F_L \subseteq G(n,p)$ to obtain dense models $D_1,\dots,D_L$. In order to ensure that $D = D_1 \cup \dots \cup D_L$ is still a useful approximation of $F = F_1 \cup \dots \cup F_L$, we need a bit more insight into how these dense models are obtained. Informally, Conlon and Gowers [3] construct a norm $\|\cdot\|$ on the set of weighted graphs $\mathbb{R}^{[n]^{(2)}}$ so that the following holds: If $\mathbb{1}_F$ is the characteristic function of the edges of some $F \subseteq G(n,p)$ and $\|p^{-1}\mathbb{1}_F - \mathfrak{d}\|$ is sufficiently small for some dense model $\mathfrak{d}$ with $\|\mathfrak{d}\|_\infty \leqslant 1$, then $p^{-1}\mathbb{1}_F$ and $\mathfrak{d}$ have a "similar" distribution of edges and copies of H. A major contribution of [3] is precisely finding a norm which is sufficiently weak to allow a dense model which is arbitrarily close to $p^{-1}\mathbb{1}_F$, and sufficiently strong to preserve the relevant properties of F. However, the norm $\|\cdot\|$ actually depends on the random graph $G(n,p)$ in the sense that asymptotically almost surely $G \in G(n,p)$ has the property that there is a norm $\|\cdot\|$ with the aforementioned properties for every subgraph $F \subseteq G$. Theorem 4.1 below is version of the transference principle of Conlon and Gowers, which is tailored for our proof of Lemma 2.2. It is implicit in the work in [3] and in the Appendix we discuss in more detail how it can be extracted.

Theorem 4.1. *For every strictly balanced graph H with $V(H) = [\ell]$ and every $\varepsilon > 0$ there is some constant $C > 0$ such that for $p = p(n)$ with $Cn^{-1/m_2(H)} \leqslant p \leqslant 1/C$ asymptotically almost surely the following holds for $G \in G(n,p)$.*

There exists a norm $\|\cdot\|$ on the set of weighted graphs $\mathbb{R}^{[n]^{(2)}}$ such that for every $F \subseteq G$, there is a dense model $\mathfrak{d}_F\colon [n]^{(2)} \longrightarrow [0,1]$ with $\|p^{-1}\mathbb{1}_F - (1+\varepsilon)\mathfrak{d}_F\| < \varepsilon$ and

(*i*) *for all functions $\mathfrak{f}$, $\mathfrak{d}\colon [n]^{(2)} \longrightarrow \mathbb{R}$ with $\|\mathfrak{f} - (1+\varepsilon)\mathfrak{d}\| < \varepsilon$ we have*

$$\|\mathfrak{f} - \mathfrak{d}\|_{\square} \leqslant 2\varepsilon\,.$$

(*ii*) *for every function $\mathfrak{d}\colon [n]^{(2)} \longrightarrow [0,2\ell]$ with $\|p^{-1}\mathbb{1}_F - (1+\varepsilon)\mathfrak{d}\| \leqslant \varepsilon$ we have*

$$\Lambda_H(\mathfrak{d}) \leqslant p^{-|E(H)|}\Lambda_H(F) + (4\ell)^{\ell^2} \cdot \varepsilon\,.$$

We emphasise that, while (*i*) applies to any weighted graph $\mathfrak{f}$, part (*ii*) only applies to the subgraphs F of the random graph. Anyhow, in our intended application we have $\mathfrak{f} = p^{-1}\mathbb{1}_F$. Theorem 4.1 is closely related to Theorem 3.1 where S and D in Theorem 3.1 take the rôle of F and $\mathfrak{d}_F$ in Theorem 4.1. In fact, applying (*i*) and (*ii*) with $\mathfrak{d} = \mathfrak{d}_F$ and $\mathfrak{f} = p^{-1}\mathbb{1}_F$ in Theorem 4.1 yields closely related statements (*i*) and (*ii*) of Theorem 3.1.

It will be convenient to move from the dense weighted graphs $\mathfrak{d}$ back to unweighted graphs D sampled by the edge weights of $\mathfrak{d}$. The following lemma follows directly from the sharp concentration of the binomial distribution and Chernoff's inequality.

Lemma 4.2. *For every $\varepsilon > 0$ and any sequence of functions $\mathfrak{d}\colon [n]^{(2)} \longrightarrow [0,1]$ asymptotically almost surely we have $\|\mathbb{1}_D - \mathfrak{d}\|_{\square} \leqslant \varepsilon$ for the random graph D on $[n]$ with every edge e appearing independently with probability $\mathfrak{d}(e)$.* □

We will also use the following *counting lemma* comparing the number of subgraphs of two graphs in terms of the cut-norm. This can be viewed as the global counting lemma for the weak regularity lemma of Frieze and Kannan [8] and it can be found in [14, Lemma 4.1].

Lemma 4.3. *For every graph H and all functions $\mathfrak{f}$, $\mathfrak{d}\colon [n]^{(2)} \longrightarrow [0,1]$ we have*

$$\big|\Lambda_H(\mathfrak{f}) - \Lambda_H(\mathfrak{d})\big| \leqslant 2e(H) \cdot \|\mathfrak{f} - \mathfrak{d}\|_{\square} \qquad \square$$

We conclude this section with the following fact that the cut-norm controls most vertex degrees into given subsets (see, e.g., the deduction of (*iii*) of Theorem 3.1 from (*i*)).

Lemma 4.4. *For every $\varepsilon > 0$ and all functions $\mathfrak{f}$, $\mathfrak{d}\colon [n]^{(2)} \longrightarrow [0,1]$ with $\|\mathfrak{f} - \mathfrak{d}\|_{\square} \leqslant \varepsilon$ the following holds. For all $U \subseteq [n]$ with $|U| > 2\varepsilon^{1/3}n$, all but at most $\varepsilon^{1/3}n$ vertices $v \in [n]$ satisfy*

$$|d_{\mathfrak{f}}(v,U) - d_{\mathfrak{d}}(v,U)| \leqslant \varepsilon^{1/3}|U|\,.$$

