

CANONICAL RAMSEY THEOREM FOR GRAPHS WITH CLEAN INTERSECTIONS

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ABSTRACT. Extending earlier results of Nešetřil and Rödl [*Selective graphs and hyper-graphs*, Ann. Discrete Math. **3** (1978), 181–189], we show that for every ordered graph F there exist an ordered graph H and a system \mathcal{H}_F of induced copies of F such that every colouring of the edges of H yields a canonically coloured copy of F from \mathcal{H}_F and any two copies from \mathcal{H}_F intersect either in a vertex or an edge or not at all.

As a consequence, this allows us to construct, for any given ordered graph F , canonical Ramsey graphs H enjoying additional structural properties. In particular, H can have the same clique number as F and, provided F is not bipartite, the same odd girth. Moreover, if F is connected, then the copies of F from \mathcal{H}_F are not only induced, but their pairs of vertices also have the same distances in H as in F .

§1 INTRODUCTION

1.1. **Structural Ramsey theory.** Following Erdős and Rado [12, 13] we use the shorthand notation

$$H \longrightarrow (F)_r \tag{1.1}$$

for graphs F and H and some number of colours r to signify the Ramsey property that every r -colouring of $E(H)$ yields a monochromatic and *induced* copy of F in H . Without requiring the monochromatic copy of F to be induced, the existence of H would be a consequence of Ramsey’s theorem [28] for (finite) graphs. Henson [16] asked whether the induced version holds as well. This was answered affirmatively independently by Deuber [4], by Erdős, Hajnal, and Pósa [9], and by Rödl [30, 31]. In particular, those authors established the *induced Ramsey theorem* asserting the existence of a graph H satisfying (1.1) for any given F and r .

Structural Ramsey theory aims at a better understanding of unavoidable properties that such a Ramsey graph H must display. A first general result in this direction is due to

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Folkman [14], who addressed a question of Erdős and Hajnal [6] and showed that for two colours there exist Ramsey graphs H for the clique K_k with the minimal possible clique number, i.e., $\omega(H) = k$.

Around the same time Galvin (see [7]) formulated the more general question of the extent to which restricted classes of graphs are closed under the Ramsey property. He specifically asked this question for the class of triangle-free graphs. This problem was addressed by Nešetřil and Rödl [20] by showing that for every triangle-free graph F and every number of colours r there exists a triangle-free graph H satisfying the Ramsey property (1.1). This result was further extended in two directions: firstly, for graphs with given clique number [21], which generalises Folkman’s theorem, and secondly, for graphs without odd cycles up to a given length [24].

Since then quite a few results in structural Ramsey theory have emerged and we point the interested reader to the surveys of Nešetřil [19] and of Hubička and Konečný [17] and the references therein. Many of the proofs in the area are based on the *partite construction method* introduced by Nešetřil and Rödl [25]. In more advanced applications of that method a careful analysis of the distribution of the copies of F in the constructed Ramsey graph H is essential. Putting trivial cases aside (e.g., graphs F with at most one edge), it is easy to see that the copies of F in any Ramsey graph H cannot be edge disjoint. Erdős, Nešetřil, and Rödl (see [5]) asked whether all other pairwise intersections can be avoided in carefully constructed Ramsey graphs for F being a clique; this was verified in [25]. For general graphs F this leads to the following concept of clean intersections, which appeared in [26, Proposition 2.3].

We say two graphs F and F' have a *clean intersection* (or they *intersect cleanly*) if

$$|V(F) \cap V(F')| \leq 1 \quad \text{or} \quad V(F) \cap V(F') \in E(F) \cap E(F').$$

Moreover, a family of graphs has clean intersections if any two distinct members intersect cleanly. Note that for disconnected F no Ramsey graph H can have the property that all copies of F in H pairwise intersect cleanly, since copies of the components of F may combine into copies of F violating this property. For that reason we consider *Ramsey systems*, i.e., distinguished families of copies \mathcal{H}_F of F in H witnessing the Ramsey property. Ramsey systems with clean intersections are not only interesting in their own right, as they exhibit the minimal possible pairwise intersections—they also turned out to be useful for obtaining further results in structural Ramsey theory.

In this work we employ the partite construction method to obtain systems of copies \mathcal{H}_F of any given graph F with clean intersections that possess the *canonical* Ramsey property. These constructions of systems of copies with clean intersections will allow us to establish analogues of the structural Ramsey results discussed above in the canonical setting (see Theorems 1.1 and 2.4 below).

1.2. Canonical colourings. The canonical Ramsey theorem of Erdős and Rado [11] generalises Ramsey's theorem to an unbounded number of colours. Obviously, with this relaxation we can no longer ensure the appearance of monochromatic copies of the target graph F . However, Erdős and Rado characterised all canonical colour patterns that are unavoidable in such colourings of edge sets. As it turns out, the ordering of the underlying vertex set plays a rôle in that context and, hence, we consider *ordered graphs* with totally ordered vertex sets. We recall that a colouring $\varphi: E(F) \rightarrow \mathbb{N}$ of an ordered graph F is *canonical* if one of the following four patterns arises:

- (i) φ is *monochromatic*, i.e., all edges of F have the same colour;
- (ii) φ is *min-coloured*, i.e., for $e, e' \in E(F)$ we have $\varphi(e) = \varphi(e') \iff \min e = \min e'$;
- (iii) φ is *max-coloured*, i.e., for $e, e' \in E(F)$ we have $\varphi(e) = \varphi(e') \iff \max e = \max e'$;
- (iv) φ is *rainbow*, i.e., all edges of F have distinct colours.

For graphs F and H we denote by $\binom{H}{F}$ the set of all *induced* copies of F in H . If H and F are ordered, then we naturally restrict $\binom{H}{F}$ to the ordered induced copies of F in H .

For ordered graphs F and H we write

$$H \xrightarrow{*} (F)$$

to signify that H has the *canonical Ramsey property* for F , i.e., for every colouring $\varphi: E(H) \rightarrow \mathbb{N}$ there exists a copy $F_\star \in \binom{H}{F}$ such that φ restricted to $E(F_\star)$ is canonical. Moreover, we write

$$\mathcal{H}_F \xrightarrow{*} (F)$$

for a system of copies $\mathcal{H}_F \subseteq \binom{H}{F}$ to signify that the guaranteed canonically coloured copy is always an element of \mathcal{H}_F .

An induced version of the canonical Ramsey theorem with preserved clique number was obtained by Nešetřil and Rödl [23]. We strengthen this result by showing that we can further restrict to a system of copies $\mathcal{H}_F \subseteq \binom{H}{F}$ with clean intersections. When F is a clique, such a result was recently obtained by Kamčev and Schacht [18] by considering (appropriate subgraphs of) sparse random graphs. The proof presented here is constructive and in addition we also maintain the same odd girth. We recall that $\text{og}(F)$, the *odd girth* of a graph F , is defined as the length of a shortest odd cycle in F if such a cycle exists and we set $\text{og}(F) = \infty$ for bipartite graphs F . With this notation at hand our main result can be stated as follows.

Theorem 1.1. *For every ordered graph F there exist an ordered graph H and a system of copies $\mathcal{H}_F \subseteq \binom{H}{F}$ with clean intersections such that $\mathcal{H}_F \xrightarrow{*} (F)$. Moreover, the graph H has the same clique number as F and, in case F is not bipartite, it also has the same odd girth. Finally, if F is connected, then any two vertices of any copy $F_\star \in \mathcal{H}_F$ have the same distance in H as in F_\star .*

We remark that in general for ordered bipartite graphs F we cannot insist that H is bipartite as well. For example, the path P of length three with vertices $u < v < w < w'$ and edges uv, vw, ww' is clearly bipartite, but for any given ordered bipartite graph H with vertex partition $X \cup Y$ we may colour the edges xy with $x \in X$ and $y \in Y$ by the colour x . Since every copy of P in H has two vertices in X , this colouring induces exactly two colours on any copy of P , while every canonical colouring of P is either monochromatic or uses exactly three colours. However, our proof shows that we can require the odd girth of H to be arbitrarily large, if F is bipartite (see Theorem 2.4 below).

Our proof of Theorem 1.1 falls short of maintaining the girth itself. Such a result likely requires more delicate control over the systems of copies arising throughout the construction. For example, such copies would need to expand and not create short cycles themselves. In the non-canonical context, this level of control was recently established by Reiher and Rödl [29] and we believe that similar results also hold for the canonical Ramsey property.

Furthermore, Theorem 1.1 provides a system \mathcal{H}_F of copies of F and leaves open when this system can be taken to consist of all copies, that is, when $\mathcal{H}_F = \binom{H}{F}$ can be achieved. For cliques such a result follows from the probabilistic proof of Kamčev and Schacht [18] and is also delivered by the constructive proof presented here. In general, it is plausible that such a strengthening of Theorem 1.1 holds for most sufficiently connected graphs F .

§2 OVERVIEW OF THE PROOF

In this section we decompose the proof of our main result into two steps—a preliminary construction and a statement from structural Ramsey theory, where instead of edges we colour subgraphs of a fixed isomorphism type.

For a fixed graph F , by an F -system we mean a pair (B, \mathcal{B}_F) consisting of a graph B and a set \mathcal{B}_F of induced copies of F in B . Similarly, for an ordered graph F an ordered F -system (B, \mathcal{B}_F) has an ordered underlying graph B and the copies in \mathcal{B}_F need to respect the vertex ordering.

In the first step of our proof we construct for every ordered graph F an auxiliary ordered F -system (B, \mathcal{B}_F) with several desirable properties. We begin with a property of edge colourings, which connects the two parts of our argument.

Definition 2.1. Given an ordered graph F and an ordered F -system (B, \mathcal{B}_F) , an edge colouring $\varphi: E(B) \rightarrow \mathbb{N}$ is \mathcal{B}_F -homogeneous if for any two copies $F_1, F_2 \in \mathcal{B}_F$ the strictly increasing map $\eta: V(F_1) \rightarrow V(F_2)$ (which is an isomorphism between F_1 and F_2) preserves the relation of having the same colour, i.e., for all edges $e, f \in E(F_1)$ the following holds:

$$\varphi(e) = \varphi(f) \iff (\varphi \circ \eta)(e) = (\varphi \circ \eta)(f).$$

So, intuitively speaking, φ is \mathcal{B}_F -homogeneous if φ induces the same colour pattern on any two copies of F in \mathcal{B}_F .

In order to preserve the odd girth in our main result, we sometimes need to work with copies that are not only induced but also inherit bounded distances from their host graph. This is made precise by the following concepts.

Definition 2.2. Let F be a subgraph of another graph B .

- (i) We call F a *distance preserving* subgraph of B if any two vertices of F have the same distance in B as in F ; the set of distance preserving copies of F in B is denoted by $\binom{B}{F}_{\text{dp}}$.
- (ii) For a natural number n , we say that F is *n -induced* in B provided that the following holds: If for two distinct vertices $x, y \in V(F)$ there is an x - y -path P in B which is not contained in F and whose length is at most n , then the distance of x and y in F is strictly smaller than the length of P . Moreover, we write $\binom{B}{F}_n$ for the set of n -induced copies of F in B .

Clearly, distance preserving copies must be induced and we have the monotone inclusions

$$\binom{B}{F} = \binom{B}{F}_1 \supseteq \cdots \supseteq \binom{B}{F}_n \supseteq \binom{B}{F}_{n+1} \supseteq \cdots .$$

Moreover,

$$\bigcap_{n \in \mathbb{N}} \binom{B}{F}_n \subseteq \binom{B}{F}_{\text{dp}} \subseteq \binom{B}{F}$$

and, if F is connected, then we also have

$$\binom{B}{F}_{\text{diam}(F)} \subseteq \binom{B}{F}_{\text{dp}} . \tag{2.1}$$

The following definition encapsulates the desired structural properties for the ordered F -system (B, \mathcal{B}_F) we intend to construct.

Definition 2.3. For an ordered graph F and an integer $n \geq 1$, we say that an ordered F -system (B, \mathcal{B}_F) is *n -conform* if

- (i) the copies in \mathcal{B}_F have clean intersections,
- (ii) $\omega(B) = \omega(F)$,
- (iii) $\text{og}(B) \geq \min\{\text{og}(F), 2n + 1\}$,
- (iv) and $\mathcal{B}_F \subseteq \binom{B}{F}_n$.

For preserving the odd girth one may expect that part (iii) in the definition above should simply be $\text{og}(B) = \text{og}(F)$. However, as discussed before, such a strong version of Theorem 1.1 is false for bipartite graphs, which have infinite odd girth. In this case we can still guarantee canonical Ramsey graphs of arbitrarily large odd girth and this leads to the

formulation chosen in part (iii) of the definition. The strong version of Theorem 1.1 can now be stated as follows.

Theorem 2.4. *For every ordered graph F and every integer $n \geq 1$ there exists an n -conform F -system (H, \mathcal{H}_F) such that $\mathcal{H}_F \xrightarrow{*} (F)$.*

In view of the inclusion (2.1), Theorem 2.4 applied for connected graphs F and with $n = \text{diam}(F)$ yields a system \mathcal{H}_F of distance preserving copies of F . In the context of Ramsey's theorem such a structural result was obtained by Dellamonica and Rödl [3].

Clearly, Theorem 2.4 implies Theorem 1.1: it suffices to apply it with

$$n = \max\left\{\text{diam}(F), \frac{\text{og}(F)-1}{2}, 1\right\},$$

where the terms $\text{diam}(F)$ and $\frac{\text{og}(F)-1}{2}$ are only taken into account if F is connected or not bipartite, respectively. Moreover, $\mathcal{H}_F \xrightarrow{*} (F)$ entails $\mathcal{H}_F \neq \emptyset$, so H contains an induced copy of F and, hence, also $\text{og}(H) \leq \text{og}(F)$. The rest of this work is devoted to the proof of Theorem 2.4. For that we first establish the following result in §3.

Proposition 2.5. *For every ordered graph F and every integer $n \geq 1$ there exists an ordered n -conform F -system (B, \mathcal{B}_F) with $\mathcal{B}_F \neq \emptyset$ such that for every \mathcal{B}_F -homogeneous colouring $\varphi: E(B) \rightarrow \mathbb{N}$ all copies in \mathcal{B}_F are canonically coloured.*

Roughly speaking, our intended way of using this result is that if for any ordered graph H we have an arbitrary edge colouring $\varphi: E(H) \rightarrow \mathbb{N}$, then each copy in $\binom{H}{F}$ receives one of boundedly many colour patterns. Indeed, the number of possible colour patterns agrees with the number of equivalence relations on $E(F)$, which is a function of $e(F)$. Thus when H is constructed in such a way that $\binom{H}{B}$ is “sufficiently rich”, then there exists a copy $B^* \in \binom{H}{B}$ such that the corresponding F -system (B^*, \mathcal{B}_F^*) isomorphic to (B, \mathcal{B}_F) receives a \mathcal{B}_F^* -homogeneous colouring, meaning that its copies of F are canonically coloured.

We shall now introduce some language that will allow us to present this plan in a more precise form. An F -system (B, \mathcal{B}_F) is said to be an *induced F -subsystem* of another F -system (H, \mathcal{H}_F) if B is an induced subgraph of H and, furthermore,

$$\mathcal{B}_F = \mathcal{H}_F \cap \binom{B}{F}.$$

We write $\binom{(H, \mathcal{H}_F)}{(B, \mathcal{B}_F)}$ for the set of all induced F -subsystems of (H, \mathcal{H}_F) which are, in the obvious sense, isomorphic to (B, \mathcal{B}_F) . By a (B, \mathcal{B}_F) -conglomerate we mean a triple $(H, \mathcal{H}_B, \mathcal{H}_F)$ such that (H, \mathcal{H}_F) is an F -system and $\mathcal{H}_B \subseteq \binom{(H, \mathcal{H}_F)}{(B, \mathcal{B}_F)}$. Given such a conglomerate and a number of colours r the partition relation

$$\mathcal{H}_B \rightarrow (B, \mathcal{B}_F)_r^F$$

indicates that for every colouring $\varphi: \mathcal{H}_F \rightarrow [r]$ there exists a copy $(B^*, \mathcal{B}_F^*) \in \mathcal{H}_B$ such that \mathcal{B}_F^* is monochromatic with respect to φ .

These concepts extend to ordered graphs and systems in the expected way. Most importantly, if F is an ordered graph and (B, \mathcal{B}_F) , (H, \mathcal{H}_F) are ordered F -systems, then $\binom{(H, \mathcal{H}_F)}{(B, \mathcal{B}_F)}$ only contains copies of (B, \mathcal{B}_F) that are ordered correctly; similarly, for an ordered (B, \mathcal{B}_F) -conglomerate $(H, \mathcal{H}_B, \mathcal{H}_F)$ the copies in \mathcal{H}_F and \mathcal{H}_B need to respect the vertex orderings of F , B , and H .

Proposition 2.6. *For every ordered graph F , every $n \geq 1$, every ordered n -conform F -system (B, \mathcal{B}_F) , and every number of colours r , there exists an ordered (B, \mathcal{B}_F) -conglomerate $(H, \mathcal{H}_B, \mathcal{H}_F)$ such that*

- (a) $\mathcal{H}_B \rightarrow (B, \mathcal{B}_F)_r^F$
- (b) and (H, \mathcal{H}_F) is an n -conform F -system.

The proof of this proposition employs the partite construction method and is deferred to §§4–6. We conclude this section with the proof of Theorem 2.4 assuming Propositions 2.5 and 2.6.

Proof of Theorem 2.4. Fix an ordered graph F and an integer $n \geq 1$. By Proposition 2.5, there exists an n -conform ordered F -system (B, \mathcal{B}_F) with $\mathcal{B}_F \neq \emptyset$ such that every \mathcal{B}_F -homogeneous colouring of $E(B)$ colours all copies of F in \mathcal{B}_F canonically.

Let r be the number of equivalence relations on $E(F)$, which equals the number of colour patterns that any edge colouring can induce on a copy of F . We apply Proposition 2.6 with (B, \mathcal{B}_F) and r to obtain an ordered (B, \mathcal{B}_F) -conglomerate $(H, \mathcal{H}_B, \mathcal{H}_F)$ satisfying properties (a) and (b) of the proposition. In particular, (H, \mathcal{H}_F) is an n -conform F -system by assertion (b). We now argue that $\mathcal{H}_F \xrightarrow{*} (F)$ holds.

Given any edge colouring $\varphi: E(H) \rightarrow \mathbb{N}$, each copy $F_\star \in \mathcal{H}_F$ inherits from φ an equivalence relation on $E(F_\star)$, determining one of the r colour patterns. This defines a colouring $\psi: \mathcal{H}_F \rightarrow [r]$ of the copies of F . By assertion (a) of Proposition 2.6, there exists a copy $(B^*, \mathcal{B}_F^*) \in \mathcal{H}_B$ such that \mathcal{B}_F^* is monochromatic under ψ . Consequently, φ induces the same colour pattern on every $F_\star \in \mathcal{B}_F^*$, i.e., φ is \mathcal{B}_F^* -homogeneous. Since (B^*, \mathcal{B}_F^*) is isomorphic to (B, \mathcal{B}_F) , Proposition 2.5 implies that every copy of F in $\mathcal{B}_F^* \subseteq \mathcal{H}_F$ is canonically coloured by φ . In particular, as $\mathcal{B}_F \neq \emptyset$, the set \mathcal{B}_F^* is non-empty and, hence, \mathcal{H}_F indeed contains a canonically coloured copy of F , i.e., $\mathcal{H}_F \xrightarrow{*} (F)$. \square

§3 CONFORM SYSTEMS BEING CANONICAL ON HOMOGENEOUS COLOURINGS

This section is devoted to a constructive proof of Proposition 2.5. For that we first study in §3.1 simple operations on systems of copies that preserve conformity. In §3.2 and §3.3

we utilise these operations to obtain the desired conform F -system (B, \mathcal{B}_F) , for which homogeneous colourings induce canonical copies of F .