Proof of Lemma 4.4. Let S be the set of vertices v with $d_{\mathfrak{f}}(v,U) - d_{\mathfrak{d}}(v,U) \geqslant \varepsilon^{1/3}|U|$. We have

$$n^2 \,\|\mathfrak{f} - \mathfrak{d}\|_{\square} \geqslant \sum_{v \in S} d_{\mathfrak{f}}(v,U) - d_{\mathfrak{d}}(v,U) \geqslant \varepsilon^{1/3}|U||S| \geqslant 2\varepsilon^{2/3}n|S|$$

and $|S| \leqslant \varepsilon^{1/3}n/2$ folllows. Similarly, one can show that there are at most $\varepsilon^{1/3}n/2$ vertices v such that $d_{\mathfrak{d}}(v,U) - d_{\mathfrak{f}}(v,U) \geqslant \varepsilon^{1/3}|U|$ and the claimed bound follows. □

4.3. **Proof of Lemma 2.2.** Given $\ell \geqslant 3$ and $\delta > 0$ we fix auxiliary constants

$$L = 2(\ell-1)(\ell-2)+2\,, \quad \nu = \frac{\delta^{4\ell^4+2\ell^2}}{10^{4\ell^2}}\,, \quad \alpha = \frac{\nu}{4\ell^2}\,, \quad \text{and} \quad \zeta = \delta^{10\ell^2+1}\,.$$

Moreover we fix auxiliary constants ε_1 and ε_2 and the desired C to satisfy the hierarchy

$$\delta\,, \ell^{-1} \gg \varepsilon_2 \gg \varepsilon_1 \gg C^{-1}\,.$$

Assume that $G \in G(n,p)$ satisfies the conclusion of Theorem 4.1 and Theorem 3.1 *(iv)* for $\varepsilon = \varepsilon_1$. Let $\|\cdot\|$ denote the norm given by Theorem 4.1. Moreover, we may assume that all vertices in G have degree $(1 \pm \varepsilon_1)pn$, since all these properties hold asymptotically almost surely.

For the moreover-part of the lemma, we have in addition $p = p(n) \leqslant n^{-\frac{2\ell-2}{\ell^2+\ell-4}}/\omega(n)$ for some function ω tending to infinity. Similarly as in the proof of Lemma 2.1 this allows us to assume

$$\big|E(G) \smallsetminus E(G_\ell)\big| \leqslant \zeta p n^2\,. \tag{4.2}$$

By our choice of ζ this implies that the crucial assumption (2.3) for G in Lemma 2.2 extends to G_ℓ on the price of a small change of the constants and the proof given below can be carried out in G_ℓ as well.

For a comparability sign $\diamond \in \{<, >\}$ set

$$B^{\diamond}(U) = \{v \in U \colon d_c^{\diamond}(v,U) \geqslant 4\delta p|U| \text{ for some colour } c\}\,.$$

The assumption (2.3) implies that for every set U with $|U| \geqslant \delta^{5\ell^2}n$, we have

$$|B^{<}(U)| \geqslant \frac{|U|}{4} \qquad \text{or} \qquad |B^{>}(U)| \geqslant \frac{|U|}{4}. \tag{4.3}$$

The condition (4.3) will be iterated to inductively build some structures in G, as detailed in the following claim. We will build subgraphs F_t of G for $t \in [L]$, which are non-strictly min-coloured or max-coloured and *relatively dense* to G. Moreover, we consider the dense models D_t of those F_t given by Theorem 4.1. The hypergraph $\mathcal{H}$ is used to keep track of cliques in $\bigcup_t D_t$. In the statement and proof, we usually identify a hypergraph $\mathcal{H}$ with its set of edges. In particular, $|\mathcal{H}|$ is the number of edges in $\mathcal{H}$.

Claim 4.5. *For every $t \in [L]$ there is a set of vertices V_t with $|V_t| \geqslant \frac{1}{25}\alpha\delta^{2t}n$, a comparability sign $\diamond_t \in \{<, >\}$, and a colour index $\psi_t \in \mathbb{N} \cup \{\star\}$ such that the following holds:*

(*a*) *Each* $v \in V_t$ *is assigned a colour* $c(v)$ *and there is a graph* $F_t \subseteq G$ *whose edges* vu *satisfy* $v \in V_t$ *and* $u \in N^{\diamond_t}_{c(v)}(v) \smallsetminus (V_1 \cup \cdots \cup V_t)$.

(*b*) *If* $\psi_t \in \mathbb{N}$, *then* $c(v) = \psi_t$ *for all* $v \in V_t$. *Otherwise, if* $\psi_t = \star$, *then we have* $|\{v \in V_t \colon c(v) = j\}| \leqslant \alpha |V_t|$ *for every colour* $j \in \mathbb{N}$.

(*c*) *There is a function* $\mathfrak{d}_t \colon [n]^{(2)} \longrightarrow [0,1]$ *with*

$$\|p^{-1}\mathbb{1}_{F_t} - (1+\varepsilon_1)\mathfrak{d}_t\| \leqslant \varepsilon_1\,,$$

and a graph D_t *with*

$$\|D_t - \mathfrak{d}_t\|_{\square} \leqslant 7\varepsilon_1 \quad \text{and} \quad \|D_t - p^{-1}\mathbb{1}_{F_t}\|_{\square} \leqslant 9\varepsilon_1\,.$$

(*d*) *There is a* t*-partite* t*-uniform hypergraph* $\mathcal{H}_t$ *on* $V_1 \cup \cdots \cup V_t$ *with*

$$|\mathcal{H}_t| \geqslant 10^{-2t}\delta^{t^2}|V_1|\cdots|V_t|$$

and for any $(v_1, \dots, v_t) \in \mathcal{H}_t$, *there is a set* $S(v_1, \dots, v_t)$ *disjoint from* $V_1, \dots, V_t$ *of size at least* $\delta^t n$ *such that for* $i \leqslant t$,

$$N_{D_i}(v_i) \supseteq \{v_{i+1}, \dots, v_t\} \cup S(v_1, \dots, v_t)\,.$$

Proof of Claim 4.5. We start the induction with $t = 1$. Let $B = B^{\diamond_1}(V(G))$ be the set of $n/4$ vertices given by (4.3). For $v \in B$, let $c(v)$ be a colour in which

$$d^{\diamond_1}_{c(v)}(v) \geqslant 4\delta pn,$$

Moreover, we wish to transfer a *bipartite* graph F_1. To this end, let each vertex of B be placed into a set Z_1 independently at random with probability $1/2$, and let $Y_1 = V(G) \smallsetminus Z_1$. We may assume that $|Z_1| \geqslant n/10$, and each $v \in Z_1$ satisfies

$$d^{\diamond_1}_{c(v)}(v, Y_1) \geqslant 1.5\delta pn \tag{4.4}$$

as this happens asymptotically almost surely.

If there is a colour j such that $|c^{-1}(j) \cap Z_1| \geqslant \alpha |Z_1|$, then we set $V_1 = c^{-1}(j)$ and $\psi_1 = j$. Otherwise take $V_1 = Z_1$ and set $\psi_1 = \star$. Note that in either case,

$$|V_1| \geqslant \frac{\alpha n}{10}\,.$$

Let F_1 be the subgraph of G with

$$E(F_1) = \{vu \colon v \in V_1\,,\ u \in Y_1\,,\ v \diamond_1 u\,,\ \text{and } \varphi(vu) = c(v)\}\,.$$

Let $\mathbb{1}_{F_1}$ be the characteristic function of F_1. By Theorem 4.1, there is a weighted graph $\mathfrak{d}_1 \colon [n]^{(2)} \longrightarrow [0,1]$ with $\|p^{-1}\mathbb{1}_{F_1} - \mathfrak{d}_1\| \leqslant \varepsilon_1$. In particular, $\|p^{-1}\mathbb{1}_{F_1} - \mathfrak{d}_1\|_{\square} \leqslant 2\varepsilon_1$ by Theorem 4.1 (*i*).

Let $\mathfrak{d}_1' = \mathfrak{d}_1\restriction_{V_1 \times ([n] \smallsetminus V_1)}$. Passing to a restriction of $\mathfrak{d}_1$ is just a technicality to circumvent small overlaps between $\mathfrak{d}_1, \dots, \mathfrak{d}_t$, and we will now show that $\|\mathfrak{d}_1 - \mathfrak{d}_1'\|_{\square} \leqslant 6\varepsilon_1$. Denote

$\mathfrak{f}_1 = p^{-1}\mathbb{1}_{F_1}$. To bound $\mathfrak{d}_1 - \mathfrak{d}'_1$, notice that, by definition of $\mathfrak{d}'_1$ and since all edges of F_1 lie in $V_1 \times ([n] \smallsetminus V_1)$,

$$\frac{1}{n^2}\left(e_{\mathfrak{d}'_1}([n]) - e_{\mathfrak{f}_1}([n])\right) = \frac{2}{n^2}\left(e_{\mathfrak{d}_1}(V_1, [n] \smallsetminus V_1) - e_{\mathfrak{f}_1}(V_1, [n] \smallsetminus V_1)\right) \leqslant 2\left\|\mathfrak{d}_1 - \mathfrak{f}_1\right\|_{\square} \leqslant 4\varepsilon_1 .$$

Hence $e_{\mathfrak{d}_1}([n]) - e_{\mathfrak{d}'_1}([n]) = e_{\mathfrak{d}_1}([n]) - e_{\mathfrak{f}_1}([n]) + e_{\mathfrak{f}_1}([n]) - e_{\mathfrak{d}'_1}([n]) \leqslant 6\varepsilon_1 n^2$. Since $\mathfrak{d}_1 - \mathfrak{d}'_1 \geqslant 0$, we have

$$\left\|\mathfrak{d}_1 - \mathfrak{d}'_1\right\|_{\square} = \frac{1}{n^2}(e_{\mathfrak{d}_1}([n]) - e_{\mathfrak{d}'_1}([n])) \leqslant 6\varepsilon_1.$$

Moreover,

$$\left\|\mathfrak{d}'_1 - p^{-1}\mathbb{1}_{F_1}\right\|_{\square} \leqslant 8\varepsilon_1$$

using the triangle inequality.

Let D_1 be a graph sampled from $\mathfrak{d}'_1$. Asymptotically almost surely, $\left\|\mathbb{1}_{D_1} - \mathfrak{d}'_1\right\|_{\square} \leqslant \varepsilon_1$ by Lemma 4.2. Therefore, using the triangle inequality, we may assume that

$$\left\|D_1 - \mathfrak{d}_1\right\|_{\square} \leqslant 7\varepsilon_1 \quad \text{and} \quad \left\|D_1 - p^{-1}\mathbb{1}_{F_1}\right\|_{\square} \leqslant 9\varepsilon_1 .$$

Let $\mathcal{H}_1$ be the set of vertices $v \in V_1$ with

$$d_{D_1}(v, Y_1) \geqslant \delta n .$$

By Lemma 4.4, using (4.4), $\left\|\mathbb{1}_{D_1} - p^{-1}\mathbb{1}_{F_1}\right\|_{\square} \leqslant 9\varepsilon_1$, and taking $\varepsilon_1 < \varepsilon_2^4$, we have that $|\mathcal{H}_1| \geqslant |V_1| - \varepsilon_2 n \geqslant \frac{|V_1|}{2}$. This completes the case $t = 1$.

Suppose the Claim holds for $1, 2, \ldots, t-1$. Let $X = [n] \smallsetminus (V_1 \cup \cdots \cup V_{t-1})$. For $(v_1, \ldots, v_{t-1}) \in \mathcal{H}_{t-1}$, consider the set $S(v_1, \ldots, v_{t-1}) \subseteq X$ as stated in *(d)*. Specifically, $|S(v_1, \ldots, v_{t-1})| \geqslant \delta^{t-1} n$. Denote $\xi = \xi(t) := \delta^{t-1}$. Applying the assumption (4.3), we obtain a set $B^{\diamond}(S(v_1, \ldots, v_{t-1}))$ of order at least $\frac{\xi n}{10}$ for some $\diamond = \diamond(v_1, \ldots v_{t-1})$ such that for every $v \in B^{\diamond}(S(v_1, \ldots, v_{t-1}))$ and some colour $c = c(v_1, \ldots, v_{t-1}, v)$,

$$d_c^{\diamond}(v, S(v_1, \ldots, v_{t-1})) \geqslant 4\delta\xi pn . \tag{4.5}$$

The next step is to remove the dependency of $\diamond$ and c on $(v_1, \ldots, v_{t-1})$. Firstly, let $\mathcal{H}'_{t-1}$ be a subhypergraph of $\mathcal{H}_{t-1}$ of order at least $\frac{1}{2}|\mathcal{H}_{t-1}|$ such that $\diamond(v_1, \ldots v_{t-1}) = \diamond_t$ for all $(v_1, \ldots, v_{t-1}) \in \mathcal{H}'_{t-1}$.