3.1. Extensions of conform systems. We study simple operations on conform systems that keep conformity intact. For example, it is easy to check that removing a copy F' of F from an n -conform F -system (consisting of at least two copies), where we only remove vertices and edges that are only contained in F' and no other copy of F , results in an n -conform system. Below we analyse the situation when an n -conform system is enlarged.

The first observation shows that the union of conform F -systems intersecting in exactly one copy of F is again conform.

Lemma 3.1. *Suppose (B, \mathcal{B}_F) and (B', \mathcal{B}'_F) are two n -conform F -systems satisfying $B \cap B' \in \mathcal{B}_F \cap \mathcal{B}'_F$. Then the union $(B \cup B', \mathcal{B}_F \cup \mathcal{B}'_F)$ is again an n -conform F -system.*

Proof. Obviously, $(B \cup B', \mathcal{B}_F \cup \mathcal{B}'_F)$ is still an F -system with

$$\omega(B \cup B') = \max \{ \omega(B), \omega(B') \} = \omega(F)$$

and the copies of F in $\mathcal{B}_F \cup \mathcal{B}'_F$ have clean intersections.

Concerning the odd girth, suppose C is a shortest odd cycle in $B \cup B'$. We may assume that C is neither contained in B nor in B' . Let v_1, \dots, v_k be the vertices of the unique shared copy $F_\star \in \mathcal{B}_F \cap \mathcal{B}'_F$ appearing on the cycle C in cyclic order. For $i \in \mathbb{Z}/k\mathbb{Z}$ let P_i be the corresponding v_i - v_{i+1} -subpath in C . By our choice of C there are $i \neq j$ such that P_i contains a vertex from $V(B) \setminus V(B')$ and P_j contains a vertex from $V(B') \setminus V(B)$. Let ℓ_i and ℓ_j denote the lengths of these two paths and suppose $\ell_i \leq \ell_j$.

If $\ell_i \geq n$, then the length of C is at least $2n + 1$ and we are done. Assuming $\ell_i < n$ allows us to appeal to the n -inducedness of (B, \mathcal{B}_F) , which yields a shorter v_i - v_{i+1} -path Q_i within F_\star . If Q_i and P_i have the same parity, then replacing P_i by Q_i in C leads to a shorter closed odd walk in $B \cup B'$ contradicting the minimal choice of C . Otherwise the union of P_i and Q_i forms a closed odd walk W with

$$\text{length of } W = \text{length of } Q_i + \text{length of } P_i < 2\ell_i \leq \ell_i + \ell_j \leq \text{length of } C,$$

which leads to the same contradiction.

For the n -inducedness we consider an x - y -path P of length $m \leq n$ in $B \cup B'$ connecting two vertices in a copy $F_0 \in \mathcal{B}_F$, but such that $P \not\subseteq F_0$. If P is contained in B , then the n -inducedness of (B, \mathcal{B}_F) implies that the distance in F_0 is less than m . Otherwise, any maximal segment aPb with all inner vertices from $V(B') \setminus V(B)$ has the property that a and b are both contained in the unique shared copy $F_\star \in \mathcal{B}_F \cap \mathcal{B}'_F$. Consequently, the n -inducedness of (B', \mathcal{B}'_F) asserts that the segment aPb can be replaced by a shorter path contained in $F_\star \subseteq B$. \square

Other simple extension operations concern adding a single copy of F to a conform system with restricted intersections. For example, it is easy to see that adding a single copy of F , which only intersects the given system in a single vertex or a single edge, has no effect on the conformity, which we summarise in the following fact.

Fact 3.2. *Suppose (B, \mathcal{B}_F) is an n -conform F -system and F' is a copy of F such that*

$$|V(F') \cap V(B)| \leq 1 \quad \text{or} \quad V(F') \cap V(B) \in E(F') \cap E(B).$$

Then the extension $(B \cup F', \mathcal{B}_F \cup \{F'\})$ is also an n -conform F -system. \square

The following lemmata analyse the situation when the added copy intersects a conform system in three or four vertices. For that we recall the following standard graph-theoretic concepts. For a graph $B = (V, E)$ and $x, y \in V$ we denote by $\text{dist}_B(x, y)$ the length of a shortest x - y -path in B if such a path exists and ∞ otherwise. More generally, for sets of vertices $X, Y \subseteq V$ we write $\text{dist}_B(X, Y)$ for the minimum of $\text{dist}_B(x, y)$ over all $x \in X$ and $y \in Y$. In particular, $\text{dist}_B(X, Y) = \infty$ if one of the sets X or Y is empty. For vertices $v \in V$, edges $xy \in E$, and subgraphs $F_\star \subseteq B$ we also simply write $\text{dist}_B(v, F_\star)$ and $\text{dist}_B(xy, F_\star)$ instead of $\text{dist}_B(\{v\}, V(F_\star))$ and $\text{dist}_B(\{x, y\}, V(F_\star))$, respectively.

The first lemma addresses the case when the added copy of F intersects the given conform system in a vertex and an edge.

Lemma 3.3. *Suppose (B, \mathcal{B}_F) is an n -conform F -system with an edge $ab \in E(B)$ and a vertex $y \in V(B) \setminus \{a, b\}$ satisfying $B = \bigcup \mathcal{B}_F$ and*

$$\text{dist}_B(ab, F_\star) + \text{dist}_B(y, F_\star) \geq \max \{3, \min\{\text{og}(F) - 1, 2n\}, n + 1\} \quad (3.1)$$

for every $F_\star \in \mathcal{B}_F$.

Then for every copy F' of F satisfying $V(F') \cap V(B) = \{a, b, y\}$ and $ab \in E(F')$ the extension $(B \cup F', \mathcal{B}_F \cup \{F'\})$ is an n -conform F -system.

The next lemma deals with a very similar case, when the single vertex y is replaced by an edge xy disjoint from ab . Below we shall only prove that version of the lemma, since the same reasoning yields Lemma 3.3.

Lemma 3.4. *Suppose (B, \mathcal{B}_F) is an n -conform F -system with disjoint edges $ab, xy \in E(B)$ satisfying $B = \bigcup \mathcal{B}_F$ and*

$$\text{dist}_B(ab, F_\star) + \text{dist}_B(xy, F_\star) \geq \max \{3, \min\{\text{og}(F) - 1, 2n\}, n + 1\}$$

for every $F_\star \in \mathcal{B}_F$.

Then for every copy F' of F satisfying $V(F') \cap V(B) = \{a, b, x, y\}$ and $ab, xy \in E(F')$ the extension $(B \cup F', \mathcal{B}_F \cup \{F'\})$ is an n -conform F -system.

Proof. Property (i) of Definition 2.3 concerns clean intersections and for the extended system $\mathcal{B}_F \cup \{F'\}$ this is a direct consequence of the distance assumption, since it prevents any copy $F_\star \in \mathcal{B}_F$ from containing a vertex from $\{a, b\}$ and $\{x, y\}$ at the same time. In particular, this also shows that all copies of F in the extended system $\mathcal{B}_F \cup \{F'\}$ are still induced in $B \cup F'$, regardless of whether F' contains an edge with one vertex in $\{a, b\}$ and one in $\{x, y\}$.

Next we address property (ii) of Definition 2.3. For that we suppose for a contradiction that there is a clique $K_m \subseteq B \cup F'$ with $m > \omega(B) = \omega(F)$.

If $V(K_m) \subseteq V(B)$, then K_m contains an edge $uv \in E(F')$ with $u \in \{a, b\}$ and $v \in \{x, y\}$ and another vertex $w \in V(B) \setminus \{a, b, x, y\}$. Since $B = \bigcup \mathcal{B}_F$ there is some copy $F_w \in \mathcal{B}_F$ containing w and owing to the distance assumption we have

$$\text{dist}_B(u, w) + \text{dist}_B(v, w) \geq \text{dist}_B(ab, F_w) + \text{dist}_B(xy, F_w) \geq 3.$$

Hence, one of the edges uw or vw must be missing in B , as well as in $B \cup F'$, ruling out this case.

In the other case K_m must contain vertices $w \in V(B) \setminus V(F')$ and $w' \in V(F') \setminus V(B)$, which contradicts the fact that $ww' \notin E(B \cup F')$.

For property (iii) of Definition 2.3 we consider a shortest odd cycle C in $B \cup F'$ not contained in B and not contained in F' . Such a cycle C must contain at least two distinct vertices from the set $\{a, b, x, y\}$. In fact, C contains a u - v -path P with $u, v \in \{a, b, x, y\}$ and all internal vertices from $V(B) \setminus V(F')$. If uv equals the edge ab , then either $C = P + ab$ contradicting that C is not contained in B or ab is a chord in C . However, such a chord ab either contradicts the minimal choice of C or it reveals that the length of C is at least $\min\{\text{og}(B), \text{og}(F)\}$ and we are done.

Consequently, it remains to address the case $u \in \{a, b\}$ and $v \in \{x, y\}$. In this case P contains some internal vertex w contained in some copy $F_w \in \mathcal{B}_F$ and we arrive at

$$\begin{aligned} |E(C)| &> |E(P)| \geq \text{dist}_B(u, w) + \text{dist}_B(v, w) \\ &\geq \text{dist}_B(ab, F_w) + \text{dist}_B(xy, F_w) \geq \min\{\text{og}(F) - 1, 2n\}, \end{aligned}$$

which concludes the verification of property (iii) of Definition 2.3.

Finally we turn our attention to the inducedness properties of the extended system. For that we first consider a copy $F_\star \in \mathcal{B}_F$ and two vertices $u, v \in V(F_\star)$ for which there exists a shortest u - v -path P in $B \cup F'$ of length at most n . Moreover, we can assume that P contains some edge from $E(F') \setminus E(B)$. In particular, P must contain two vertices from $\{a, b, x, y\}$ and appealing to the minimality of P we infer that P contains a vertex from $\{a, b\}$ and a vertex from $\{x, y\}$, since otherwise we may use one of the edges ab or xy to locate a shorter path Q , which either contradicts the choice of P or it is already contained

in B allowing us to appeal to the assumption that (B, \mathcal{B}_F) is n -induced. However, with P containing a vertex from $\{a, b\}$ and from $\{x, y\}$, we can invoke the distance assumption

$$|E(P)| > \text{dist}_B(ab, F_\star) + \text{dist}_B(xy, F_\star) \geq n + 1,$$

which contradicts the choice of P .

It remains to consider the case when u, v are vertices of the added copy F' . However, in this case the reasoning above applies again: a shortest u - v -path of length at most n not contained in F' either admits a shortcut through one of the edges $ab, xy \in E(F')$, or it meets both $\{a, b\}$ and $\{x, y\}$ and has length at least $n + 1$. \square

The last lemma addresses the situation when the extended copy intersects the given system in two edges sharing a vertex y . For the statement we consider distances in the given graph with y removed. For a graph $B = (V, E)$, distinct vertices $x, y \in V$, and a subgraph $F \subseteq B$ we set

$$\text{dist}_{B-y}(x, F) = \text{dist}_{B'}(x, V(F) \setminus \{y\})$$

for the induced subgraph B' of B induced on $V \setminus \{y\}$. Note that we make no assumption on whether the considered subgraph $F \subseteq B$ contains the vertex y or not.

Lemma 3.5. *Suppose (B, \mathcal{B}_F) is an n -conform F -system with edges $xy, yz \in E(B)$ satisfying $B = \bigcup \mathcal{B}_F$ and*

$$\text{dist}_{B-y}(x, F_\star) + \text{dist}_{B-y}(z, F_\star) \geq \max\{3, n + 1\} \tag{3.2}$$

for every $F_\star \in \mathcal{B}_F$.

Then for every copy F' of F satisfying $V(F') \cap V(B) = \{x, y, z\}$ and $xy, yz \in E(F')$ the extension $(B \cup F', \mathcal{B}_F \cup \{F'\})$ is an n -conform F -system.

The proof of Lemma 3.5 parallels the proof of Lemma 3.4 and the main difference concerns proving property (iii) of Definition 2.3.

Proof. Again the distance assumption ensures that no copy of F in \mathcal{B}_F can contain both vertices x and z . As a result all copies of F in the extended system $\mathcal{B}_F \cup \{F'\}$ are induced in $B \cup F'$. Moreover, since no copy of F in \mathcal{B}_F contains x and z , the extended system still enjoys clean intersections.

For property (ii) of Definition 2.3 we again have only to consider cliques K_m contained in $V(B)$ that appeared by adding the edge xz to B . However, the distance assumption combined with $B = \bigcup \mathcal{B}_F$ ensures that at least one of the edges wx or wz must be missing in B for every $w \in V(B) \setminus \{x, y, z\}$.

Regarding the odd girth in form of property (iii) of Definition 2.3, we may assume that $xz \notin E(F')$, since otherwise $\text{og}(F) = 3$ and property (iii) is trivial. We consider a shortest odd cycle C in $B \cup F'$ and we may assume that it is neither contained in B nor in F' . The

vertices of $\{x, y, z\}$ traversed by C split it into at least two segments whose internal vertices avoid $\{x, y, z\}$. Consequently, every such segment lies completely in B or completely in F' and its endvertices form one of the pairs $\{x, y\}$, $\{y, z\}$, or $\{x, z\}$. Now we fix one segment and replace every other segment by the edge xy , the edge yz , or the 2-edge path $x-y-z$, respectively, all of which are available in both B and F' . This results in a closed walk, which is not longer than C and which is completely contained in F' or in B , and a parity check shows that the kept segment can be chosen such that this walk is odd.

It is left to show that $(B \cup F', \mathcal{B}_F \cup \{F'\})$ is n -induced. For that we first consider a copy $F_\star \in \mathcal{B}_F$ and two vertices $u, v \in V(F_\star)$ for which there exists a shortest u - v -path P in $B \cup F'$ of length at most n . Moreover, we can assume that P contains some edge from $E(F') \setminus E(B)$. In particular, P must contain two vertices from $\{x, y, z\}$ and owing to the minimality of P we infer that $V(P) \cap \{x, y, z\} = \{x, z\}$, since otherwise we locate a shorter path contained in B allowing us to appeal to $\mathcal{B}_F \subseteq \binom{B}{F}_n$. Consequently, the distance assumption yields the contradiction $|E(P)| > \text{dist}_{B-y}(x, F_\star) + \text{dist}_{B-y}(z, F_\star) \geq n + 1$.

It remains to consider the case when u, v are vertices of the added copy F' . However, in this case the reasoning above applies again: a shortest u - v -path of length at most n not contained in F' either admits a shortcut through one of the edges $xy, yz \in E(F')$, or it contains a segment from x to z avoiding y that passes through a vertex $w \in V(B) \setminus V(F')$ and, considering the copy F_\star containing w , has length at least

$$\text{dist}_{B-y}(x, F_\star) + \text{dist}_{B-y}(z, F_\star) \geq n + 1,$$

which is absurd. This concludes the proof of Lemma 3.5. \square

3.2. Conformity of cycles of copies. Next we consider special systems resembling the structure of a cycle of copies of a given graph F .

Definition 3.6 (cycles of copies). For $\ell \geq 3$ let $S = (V_S, E_S)$ be a graph with no isolated vertices and ℓ edges, which is either an ℓ -cycle or a family of pairwise vertex disjoint paths. Let $E_S = \{e_1, \dots, e_\ell\}$ be an enumeration of the edges such that only consecutive edges (modulo ℓ) share a vertex.

We say an F -system (Z, \mathcal{Z}_F) with $Z = \bigcup \mathcal{Z}_F$ is an S -cycle if there is an enumeration $\mathcal{Z}_F = \{F_1, \dots, F_\ell\}$ such that we have

- (i) $E(F_i) \cap E_S = \{e_i, e_{i+1}\}$ and $V(F_i) \cap V_S = e_i \cup e_{i+1}$ for all $i \in \mathbb{Z}/\ell\mathbb{Z}$
- (ii) and $V(F_i) \cap V(F_j) = (e_i \cup e_{i+1}) \cap (e_j \cup e_{j+1})$ for all distinct $i, j \in \mathbb{Z}/\ell\mathbb{Z}$.

For an S -cycle (Z, \mathcal{Z}_F) we refer to $S \subseteq Z$ as its *skeleton*.

It follows from part (ii) of this definition and the structural requirement on S that for all $i, j \in \mathbb{Z}/\ell\mathbb{Z}$ we have

$$V(F_i) \cap V(F_j) \neq \emptyset \implies j \in \{i - 2, i - 1, i, i + 1, i + 2\}.$$

Moreover, for $\ell \geq 5$ part (i) implies for all distinct $i, j \in \mathbb{Z}/\ell\mathbb{Z}$

$$|V(F_i) \cap V(F_j)| \geq 2 \implies j \in \{i - 1, i + 1\} \text{ and } V(F_i) \cap V(F_j) \in E_S.$$

In particular, for $\ell \geq 5$ such S -cycles have clean intersections. Moreover, for sufficiently large ℓ such systems will be n -conform.

Lemma 3.7. *For every integer $n \geq 1$ and for every S -cycle (Z, \mathcal{Z}_F) the following holds. If $|E(S)| \geq \max\{10, \min\{2\log(F) + 2, 4n + 4\}, 2n + 6\}$, then (Z, \mathcal{Z}_F) is n -conform.*

Proof. For the proof we assume that the skeleton S of the given S -cycle (Z, \mathcal{Z}_F) is a graph cycle of length $\ell = |E(S)|$. The proof for the case when S is a collection of paths is almost identical.

Let $E_S = \{e_1, \dots, e_\ell\}$ and $\mathcal{Z}_F = \{F_1, \dots, F_\ell\}$ be the enumerations exemplifying that (Z, \mathcal{Z}_F) is an S -cycle. Appealing to $\ell - 3$ applications of Fact 3.2 tells us that the F -system $(Z_{--}, \mathcal{Z}_F^{--})$ given by $\mathcal{Z}_F^{--} = \{F_1, \dots, F_{\ell-2}\}$ and $Z_{--} = \bigcup \mathcal{Z}_F^{--}$ is n -conform.

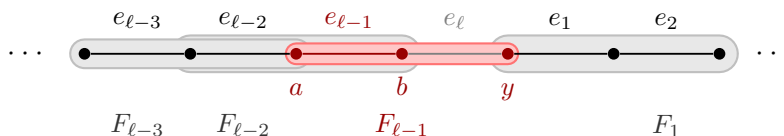


FIGURE 3.1. Adding $F_{\ell-1}$ to $(Z_{--}, \mathcal{Z}_F^{--})$ in the proof of Lemma 3.7.

For the extension of \mathcal{Z}_F^{--} by $F_{\ell-1}$ we shall appeal to Lemma 3.3 applied for the unique vertex y in $e_1 \cap e_\ell$ and the edge $ab = e_{\ell-1}$. For that we consider a copy F_i with $i \in [\ell - 2]$ and verify that it meets the distance assumption given in inequality (3.1). Since the vertex set of every edge e_j with $j \in \{2, \dots, i - 2\}$ separates the vertex y from the vertex set of F_i in Z_{--} , every y - F_i -path in Z_{--} has at least $(i - 3)/2$ internal vertices, i.e.,

$$\text{dist}_{Z_{--}}(y, F_i) \geq \frac{i - 1}{2}.$$

Applying the same reasoning for paths connecting F_i with $e_{\ell-1}$ in Z_{--} implies

$$\text{dist}_{Z_{--}}(y, F_i) + \text{dist}_{Z_{--}}(ab, F_i) \geq \frac{i - 1}{2} + \frac{\ell - 3 - i}{2} = \frac{\ell}{2} - 2$$

and the assumption on $\ell = |E(S)|$ yields inequality (3.1). Consequently, the F -system (Z_-, \mathcal{Z}_F^-) given by $Z_- = Z_{--} \cup F_{\ell-1}$ and $\mathcal{Z}_F^- = \{F_1, \dots, F_{\ell-1}\}$ is n -conform.

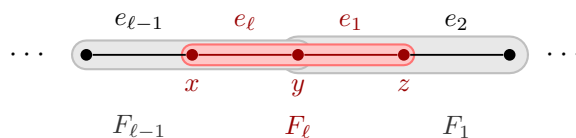


FIGURE 3.2. Closing the S -cycle in the proof of Lemma 3.7. The copy F_ℓ added last is drawn in red.

Finally, we add F_ℓ to (Z_-, \mathcal{Z}_F^-) (see Figure 3.2) and for that we shall appeal to Lemma 3.5 applied for the edges $e_\ell = xy$ and $e_1 = yz$. In fact, the distance assumption (3.2) follows by the same argument, since in $Z_- - y$ still the vertex set of every edge e_j with $j \in \{2, \dots, i-2\}$ separates z from the vertex set of the copy F_i . \square

3.3. Forcing canonical patterns from homogeneous colourings. For the proof of Proposition 2.5 we shall consider several constructions—one for each of the six ordered two-edge graphs. Roughly speaking, for $T \in \{\dots, \curvearrowright, \curvearrowleft, \dots\}$ we will add a family of copies of F such that if the edges of some copy of T in a homogeneous colouring have the same colour, then every copy of F will be monochromatic. Moreover, for T being \curvearrowright or \curvearrowleft it may result in a common relaxation of monochromatic or min- or max-coloured copies. However, this restriction will be strong enough to guarantee canonical copies in combination with the other cases (see proof of Proposition 2.5 at the end of this section). In the complementary case, when no two edges in F are monochromatic, the homogeneous colouring restricted to any copy of F must be rainbow.

We have to ensure that the final F -system is n -conform. However, the constructions for different pairs of edges can be carried out “independently” and we shall only consider unions of systems that intersect in a copy of F . Owing to Lemma 3.1 the union of all those systems will preserve n -conformity, if we ensure that each single system is n -conform.

3.3.1. Forcing equal colours with \dots and \curvearrowright . We start with the case $T = \dots$, which pivots on the following construction.

Definition 3.8. Let F_0 be a copy of an ordered graph F with edges $e, e' = x_1y_1 \in E(F_0)$ forming an ordered copy of \dots , i.e., $\max e < x_1 < y_1$. For an edge $f \in E(F_0) \setminus \{e, e'\}$ and an integer $k \geq 1$ we define the system $\mathcal{Z}_F^k(e, e'; f)$ on F_0 as follows:

$(\dots)_1$ Add new vertices x_2, \dots, x_{k+1} and y_2, \dots, y_{k+1} defining the edges $e_i = x_iy_i$ for $i \in \{2, \dots, k+1\}$ and obeying the ordering

$$\max V(F_0) < x_2 < y_2 < x_3 < y_3 < \dots < x_{k+1} < y_{k+1}.$$

$(\dots)_2$ Let $S = (V_S, E_S)$ be the graph with enumerated edge set E_S given by

$$E_S = \{e_0 = f, e_1 = e', e_2, e_3, \dots, e_{k+1}\}.$$

$(\dots)_3$ Let $(Z, \mathcal{Z}_F^k(e, e'; f))$ be an S -cycle with $\mathcal{Z}_F^k(e, e'; f) = \{F_0, F_1, \dots, F_{k+1}\}$ (see Figure 3.3), where for every $i \neq k+1$ the edges e_i and e_{i+1} are contained in F_i playing the rôles of e and e' in F_0 , while for F_{k+1} the edge $e_0 = f$ plays the rôle of e and e_{k+1} plays the rôle of e' . So, with an obvious notation, we ensure

$$(F_0, e, e') \cong (F_i, e_i, e_{i+1}) \cong (F_{k+1}, f, e_{k+1})$$

for every $i \in [k]$.

The ordering of the remaining vertices of the copies of F is not essential and any linear extension is suitable.

We also define the union of these systems over all suitable choices of f in the fixed ordered graph F_0 and set

$$\mathcal{B}_F^k(F_0, e, e') = \bigcup_{f \in E(F_0) \setminus \{e, e'\}} \mathcal{Z}_F^k(e, e'; f).$$

This is meant to be a free amalgamation over F_0 , i.e., for different edges $f, f' \in E(F_0) \setminus \{e, e'\}$ the cycles $\mathcal{Z}_F^k(e, e'; f)$ and $\mathcal{Z}_F^k(e, e'; f')$ are supposed to intersect only in the copy F_0 .

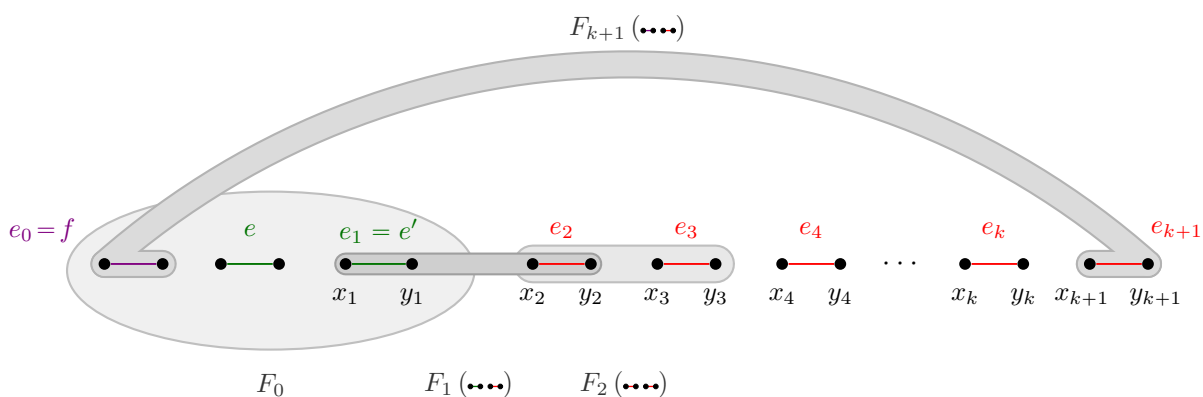


FIGURE 3.3. Construction of $\mathcal{Z}_F^k(e, e'; f)$ for e and e' forming a copy of \dots (see Definition 3.8) with the copies $F_0, F_1, F_2,$ and F_{k+1} shaded in grey. The coloured copies of \dots next to the labels indicate the rôles played by the corresponding pairs of skeleton edges.

The relevance of this construction is given by the following lemma.

Lemma 3.9. *For every ordered graph F and all integers $k, n \geq 1$ the following holds: If the edges e and e' form a monochromatic copy of \dots in a $\mathcal{B}_F^k(F_0, e, e')$ -homogeneous colouring $\varphi: E(B) \rightarrow \mathbb{N}$ for $B = \bigcup \mathcal{B}_F^k(F_0, e, e')$, then φ is monochromatic on every copy of F in $\mathcal{B}_F^k(F_0, e, e')$.*

Moreover, if we have $k \geq \max\{8, \min\{2\log(F), 4n + 2\}, 2n + 4\}$, then the F -system $(B, \mathcal{B}_F^k(F_0, e, e'))$ is n -conform.

Proof. Combining the assumption $\varphi(e) = \varphi(e')$ and invoking the $\mathcal{B}_F^k(F_0, e, e')$ -homogeneity restricted to $\mathcal{Z}_F^k(e, e'; f)$ on the skeleton S_f with edge set $\{e_0^f = f, e_1^f = e', e_2^f, \dots, e_{k+1}^f\}$ for a fixed edge f yields

$$\varphi(e) = \varphi(e') = \varphi(e_1^f) = \varphi(e_2^f) = \dots = \varphi(e_{k+1}^f) = \varphi(e_0^f) = \varphi(f).$$

Since $\mathcal{B}_F^k(F_0, e, e') = \bigcup_{f \in E(F_0) \setminus \{e, e'\}} \mathcal{Z}_F^k(e, e'; f)$, we infer that φ is monochromatic on F_0 , and by homogeneity, on every copy of F in $\mathcal{B}_F^k(F_0, e, e')$.

For the moreover-part, we note that the skeletons of the cycles $\mathcal{Z}_F^k(e, e'; f)$ have size $k + 2$ and, hence, Lemma 3.7 implies that all those cycles are n -conform. Since any two of these cycles in $\mathcal{B}_F^k(F_0, e, e')$ share only F_0 , Lemma 3.1 tells us that also $\mathcal{B}_F^k(F_0, e, e')$ is n -conform. \square

We remark that the proof above also shows that φ is monochromatic on all edges of B , i.e., every copy of F in $\mathcal{B}_F^k(F_0, e, e')$ is monochromatic with the same colour, but we make no use of this information.

Next we turn to the ordered matching of type \curvearrowright . In fact the corresponding construction of $\mathcal{Z}_F^k(e, e'; f)$ is almost identical and the only difference appears in the ordering of the additional vertices $x_2, y_2, \dots, x_{k+1}, y_{k+1}$. More precisely, given a copy of \curvearrowright by edges e and $e' = x_1 y_1$ with

$$x_1 < \min e < \max e < y_1,$$

and another edge f of F_0 , we replace $(\curvearrowright)_1$ in Definition 3.8 by the following:

$(\curvearrowright)_1$ Add new vertices x_2, \dots, x_{k+1} and y_2, \dots, y_{k+1} defining the edges $e_i = x_i y_i$ for $i \in \{2, \dots, k+1\}$ and obeying the ordering

$$x_{k+1} < \dots < x_3 < x_2 < \min V(F_0) < \max V(F_0) < y_2 < y_3 < \dots < y_{k+1}.$$

Having defined the cycle of copies $\mathcal{Z}_F^k(e, e'; f)$ this way, for e and e' fixed we again set

$$\mathcal{B}_F^k(F_0, e, e') = \bigcup_{f \in E(F_0) \setminus \{e, e'\}} \mathcal{Z}_F^k(e, e'; f).$$

This way we obtain a version of Lemma 3.9 for copies of \curvearrowright by an identical proof and we omit the details.

Lemma 3.10. *For every ordered graph F and all integers $k, n \geq 1$ the following holds: If the edges e and e' form a monochromatic copy of \curvearrowright in a $\mathcal{B}_F^k(F_0, e, e')$ -homogeneous colouring $\varphi: E(B) \rightarrow \mathbb{N}$ for $B = \bigcup \mathcal{B}_F^k(F_0, e, e')$, then φ is monochromatic on every copy of F in $\mathcal{B}_F^k(F_0, e, e')$.*

Moreover, if we have $k \geq \max\{8, \min\{2\log(F), 4n + 2\}, 2n + 4\}$, then the F -system $(B, \mathcal{B}_F^k(F_0, e, e'))$ is n -conform. \square

3.3.2. *Forcing equal colours with \curvearrowright .* Next we focus on the ordered matching of type \curvearrowright . As in the previous two matching cases we construct a cycle of copies.

Definition 3.11. Let F_0 be a copy of an ordered graph F whose edges $e, e' = x_1 y_1 \in E(F_0)$ form an ordered copy of \curvearrowright , i.e., $\min e < x_1 < \max e < y_1$. For an edge $f \in E(F_0) \setminus \{e, e'\}$ and an integer $k \geq 1$ we define the system $\mathcal{Z}_F^k(e, e'; f)$ on F_0 as follows:

(\curvearrowright)₁ Add new vertices x_2, \dots, x_{k+1} and y_2, \dots, y_{k+1} defining the edges $e_i = x_i y_i$ for $i \in \{2, \dots, k+1\}$ and obeying the ordering

$$x_1 < x_2 < y_1 \leq \max V(F_0) < x_3 < y_2 < x_4 < y_3 < \dots < x_{k+1} < y_k < y_{k+1}.$$

In addition we add the edge $e_{k+2} = x_{k+2} y_{k+2}$ satisfying

$$\min f < x_{k+2} < \max f \quad \text{and} \quad x_{k+1} < y_{k+2} < y_{k+1}.$$

(\curvearrowright)₂ Let $S = (V_S, E_S)$ be the graph with enumerated edge set E_S given by

$$E_S = \{e_0 = f, e_1 = e', e_2, e_3, \dots, e_{k+1}, e_{k+2}\}.$$

(\curvearrowright)₃ Let $(Z, \mathcal{Z}_F^k(e, e'; f))$ be an S -cycle with $\mathcal{Z}_F^k(e, e'; f) = \{F_0, F_1, \dots, F_{k+2}\}$ (see Figure 3.4), where for every $i \in [k]$ the edges e_i and e_{i+1} are contained in F_i playing the rôles of e and e' in F_0 while for F_{k+1} the edge e_{k+2} plays the rôle of e and e_{k+1} plays the rôle of e' , and for F_{k+2} the edge $f = e_0$ plays the rôle of e and e_{k+2} plays the rôle of e' .[†] In other words, for every $i \in [k]$ we ensure

$$(F_0, e, e') \cong (F_i, e_i, e_{i+1}) \cong (F_{k+1}, e_{k+2}, e_{k+1}) \cong (F_{k+2}, f, e_{k+2}).$$

Moreover, fix some linear extension of the partial order of the vertices defined above. Finally, we define the union of these systems over all suitable choices of f in the fixed ordered graph F_0 and set

$$\mathcal{B}_F^k(F_0, e, e') = \bigcup_{f \in E(F_0) \setminus \{e, e'\}} \mathcal{Z}_F^k(e, e'; f).$$

This construction yields an analogous variant of Lemmata 3.9 and 3.10 for \curvearrowright .

Lemma 3.12. *For every ordered graph F and all integers $k, n \geq 1$ the following holds: If the edges e and e' form a monochromatic copy of \curvearrowright in a $\mathcal{B}_F^k(F_0, e, e')$ -homogeneous colouring $\varphi: E(B) \rightarrow \mathbb{N}$ for $B = \bigcup \mathcal{B}_F^k(F_0, e, e')$, then φ is monochromatic on every copy of F in $\mathcal{B}_F^k(F_0, e, e')$.*

Moreover, if we have $k \geq \max\{8, \min\{2\log(F), 4n + 2\}, 2n + 4\}$, then the F -system $(B, \mathcal{B}_F^k(F_0, e, e'))$ is n -conform.

Proof. Owing to the assumption $\varphi(e) = \varphi(e')$, the $\mathcal{B}_F^k(F_0, e, e')$ -homogeneity of φ restricted to the skeleton S_f of $\mathcal{Z}_F^k(e, e'; f)$ with edge set $\{f = e_0^f, e' = e_1^f, \dots, e_{k+2}^f\}$ yields

$$\varphi(e) = \varphi(e') = \varphi(e_1^f) = \varphi(e_2^f) = \dots = \varphi(e_{k+2}^f) = \varphi(e_0^f) = \varphi(f).$$

Since $\mathcal{B}_F^k(F_0, e, e') = \bigcup_{f \in E(F_0) \setminus \{e, e'\}} \mathcal{Z}_F^k(e, e'; f)$, we infer that φ is monochromatic on F_0 and, by homogeneity, it is monochromatic on every copy of F in $\mathcal{B}_F^k(F_0, e, e')$.

[†]Comparing with the constructions for \dots and \curvearrowright , we note that the cycle $\mathcal{Z}_F^k(e, e'; f)$ for \curvearrowright consists of $k + 3$ copies of F (instead of $k + 2$ copies) and the rôles of the edges are “reversed” twice, for the copies F_{k+1} and F_{k+2} (instead of just once).

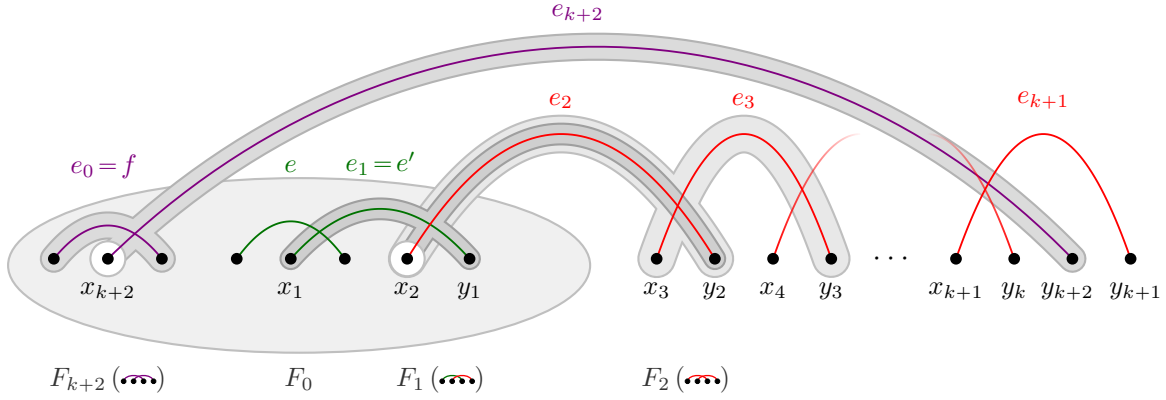


FIGURE 3.4. Construction of $\mathcal{Z}_F^k(e, e'; f)$ for e and e' forming a copy of \curvearrowright (see Definition 3.11) with F_0, F_1, F_2 , and F_{k+2} shaded in grey. The white circles indicate that x_2 and x_{k+2} are not vertices of F_0 and the coloured crossings next to the labels indicate the rôles played by the skeleton edges.

Since $(B, \mathcal{B}_F^k(F_0, e, e'))$ is the union of sufficiently long cycles of copies of F intersecting in the fixed copy F_0 , the conformity follows from Lemmata 3.1 and 3.7. \square

This concludes the discussion for the pairs of edges forming a matching and it remains to address the situation when the two edges share a vertex.

3.3.3. *Forcing equal colours with \curvearrowright .* The definition of the construction in this case is somewhat similar to that for \curvearrowright . Roughly speaking, the following construction arises from Definition 3.11 by identifying x_{i+1} with y_i for every $i \in [k]$ and, in addition, $y_k = x_{k+1} = y_{k+2}$ and $\max f = x_{k+2}$. However, due to these identifications the resulting system is not a cycle of copies in the sense of Definition 3.6 and we therefore omit the second step in the construction below.

Definition 3.13. Let F_0 be a copy of an ordered graph F with edges $e, e' = x_1x_2 \in E(F_0)$ forming an ordered copy of \curvearrowright , i.e., $\max e = x_1 < x_2$. For an edge $f \in E(F_0) \setminus \{e, e'\}$ and an integer $k \geq 1$ we define the system $\mathcal{Y}_F^k(e, e'; f)$ on F_0 as follows:

- (\curvearrowright)₁ Add new vertices x_3, \dots, x_{k+2} and edges $e_i = x_i x_{i+1}$ for $i \in \{2, \dots, k+1\}$ obeying the ordering

$$\max V(F_0) < x_3 < x_4 < \dots < x_{k+1} < x_{k+2}.$$

We also add the edge e_{k+2} joining $\max f$ with x_{k+1} and we set $e_0 = f$ and $e_1 = e'$.