Now form an auxiliary bipartite graph J with parts $\mathcal{H}'_{t-1}$ and $X \times \mathbb{N}$ as follows: an edge $((v_1, v_2, \ldots v_{t-1}), (v, c))$ in J means that

$$d_c^{\diamond_t}(v, S(v_1, \ldots, v_{t-1})) \geqslant 4\delta\xi pn . \tag{4.6}$$

Since every $(v_1, \ldots v_{t-1}) \in \mathcal{H}'_{t-1}$ is contained in at least $\xi n/10$ edges in J (one for each element of $B^{\diamond_t}(v_1, \ldots, v_{t-1})$), we have

$$|J| \geqslant \frac{1}{2}|\mathcal{H}_{t-1}| \cdot \frac{\xi n}{10} .$$

Let $X' \subset X$ be the set of vertices v such that some (v,c) is incident to an edge of J, and note that for each v, there are at most $(3\delta\xi)^{-1}$ such colours c – this follows from $d_c(v) \geqslant 4\delta\xi pn$ and $d_G(v) \leqslant (1+\varepsilon_1)pn$. For each $v \in X'$, let $c(v)$ be the colour which maximises the degree of (v,c) in J. Form J' from J by deleting all the vertices $(v,c^\dagger)$ with $c^\dagger \neq c(v)$; we have

$$|J'| \geqslant |J| \cdot 3\delta\xi \geqslant \frac{3}{20}|\mathcal{H}_{t-1}|\delta\xi^2 n\,.$$

Since now each vertex $v \in X'$ is associated with a unique colour $c(v)$, we may assume that one vertex part of J' is just X'

We wish the graph F_t to be bipartite, so let us split the set X as follows. Let $W_t \subseteq X'$ consist of vertices sampled from X' independently at random with probability 1/2. Let $Y_t = X \smallsetminus W_t$. With positive probability,

$$|J'[E(\mathcal{H}'_{t-1}), W_t]| \geqslant \frac{3}{50}|\mathcal{H}_{t-1}|\delta\xi^2 n\,,$$

and for each $((v_1,\dots,v_{t-1}),v) \in J'$ (recalling (4.6)),

$$d^{\diamond_t}_{c(v)}(v, Y_t \cap S(v_1,\dots,v_{t-1})) \geqslant 1.5\delta\xi pn\,, \tag{4.7}$$

where we used Chernoff bounds and the union bound. Thus we may assume that these two inequalities are satisfied.

Now, let $Z_t \subseteq W_t$ be the set of vertices of degree at least $\frac{1}{50}|\mathcal{H}_{t-1}|\delta\xi^2$ in J', and let J^* be the induced subgraph of J' on $(E(\mathcal{H}'_{t-1}), Z_t)$. Since the vertices in $W_t \smallsetminus Z_t$ were incident to at most $\frac{1}{50}|\mathcal{H}_{t-1}|\delta\xi^2 n$ edges in total, we have

$$|J^*| \geqslant \frac{1}{25}|\mathcal{H}_{t-1}|\delta\xi^2 n\,.$$

Recalling that J^* is a bipartite graph on the vertex sets $(E(\mathcal{H}'_{t-1}), Z_t)$, we have the lower bound

$$|Z_t| \geqslant \frac{|J^*|}{|\mathcal{H}_{t-1}|} \geqslant \frac{1}{25}\delta\xi^2 n = \frac{1}{25}\delta^{1+2(t-1)}n \geqslant \frac{1}{25}\delta^{2t}n\,. \tag{4.8}$$

Now, to ensure (*b*), if there is a colour j such that

$$Z_{t,j} = \{v \in Z_t \colon c(v) = j\}$$

contains at least $\alpha|Z_t|$ vertices, let $V_t = Z_{t,j}$ and set $\psi_t = j$ (recalling that $\alpha \ll \delta, \ell^{-1}$ is a constant). Otherwise, set $V_t = Z_t$ and $\psi_t = \star$. Using the minimum degree of the vertices from V_t, we have

$$\left|J^*[E(\mathcal{H}'_{t-1}), V_t]\right| \geqslant \frac{1}{50}|\mathcal{H}_{t-1}|\delta\xi^2|V_t|\,.$$

Moreover, $|V_t| \geqslant \alpha|Z_t| \geqslant \frac{1}{25}\alpha\delta^{2t}n$, as required by the claim.

Let F_t be the subgraph of G with

$$E(F_t) = \{vy \colon v \in V_t, y \in Y_t, v \diamond_t y\}\,.$$

Recall that if $((v_1,\dots,v_{t-1}),v_t)\in J^*$, then by (4.7),

$$d_{F_t}(v_t,S(v_1,\dots,v_{t-1})\cap Y_t)\geqslant 1.5\delta\xi pn=1.5\delta^t pn\,. \tag{4.9}$$

Let $\mathfrak{f}_t=p^{-1}\mathbb{1}_{F_t}$. By Theorem 4.1, there is a function $\mathfrak{d}_t$ with $0\leqslant\mathfrak{d}_t\leqslant 1$ and

$$\|\mathfrak{f}_t-(1+\varepsilon_1)\mathfrak{d}_t\|\leqslant\varepsilon_1\,.$$

Let $\mathfrak{d}'_t=\mathfrak{d}_t\restriction_{V_t\times([n]\setminus(V_1\cup\dots V_t))}$, and let D_t be a graph sampled from $\mathfrak{d}'_t$. By the same argument as in the induction basis, we may assume that

$$\|D_t-\mathfrak{d}_t\|_{\square}\leqslant 7\varepsilon_1\quad\text{and}\quad\left\|D_t-p^{-1}\mathbb{1}_{F_t}\right\|_{\square}\leqslant 9\varepsilon_1\,,$$

where D_t stands for the indicator function $\mathbb{1}_{D_t}$.