- (\curvearrowright)₃ Let $\mathcal{Y}_F^k(e, e'; f) = \{F_0, F_1, \dots, F_{k+1}, F_{k+2}\}$ be a system of copies of F (see Figure 3.5), where for every $i \in [k]$ the edges e_i and e_{i+1} are contained in F_i playing the rôles of e and e' in F_0 , while for F_{k+1} the edge e_{k+2} plays the rôle of e and e_{k+1}

plays the rôle of e' , and for F_{k+2} the edge $f = e_0$ plays the rôle of e and e_{k+2} plays the rôle of e' . So we guarantee

$$(F_0, e, e') \cong (F_i, e_i, e_{i+1}) \cong (F_{k+1}, e_{k+2}, e_{k+1}) \cong (F_{k+2}, f, e_{k+2})$$

for all $i \in [k]$. Moreover, besides the necessary intersections given above, all those copies are as vertex disjoint as possible.

The ordering of the remaining vertices of the copies of F is not essential and any linear extension is suitable.

We also define the union of these systems over all suitable choices of f in the fixed ordered graph F_0 and set

$$\mathcal{B}_F^k(F_0, e, e') = \bigcup_{f \in E(F_0) \setminus \{e, e'\}} \mathcal{Y}_F^k(e, e'; f).$$

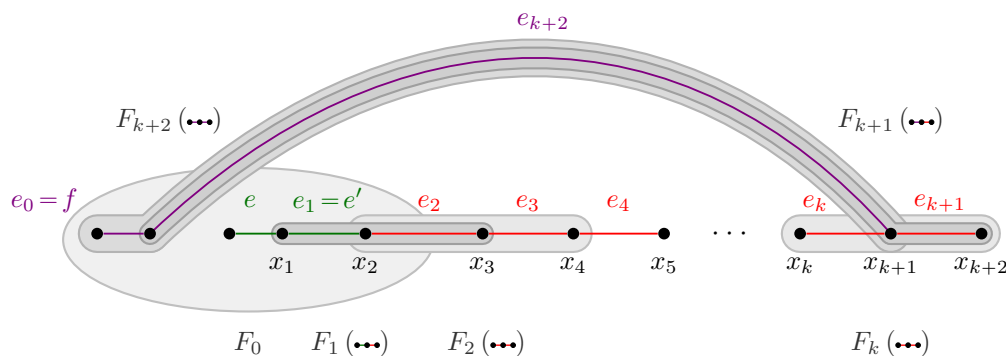


FIGURE 3.5. Construction of $\mathcal{Y}_F^k(e, e'; f)$ for e and e' forming a copy of $\bullet\bullet\bullet$ (see Definition 3.13) with the copies $F_0, F_1, F_2, F_k, F_{k+1}$, and F_{k+2} shaded in grey. The coloured copies of $\bullet\bullet\bullet$ next to the labels indicate the rôles played by the corresponding pairs of skeleton edges.

Even though the construction above is not a cycle of copies of F , this still yields an n -conform F -system for sufficiently large k . In fact we shall derive the following lemma for this construction.

Lemma 3.14. *For every ordered graph F and all integers $k, n \geq 1$ the following holds: If the edges e and e' form a monochromatic copy of $\bullet\bullet\bullet$ in a $\mathcal{B}_F^k(F_0, e, e')$ -homogeneous colouring $\varphi: E(B) \rightarrow \mathbb{N}$ for $B = \bigcup \mathcal{B}_F^k(F_0, e, e')$, then φ is monochromatic on every copy of F in $\mathcal{B}_F^k(F_0, e, e')$.*

Moreover, if we have $k \geq \max \{9, \min \{2\log(F) + 1, 4n + 3\}, 2n + 5\}$, then the F -system $(B, \mathcal{B}_F^k(F_0, e, e'))$ is n -conform.

Proof. The proof of the first part is identical to that part of the proof of Lemma 3.12. In fact, owing to the assumption $\varphi(e) = \varphi(e')$ the fact that φ is $\mathcal{B}_F^k(F_0, e, e')$ -homogeneous restricted to the set $\{f = e_0^f, e' = e_1^f, \dots, e_{k+2}^f\}$ in $\mathcal{Y}_F^k(e, e'; f)$ yields

$$\varphi(e) = \varphi(e') = \varphi(e_1^f) = \varphi(e_2^f) = \dots = \varphi(e_{k+2}^f) = \varphi(e_0^f) = \varphi(f).$$

Since $\mathcal{B}_F^k(F_0, e, e') = \bigcup_{f \in E(F_0) \setminus \{e, e'\}} \mathcal{Y}_F^k(e, e'; f)$, we infer that φ is monochromatic on F_0 and, by homogeneity, it is monochromatic on every copy of F in $\mathcal{B}_F^k(F_0, e, e')$.

For the proof of the moreover-part we first note that in view of Lemma 3.1 and the definition of $\mathcal{B}_F^k(F_0, e, e')$ it suffices to establish the conformity of $\mathcal{Y}_F^k(e, e'; f)$. Even though $\mathcal{Y}_F^k(e, e'; f)$ is not a cycle of copies, below we shall parallel the proof of Lemma 3.7 to derive the conformity with an adjusted choice of k .

Let $f \in E(F_0) \setminus \{e, e'\}$ be fixed. Clearly, $\{F_0\}$ is an n -conform F -system. Owing to the construction in step $(\bullet \leftrightarrow)_3$ in Definition 3.13, we note that by adding F_i sequentially for every $i \in [k]$ we can invoke Fact 3.2. Consequently, the system $\{F_0, \dots, F_k\}$ is n -conform.

Next we employ Lemma 3.3 to extend $\{F_0, \dots, F_k\}$ by F_{k+1} , where $ab = e_{k+1}$ and $y = \max f$. For that we appeal to the assumed lower bound on k , to verify the required distance assumption (3.1) of the lemma in a similar way as in the proof of Lemma 3.7. We remark that the worst case for the assumption on k occurs when the vertex $\max f$ coincides with x_2 .

Finally, we extend $\{F_0, \dots, F_k, F_{k+1}\}$ by F_{k+2} in a similar way by appealing to Lemma 3.5 applied to the edges f and e_{k+2} , which share the vertex $\max f$. \square

3.3.4. *Forcing non-strict patterns with \curvearrowright and \curvearrowleft .* In the remaining case we consider monochromatic pairs of edges isomorphic to \curvearrowright or \curvearrowleft and by symmetry it will be sufficient to address it for \curvearrowright . While the general strategy is very similar, there are two main differences: firstly, the construction yields a family of copies more resembling the structure of a *star* instead of a cycle (see the construction in Definition 3.15 below). Secondly, homogeneous colourings of the system may result in non-strict min-coloured copies of F , which is a common relaxation of monochromatic and min-coloured copies. More precisely we relax the equivalence in the definition of min-colourings and we say a colouring $\varphi: E(F) \rightarrow \mathbb{N}$ is *non-strictly min-coloured* if for all edges $e, e' \in E(F)$ we have

$$\min e = \min e' \implies \varphi(e) = \varphi(e').$$

Roughly speaking, in non-strictly min-coloured copies of F all stars whose centres are smaller than all their leaves are monochromatic, but some of these stars may use the same colour. The notion of *non-strictly max-coloured* colourings is defined analogously (with \max in place of \min).

Definition 3.15. Let F_0 be a copy of an ordered graph F with edges $e_0 = yx_0$ and $e_1 = yx_1 \in E(F_0)$ forming an ordered copy of \blacktriangleright , i.e., $y < x_0 < x_1$. For another copy of \blacktriangleright given by $f_0 = ab_0$, $f_1 = ab_1 \in E(F_0)$ with $a < b_0 < b_1$ and an integer $k \geq 1$ we define the system $\mathcal{S}_F^k(e_0, e_1; f_0, f_1)$ on F_0 as follows:

(\blacktriangleright)₁ Add new vertices x_2, \dots, x_{k+1} and edges $e_{i+1} = yx_{i+1}$ for $i \in [k]$ obeying the ordering

$$\max V(F_0) < x_2 < \dots < x_{k+1}.$$

(\blacktriangleright)₃ Let $\mathcal{S}_F^k(e_0, e_1; f_0, f_1) = \{F_0, F_1, \dots, F_{k+1}\}$ be a system of copies of F where for every $i \in [k]$ the edges e_i and e_{i+1} are contained in F_i playing the rôles of e_0 and e_1 in F_0 while for F_{k+1} the edge e_0 plays the rôle of f_0 and e_{k+1} plays the rôle of f_1 . In other words, we ensure

$$(F_0, e_0, e_1) \cong (F_i, e_i, e_{i+1}) \text{ for every } i \in [k] \text{ and } (F_{k+1}, e_0, e_{k+1}) \cong (F_0, f_0, f_1).$$

Moreover, besides the necessary intersections given above, all those copies are as vertex disjoint as possible.

The ordering of the remaining vertices of the copies of F is not essential and any linear extension is suitable.

We also define the union of these systems over all choices of copies of \blacktriangleright given by suitable edges $f_0 = ab_0$ and $f_1 = ab_1$ in the fixed ordered graph F_0 and set

$$\mathcal{B}_F^k(F_0, e_0, e_1) = \bigcup_{\substack{ab_0, ab_1 \in E(F_0) \\ a < b_0 < b_1}} \mathcal{S}_F^k(e_0, e_1; ab_0, ab_1).$$

As before, the construction is not a cycle of copies of F in the sense of Definition 3.6, but for sufficiently large k it still yields an n -conform F -system:

Lemma 3.16. *For every ordered graph F and all integers $k, n \geq 1$ the following holds: If the edges e_0 and e_1 form a monochromatic copy of \blacktriangleright in a $\mathcal{B}_F^k(F_0, e_0, e_1)$ -homogeneous colouring $\varphi: E(B) \rightarrow \mathbb{N}$ for $B = \bigcup \mathcal{B}_F^k(F_0, e_0, e_1)$, then φ is non-strictly min-coloured on every copy of F in $\mathcal{B}_F^k(F_0, e_0, e_1)$.*

Moreover, if we have $k \geq \max\{3, n + 1\}$, then the F -system $(B, \mathcal{B}_F^k(F_0, e_0, e_1))$ is n -conform.

Proof. Owing to the homogeneity of φ we may assume that the monochromatic copy of \blacktriangleright is contained in F_0 , say on edges $e_0 = yx_0$ and $e_1 = yx_1$ with $y < x_0 < x_1$. Note that being non-strictly min-coloured is equivalent to the property that all copies of \blacktriangleright in F_0 are monochromatic.

For any given pair of edges $f_0 = ab_0$ and $f_1 = ab_1$ in F_0 with $a < b_0 < b_1$ we consider the system $\mathcal{S}_F^k(e_0, e_1; f_0, f_1)$. It follows from the construction in Definition 3.15 combined

with the homogeneity of φ that all edges e_0, \dots, e_{k+1} in $\mathcal{S}_F^k(e_0, e_1; f_0, f_1)$ have the same colour. In particular, the pair of edges playing the rôles of f_0 and f_1 in the $(k+1)$ -st copy of F in $\mathcal{S}_F^k(e_0, e_1; f_0, f_1)$ have the same colour. Appealing once more to the homogeneity of φ , this implies $\varphi(f_0) = \varphi(f_1)$. Therefore, φ is non-strictly min-coloured on F_0 and, by homogeneity, on every copy of F in $\mathcal{B}_F^k(F_0, e_0, e_1)$.

For the moreover-part we again exploit Lemma 3.1 and the definition of $\mathcal{B}_F^k(F_0, e_0, e_1)$, which reduces the conformity of $\mathcal{B}_F^k(F_0, e_0, e_1)$ to the conformity of $\mathcal{S}_F^k(e_0, e_1; f_0, f_1)$. For that we start with $\{F_0\}$, which clearly is an n -conform F -system. Owing to the construction in step $(\heartsuit)_3$ in Definition 3.15, we note that by adding F_i sequentially for every $i \in [k]$ we can invoke Fact 3.2. Consequently, the system $\{F_0, \dots, F_k\}$ is n -conform.

Finally, we extend the system $\{F_0, \dots, F_k\}$ by F_{k+1} by applying Lemma 3.5 over the edges $e_0 = yx_0$ and $e_{k+1} = yx_{k+1}$, which share the vertex y . For the verification of the required distance assumption (3.2) fix two vertices $u, v \in V(F_i)$ for some $i \in \{0, \dots, k\}$. Observe that every u - x_0 -path avoiding y must contain the vertices x_0, x_1, \dots, x_i . Consequently, it has length at least i . Similarly, every v - x_{k+1} -path avoiding y must pass through the vertices $x_{i+1}, \dots, x_k, x_{k+1}$ and has length at least $k - i$. Consequently, for $S' = F_0 \cup \dots \cup F_k$ the lower bound on k yields

$$\text{dist}_{S'-y}(x_0, F_i) + \text{dist}_{S'-y}(x_{k+1}, F_i) \geq k \geq \max\{3, n+1\}$$

and, hence, $\mathcal{S}_F^k(e_0, e_1; f_0, f_1)$ is n -conform by Lemma 3.5. \square

We close this section with a final construction dealing with the situation when both a copy of \heartsuit and a copy of \spadesuit in F_0 are monochromatic. In this situation, by alternating between both types of cherries, we can transfer the colour to other edges of F , since the centres of the stars are no longer fixed. Consequently, we can enforce monochromatic copies of F again (see Lemma 3.18 below).

Definition 3.17. Let F_0 be a copy of an ordered graph F with edges $e_0 = x_0x_1$, $e_1 = x_0x_2$, $e'_0 = x'_0x'_1$, and $e'_1 = x'_0x'_2 \in E(F_0)$ such that e_0, e_1 form an ordered copy of \heartsuit , and e'_0, e'_1 form a copy of \spadesuit , meaning $x_0 < x_1 < x_2$ and $x'_2 < x'_1 < x'_0$. Additionally, for an edge $f \in E(F_0) \setminus \{e_0, e_1, x_1x_2\}$ with $\min f \neq x_0$ and an integer $k \geq 1$ we define the system $\mathcal{L}_F^k(e_0, e_1, e'_0, e'_1; f)$ on F_0 as follows:

- $(\heartsuit)_1$ Fix a vertex y_1 in $\{x_1, x_2\} \setminus f$ and add new vertices y_2, \dots, y_{2k-1} obeying the ordering

$$y_{2k-2} < \dots < y_4 < y_2 < \min V(F_0) < y_1 \leq \max V(F_0) < y_3 < y_5 < \dots < y_{2k-1}.$$

Moreover, we introduce edges $\tilde{e}_1 = x_0y_1$ (which coincides with either e_0 or e_1), $\tilde{e}_{i+1} = y_iy_{i+1}$ for $i \in [2k-2]$, \tilde{e}_{2k} joining $\min f$ with y_{2k-1} , and $\tilde{e}_{2k+1} = f$.

- $(\spadesuit)_2$ Let S be the graph with enumerated edge set $E_S = \{\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_{2k-1}, \tilde{e}_{2k}, \tilde{e}_{2k+1}\}$.

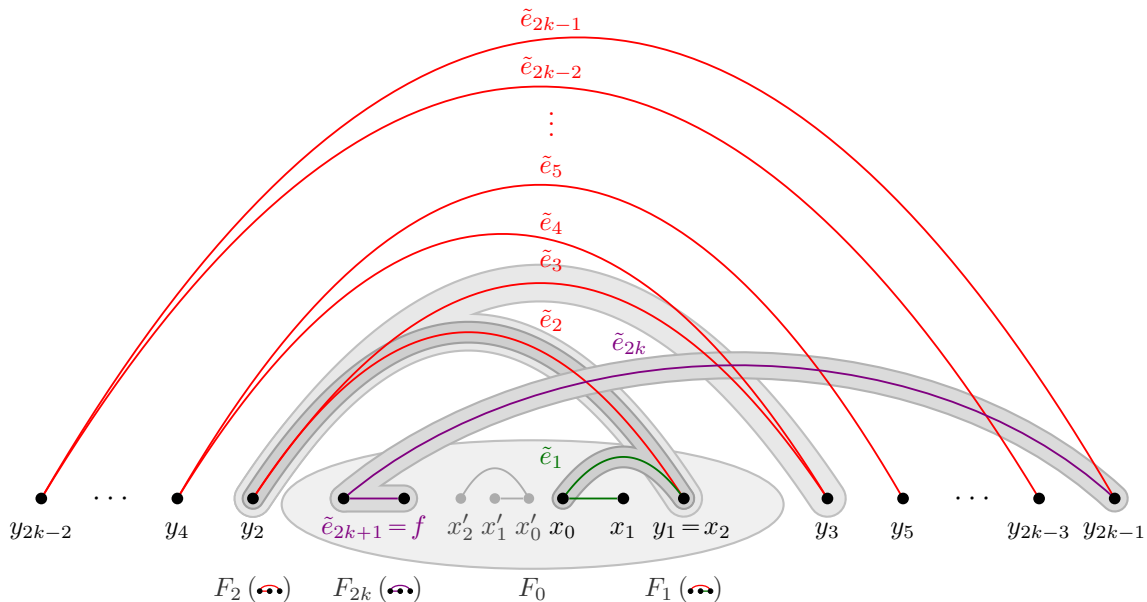


FIGURE 3.6. Construction of $\mathcal{Z}_F^k(e_0, e_1, e'_0, e'_1; f)$ with the copies $F_0, F_1, F_2,$ and F_{2k} shaded in grey. The coloured cherries next to their labels indicate the rôles played by the corresponding pairs of consecutive skeleton edges and here $y_1 = x_2$ is a suitable choice.

(\leftarrow)₃ Let $\{F_1, \dots, F_{2k}\}$ be a system of copies of F where for every $i \in [2k - 2]$ the edges \tilde{e}_i and \tilde{e}_{i+1} are contained in F_i playing the rôles of

- e'_0 and e'_1 in F_0 for odd i
- and e_0 and e_1 in F_0 for even i ,

while for F_{2k-1} the edges \tilde{e}_{2k-1} and \tilde{e}_{2k} play the rôles of e'_1 and e'_0 , and for F_{2k} we have that \tilde{e}_{2k} and f play the rôles of e_1 and e_0 . In other words, we ensure

$$(F_0, e_0, e_1) \cong (F_{2i}, \tilde{e}_{2i}, \tilde{e}_{2i+1}) \cong (F_{2k}, f, \tilde{e}_{2k}),$$

and

$$(F_0, e'_0, e'_1) \cong (F_{2i-1}, \tilde{e}_{2i-1}, \tilde{e}_{2i}) \cong (F_{2k-1}, \tilde{e}_{2k}, \tilde{e}_{2k-1})$$

for all $i \in [k - 1]$. Moreover, besides the necessary intersections given above, all those copies are as vertex disjoint as possible from each other and from F_0 . Finally, note that adding F_0 to the family $\{F_1, \dots, F_{2k}\}$ defines an S -cycle, which we denote by $\mathcal{Z}_F^k(e_0, e_1, e'_0, e'_1; f)$ (see Figure 3.6).

The ordering of the remaining vertices of the copies of F is not essential and any linear extension is suitable.

We also define the union of these systems over all suitable edges $f = ab$ in the fixed ordered graph F_0 , i.e., $f \notin \{e_0 = x_0x_1, e_1 = x_0x_2, x_1x_2\}$ and $\min f \neq x_0$ and set

$$\mathcal{B}_F^k(F_0, e_0, e_1, e'_0, e'_1) = \bigcup_{\substack{ab \in E(F_0) \\ a < b}} \mathcal{Z}_F^k(e_0, e_1, e'_0, e'_1; ab).$$

Lemma 3.18. *For every ordered graph F and all integers $k, n \geq 1$ the following holds: If the edges e_0 and e_1 form a monochromatic copy of \curvearrowright , and the edges e'_0 and e'_1 form a monochromatic copy of \curvearrowleft in a $\mathcal{B}_F^k(F_0, e_0, e_1, e'_0, e'_1)$ -homogeneous colouring $\varphi: E(B) \rightarrow \mathbb{N}$ for the graph $B = \bigcup \mathcal{B}_F^k(F_0, e_0, e_1, e'_0, e'_1)$, and if φ is in addition non-strictly min- and non-strictly max-coloured on F_0 , then φ is monochromatic on every copy of F from $\mathcal{B}_F^k(F_0, e_0, e_1, e'_0, e'_1)$.*

Moreover, if $k \geq \max\{5, \min\{\text{og}(F) + 1, 2n + 2\}, n + 3\}$, then the resulting F -system $(B, \mathcal{B}_F^k(F_0, e_0, e_1, e'_0, e'_1))$ is n -conform.

Proof. Owing to the homogeneity of φ we may assume that the monochromatic copies of \curvearrowright and \curvearrowleft are contained in F_0 , say on edges $e_0 = x_0x_1$ and $e_1 = x_0x_2$ with $x_0 < x_1 < x_2$, and $e'_0 = x'_0x'_1$ and $e'_1 = x'_0x'_2$ with $x'_2 < x'_1 < x'_0$.

For any suitable edge $f = ab$ in F_0 with $a < b$ we consider the system $\mathcal{Z}_F^k(e_0, e_1, e'_0, e'_1; f)$. It follows from the construction in Definition 3.17 combined with the homogeneity of φ that all edges $e_0, \tilde{e}_1, \dots, \tilde{e}_{2k}, \tilde{e}_{2k+1} = f$ have the same colour, since any consecutive pair either forms a copy of e_0, e_1 or a copy of e'_0, e'_1 in some copy of F from $\mathcal{Z}_F^k(e_0, e_1, e'_0, e'_1; f)$. In particular, this implies $\varphi(e_0) = \varphi(f)$ for every suitable edge f .

For the remaining cases, when $\min f = x_0$ or $f = x_1x_2$ (which are excluded in Definition 3.17) we appeal to the additional assumption: if $\min f = x_0$, then $\varphi(f) = \varphi(e_0)$, since φ is non-strictly min-coloured, and for $f = x_1x_2$ we have $\max f = x_2 = \max e_1$ and the non-strictly max-coloured assumption yields $\varphi(f) = \varphi(e_1) = \varphi(e_0)$. Consequently, φ is monochromatic on F_0 and, by homogeneity, on every copy of F in $\mathcal{B}_F^k(F_0, e_0, e_1, e'_0, e'_1)$.

For the moreover-part, note that $\mathcal{Z}_F^k(e_0, e_1, e'_0, e'_1; f)$ are S -cycles and that their skeletons (as described in $(\curvearrowright)_2$) have size $|E(S)| = 2k + 1$ for any fixed edge f . Hence, Lemma 3.7 implies that all those cycles are n -conform. Since any two of these cycles in $\mathcal{B}_F^k(F_0, e_0, e_1, e'_0, e'_1)$ share only F_0 , Lemma 3.1 tells us that also $\mathcal{B}_F^k(F_0, e_0, e_1, e'_0, e'_1)$ is n -conform. \square

We close this section with the proof of Proposition 2.5, which is a simple consequence of Lemmata 3.9, 3.10, 3.12, 3.14, 3.16, and 3.18.

Proof of Proposition 2.5. If $e(F) \leq 1$, then every colouring of a copy of F is canonical and the F -system $(F, \{F\})$ has all required properties. Hence, we may assume $e(F) \geq 2$.

Given F and $n \geq 1$, let k be sufficiently large so that we can appeal to the lemmata from this section. In fact, in that direction Lemma 3.14 is most restrictive and we set

$$k = \max \{9, \min\{2\log(F) + 1, 4n + 3\}, 2n + 5\}.$$

We fix some copy F_0 of F . If F_0 contains both a copy of \curvearrowright on edges e_0, e_1 and a copy of \curvearrowleft on edges e'_0, e'_1 , then let

$$\mathcal{B}_F(\curvearrowright, \curvearrowleft) = \mathcal{B}_F^k(F_0, e_0, e_1, e'_0, e'_1),$$

which is provided by Definition 3.17, and otherwise $\mathcal{B}_F(\curvearrowright, \curvearrowleft) = \emptyset$. Next we define the F -system

$$\mathcal{B}_F = \bigcup \{ \mathcal{B}_F^k(F_0, e, e') : e, e' \in E(F_0) \text{ and } e \neq e' \} \cup \mathcal{B}_F(\curvearrowright, \curvearrowleft),$$

where we appeal to the other constructions for all $T \in \{\dots, \curvearrowright, \curvearrowleft, \dots, \curvearrowright, \curvearrowleft\}$ laid out in this section. Since F_0 is fixed, it follows from Lemma 3.1 combined with Lemmata 3.9, 3.10, 3.12, 3.14, 3.16, and 3.18 that the F -system (B, \mathcal{B}_F) with $B = \bigcup \mathcal{B}_F$ is n -conform.

It remains to consider a \mathcal{B}_F -homogeneous colouring $\varphi: E(B) \rightarrow \mathbb{N}$. Obviously, if no two edges in F_0 have the same colour, then φ is rainbow on F_0 and we are done. Hence, we assume there are two edges e and e' in F_0 such that $\varphi(e) = \varphi(e')$.

In case e and e' form a copy from $\{\dots, \curvearrowright, \curvearrowleft, \dots\}$, then φ is monochromatic on F_0 owing to the corresponding lemma among Lemmata 3.9, 3.10, 3.12, and 3.14.

In the remaining case, we may assume that F_0 contains a monochromatic copy of \curvearrowright (the case of a monochromatic copy of \curvearrowleft is symmetric) and in view of Lemma 3.16 the colouring φ is, possibly non-strictly, min-coloured. We shall show that either F_0 is in fact (strictly) min-coloured or it is monochromatic.

Suppose F_0 is not min-coloured. This gives rise to two edges f and f' of the same colour, which do not form a copy of \curvearrowright . From the discussion above we can also assume that f and f' do not form a copy from the set $\{\dots, \curvearrowright, \curvearrowleft, \dots\}$ and, therefore, they form a copy of \curvearrowright . Consequently, F_0 contains a monochromatic copy of \curvearrowright and a monochromatic copy of \curvearrowleft . Therefore, by Lemma 3.16 the colouring φ is non-strictly min- and non-strictly max-coloured on F_0 . Moreover, the system $\mathcal{B}_F(\curvearrowright, \curvearrowleft) \neq \emptyset$ and, therefore, we conclude with Lemma 3.18 that φ is monochromatic on F_0 . \square

§4 THE HALES–JEWETT CONSTRUCTION

Our proof of Proposition 2.6 is based on the *partite construction method* introduced by Nešetřil and Rödl [25], which has found many applications since then (see, e.g., [17, 29] and the references therein). This method allows us to decompose the proof of Proposition 2.6 into several easier problems. One of these subproblems, treated in the present section, deals with partite structures. Therefore all systems and conglomerates appearing in this

section will be “partite” in some sense, while vertex orderings will be irrelevant for now. In fact, the first moment they become relevant again is in §5.3.

The first partite lemma we obtain will be derived from the Hales–Jewett theorem [15], which we recall below. Roughly speaking, it asserts that for every set A and every $r \in \mathbb{N}$ every r -colouring of a sufficiently high-dimensional discrete space over A has a monochromatic line. It will be convenient to identify for every set A and every dimension $n \in \mathbb{N}$ the Cartesian power A^n with the set of all functions from $[n]$ to A .

Definition 4.1. Given a set A and a dimension n let $[n] = C \cup M$ be a partition with $M \neq \emptyset$ and let $g: C \rightarrow A$ be a function. Defining $\bar{g}: [n] \times A \rightarrow A$ by

$$\bar{g}(i, a) = \begin{cases} g(i) & \text{if } i \in C \\ a & \text{if } i \in M \end{cases}$$

we call the map

$$\eta_g: A \rightarrow A^n, \quad a \mapsto (\bar{g}(1, a), \dots, \bar{g}(n, a))$$

a *combinatorial embedding* and its image $L_g = \eta_g[A]$ the associated *combinatorial line*.

We think of the elements of C as “constant coordinates”, while the coordinates in M are “moving”. For instance, if $A = [3]$, $n = 4$, and $M = \{2, 4\}$, then $L_g = \{2111, 2212, 2313\}$ is the line corresponding to the function $g: \{1, 3\} \rightarrow [3]$ defined by $g(1) = 2$ and $g(3) = 1$.

Theorem 4.2 (Hales & Jewett). *For every finite set A and every number of colours r there exists a dimension n such that for every colouring $\varphi: A^n \rightarrow [r]$ there exists a monochromatic combinatorial line.* \square

Let us fix any (not necessarily ordered) graph F . By an *F -partite graph* we mean a graph L together with a distinguished partition

$$V(L) = \bigcup_{x \in V(F)} V_x(L)$$

of its vertex set into independent sets such that for every edge $v_x v_y$ of L with $v_x \in V_x(L)$ and $v_y \in V_y(L)$ the pair xy is an edge of F . In other words, the projection from L to F sending each vertex class $V_x(L)$ to x is demanded to be a graph homomorphism.

When viewing F itself as an F -partite graph we always have the partition $V_x(F) = \{x\}$ in mind. For two F -partite graphs L and N the former is said to be an *F -partite subgraph* of the latter if it is a subgraph in the ordinary sense and, moreover, $V_x(L) \subseteq V_x(N)$ holds for every $x \in V(F)$. We write $\binom{N}{L}^{\text{pt}}$ for the set of all F -partite copies of L in N , i.e., for the set of all induced F -partite subgraphs of N which are isomorphic to L as F -partite graphs. In particular, this defines $\binom{N}{F}^{\text{pt}}$ for every F -partite graph N .

By a *partite F -system* we mean a pair (L, \mathcal{L}_F) consisting of an F -partite graph L and a subset $\mathcal{L}_F \subseteq \binom{L}{F}^{\text{pt}}$. In other words, an F -system (L, \mathcal{L}_F) is called a partite F -system if both L and \mathcal{L}_F come with an F -partite structure. As expected, (L, \mathcal{L}_F) is called an *induced partite F -subsystem* of another partite F -system (N, \mathcal{N}_F) if

- L is an induced F -partite subgraph of N
- and $\mathcal{L}_F = \mathcal{N}_F \cap \binom{L}{F}^{\text{pt}}$.

Furthermore we write $\left(\binom{(N, \mathcal{N}_F)}{(L, \mathcal{L}_F)}\right)^{\text{pt}}$ for the collection of the induced partite F -subsystems of (N, \mathcal{N}_F) which are—in an obvious sense—isomorphic to (L, \mathcal{L}_F) . Finally, a *partite (L, \mathcal{L}_F) -conglomerate* is an (L, \mathcal{L}_F) -conglomerate $(N, \mathcal{N}_L, \mathcal{N}_F)$ such that $\mathcal{N}_L \subseteq \left(\binom{(N, \mathcal{N}_F)}{(L, \mathcal{L}_F)}\right)^{\text{pt}}$.

The *Hales–Jewett construction*, which we describe in steps (1)–(5) below, associates to every partite F -system (L, \mathcal{L}_F) and every number of colours r a partite (L, \mathcal{L}_F) -conglomerate $\text{HJ}_r(L, \mathcal{L}_F) = (N, \mathcal{N}_L, \mathcal{N}_F)$ such that $\mathcal{N}_L \rightarrow (L, \mathcal{L}_F)_r^F$.

In the degenerate case $|\mathcal{L}_F| \leq 1$ we simply set $(N, \mathcal{N}_F) = (L, \mathcal{L}_F)$ and $\mathcal{N}_L = \{(L, \mathcal{L}_F)\}$. Assuming $|\mathcal{L}_F| \geq 2$ from now on, we intend to satisfy the desired partition relation, roughly speaking, by taking (N, \mathcal{N}_F) to be a sufficiently high-dimensional power of (L, \mathcal{L}_F) and letting \mathcal{N}_L correspond to the combinatorial lines in the associated discrete space over \mathcal{L}_F .

- (1) Let n denote the dimension delivered by Theorem 4.2 applied to $A = \mathcal{L}_F$ and r . In other words, n is chosen so large that for every r -colouring of \mathcal{L}_F^n there exists a monochromatic combinatorial line.
- (2) The vertex classes of the new graph N are the n^{th} Cartesian powers of the corresponding vertex classes of L , i.e., we set $V_x(N) = V_x(L)^n$ for every $x \in V(F)$.
- (3) Similarly, the edges of N are defined such that the entire graph N is, in the following sense, a Cartesian power of L . Two vertices (u_1, \dots, u_n) and (v_1, \dots, v_n) of N are declared to be adjacent if $u_i v_i \in E(L)$ holds for every $i \in [n]$. In particular, N is indeed an F -partite graph.
- (4) The set \mathcal{N}_F is constructed together with a *canonical bijection* $\lambda: \mathcal{L}_F^n \rightarrow \mathcal{N}_F$. Given an n -tuple $\vec{F} = (F_1, \dots, F_n) \in \mathcal{L}_F^n$ we need to specify a corresponding copy $\lambda(\vec{F}) \in \binom{N}{F}^{\text{pt}}$, which we intend to put into \mathcal{N}_F . To this end we denote for all pairs $(i, x) \in [n] \times V(F)$ the unique vertex in $V(F_i) \cap V_x(L)$ by u_{ix} . This defines the vertex $(u_{1x}, \dots, u_{nx}) \in V_x(L)^n$ and we define $\lambda(\vec{F})$ to have the vertex set $\{(u_{1x}, \dots, u_{nx}): x \in V(F)\} \subseteq V(N)$. The definition of $E(L)$ in Step (3) immediately implies that this set induces a partite copy of F in N and justifies the definition $\mathcal{N}_F = \{\lambda(\vec{F}): \vec{F} \in \mathcal{L}_F^n\}$.
- (5) Finally, we implement the idea that the copies in \mathcal{N}_L should correspond to combinatorial lines. Resuming the notation of Definition 4.1 we suppose that $\eta: \mathcal{L}_F \rightarrow \mathcal{L}_F^n$ denotes the combinatorial embedding corresponding to the partition $[n] = C \cup M$ with $M \neq \emptyset$ and the function $g: C \rightarrow \mathcal{L}_F$. For each $i \in C$ the graph $g(i)$ is

a partite copy F_i of F in L and thus its vertex set can be written in the form $V(g(i)) = \{u_{ix} : x \in V(F)\}$ with $u_{ix} \in V_x(L)$ for every $x \in V(F)$. These vertices can be used to define combinatorial embeddings $\eta_x : V_x(L) \rightarrow V_x(N)$ associated again to the partition $[n] = C \cup M$ and to the functions

$$g_x : C \rightarrow V_x(L), \quad i \mapsto u_{ix}.$$

In more explicit terms this means that we set

$$\eta_x(v_x) = (\bar{g}_x(1, v_x), \dots, \bar{g}_x(n, v_x))$$

for every $v_x \in V_x(L)$, where

$$\bar{g}_x(i, v_x) = \begin{cases} u_{ix}, & \text{if } i \in C, \\ v_x, & \text{if } i \in M. \end{cases}$$

As we shall verify in the claim below, for every combinatorial embedding η the union $\bigcup_{x \in V(F)} \eta_x[V_x(L)]$ induces an F -partite copy of the system (L, \mathcal{L}_F) in (N, \mathcal{N}_F) . We write \mathcal{N}_L for the collection of all such copies as η varies over the combinatorial embeddings from \mathcal{L}_F to \mathcal{N}_F .

Having thus completed our description of the construction HJ, it remains to address the assertion in Step (5) on induced subsystems. The following claim provides more information than would be required for this purpose.

Claim 4.3. In the notation of Step (5), the vertex set $\bigcup_{x \in V(F)} \eta_x[V_x(L)]$ induces a copy $(L^\eta, \mathcal{L}_F^\eta) \in \left(\binom{(N, \mathcal{N}_F)}{(L, \mathcal{L}_F)}\right)^{\text{pt}}$ such that L^η is distance preserving in N . Moreover, for every $F_\star \in \mathcal{N}_F$ there is some $F_\star^\eta \in \mathcal{L}_F^\eta$ such that $V(L^\eta) \cap V(F_\star) \subseteq V(F_\star^\eta)$.

Proof. Stage A: First we note that, since the set of moving coordinates is non-empty, the mapping $\eta_\bullet = \bigcup_{x \in V(F)} \eta_x$ is an injection from $V(L)$ into $V(N)$. Next we convince ourselves that η_\bullet sends edges of L to edges of N . So given any two adjacent vertices $v_x \in V_x(L)$ and $v_y \in V_y(L)$ we need to show $\eta_x(v_x)\eta_y(v_y) \in E(N)$, which due to the definition of N in Step (3) means that $\bar{g}_x(i, v_x)\bar{g}_y(i, v_y) \in E(L)$ holds for every $i \in [n]$. Indeed for $i \in M$ this is equivalent to $v_x v_y \in E(L)$ and for $i \in C$ we have

$$\bar{g}_x(i, v_x)\bar{g}_y(i, v_y) = u_{ix}u_{iy} \in E(F_i) \subseteq E(L),$$

where $F_i \in \mathcal{L}_F$ is given by $F_i = g(i)$. Thus η_\bullet maps L to an isomorphic subgraph L^η of N , but we have not checked yet that this graph L^η is induced in N .

Stage B: Our next step is to show $L^\eta \in \binom{N}{L}_{\text{dp}}$, which implies, in particular, that L^η is an induced subgraph of N . To this end we pick an arbitrary moving coordinate $i \in M$,

let $\pi_i: V(N) \longrightarrow V(L)$ be the projection to the i^{th} coordinate, and define the map $\varrho: V(N) \longrightarrow V(L^\eta)$ by

$$\varrho(v_x) = (\eta_x \circ \pi_i)(v_x)$$

whenever $v_x \in V_x(N)$ and $x \in V(F)$. This is the composition of the graph homomorphisms

$$N \xrightarrow{\pi_i} L \xrightarrow{\eta_\bullet} L^\eta.$$

Therefore, ϱ maps paths in N to walks in L^η . In particular, any shortest path in N connecting two vertices of $L^\eta \subseteq N$ is mapped to a walk within L^η . Combined with the observation that ϱ is the identity on L^η , this implies that L^η is distance preserving in N .

Stage C: Next, for every copy $F' \in \mathcal{L}_F$ the copy $(\lambda \circ \eta)(F') \in \mathcal{N}_F$ (see Step (4)) has the set of vertices

$$V((\lambda \circ \eta)(F')) = \{\eta_x(v_x) : x \in V(F) \text{ and } v_x \in V(F') \cap V_x(L)\} = \eta_\bullet[V(F')]. \quad (4.1)$$

Setting $\mathcal{L}_F^\eta = (\lambda \circ \eta)[\mathcal{L}_F]$ this tells us that $(L^\eta, \mathcal{L}_F^\eta)$ is a partite F -subsystem of (N, \mathcal{N}_F) , which is isomorphic to (L, \mathcal{L}_F) , but we have not checked yet that it is induced, i.e., that $\mathcal{L}_F^\eta \supseteq \binom{L^\eta}{F}^{\text{pt}} \cap \mathcal{N}_F$ holds as well. As this follows from the moreover-part of our claim by applying it to any $F_\star \in \binom{L^\eta}{F}^{\text{pt}} \cap \mathcal{N}_F$, it suffices to establish the latter statement.