Let $\mathcal{H}_t$ consist of t-tuples $(v_1,\dots,v_t)$ such that $v_t\in N_{J^*}(v_1,\dots,v_{t-1})$ and

$$d_{D_t}(v_t,S(v_1,\dots,v_{t-1})\cap Y_t)\geqslant\delta^t n\,. \tag{4.10}$$

Using Lemma 4.4 and (4.9), for each $(v_1,\dots,v_{t-1})\in\mathcal{H}'_{t-1}$, there are at most $\varepsilon_2 n$ vertices $v_t\in N_{J^*}(v_1,\dots,v_{t-1})$ violating (4.10) (recalling that $\varepsilon_2>\varepsilon_1^4$), so indeed

$$\begin{aligned}|\mathcal{H}_t|&\geqslant|J^*[E(\mathcal{H}'_{t-1}),V_t]|-\varepsilon_2 n^t\\&\geqslant\frac{1}{100}|\mathcal{H}_{t-1}|\delta\xi^2|V_t|\\&\geqslant 10^{-2t}\delta^{(t-1)^2+2t-1}|V_1|\dots|V_t|\\&=10^{-2t}\delta^{t^2}|V_1|\dots|V_t|\,,\end{aligned}$$

where we used the inductive hypothesis in the second line.

Finally, we set $S(v_1,\dots v_t)=N_{D_t}(v_t)\cap S(v_1,\dots,v_{t-1})\cap Y_t$ to obtain a set which satisfies asserting (d) of Claim 4.5. □

For the remainder of the proof, we do not need properties (c) and (d), but only the following consequences. We remark that it is crucial that the relative density of K_ℓ-copies mentioned in (d) (denoted ν) does not depend on α, but only on δ and ℓ. On the other hand $|V_i|/n$ may depend on α.

For every subset $M\subseteq[L]$ below we show

(A) The graph $\bigcup_{i\in M}D_i$ contains at least $\nu n\prod_{i\in M}|V_i|$ copies of $K_{|M|+1}$.

(B) If $|M|\leqslant\ell$, then

$$p^{-\binom{\ell}{2}}\cdot\Lambda_{K_\ell}\left(\bigcup_{i\in M}F_i\right)\geqslant\Lambda_{K_\ell}\left(\bigcup_{i\in M}D_i\right)-\varepsilon_2\,.$$

We first show that part (d) of Claim 4.5 implies (A). Fix any $(v_1,\dots,v_L)\in\mathcal{H}_L$, and let $|V^*_{L+1}|=S(v_1,\dots,v_L)$, so $|V^*_{L+1}|\geqslant\delta^L n$. Now, for each $v_{L+1}\in V^*_{L+1}$, the vertices

$\{v_i\colon i\in M\}\cup\{v_{L+1}\}$ form a clique in $\bigcup_{i\in M} D_i$, since $N_{D_i}(v_i)$ contains v_j for $j>i$ by (d). The number of choices for $(v_i\colon i\in M)$ contained in some edge $(v_1,\dots,v_L)\in\mathcal{H}_L$ is at least

$$|\mathcal{H}_L|\left(\prod_{i\in[L]\smallsetminus M}|V_i|\right)^{-1}\geqslant 10^{-2L}\delta^{L^2}\prod_{i\in M}|V_i|.$$

Putting these two bounds together, we obtain at least $n\delta^L\cdot 10^{-2L}\delta^{L^2}\prod_{i\in M}|V_i|$ copies of $K_{|M|+1}$, which implies (A) since $L\leqslant 2\ell^2$.

Secondly, we claim that D_i satisfy (B). Let $D=\bigcup_{i\in M} D_i$, $\mathfrak{d}=\sum_{i\in M}\mathfrak{d}_i$, and $F=\bigcup_{i\in M}F_i$, so that $\mathbb{1}_F=\sum_{i\in M}\mathbb{1}_{F_i}$. By the triangle inequality and part (c) of Claim 4.5, we have

$$\left\|p^{-1}\mathbb{1}_F-(1+\varepsilon_1)\sum_{i\in M}\mathfrak{d}_i\right\|\leqslant\varepsilon_1\ell.$$

Hence, by Theorem 4.1, and taking ε_1 sufficiently small depending on ε_2, we have

$$\Lambda_{K_\ell}(p^{-1}\mathbb{1}_F)\geqslant\Lambda_{K_\ell}(\mathfrak{d})-\varepsilon_1\ell^{3\ell^2}\geqslant\Lambda_{K_\ell}(\mathfrak{d})-\frac{\varepsilon_2}{2}.$$

Moreover, $\|\mathfrak{d}_t-D_t\|_\square\leqslant 7\varepsilon_1$ for $t\in[L]$, so using Lemma 4.3, we have $\Lambda_{K_\ell}(\mathfrak{d})\geqslant\Lambda_{K_\ell}(D)-\frac{\varepsilon_2}{2}$. It follows that

$$\Lambda_{K_\ell}(p^{-1}\mathbb{1}_F)\geqslant\Lambda_{K_\ell}(D)-\varepsilon_2\,,$$

as required for the proof of (B).

We now complete the proof of Lemma 2.2. Let $I'\subset[L]$ be a set of order $L/2$ such that $\diamond$ is constant on I' and, without loss of generality, we may assume that $\diamond_i = <$ for $i\in I'$. Moreover, let $I\subset I'$ be a set of order $\ell-1$ such that

(i) either $\psi_i\neq\star$ for $i\in I$ and ψ is constant or injective on I,

(ii) or $\psi_i=\star$ for $i\in I$.

Let

$$\vartheta=\prod_{i\in I}\frac{|V_i|}{n},\quad F=\bigcup_{i\in I}F_i\quad\text{ and }\quad D=\bigcup_{i\in I}D_i\,,$$

and note that ϑ is bounded from below by a constant depending on α,δ,ℓ, due to the lower bound on $|V_i|$ in Claim 4.5. By assertion (A), D contains at least $\nu\vartheta n^\ell$ copies of K_ℓ. Hence, owing to (B) and $\varepsilon_2\leqslant\frac{1}{10}\nu\vartheta$, the graph F contains at least $\frac12\nu\vartheta n^\ell p^{\binom{\ell}{2}}$ copies of K_ℓ.

All these copies are non-strictly min-coloured by construction of F (i.e., $\varphi(uv)=c(u)$ for $u<v$, $uv\in F$), and now we will use (b) and the choice of I to show that there is actually a strictly min-coloured or a monochromatic copy. We first show that each copy of K_ℓ in F has exactly one vertex in V_i for $i\in I$. Let $v_1<v_2<\dots<v_\ell$ be the vertex set of a K_ℓ in F, and recall the property (a) of Claim 4.5 for F. Since all the edges in F have the starting point in $\bigcup_{i\in I}V_i$, we have that $\{v_1,\dots,v_{\ell-1}\}\subseteq\bigcup_{i\in I}V_i$. But each V_i is an independent set in F, so it contains at most one (and hence exactly one) vertex from $\{v_1,\dots,v_{\ell-1}\}$.

If $\psi_i \neq \star$ for $i \in I$, any copy of K_ℓ in F is min-coloured (in case ψ is injective on I) or monochromatic (in case ψ is constant on I); to see this, recall that by (*b*) of Claim 4.5, if $uv \in E(F_i)$, $\varphi(uv) = \psi_i$.

Suppose that $\psi_i = \star$ for $i \in I$. For $i, j \in I$, let $\mathcal{K}_{ij}$ be the collection of K_ℓ-copies containing vertices $v_i \in V_i$ and $v_j \in V_j$ with $c(v_i) = c(v_j)$. We will show that for all $i \neq j \in I$

$$|\mathcal{K}_{ij}| \leqslant 3\alpha\vartheta p^{\binom{\ell}{2}} n^\ell. \tag{4.11}$$

(This follows easily from Theorem 3.1 (*iv*) when each colour class in V_i is of size $\alpha|V_i|$, but we need to be slightly more careful about smaller vertex classes.)

Partition the colours in $c[V_i]$ into *clusters* $1, \dots, m$ with $m \leqslant 2\alpha^{-1}$ such that for each $k \in [m]$, the proportion of vertices in cluster k in V_i lies in $[\alpha, 2\alpha]$. Note that such a partition exists since $|c[V_i]| \leqslant \alpha|V_i|$ by (*b*) of Claim 4.5. For each cluster $k \in [m]$, let $\beta(k)$ (resp. $\gamma(k)$) be the proportion of vertices v in V_i (resp. V_j) such that $c(v)$ is in cluster k, so $\alpha \leqslant \beta(k) \leqslant 2\alpha$. By Theorem 3.1 (*iv*), the number of K_ℓ-copies with vertices in cluster k in both V_i and V_j is at most

$$(\beta(k)\gamma(k)\vartheta + \varepsilon_1)p^{\binom{\ell}{2}} n^\ell \leqslant (2\alpha\gamma(k)\vartheta + \varepsilon_1)p^{\binom{\ell}{2}} n^\ell.$$

Summing over $k \in [m]$, corresponding to clusters $1, \dots, m$, and using $\sum_{k\in[m]} \gamma(k) \leqslant 1$, we obtain

$$|\mathcal{K}_{ij}| \leqslant \sum_{k\in[m]} (2\alpha\gamma(k)\vartheta + \varepsilon_1)p^{\binom{\ell}{2}} n^\ell \leqslant (2\alpha\vartheta + 2\alpha^{-1}\varepsilon_1)p^{\binom{\ell}{2}} n^\ell.$$

Taking $\varepsilon_1 \leqslant \frac{1}{3}\alpha^2\vartheta$ implies (4.11).

The bound (4.11) holds for any i, j, so taking $\alpha < \nu/(4\ell^2)$, we obtain

$$\left|\bigcup_{i<j\in I} \mathcal{K}_{ij}\right| \leqslant 3\ell^2\alpha\vartheta p^{\binom{\ell}{2}} n^\ell \leqslant \frac{1}{4}\nu\vartheta p^{\binom{\ell}{2}} n^\ell.$$

Recalling that F contains at least $\frac{1}{2}\nu\vartheta p^{\binom{\ell}{2}} n^\ell$ copies of K_ℓ, it follows that there is a copy outside $\bigcup_{i<j\in I} \mathcal{K}_{ij}$, which is then strictly min-coloured.

This completes the proof of Lemma 2.2. □

Remark 4.6. The fact that K_ℓ is a clique was only used to show that each copy of K_ℓ in F has at most one vertex in each V_i for $i \in I$. In the concluding remarks, we will discuss to what extent our proof extends to general graphs H.

§5. Concluding remarks

5.1. **Thresholds for canonical Ramsey properties for general graphs.** Recall that for an ordered graph H, we defined $\hat{p}_H$ as the threshold for the property $G(n,p) \xrightarrow{*} (H)$ and Theorem 1.3 establishes $\hat{p}_{K_\ell} = n^{-\frac{2}{\ell+1}}$. The problem of determining the threshold $\hat{p}_H$

for ordered graphs H which are not complete is still open, but there are some partial results.

Firstly, Alvarado, Kohayakawa, Morris, and Mota [17] studied a closely related problem for even cycles $C_{2\ell}$. Their result implies that for $p = Cn^{-1/m_2(C_{2\ell})} \log n$, any colouring of $G(n,p)$ contains a canonical copy of the cycle $C_{2\ell}$. However, in their work the ordering of the random graph $G(n,p)$ is determined after the colouring.

Secondly, for a strictly balanced graph H, our proof guarantees for $p \gg n^{-1/m_2(H)}$ a canonical copy of H, but one cannot require a specific vertex ordering of H. This statement is shown using the following modification of Theorem 4.1, which actually slightly simplifies the present proof of Lemma 2.2 for K_ℓ as well, but at the expense of introducing some additional formalism. For a collection of functions $f = (f_e)_{e \in H}$, define

$$\Lambda^\dagger((f_e)_{e\in H}) = \sum_{u_1,\dots,u_\ell} \prod_{ij \in H} f_{ij}(u_i, u_j),$$

that is, the density of H-copies in which the image of each edge $e \in H$ is weighted by f_e. A small modification of Corollary 3.7 in [3] (which appears in [4] in order to prove Theorem 3.1) implies that if $\|\mathfrak{f}_e - \mathfrak{d}_e\| = o(1)$ for $e \in E(H)$, then $|\Lambda^\dagger((\mathfrak{f}_e)_{e\in H}) - \Lambda^\dagger((\mathfrak{d}_e)_{e\in H})| = o(1)$. Hence, using our proof, we can find *many* embeddings $\zeta\colon H \longrightarrow G$ such that if $ij \in E(H)$ with $i < j$, then $\zeta(i)\zeta(j)$ lies in F_i, so its colour is determined by $\min(\zeta(i), \zeta(j))$.

5.2. **Canonical colourings in random hypergraphs.** Furthermore, it would be interesting to investigate extensions of Theorem 1.3 to k-uniform hypergraphs for $k \geqslant 3$. Namely, in their original work Erdős and Rado [6] established a canonical Ramsey theorem for k-uniform hypergraphs. However, their proof for k-uniform hypergraphs used Ramsey's theorem for $2k$-uniform hypergraphs and this seems to be an obstacle for transferring it to random hypergraphs at the right threshold. Hence, for transferring their result to the random setting, it seems necessary to start with a proof which avoids the use of hypergraphs with larger uniformity. Such proofs can be found in [22, 27].