Stage D: So let any copy $F_\star \in \mathcal{N}_F$ be given. Owing to Step (4) there exists an n -tuple $\vec{F} = (F_1, \dots, F_n) \in \mathcal{L}_F^n$ such that $F_\star = \lambda(\vec{F})$. Picking an arbitrary index $i \in M$ we contend that $F_\star^\eta = (\lambda \circ \eta)(F_i)$ has the desired property $V(L^\eta) \cap V(F_\star) \subseteq V(F_\star^\eta)$. To see this, we consider any vertex in $V(L^\eta) \cap V(F_\star)$ and write it in the form $\eta_x(v_x)$ with $x \in V(F)$ and $v_x \in V_x(L)$. In the i^{th} coordinate of $\eta_x(v_x) \in V(F_\star)$ we see the statement $v_x \in V(F_i)$. By the first equality in (4.1) applied to F_i in place of F' this leads to $\eta_x(v_x) \in V(F_\star^\eta)$. \square

The next proposition summarises all properties of the Hales–Jewett construction we need in the sequel.

Proposition 4.4. *For every partite F -system (L, \mathcal{L}_F) and every number of colours r the partite conglomerate $\text{HJ}_r(L, \mathcal{L}_F) = (N, \mathcal{N}_L, \mathcal{N}_F)$ satisfies*

- (a) $\mathcal{N}_L \longrightarrow (L, \mathcal{L}_F)_r^F$,
- (b) $\mathcal{N}_L \subseteq \binom{N}{L}_{\text{dp}}$,
- (c) $\omega(N) = \omega(L)$,
- (d) $\text{og}(N) = \text{og}(L)$,
- (e) and if for some $m \in \mathbb{N}$ we have $\mathcal{L}_F \subseteq \binom{L}{F}_m$, then $\mathcal{N}_F \subseteq \binom{N}{F}_m$ follows.

Furthermore, if the copies in \mathcal{L}_F have clean intersections, then

- (f) so do the copies in \mathcal{N}_F ;
- (g) and for every $(L^\eta, \mathcal{L}_F^\eta) \in \mathcal{N}_L$ and every $F_\star \in \mathcal{N}_F \setminus \mathcal{L}_F^\eta$ the intersection of F_\star and L^η is clean.

Strictly speaking, there is some abuse of language in clause (b), because in reality \mathcal{N}_L consists of copies of the form (L^n, \mathcal{L}_F^n) , and what we claim is that their underlying graphs L^n are distance preserving in N .

Proof. In the degenerate case $|\mathcal{L}_F| \leq 1$ we have $(N, \mathcal{N}_F) = (L, \mathcal{L}_F)$ and $\mathcal{N}_L = \{(L, \mathcal{L}_F)\}$, so that all seven statements are clear. It remains to discuss the case $|\mathcal{L}_F| \geq 2$, where the conglomerate $(N, \mathcal{N}_L, \mathcal{N}_F)$ is defined according to the five Steps (1)–(5) described above.

The partition relation (a) follows from the fact that every colouring $\varphi: \mathcal{N}_F \rightarrow [r]$ induces, via λ^{-1} , an r -colouring of the discrete space \mathcal{L}_F^n . By our choice of n in Step (1) this colouring contains a monochromatic combinatorial line, which translates back to a copy $(L^n, \mathcal{L}_F^n) \in \mathcal{N}_L$ such that \mathcal{L}_F^n is monochromatic with respect to φ .

Part (b) was already obtained in Claim 4.3. The statements (c) and (d) follow from the fact, implicit in Stage B of the proof of Claim 4.3, that there exists a graph homomorphism from N to L combined with $\mathcal{N}_L \neq \emptyset$.

Proceeding with (e) we assume $\mathcal{L}_F \subseteq \binom{L}{F}_m$ for some $m \in \mathbb{N}$ and consider any copy $F_\star = \lambda(F_1, \dots, F_n) \in \mathcal{N}_F$. Let $P \subseteq N$ be a path which connects two vertices $u, w \in V(F_\star)$, which is not completely contained in F_\star , and which has length at most m . Take an arbitrary vertex $v \in V(P) \setminus V(F_\star)$, write it as an n -tuple $v = (v_1, \dots, v_n)$, and notice that due to $v \notin V(F_\star)$ there is a coordinate direction $i \in [n]$ such that $v_i \notin V(F_i)$.

Now for each $j \in [n]$ the projection $\pi_j: V(N) \rightarrow V(L)$ to the j^{th} coordinate is a graph homomorphism from N to L . In particular, $\pi_i[P] \subseteq L$ is a walk connecting the vertices $\pi_i(u)$ and $\pi_i(w)$ of F_i and passing through v_i . Since $F_i \in \binom{L}{F}_m$, this leads to a $\pi_i(u)$ - $\pi_i(w)$ -path $Q_i \subseteq F_i$, which is strictly shorter than P .

Next, the projection $\psi: V(L) \rightarrow V(F)$ mapping each vertex class $V_x(L)$ to x is a graph homomorphism as well. As $\psi|_{V(F_i)}$ is a graph isomorphism between F_i and F , it sends Q_i to a $(\psi \circ \pi_i)(u)$ - $(\psi \circ \pi_i)(w)$ -path $Q \subseteq F$. For each $j \in [n]$ the graph isomorphism $\psi|_{V(F_j)}$ between F_j and F pulls Q back to a $\pi_j(u)$ - $\pi_j(w)$ -path $Q_j \subseteq F_j$. In the same way as the n -tuple (F_1, \dots, F_n) corresponds to the copy $F_\star \in \binom{N}{F}^{\text{pt}}$, the n -tuple (Q_1, \dots, Q_n) corresponds to a u - w -path $Q_\star \subseteq F_\star$ of the same length as Q . This completes the proof of $F_\star \in \binom{N}{F}_m$ and, hence, of clause (e).

Throughout the remainder of the argument we suppose that the copies in \mathcal{L}_F have clean intersections. For Part (f) we consider two distinct n -tuples (F_1, \dots, F_n) and (F'_1, \dots, F'_n) in \mathcal{L}_F^n such that the copies $F_\star = \lambda(F_1, \dots, F_n)$ and $F_{\star\star} = \lambda(F'_1, \dots, F'_n)$ in \mathcal{N}_F have a set e of at least two vertices in common. We need to show that e is a common edge of these two copies.

To this end we pick an index $i \in [n]$ such that $F_i \neq F'_i$. Every $x \in V(F)$ with $e \cap V_x(N) \neq \emptyset$ needs to satisfy $V(F_i) \cap V(F'_i) \cap V_x(L) \neq \emptyset$. This can hold for at most two vertices $x \in V(F)$, because by our hypothesis on (L, \mathcal{L}_F) the intersection of F_i and F'_i

is clean. Together with $|e| \geq 2$ this shows that there are two vertices of F , say x and y , such that $V_x(N)$ and $V_y(N)$ are the only vertex classes of N intersected by e .

Using again that the intersection of F_i and F'_i is clean, we see that the vertices in $V(F_i) \cap V(F'_i) \cap V_x(L)$ and $V(F_i) \cap V(F'_i) \cap V_y(L)$ are adjacent. In particular, xy is an edge of F . Exploiting the product structure of $E(N)$ (cf. Step (3)) it is now easy to confirm that e is an edge from $V_x(N)$ to $V_y(N)$ belonging to $E(F_\star) \cap E(F_{\star\star})$.

Concluding with (g) we refer to the moreover-part of Claim 4.3, which yields a copy F_\star^η in \mathcal{L}_F^η such that $V(L^\eta) \cap V(F_\star) \subseteq V(F_\star^\eta)$. Due to $F_\star \notin \mathcal{L}_F^\eta$ we have $F_\star \neq F_\star^\eta$, so according to (f) the intersection of F_\star and F_\star^η is clean. It follows that the intersection of F_\star and L^η is clean as well. \square

§5 CLEAN RAMSEY CONGLOMERATES

Our goal in this section is to establish a version of Proposition 2.6 not addressing distances and odd girth yet, but preserving clean intersections and the clique number. For that we define an adequate notion of clean intersections for conglomerates.

Definition 5.1. Given a graph F and an F -system (B, \mathcal{B}_F) with clean intersections, a (B, \mathcal{B}_F) -conglomerate $(H, \mathcal{H}_B, \mathcal{H}_F)$ is said to be *clean* provided that

- (i) the copies in \mathcal{H}_F have clean intersections;
- (ii) for every $(B^\star, \mathcal{B}_F^\star) \in \mathcal{H}_B$ and every $F_{\star\star} \in \mathcal{H}_F \setminus \mathcal{B}_F^\star$ the intersection of B^\star and $F_{\star\star}$ is clean;
- (iii) and for all distinct $(B^\star, \mathcal{B}_F^\star), (B^{\star\star}, \mathcal{B}_F^{\star\star}) \in \mathcal{H}_B$ the intersection of B^\star and $B^{\star\star}$ is either clean or a copy of F belonging to $\mathcal{B}_F^\star \cap \mathcal{B}_F^{\star\star}$.

The following proposition yields clean conglomerates that preserve the clique number and enjoy the Ramsey property for colouring copies of F .

Proposition 5.2. *For every ordered F -system (B, \mathcal{B}_F) with clean intersections and every number of colours r there exists a clean ordered (B, \mathcal{B}_F) -conglomerate[‡]*

$$\Upsilon_r^{(1)}(B, \mathcal{B}_F) = (H, \mathcal{H}_B, \mathcal{H}_F)$$

such that $\mathcal{H}_B \longrightarrow (B, \mathcal{B}_F)_r^F$ and $\omega(H) = \omega(B)$.

The proof involves two applications of the partite construction method. In a first major step we obtain the so-called *clean partite lemma*—an improved version of the Hales–Jewett construction which yields clean partite conglomerates (see Proposition 5.6). Armed with this result and utilising the well-known fact that ordered graphs and induced embeddings form a Ramsey category (quoted as Theorem 5.8 below) we can then derive Proposition 5.2 by means of a second partite construction.

[‡]The superscript in $\Upsilon_r^{(1)}$ indicates it is the first member of a family of constructions $\Upsilon_r^{(n)}$ defined in §6.

5.1. Pictures. Before starting a partite construction, one needs to define an appropriate notion of pictures and clarify how they amalgamate. For reasons that will become apparent later, we need to work with a slightly relaxed version of conglomerates to this end.

For a fixed graph F , an F -system (B, \mathcal{B}_F) is said to be a *semi-induced subsystem* of another F -system (H, \mathcal{H}_F) if B is an induced subgraph of H and $\mathcal{B}_F \subseteq \mathcal{H}_F$. So in contrast to induced subsystems the reverse inclusion $\mathcal{B}_F \supseteq \mathcal{H}_F \cap \binom{B}{F}$ is not necessarily satisfied. Triples $(H, \mathcal{H}_B, \mathcal{H}_F)$ such that (H, \mathcal{H}_F) is an F -system and \mathcal{H}_B is a collection of semi-induced copies of (B, \mathcal{B}_F) in (H, \mathcal{H}_F) will be referred to as *weak (B, \mathcal{B}_F) -conglomerates*.

Definition 5.3. Suppose that F is a graph, (B, \mathcal{B}_F) is an F -system, and $(G, \mathcal{G}_B, \mathcal{G}_F)$ is a weak (B, \mathcal{B}_F) -conglomerate. A *picture* over $(G, \mathcal{G}_B, \mathcal{G}_F)$ is a triple $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ consisting of a G -partite graph Π and subsets $\mathcal{P}_F \subseteq \binom{\Pi}{F}$, $\mathcal{P}_B \subseteq \binom{(\Pi, \mathcal{P}_F)}{(B, \mathcal{B}_F)}$ such that the projection $\psi: V(\Pi) \rightarrow V(G)$ sending each vertex class $V_x(\Pi)$ to x maps each copy in \mathcal{P}_F to a copy in \mathcal{G}_F and each copy in \mathcal{P}_B to a copy in \mathcal{G}_B .

We would like to emphasise that the map ψ , called the *canonical projection* of Π , is uniquely determined by the G -partite structure of Π . The vertex classes $V_x(\Pi)$ of a picture are also called its *music lines*. In figures, they are usually drawn horizontally. For each copy $F_\bullet \in \mathcal{G}_F$ we define the *constituent* of $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ over F_\bullet to be the following partite F_\bullet -system $(\Pi^{F_\bullet}, \mathcal{P}^{F_\bullet})$.

- The underlying graph of Π^{F_\bullet} is the subgraph of Π induced by $\bigcup_{x \in V(F_\bullet)} V_x(\Pi)$ and the vertex classes determining its F_\bullet -partite structure are the music lines $V_x(\Pi)$.
- The set \mathcal{P}^{F_\bullet} consists of all members of \mathcal{P}_F projected by ψ onto F_\bullet . Equivalently, we set $\mathcal{P}^{F_\bullet} = \{F_\circ \in \mathcal{P}_F: V(F_\circ) \subseteq V(\Pi^{F_\bullet})\}$.

Notice that each copy in \mathcal{P}_F belongs to a unique constituent, while vertices and edges of Π can belong to many constituents.

The picture every partite construction begins with, called *picture zero*, can be defined over every weak (B, \mathcal{B}_F) -conglomerate $(G, \mathcal{G}_B, \mathcal{G}_F)$. This picture $(\Pi_0, \mathcal{P}_{0,B}, \mathcal{P}_{0,F})$ is obtained by taking vertex disjoint copies of (B, \mathcal{B}_F) , one for each member of \mathcal{G}_B , and placing them onto the music lines in such a way that the canonical projection ψ_0 of Π_0 sets up a bijection between the set $\mathcal{P}_{0,B}$ of these copies and \mathcal{G}_B . As an F -system $(\Pi_0, \mathcal{P}_{0,F})$ is the disjoint union of the F -systems constituting $\mathcal{P}_{0,B}$. For clarity we point out that picture zero satisfies

$$|V(\Pi_0)| = |V(B)| \cdot |\mathcal{G}_B|, \quad |E(\Pi_0)| = |E(B)| \cdot |\mathcal{G}_B|,$$

and

$$|\mathcal{P}_{0,B}| = |\mathcal{G}_B|, \quad |\mathcal{P}_{0,F}| = |\mathcal{B}_F| \cdot |\mathcal{G}_B|.$$

Given a picture $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ over a weak (B, \mathcal{B}_F) -conglomerate $(G, \mathcal{G}_B, \mathcal{G}_F)$, a copy $F_\bullet \in \mathcal{G}_F$ and a partite $(\Pi^{F_\bullet}, \mathcal{P}^{F_\bullet})$ -conglomerate $(H, \mathcal{H}_{\Pi^{F_\bullet}}, \mathcal{H}_{F_\bullet})$ we define the picture

$$(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F) = (\Pi, \mathcal{P}_B, \mathcal{P}_F) * (H, \mathcal{H}_{\Pi^{F_\bullet}}, \mathcal{H}_{F_\bullet})$$

to be the result of the following amalgamation procedure.

- Recall that each copy $(\Pi_\circ^{F_\bullet}, \mathcal{P}_\circ^{F_\bullet}) \in \mathcal{H}_{\Pi^{F_\bullet}}$ is a partite F_\bullet -system isomorphic to the constituent $(\Pi^{F_\bullet}, \mathcal{P}^{F_\bullet})$ of $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ over F_\bullet . Thus each of these copies can be extended to a copy $(\Pi_\circ, \mathcal{P}_B^\circ, \mathcal{P}_F^\circ)$ of this picture. We suppose that these extensions are constructed as disjoint as possible, so that we have

$$V(\Pi_\circ) \cap V(\Pi_{\circ\circ}) = V(\Pi_\circ^{F_\bullet}) \cap V(\Pi_{\circ\circ}^{F_\bullet})$$

for all distinct $(\Pi_\circ^{F_\bullet}, \mathcal{P}_\circ^{F_\bullet}), (\Pi_{\circ\circ}^{F_\bullet}, \mathcal{P}_{\circ\circ}^{F_\bullet}) \in \mathcal{H}_{\Pi^{F_\bullet}}$.

- As a graph, Σ is the union of H with all graphs Π_\circ obtained in this way. Moreover, the G -partite structure of Σ is defined by setting

$$V_x(\Sigma) = \begin{cases} \bigcup \{V_x(\Pi_\circ) : (\Pi_\circ^{F_\bullet}, \mathcal{P}_\circ^{F_\bullet}) \in \mathcal{H}_{\Pi^{F_\bullet}}\}, & \text{if } x \notin V(F_\bullet) \\ V_x(H), & \text{if } x \in V(F_\bullet). \end{cases}$$

- Put

$$\mathcal{Q}_F = \bigcup \{ \mathcal{P}_F^\circ : (\Pi_\circ^{F_\bullet}, \mathcal{P}_\circ^{F_\bullet}) \in \mathcal{H}_{\Pi^{F_\bullet}} \} \cup \mathcal{H}_{F_\bullet}$$

and

$$\mathcal{Q}_B = \bigcup \{ \mathcal{P}_B^\circ : (\Pi_\circ^{F_\bullet}, \mathcal{P}_\circ^{F_\bullet}) \in \mathcal{H}_{\Pi^{F_\bullet}} \}.$$

It is straightforward to check that $(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F)$ is indeed a picture over $(G, \mathcal{G}_B, \mathcal{G}_F)$. The copies $(\Pi_\circ, \mathcal{P}_B^\circ, \mathcal{P}_F^\circ)$ of the given picture $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ constructed in the first bullet are called the *standard copies* of $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ in $(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F)$.

5.2. The clean partite lemma. The partite conglomerates produced by the Hales–Jewett construction have properties (i) and (ii) of Definition 5.1, but they are capable of violating (iii), so that they are usually not clean. Our first application of the partite construction method leads to another partite lemma, which gives better control over the intersections of two copies.

Let us point out that when we have a picture $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ over a weak (B, \mathcal{B}_F) -conglomerate $(G, \mathcal{G}_B, \mathcal{G}_F)$, then by “forgetting” the G -partite structure of Π we can view this picture as a (B, \mathcal{B}_F) -conglomerate (which, however, by Definition 5.3 cannot be “only a weak one”). Accordingly, we can also speak of clean pictures. For instance, picture zero is clean as soon as the copies in \mathcal{B}_F have clean intersections. Next we show that, under suitable assumptions, which are satisfied when $(G, \mathcal{G}_B, \mathcal{G}_F)$ is obtained by means of

the Hales–Jewett construction, being clean is preserved by partite amalgamations. This observation can be traced back to [2, Lemma 2.12] (see also [29, Lemma 4.5]).

Lemma 5.4. *Let F be a graph, (B, \mathcal{B}_F) an F -system with clean intersections, and $(G, \mathcal{G}_B, \mathcal{G}_F)$ a (B, \mathcal{B}_F) -conglomerate such that*

- (a) *the copies in \mathcal{G}_F have clean intersections;*
- (b) *and for every $(B', \mathcal{B}'_F) \in \mathcal{G}_B$ and every $F_\star \in \mathcal{G}_F \setminus \mathcal{B}'_F$ the intersection of B' and F_\star is clean.*

If we have

$$(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F) = (\Pi, \mathcal{P}_B, \mathcal{P}_F) * (H, \mathcal{H}_{\Pi F_\bullet}, \mathcal{H}_{F_\bullet})$$

for two pictures $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$, $(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F)$ over $(G, \mathcal{G}_B, \mathcal{G}_F)$, the copies in \mathcal{H}_{F_\bullet} have clean intersections, and the picture $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ is clean, then so is $(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F)$.

Proof. Stage A: Let $\psi: V(\Sigma) \rightarrow V(G)$ be the canonical projection of $(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F)$. The following projection argument will often be used implicitly.

Claim 5.5. Let Z be a subgraph of Σ that either belongs to \mathcal{Q}_F or underlies a member of \mathcal{Q}_B . If for some copy $F' \in \mathcal{G}_F$ we have $V(F') \not\subseteq \psi[V(Z)]$, then the constituent of Σ over F' has a clean intersection with Z .

Proof. By our assumptions (a) and (b), the intersection of the graphs $\psi[Z]$ and F' is clean, whence $\psi[V(Z)] \cap V(F')$ contains at most two vertices, and if there are two, then they form an edge of $\psi[Z]$. Since $\psi|_{V(Z)}$ is a graph isomorphism, this entails our claim. \square

We need to show that the new picture $(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F)$ satisfies clauses (i)–(iii) of Definition 5.1.

Stage B: Beginning with (i), we consider two distinct copies $F_\star, F_{\star\star} \in \mathcal{Q}_F$. If their projections $\psi[F_\star], \psi[F_{\star\star}] \in \mathcal{G}_F$ are distinct, they have a clean intersection by assumption (a) and, hence, the intersection of $F_\star, F_{\star\star}$ is clean. So we may assume $F' = \psi[F_\star] = \psi[F_{\star\star}]$ for some $F' \in \mathcal{G}_F$ from now on. If $F' = F_\bullet$ is the copy over which the amalgamation occurs, we appeal to the assumption that the copies in \mathcal{H}_{F_\bullet} have clean intersections. Turning now to the case $F' \neq F_\bullet$ we let $(\Pi_\circ, \mathcal{P}_B^\circ, \mathcal{P}_F^\circ)$, $(\Pi_{\circ\circ}, \mathcal{P}_B^{\circ\circ}, \mathcal{P}_F^{\circ\circ})$ be the standard copies of $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ satisfying $F_\star \in \mathcal{P}_F^\circ$ and $F_{\star\star} \in \mathcal{P}_F^{\circ\circ}$, respectively. If $\Pi_\circ = \Pi_{\circ\circ}$ our claim follows from \mathcal{P}_F having clean intersections, so the interesting case is $\Pi_\circ \neq \Pi_{\circ\circ}$. Now we have

$$V(F_\star) \cap V(F_{\star\star}) \subseteq V(\Pi_\circ) \cap V(\Pi_{\circ\circ}) \subseteq V(H),$$

whence $\psi[V(F_\star) \cap V(F_{\star\star})] \subseteq V(F') \cap V(F_\bullet)$. The clean intersections of \mathcal{G}_F imply that this intersection has at most two elements. Moreover, Claim 5.5 applied to $Z = F_\star, F_{\star\star}$ and F_\bullet here in place of F' implies that $V(F_\star) \cap V(F_{\star\star}) \in E(F_\star) \cap E(F_{\star\star})$, if $|V(F_\star) \cap V(F_{\star\star})| = 2$.

Stage C: Continuing with (ii) we need to show that for every F -system $(B^*, \mathcal{B}_F^*) \in \mathcal{Q}_B$ and every copy $F_{**} \in \mathcal{Q}_F \setminus \mathcal{B}_F^*$ the intersection of B^* and F_{**} is clean. Let $(B', \mathcal{B}'_F) \in \mathcal{G}_B$ and $F' \in \mathcal{G}_F$ denote the projections of (B^*, \mathcal{B}_F^*) and F_{**} , respectively. If $F' \notin \mathcal{B}'_F$ we just need to invoke assumption (b) and Claim 5.5, so suppose $F' \in \mathcal{B}'_F$ from now on.

First case: $F' = F_\bullet$.

This is equivalent to $F_{**} \in \mathcal{H}_{F_\bullet}$. Now $F_\bullet = F' \in \mathcal{B}'_F$ leads to a copy $F_\star \in \mathcal{B}_F^* \cap \mathcal{H}_{F_\bullet}$; due to $F_{**} \notin \mathcal{B}_F^*$ this copy is distinct from F_{**} and, therefore, the intersection of $F_\star, F_{**} \in \mathcal{H}_{F_\bullet}$ is clean. Together with $V(B^*) \cap V(F_{**}) = V(F_\star) \cap V(F_{**})$ this shows that the intersection of B^* and F_{**} is clean.

Second case: $F' \neq F_\bullet$.

Now there are standard copies $(\Pi_\circ, \mathcal{P}_B^\circ, \mathcal{P}_F^\circ)$ and $(\Pi_{\circ\circ}, \mathcal{P}_B^{\circ\circ}, \mathcal{P}_F^{\circ\circ})$ such that $F_{**} \in \mathcal{P}_F^{\circ\circ}$ and $(B^*, \mathcal{B}_F^*) \in \mathcal{P}_B^\circ$. In the special case $\Pi_\circ = \Pi_{\circ\circ}$ our claim follows from the fact that $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ is a clean picture. On the other hand, if $\Pi_\circ \neq \Pi_{\circ\circ}$, then

$$\psi[V(B^*) \cap V(F_{**})] \subseteq V(F') \cap V(F_\bullet)$$

has at most two elements and by another application of Claim 5.5 the intersection of B^* and F_{**} is indeed clean. This proves assertion (ii) of Definition 5.1.

Stage D: Proceeding with part (iii) of Definition 5.1 we consider two F -systems (B^*, \mathcal{B}_F^*) and $(B^{**}, \mathcal{B}_F^{**})$ in \mathcal{Q}_B and standard copies $(\Pi_\circ, \mathcal{P}_B^\circ, \mathcal{P}_F^\circ)$, $(\Pi_{\circ\circ}, \mathcal{P}_B^{\circ\circ}, \mathcal{P}_F^{\circ\circ})$ such that $(B^*, \mathcal{B}_F^*) \in \mathcal{P}_B^\circ$ and $(B^{**}, \mathcal{B}_F^{**}) \in \mathcal{P}_B^{\circ\circ}$. If these standard copies coincide, we just need to exploit that $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ is clean. So suppose $\Pi_\circ \neq \Pi_{\circ\circ}$ from now on, which yields

$$V(B^*) \cap V(B^{**}) \subseteq V(H).$$

Let (B', \mathcal{B}'_F) and (B'', \mathcal{B}''_F) be the projections of (B^*, \mathcal{B}_F^*) and $(B^{**}, \mathcal{B}_F^{**})$ in \mathcal{G}_B . In the special case $F_\bullet \notin \mathcal{B}'_F$ hypothesis (b) shows that B^* meets H in at most two vertices and thus the intersection of B^* and B^{**} is clean, by another application of Claim 5.5.

So we can henceforth suppose $F_\bullet \in \mathcal{B}'_F$ and, for the same reason, $F_\bullet \in \mathcal{B}''_F$. This means that there are copies $F_\star \in \mathcal{B}_F^*$ and $F_{**} \in \mathcal{B}_F^{**}$ projecting to F_\bullet . Moreover, due to $\Pi_\circ \neq \Pi_{\circ\circ}$ the graphs B^*, B^{**} have the same intersection as F_\star, F_{**} . So either $F_\star \neq F_{**}$ and this intersection is clean, or the intersection in question is $F_\star = F_{**}$, and thus a member of $\mathcal{B}_F^* \cap \mathcal{B}_F^{**}$. This proves (iii) and, hence, the lemma. \square

Suppose now that a graph F , a partite F -system (B, \mathcal{B}_F) with clean intersections, and a number of colours r are given. The partite construction we have in mind produces a clean partite (B, \mathcal{B}_F) -conglomerate $(H, \mathcal{H}_B, \mathcal{H}_F)$ with $\mathcal{H}_B \longrightarrow (B, \mathcal{B}_F)_r^F$. This conglomerate will be denoted by $\text{CPL}_r(B, \mathcal{B}_F)$, where CPL abbreviates ‘‘clean partite lemma’’; it is constructed as follows.

- Set $(G, \mathcal{G}_B, \mathcal{G}_F) = \text{HJ}_r(B, \mathcal{B}_F)$ and enumerate \mathcal{G}_F arbitrarily as

$$\mathcal{G}_F = \{F(1), \dots, F(N)\}.$$

- Starting with picture zero we recursively define a sequence $(\Pi_\alpha, \mathcal{P}_{\alpha,B}, \mathcal{P}_{\alpha,F})_{\alpha \leq N}$ of pictures over $(G, \mathcal{G}_B, \mathcal{G}_F)$. If for some $\alpha \in [N]$ the picture $(\Pi_{\alpha-1}, \mathcal{P}_{\alpha-1,B}, \mathcal{P}_{\alpha-1,F})$ has just been constructed, we subject its constituent over $F(\alpha)$ to the construction $\text{HJ}_r(\cdot)$, thereby obtaining a conglomerate $(H_\alpha, \mathcal{H}_{\Pi_{\alpha-1}^{F(\alpha)}}, \mathcal{H}_{F(\alpha)})$, and then we set

$$(\Pi_\alpha, \mathcal{P}_{\alpha,B}, \mathcal{P}_{\alpha,F}) = (\Pi_{\alpha-1}, \mathcal{P}_{\alpha-1,B}, \mathcal{P}_{\alpha-1,F}) * (H_\alpha, \mathcal{H}_{\Pi_{\alpha-1}^{F(\alpha)}}, \mathcal{H}_{F(\alpha)}), \quad (5.1)$$

where the amalgamation occurs over $F(\alpha)$.

- This recursive construction ends with a terminal picture $(\Pi_N, \mathcal{P}_{N,B}, \mathcal{P}_{N,F})$. As Π_N is a G -partite graph, while G itself is an F -partite graph, we can define an F -partite structure on Π_N by setting

$$V_x(\Pi_N) = \bigcup \{V_y(\Pi_N) : y \in V_x(G)\}$$

for every $x \in V(F)$. Owing to the fact that (G, \mathcal{G}_F) is a partite F -system, we can view $(\Pi_N, \mathcal{P}_{N,F})$ as a partite F -system as well. For a similar reason we can regard $(\Pi_N, \mathcal{P}_{N,B}, \mathcal{P}_{N,F})$ as a partite (B, \mathcal{B}_F) -conglomerate. This partite conglomerate will henceforth be denoted by $\text{CPL}_r(B, \mathcal{B}_F)$.

Proposition 5.6. *For every graph F , every partite F -system (B, \mathcal{B}_F) with clean intersections, and every number of colours r the partite (B, \mathcal{B}_F) -conglomerate*

$$\text{CPL}_r(B, \mathcal{B}_F) = (H, \mathcal{H}_B, \mathcal{H}_F)$$

is clean and satisfies

- (a) $\mathcal{H}_B \longrightarrow (B, \mathcal{B}_F)_r^F$,
- (b) $\mathcal{H}_B \subseteq \binom{H}{B}_{\text{dp}}$,
- (c) $\omega(H) = \omega(B)$,
- (d) *as well as* $\text{og}(H) = \text{og}(B)$.

Proof. Clearly, picture zero is clean. Repeated applications of Lemma 5.4 show that all pictures $(\Pi_\alpha, \mathcal{P}_{\alpha,B}, \mathcal{P}_{\alpha,F})$ defined in the course of the partite construction are clean. Indeed, the vertical system $(G, \mathcal{G}_B, \mathcal{G}_F)$ fulfils the assumptions (a) and (b) of Lemma 5.4 due to Proposition 4.4. Similarly, for every $\alpha \in [N]$ the copies in the sets $\mathcal{H}_{F(\alpha)}$ appearing horizontally have clean intersections, provided that in the previous picture the copies in $\mathcal{P}_{\alpha-1,F}$ have clean intersections. In particular, the final picture $(\Pi_N, \mathcal{P}_{N,B}, \mathcal{P}_{N,F})$ and thus the conglomerate $(H, \mathcal{H}_B, \mathcal{H}_F)$ is clean.

The partition relation $\mathcal{P}_{N,B} \longrightarrow (B, \mathcal{B}_F)_r^F$ follows from a standard argument in the area, so we shall be brief about it. The main point is that an easy induction on $\alpha \in \{0, 1, \dots, N\}$ discloses that

for every colouring $\varphi: \mathcal{P}_{\alpha,F} \longrightarrow [r]$ there is a copy $(\tilde{\Pi}_0, \tilde{\mathcal{P}}_{0,B}, \tilde{\mathcal{P}}_{0,F})$ of picture zero in $(\Pi_\alpha, \mathcal{P}_{\alpha,B}, \mathcal{P}_{\alpha,F})$ such that for every $\beta \in [\alpha]$ the constituent of $(\tilde{\Pi}_0, \tilde{\mathcal{P}}_{0,B}, \tilde{\mathcal{P}}_{0,F})$ over $F(\beta)$ is monochromatic[§].

This statement is vacuously true for $\alpha = 0$ and in the induction step from $\alpha - 1$ to α one exploits that the constituent of picture α over $F(\alpha)$ is obtained by means of the Hales–Jewett construction (cf. (5.1) and Proposition 4.4(a)).

In particular, for $\alpha = N$ we learn that for every colouring $\varphi: \mathcal{P}_{N,F} \longrightarrow [r]$ there is a copy $(\tilde{\Pi}_0, \tilde{\mathcal{P}}_{0,B}, \tilde{\mathcal{P}}_{0,F})$ of picture zero all of whose constituents are monochromatic. This gives rise to an auxiliary colouring $\chi: [N] \longrightarrow [r]$ such that for every $\beta \in [N]$ all copies of F belonging to the constituent of $(\tilde{\Pi}_0, \tilde{\mathcal{P}}_{0,B}, \tilde{\mathcal{P}}_{0,F})$ over $F(\beta)$ have the colour $\chi(\beta)$. Now we apply the partition relation $\mathcal{G}_B \longrightarrow (B, \mathcal{B}_F)_r^F$, which follows from the construction of $(G, \mathcal{G}_B, \mathcal{G}_F)$ in the first bullet and from Proposition 4.4(a), to the r -colouring $F(\beta) \longmapsto \chi(\beta)$ of \mathcal{G}_F . Thereby we find a monochromatic copy $(B', \mathcal{B}'_F) \in \mathcal{G}_B$. Now, if (B^*, \mathcal{B}^*_F) denotes the copy in $\tilde{\mathcal{P}}_{0,B} \subseteq \mathcal{P}_{N,B}$ projected to (B', \mathcal{B}'_F) , then \mathcal{B}^*_F is monochromatic with respect to φ , which completes the proof of (a).

For the remaining three statements we need the canonical projection ψ from Π_N to G . Since ψ is a graph homomorphism, (c) and (d) follow from the analogous properties of the Hales–Jewett construction established in Proposition 4.4.

It remains to address (b) and to this end we consider any copy $(B^*, \mathcal{B}^*_F) \in \mathcal{P}_{N,B}$ together with a path P in Π_N connecting two vertices x and y of B^* . By Proposition 4.4(b) ψ sends B^* to some $B' \in \binom{G}{B}_{\text{dp}}$. Thus there is a $\psi(x)$ - $\psi(y)$ -path $Q \subseteq B'$ whose length is at most the length of P . The graph isomorphism $\psi|_{V(B^*)}$ between B^* and B' allows us to pull Q back to B^* . □

Remark 5.7. It seems less obvious that the clean partite lemma also satisfies property (e) of Proposition 4.4, meaning that if $\mathcal{B}_F \subseteq \binom{B}{F}_m$ holds for some natural number m , then $\mathcal{H}_F \subseteq \binom{H}{F}_m$ follows (provided that \mathcal{B}_F has clean intersections). We shall return to this statement, which is central to our ability to preserve odd girth, in Corollary 6.3 below.

5.3. The second partite construction. The following strengthening of the induced Ramsey theorem was proved independently by Abramson and Harrington [1], and by Nešetřil and Rödl [22].

[§]More precisely, if $(\tilde{\Pi}_0^{F(\beta)}, \tilde{\mathcal{P}}_{0,F}^{F(\beta)})$ denotes that constituent, we mean that all elements of $\tilde{\mathcal{P}}_{0,F}^{F(\beta)}$ have the same colour.

Theorem 5.8. *For all ordered graphs F and B and every number of colours r there exists an ordered graph G satisfying*

$$G \longrightarrow (B)_r^F,$$

which means that for every colouring $\varphi: \binom{G}{F} \longrightarrow [r]$ there exists a copy $B^\star \in \binom{G}{B}$ such that $\binom{B^\star}{F}$ is monochromatic. \square

This result is sometimes expressed by saying that ordered graphs form a Ramsey class or a Ramsey category. We do not introduce the corresponding abstract language here and refer the interested reader to the preface of [29].

By choosing G according to Theorem 5.8 and setting $\mathcal{G}_F = \binom{G}{F}$ one sees that for every ordered F -system (B, \mathcal{B}_F) and every number of colours r there exists an ordered F -system (G, \mathcal{G}_F) such that for every colouring $\varphi: \mathcal{G}_F \longrightarrow [r]$ there is a semi-induced copy $(B^\star, \mathcal{B}_F^\star)$ of (B, \mathcal{B}_F) such that \mathcal{B}_F^\star is monochromatic. In several respects this is weaker than Proposition 5.2 and thus we run another partite construction, this time with the clean partite lemma. The advantage of horizontal clean intersections is most clearly spelled out in [29, Lemma 4.7] and [29, Lemma 4.14]. Here is a straightforward adaptation of the basic ideas to our current situation.

Lemma 5.9. *Suppose that $(G, \mathcal{G}_B, \mathcal{G}_F)$ denotes a weak (B, \mathcal{B}_F) -conglomerate for some F -system (B, \mathcal{B}_F) and that we have*

$$(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F) = (\Pi, \mathcal{P}_B, \mathcal{P}_F) \ast (H, \mathcal{H}_{\Pi^{F_\bullet}}, \mathcal{H}_{F_\bullet})$$

for two pictures $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$, $(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F)$ over $(G, \mathcal{G}_B, \mathcal{G}_F)$ and a partite $(\Pi^{F_\bullet}, \mathcal{P}^{F_\bullet})$ -conglomerate $(H, \mathcal{H}_{\Pi^{F_\bullet}}, \mathcal{H}_{F_\bullet})$. If $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ and $(H, \mathcal{H}_{\Pi^{F_\bullet}}, \mathcal{H}_{F_\bullet})$ are clean, then so is $(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F)$.

Proof. Again we need to check that $(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F)$ satisfies the three clauses (i)–(iii) of Definition 5.1. As usual, we denote the canonical projection from Σ to G by ψ .

Stage A: Starting with (i) we consider two distinct copies $F_\star, F_{\star\star} \in \mathcal{Q}_F$. If $F_\star, F_{\star\star} \in \mathcal{H}_{F_\bullet}$ we appeal to the assumption that \mathcal{H}_{F_\bullet} has clean intersections. So we can assume $F_\star \notin \mathcal{H}_{F_\bullet}$ from now on, so that there is a unique standard copy $(\Pi_\circ, \mathcal{P}_B^\circ, \mathcal{P}_F^\circ)$ of $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ with $F_\star \in \mathcal{P}_F^\circ$.

First case: $F_{\star\star} \notin \mathcal{H}_{F_\bullet}$.

This means that there also exists a standard copy $(\Pi_{\circ\circ}, \mathcal{P}_B^{\circ\circ}, \mathcal{P}_F^{\circ\circ})$ with $F_{\star\star} \in \mathcal{P}_F^{\circ\circ}$. If $\Pi_\circ = \Pi_{\circ\circ}$ we use that \mathcal{P}_F° has clean intersections; so we can assume $\Pi_\circ \neq \Pi_{\circ\circ}$ from now on, which implies that the constituents $\Pi_\circ^{F_\bullet}$ and $\Pi_{\circ\circ}^{F_\bullet}$ intersect each other either cleanly or in some copy of $\tilde{F} \in \mathcal{H}_{F_\bullet}$. In the former case

$$V(F_\star) \cap V(F_{\star\star}) \subseteq V(\Pi_\circ) \cap V(\Pi_{\circ\circ}) = V(\Pi_\circ^{F_\bullet}) \cap V(\Pi_{\circ\circ}^{F_\bullet}) \quad (5.2)$$

immediately yields $|V(F_\star) \cap V(F_{\star\star})| \leq 2$ and we are done. If, on the other hand, the aforementioned copy \tilde{F} exists, then (5.2) tells us $V(F_\star) \cap V(F_{\star\star}) \subseteq V(\tilde{F})$ and our assertion follows from the fact that \tilde{F} has a clean intersection with each of $F_\star, F_{\star\star}$.