Appendix A. Transference

First we informally outline the proof of Theorem 9.3 from [3], which corresponds to our Theorem 3.1. Then we state the formal claims that we need from [3], and show how they are applied to deduce Theorem 4.1. Note that the proof of Theorem 9.3 itself (as well as its corresponding deterministic result, Theorem 4.10) is not actually spelled out in [3]. Instead, the authors prove Theorem 9.1, which is a Ramsey-type result for $G(n,p)$, and say that the proof of Theorem 9.3 is 'much the same'. Moreover, their setting is much more general – they work with random subsets of a set X, which for us is just the set of edges of a complete graph K_n. In particular, for us, $|X| = \binom{n}{2}$.

Theorem 9.3 from [3] states that asymptotically almost surely, any subgraph F of $G(n,p)$ with $p = Cn^{-1/m_2(H)}$ can be approximated by a *dense model* D which asymptotically matches the edge distribution and the number of H-copies in F.

As mentioned, instead of graphs, we work with functions from $[n]^{(2)}$ to $\mathbb{R}$, or weighted graphs. For a random graph $G \in G(n,p)$, the *associated measure* of G is defined as $\mu = \mu_G = p^{-1}\mathbb{1}_G$. Given an m-tuple of functions $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^{[n]^{(2)}}$ (which will later be taken as the associated measures of m independent copies of $G(n,p_\star)$), Conlon and Gowers introduce the set of $(\mu,1)$-*basic anti-uniform functions* $\Phi_{\mu,1}$, which have the key property that the number of H-copies in a weighted graph $\mathfrak{f} \leqslant m^{-1}(\mu_1 + \cdots + \mu_m)$ can be bounded in terms of the inner products

$$\max\left\{|\langle \mathfrak{f}, \varphi\rangle| : \varphi \in \Phi_{\mu,1}\right\} .$$

The norm $\|\cdot\|$ is defined as

$$\|\mathfrak{f}\| = \max\left\{|\langle \mathfrak{f}, \varphi\rangle| : \varphi \in \Phi_{\mu,1}\right\} \cup \left\{\|\mathfrak{f}\|_{\square}\right\},$$

where the term $\|\mathfrak{f}\|_{\square}$ is just appended to ensure that $\|\cdot\|$ also controls the edge distribution of a weighted graph. This corresponds to Definition 4.9 from [3]. Note that in order for this norm to be *useful* (that is, for the properties (P0)–(P3′)) below to be satisfied asymptotically almost surely), it is important that $\|\mathfrak{f}\|_{\square}$ can be expressed in terms of inner products of $\mathfrak{f}$ with at most $2^{O(n)}$ functions, that is,

$$\|\mathfrak{f}\|_{\square} = \max\left\{\left|\langle \mathfrak{f}, \mathbb{1}_{[U,W]} + \mathbb{1}_{[U\cap W, U\cap W]}\rangle\right| : U, W \subset [n]\right\}. \tag{A.1}$$

One caveat in this description is that $\|\cdot\|$ only controls the number of H-copies under certain deterministic conditions on $\mu_1, \ldots, \mu_m$ (in the context of graphs, these conditions, denoted (P0)–(P3′) in [3], imply the property that the corresponding random graphs have a sufficiently homogeneous edges distribution, which is a well-known necessary condition for all similar counting results in sparse random graphs). To prove Theorem 3.1, Conlon and Gowers show three statements. Firstly, asymptotically almost surely, the associated measures $\mu_1, \ldots, \mu_m$ of $G(n,p_\star)$ with $p_\star = Cn^{-1/m_2(H)}$ satisfy (P0)–(P3′)). Secondly, assuming (P0)–(P3′), for any $\mathfrak{f} \leqslant m^{-1}(\mu_1 + \cdots + \mu_m)$ there is a *dense model* $\mathfrak{d} \colon [n]^{(2)} \longrightarrow [0,1]$ with

$$\|\mathfrak{f} - (1+\varepsilon)\mathfrak{d}\| \leqslant \varepsilon,$$

and thirdly, $\mathfrak{d}$ is a useful approximation for $\mathfrak{f}$ in our context.

These three statements imply that asymptotically almost surely, any such function $\mathfrak{f} \leqslant m^{-1}(\mu_1 + \cdots + \mu_m)$ has a suitable dense model. To reach the same conclusion for subgraphs of $G(n,p)$, a small additional step is needed (cf. Proof of Theorem 9.1 in [3]). Given m independent samples of $G(n,p_\star)$ with $p_\star = Cn^{-1/m_2(H)}$, let G be the union of $U_1, U_2, \ldots, U_m$. Then G is distributed as $G(n,p)$ with $p = 1-(1-p_\star)^m$, which is slightly smaller than $p_\star m$. Thus the hypothesis $\mathfrak{f} \leqslant p^{-1}\mathbb{1}_G$ does not quite imply that $\mathfrak{f} \leqslant m^{-1}p_\star^{-1}(\mathbb{1}_{U_1} + \cdots + \mathbb{1}_{U_m}) = m^{-1}(\mu_1 \cdots + \ldots \mu_m)$. Still, since $p = p_\star m(1 + o(1))$, this caveat can be resolved by slightly rescaling $\mathfrak{f}$, which we do at the start of the proof.

Now we formally state the claims which are used for deducing Theorem 4.1, and where they can be found in [3]. Say that $\mu_1, \ldots, \mu_m$ satisfy the property $\mathcal{P}(\eta, \lambda, d, m)$ if they satisfy properties (P0)–(P3′) stated in [3, Section 4]. The following statement can be found in the proof of Theorem 9.1 in [3].

Lemma A.1. *Given η, λ, d, m, there is C such that for $p_\star = Cn^{-1/m_2(H)}$ the following holds. If $U_1, \ldots, U_m \in G(n, p_\star)$ are mutually independent and $\mu_i = p_\star^{-1}\mathbb{1}_{U_i}$ is the associated measure of U_i for $i \in [m]$, then $\mu_1, \ldots, \mu_m$ satisfy $\mathcal{P}(\eta, \lambda, d, m)$ asymptotically almost surely.*

The following lemmata can be deduced from the proof of Theorem 4.5 in [3].

Lemma A.2. *Given $\varepsilon > 0$, there are sufficiently small constants $\eta, \lambda > 0$ and large integers d, m such that if $\mu_1, \ldots, \mu_m$ satisfy $\mathcal{P}(\eta, \lambda, d, m)$ and $f \leqslant m^{-1}(\mu_1 + \ldots \mu_m)$, then the following holds.*

(i) There is $\mathfrak{d}' \colon [n]^{(2)} \longrightarrow \mathbb{R}$ with $0 \leqslant \mathfrak{d}' \leqslant 1$ and $\|f - (1 + \varepsilon/4)\mathfrak{d}'\| \leqslant \frac{\varepsilon}{2}$.

(ii) If $\mathfrak{d} \colon [n]^{(2)} \longrightarrow \mathbb{R}$ is a function with $0 \leqslant \mathfrak{d} \leqslant 1$ and $\|f - (1 + \varepsilon)\mathfrak{d}\| \leqslant \varepsilon$, then

$$\Lambda_H(f) \geqslant \Lambda_H(\mathfrak{d}) - 4|E(H)| \cdot \varepsilon.$$

Now we can deduce our desired result.

Proof outline for Theorem 4.1. Given $\varepsilon > 0$, let η, λ, d and m be as required for the conclusion of Lemma A.2 to hold. The random graph G will be sampled in m rounds – that is, we set $p_\star = Cn^{-1/m_2(H)}$ and $p = 1 - (1 - p_\star)^m \geqslant (1 - \varepsilon/4)\, p_\star m$ for sufficiently large n. Following the notation of Conlon and Gowers, let $U_1, \ldots, U_m$ be m mutually independent random graphs sampled independently with edge probability $p_\star$, where C is a sufficiently large constant. Our random graph G with edge probability p will then be sampled by taking the union of $U_1, \ldots, U_m$, which indeed has the claimed distribution. For $i \in [m]$, let $\mu_i = p_\star^{-1}\mathbb{1}_{U_i}$ be the associated measure of U_i, and define $\mu = m^{-1}(\mu_1 + \cdots + \mu_m)$. Assume that $(\mu_i)_{i\in[m]}$ satisfy the property $\mathcal{P} = \mathcal{P}(\eta, \lambda, d, m)$. By Lemma A.1, this occurs asymptotically almost surely.

We will apply Lemma A.2 to deduce the existence of $\mathfrak{d}_F$ and part *(ii)*. Let F be a subgraph of G, so $0 \leqslant \mathbb{1}_F \leqslant \mathbb{1}_G \leqslant \sum_{i\in[m]} \mathbb{1}_{U_i}$. Define

$$\tilde{\mathfrak{f}} = p^{-1}\mathbb{1}_F \quad \text{and} \quad \mathfrak{f} = \frac{1 + \varepsilon/4}{1 + \varepsilon} p^{-1}\mathbb{1}_F. \tag{A.2}$$

We claim that $\mathfrak{f} \leqslant \mu$. Indeed, recalling that $p^{-1} \cdot \frac{1+\varepsilon/4}{1+\varepsilon} \leqslant (mp_\star)^{-1}$ for large n, we have

$$\mathfrak{f} = \frac{1 + \varepsilon/4}{1 + \varepsilon} p^{-1}\mathbb{1}_F \leqslant \frac{1 + \varepsilon/4}{1 + \varepsilon} p^{-1} \sum_{i\in[m]} \mathbb{1}_{U_i} \leqslant (mp_\star)^{-1} \sum_{i\in[m]} \mathbb{1}_{U_i} = m^{-1} \sum_{i\in[m]} \mu_i = \mu\,.$$

Thus we can apply the above-mentioned claims.

Lemma A.2 *(i)* applied to $f = \mathfrak{f}$ gives a function $\mathfrak{d}_F = \mathfrak{d}'$ such that $\|\mathfrak{f} - (1+\varepsilon/4)\mathfrak{d}_F\| \leqslant \frac{\varepsilon}{2}$. Multiplying by $\frac{1+\varepsilon}{1+\varepsilon/4}$ and recalling (A.2), we obtain

$$\|\tilde{\mathfrak{f}} - (1+\varepsilon)\mathfrak{d}_F\| \leqslant \varepsilon\,,$$

as required.

To see *(ii)*, take $\mathfrak{d}$ with $\|p^{-1}\mathbb{1}_F - (1+\varepsilon)\mathfrak{d}\| \leqslant \varepsilon$ and $0 \leqslant \mathfrak{d} \leqslant 2\ell$. We have

$$\left\|\frac{\mathbb{1}_F}{2\ell p} - \frac{(1+\varepsilon)\mathfrak{d}}{2\ell}\right\| \leqslant \frac{\varepsilon}{2\ell} < \varepsilon\,.$$

We may apply Lemma A.2 *(ii)* with $f = \frac{\mathbb{1}_F}{2\ell p} \leqslant \mu$ and $\mathfrak{d}$ replaced by $\frac{\mathfrak{d}}{2\ell} \leqslant 1$ to obtain

$$\Lambda_H\left(\frac{\mathbb{1}_F}{2\ell p}\right) \geqslant \Lambda_H\left(\frac{\mathfrak{d}}{2\ell}\right) - 4|E(H)| \cdot \varepsilon\,.$$

Using the fact that $\Lambda_H(\alpha\mathfrak{f}') = \alpha^{|E(H)|}\Lambda_H(\mathfrak{f}')$ for any constant $\alpha \geqslant 0$ and any $\mathfrak{f}' \colon [n]^{(2)} \longrightarrow \mathbb{R}$, it follows that

$$p^{-|H|}\Lambda_H(F) \geqslant \Lambda_H(\mathfrak{d}) - 4\varepsilon|H|(2\ell)^{|E(H)|} \geqslant \Lambda_H(\mathfrak{d}) - 4\varepsilon(2\ell)^{\ell^2},$$

as required.

Statement *(i)* follows from the definition of $\|\cdot\|_\square$ and the triangle inequality. That is,

$$\|\mathfrak{f} - \mathfrak{d}\|_\square \leqslant \|\mathfrak{f} - (1+\varepsilon)\mathfrak{d}\|_\square + \|\varepsilon\mathfrak{d}\|_\square \leqslant \|\mathfrak{f} - (1+\varepsilon)\mathfrak{d}\|_\square + \varepsilon \leqslant 2\varepsilon\,. \qquad \square$$

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