Second case: $F_{\star\star} \in \mathcal{H}_{F_\bullet}$.

If $F_{\star\star} \in \mathcal{P}_F^\circ$ we exploit that the copies in \mathcal{P}_F° have clean intersections and otherwise property (ii) of $(H, \mathcal{H}_{\Pi^{F_\bullet}}, \mathcal{H}_{F_\bullet})$ being clean yields

$$|V(F_\star) \cap V(F_{\star\star})| \leq |V(\Pi_\circ^{F_\bullet}) \cap V(F_{\star\star})| \leq 2.$$

Stage B: Moving on to the verification of (ii) we consider two copies $(B^\star, \mathcal{B}_F^\star) \in \mathcal{Q}_B$ and $F_{\star\star} \in \mathcal{Q}_F \setminus \mathcal{B}_F^\star$, and need to show that the intersection of B^\star and $F_{\star\star}$ is clean. Let $(\Pi_\circ, \mathcal{P}_B^\circ, \mathcal{P}_F^\circ)$ be a standard copy of $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ with $(B^\star, \mathcal{B}_F^\star) \in \mathcal{P}_B^\circ$. Since $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$ is clean, only the case $F_{\star\star} \notin \mathcal{P}_F^\circ$ is interesting. If $F_{\star\star} \in \mathcal{H}_{F_\bullet} \setminus \mathcal{P}_F^\circ$ we know that $F_{\star\star}$ intersects the constituent $\Pi_\circ^{F_\bullet}$ cleanly and our claim follows again. So it remains to discuss the case that some standard copy $(\Pi_{\circ\circ}, \mathcal{P}_B^{\circ\circ}, \mathcal{P}_F^{\circ\circ})$ satisfies $\Pi_\circ \neq \Pi_{\circ\circ}$ and $F_{\star\star} \in \mathcal{P}_F^{\circ\circ} \setminus \mathcal{H}_{F_\bullet}$. Now we have

$$V(B^\star) \cap V(F_{\star\star}) \subseteq V(\Pi_\circ^{F_\bullet}) \cap V(\Pi_{\circ\circ}^{F_\bullet}).$$

So if the graphs $\Pi_\circ^{F_\bullet}, \Pi_{\circ\circ}^{F_\bullet}$ have a clean intersection we are done immediately and if they intersect in some copy $\tilde{F} \in \mathcal{P}_F^\circ \cap \mathcal{P}_F^{\circ\circ}$ we can exploit that $\tilde{F}, F_{\star\star}$ have a clean intersection due to the already established statement (i).

Stage C: Finally addressing (iii) we consider two distinct copies $(B^\star, \mathcal{B}_F^\star)$ and $(B^{\star\star}, \mathcal{B}_F^{\star\star})$ in \mathcal{Q}_B and take two standard copies $(\Pi_\circ, \mathcal{P}_B^\circ, \mathcal{P}_F^\circ), (\Pi_{\circ\circ}, \mathcal{P}_B^{\circ\circ}, \mathcal{P}_F^{\circ\circ})$ with $(B^\star, \mathcal{B}_F^\star) \in \mathcal{P}_B^\circ$ and $(B^{\star\star}, \mathcal{B}_F^{\star\star}) \in \mathcal{P}_B^{\circ\circ}$, respectively. As usual, it suffices to treat the case $\Pi_\circ \neq \Pi_{\circ\circ}$, so that we have

$$V(B^\star) \cap V(B^{\star\star}) \subseteq V(\Pi_\circ^{F_\bullet}) \cap V(\Pi_{\circ\circ}^{F_\bullet}).$$

If the intersection of $\Pi_\circ^{F_\bullet}$ and $\Pi_{\circ\circ}^{F_\bullet}$ is clean, everything is clear, and we can assume from now on that this intersection is some copy $\tilde{F} \in \mathcal{P}_F^\circ \cap \mathcal{P}_F^{\circ\circ}$. As long as $\tilde{F} \notin \mathcal{B}_F^\star \cap \mathcal{B}_F^{\star\star}$ it follows from the already established statement (ii) that \tilde{F} intersects one of B^\star and $B^{\star\star}$ cleanly, which causes the intersection of B^\star and $B^{\star\star}$ to be clean as well. In the remaining case, $\tilde{F} \in \mathcal{B}_F^\star \cap \mathcal{B}_F^{\star\star}$, the second outcome of (iii) occurs. \square

We are now ready to describe the construction $\Upsilon^{(1)}$ promised at the beginning of this section. As mentioned earlier, the proof is rendered by a partite construction and parallels the proof of the clean partite lemma—Proposition 5.6. In fact, we shall use the clean partite lemma here. The other difference concerns the initial conglomerate $(G, \mathcal{G}_B, \mathcal{G}_F)$ underlying the pictures in the construction. In the context of Proposition 5.2 the conglomerate $(G, \mathcal{G}_B, \mathcal{G}_F)$ will be provided by Theorem 5.8 on Ramsey classes. As a result we have less control over the structural properties of $(G, \mathcal{G}_B, \mathcal{G}_F)$ and this is reflected in the conclusion

of Proposition 5.2. In particular, the delivered conglomerate $(H, \mathcal{H}_B, \mathcal{H}_F)$ falls short of preserving distances and odd girth. These shortcomings will be addressed in §6 by another application of the partite construction method.

Proof of Proposition 5.2. Given an ordered F -system (B, \mathcal{B}_F) with clean intersections and a number of colours r we proceed as follows.

- By Theorem 5.8 there exists an ordered graph G satisfying $G \longrightarrow (B)_r^F$. Set $\mathcal{G}_F = \binom{G}{F}$, let \mathcal{G}_B be the collection of all semi-induced ordered copies of (B, \mathcal{B}_F) in the ordered F -system (G, \mathcal{G}_F) , and observe that $(G, \mathcal{G}_B, \mathcal{G}_F)$ is an ordered weak (B, \mathcal{B}_F) -conglomerate. Choose an arbitrary enumeration $\mathcal{G}_F = \{F(1), \dots, F(N)\}$ of \mathcal{G}_F .
- Starting with picture zero we recursively define a sequence $(\Pi_\alpha, \mathcal{P}_{\alpha,B}, \mathcal{P}_{\alpha,F})_{\alpha \leq N}$ of clean pictures over $(G, \mathcal{G}_B, \mathcal{G}_F)$. If for some $\alpha \in [N]$ we have just constructed the clean picture $(\Pi_{\alpha-1}, \mathcal{P}_{\alpha-1,B}, \mathcal{P}_{\alpha-1,F})$, then we know that the copies of F belonging to its constituent over $F(\alpha)$ have clean intersections. By Proposition 5.6 the clean partite lemma $\text{CPL}_r(\cdot)$ applied to this constituent yields a clean partite conglomerate $(H_\alpha, \mathcal{H}_{\Pi_{\alpha-1}^{F(\alpha)}}, \mathcal{H}_{F(\alpha)})$. By Lemma 5.9 the picture

$$(\Pi_\alpha, \mathcal{P}_{\alpha,B}, \mathcal{P}_{\alpha,F}) = (\Pi_{\alpha-1}, \mathcal{P}_{\alpha-1,B}, \mathcal{P}_{\alpha-1,F}) * (H_\alpha, \mathcal{H}_{\Pi_{\alpha-1}^{F(\alpha)}}, \mathcal{H}_{F(\alpha)}),$$

where the amalgamation occurs over $F(\alpha)$, is clean again and, therefore, our construction continues.

- This recursive process culminates with a final picture $(\Pi_N, \mathcal{P}_{N,B}, \mathcal{P}_{N,F})$. We order the vertices of Π_N in such a manner that each music line becomes an interval and the ordering of these intervals reflects the vertex ordering of the ordered graph G . Since the copies in $\mathcal{P}_{N,B} \cup \mathcal{P}_{N,F}$ intersect each music line at most once, this ensures that these copies are ordered correctly in Π_N . So we have converted $(\Pi_N, \mathcal{P}_{N,B}, \mathcal{P}_{N,F})$ into an ordered (B, \mathcal{B}_F) -conglomerate.

The partition relation $\mathcal{P}_{N,B} \longrightarrow (B, \mathcal{B}_F)_r^F$ follows from the same standard argument we have already presented in the proof of Proposition 5.6 and we omit the details.

It remains to address the preservation of the clique number. To this end, we prove $\omega(\Pi_\alpha) = \omega(B)$ by induction on $\alpha \leq N$. The base case $\alpha = 0$ is trivial; in the induction step we assume $\omega(\Pi_{\alpha-1}) = \omega(B)$ for some $\alpha \in [N]$ and consider a clique K in Π_α . If K is completely contained in H_α , then its order is at most

$$\omega(H_\alpha) = \omega(\Pi_{\alpha-1}^{F(\alpha)}) \leq \omega(\Pi_{\alpha-1}) = \omega(B)$$

by Proposition 5.6(c).

Otherwise there are a standard copy $\Pi_{\alpha-1}^\circ$ of $\Pi_{\alpha-1}$ and a vertex v in K such that $v \in V(\Pi_{\alpha-1}^\circ) \setminus V(H_\alpha)$. Since v is adjacent to all other vertices of K , we have $K \subseteq \Pi_{\alpha-1}^\circ$

and, consequently, the order of K is at most $\omega(\Pi_{\alpha-1}) = \omega(B)$. This proves $\omega(\Pi_\alpha) \leq \omega(B)$ and the reverse inequality is obvious. \square

§6 CONFORM RAMSEY CONGLOMERATES

For the proof of Proposition 2.6 we recall that $\Upsilon_r^{(1)}$ from Proposition 5.2 already provides a clean conglomerate $(H, \mathcal{H}_B, \mathcal{H}_F)$ preserving the clique number. In particular, this conglomerate already satisfies property (a) of Proposition 2.6 and parts (i) and (ii) from Definition 2.3 required by property (b). Consequently, only parts (iii) and (iv) of Definition 2.3, which concern the odd girth and n -inducedness, are missing. For $n = 1$ these two missing properties are trivial and, in fact, $\Upsilon_r^{(1)}$ provides the base case for an inductive proof. For the inductive step, we run a partite construction, where we employ $\Upsilon_r^{(n-1)}$ vertically and the clean partite lemma CPL_r from Proposition 5.6 horizontally. Since the standard construction of picture zero consists of disjoint copies, it clearly preserves odd girth and n -inducedness.

Roughly speaking, the following picturesque statement rendered in Lemma 6.1 shows that these properties are kept during the partite construction described above (see statements (a) and (b) of the lemma). However, we remark that in Lemma 6.1 we also require n -induced copies of F provided by the partite lemma, which is not stated to be a property of the clean partite lemma in Proposition 5.6; but we shall address this issue in Corollary 6.3.

Lemma 6.1. *Given a graph F , let (B, \mathcal{B}_F) be an F -system and let $(G, \mathcal{G}_B, \mathcal{G}_F)$ be a weak (B, \mathcal{B}_F) -conglomerate. Suppose further that we have*

$$(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F) = (\Pi, \mathcal{P}_B, \mathcal{P}_F) * (H, \mathcal{H}_{\Pi^{F_\bullet}}, \mathcal{H}_{F_\bullet})$$

for two pictures $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$, $(\Sigma, \mathcal{Q}_B, \mathcal{Q}_F)$ over $(G, \mathcal{G}_B, \mathcal{G}_F)$ and an F_\bullet -partite $(\Pi^{F_\bullet}, \mathcal{P}^{F_\bullet})$ -conglomerate $(H, \mathcal{H}_{\Pi^{F_\bullet}}, \mathcal{H}_{F_\bullet})$ with $\mathcal{H}_{\Pi^{F_\bullet}} \subseteq \binom{H}{\Pi^{F_\bullet}}_{\text{dp}}$. Finally let $n \geq 2$ be an integer such that $\mathcal{G}_F \subseteq \binom{G}{F}_{n-1}$.

- (a) We have $\mathcal{Q}_F \subseteq \binom{\Sigma}{F}_n$ provided that
 - (1) $\mathcal{P}_F \subseteq \binom{\Pi}{F}_n$ and $\mathcal{H}_{F_\bullet} \subseteq \binom{H}{F}_n$,
 - (2) the copies in \mathcal{G}_F have clean intersections,
 - (3) and for all copies $(\Pi_\circ^{F_\bullet}, \mathcal{P}_\circ^{F_\bullet}) \in \mathcal{H}_{\Pi^{F_\bullet}}$ and $F_\star \in \mathcal{H}_{F_\bullet} \setminus \mathcal{P}_\circ^{F_\bullet}$ the intersection of $\Pi_\circ^{F_\bullet}$ and F_\star is clean.
- (b) If $\text{og}(G) \geq 2n - 1$, $\text{og}(\Pi) \geq 2n + 1$, and $\text{og}(H) \geq 2n + 1$, then $\text{og}(\Sigma) \geq 2n + 1$.

Proof. Starting with property (a) we consider a copy $F_\star \in \mathcal{Q}_F$ and a path $P \not\subseteq F_\star$ of length at most n that connects two vertices x and y of F_\star . We need to find some x - y -path $R \subseteq F_\star$ which is shorter than P . The canonical projection ψ of Σ sends F_\star to some copy F' in the system \mathcal{G}_F .

First case: $F' = F_\bullet$.

Suppose first that no inner vertex of P belongs to H . Now the entire path P is contained in a single standard copy $(\Pi_\circ, \mathcal{P}_B^\circ, \mathcal{P}_F^\circ)$ of $(\Pi, \mathcal{P}_B, \mathcal{P}_F)$. So if $F_\star \in \mathcal{P}_F^\circ$ we just need to appeal to the hypothesis $\mathcal{P}_F \subseteq \binom{\Pi}{F}_n$. On the other hand, if $F_\star \notin \mathcal{P}_F^\circ$, then by (3) the intersection of Π_\circ and F_\star is clean, so that xy is an edge of F_\star and can play the rôle of the desired path R .

It remains to deal with the case that P is of the form $P = xP'zP''y$, where z is an inner vertex of P belonging to $V(H)$ connected to x and y by the paths P' and P'' . Let the music line through z intersect the copy F_\star in z_\star .

We contend that there is an x - z_\star -path $Q' \subseteq F_\star$ whose length is at most the length of P' . To see this, we look at the $\psi(x)$ - $\psi(z)$ -path $\psi[P']$ in G . If it is contained in F_\bullet , we just need to pull this path back to F_\star . Otherwise, $F_\bullet \in \mathcal{G}_F \subseteq \binom{G}{F}_{n-1}$ yields a $\psi(x)$ - $\psi(z)$ -path $S' \subseteq F_\bullet$ that is even shorter than P' and the existence of Q' follows again by pulling this path back.

The same argument also yields a z_\star - y -path $Q'' \subseteq F_\star$ whose length is at most the length of P'' . The x - y -walk $xQ'z_\star Q''y$ is contained in F_\star and has at most the length of P . If this walk fails to contain the desired path, then P', Q' have the same length, and so do P'', Q'' . But then the entire path P is contained in H and the assumption $\mathcal{H}_{F_\bullet} \subseteq \binom{H}{F}_n$ from (1) leads to the desired path.

Second case: $F' \neq F_\bullet$.

Let Π_\circ denote the unique standard copy of Π containing F_\star . If P crosses H in at most one vertex, then we have $P \subseteq \Pi_\circ$ and $F_\star \in \binom{\Pi_\circ}{F}_n$ leads to the desired path.

As we traverse P from x to y , we can therefore assume that there are a first vertex u and a last vertex v belonging to H , and that these vertices are distinct. We write $P = xP'uP''vP'''y$, where the paths $P', P''' \subseteq \Pi_\circ$ might degenerate to single vertices (in case $x = u$ or $v = y$), while P'' has at least one edge.

Suppose first that $P'' \subseteq H$. Since $\Pi_\circ^{F_\bullet}$ is distance preserving in H , there exists a u - v -path $Q'' \subseteq \Pi_\circ^{F_\bullet}$ of at most the same length as P'' . Now $xP'uQ''vP'''y$ is an x - y -walk in Π_\circ of at most the same length as P and $F_\star \in \binom{\Pi_\circ}{F}_n$ leads again to the desired path. So we can assume $P'' \not\subseteq H$ in the sequel.

Suppose next that $u = x$ and $v = y$. By (2) the intersection of F' and F_\bullet is clean; as $\psi(x), \psi(y)$ belong to both graphs, we have $\psi(x)\psi(y) \in E(G)$. Consequently, xy is an edge of F_\star and our claim is again clear. So we can assume from now on that P'' has length at most $n - 1$.

By $F_\bullet \in \binom{G}{F}_{n-1}$ applied to the $\psi(u)$ - $\psi(v)$ -walk $\psi[P'']$, which is not contained in F_\bullet , there is a $\psi(u)$ - $\psi(v)$ -path $R'' \subseteq F_\bullet$, which is shorter than P'' . Now $\psi[xP'u]R''\psi[vP'''y]$ is a $\psi(x)$ - $\psi(y)$ -walk in G and shorter than P . Due to $F' \in \mathcal{G}_F \subseteq \binom{G}{F}_{n-1}$ this entails that some

$\psi(x)$ - $\psi(y)$ -path $S \subseteq F'$ is shorter than P as well, wherefore the desired x - y -path $R \subseteq F_\star$ exists again.

Having thereby proved (a) we proceed with (b). Assume towards a contradiction that Σ contains an odd cycle C whose length is at most $2n - 1$. The projection ψ sends C to an odd closed walk D in G of the same length as C . Due to $\text{og}(G) \geq 2n - 1$ this implies that both C and D are cycles of length $2n - 1$. If C intersects H in at most one vertex, then C has to be contained in a single standard copy of Π , contrary to $\text{og}(\Pi) \geq 2n + 1$. Thus we can assume that $s = |V(C) \cap V(H)|$ is at least 2. Consequently, C is of the form

$$C = x_1 P_1 \dots x_s P_s, \tag{6.1}$$

where $V(C) \cap V(H) = \{x_1, \dots, x_s\}$ and P_ϱ is an x_ϱ - $x_{\varrho+1}$ -path for every $\varrho \in \mathbb{Z}/s\mathbb{Z}$.

Claim 6.2. If for some $\varrho \in \mathbb{Z}/s\mathbb{Z}$ the length of P_ϱ is at most $n - 1$, then this path is contained in H .

Proof. Assume contrariwise that the length of P_ϱ is at most $n - 1$, while $P_\varrho \not\subseteq H$. Since ψ is injective on $V(C)$, the image $\psi[P_\varrho]$ is a $\psi(x_\varrho)$ - $\psi(x_{\varrho+1})$ -path in G . Together with $\psi[P_\varrho] \not\subseteq F_\bullet$ and $F_\bullet \in \binom{G}{F}_{n-1}$ this shows that there exists a $\psi(x_\varrho)$ - $\psi(x_{\varrho+1})$ -path $Q \subseteq F_\bullet$ whose length is strictly smaller than the length of P_ϱ . Both

$$Q\psi[x_{\varrho+1}P_{\varrho+1} \dots x_\varrho] \quad \text{and} \quad \psi[x_\varrho P_\varrho x_{\varrho+1}]Q^{-1},$$

where Q^{-1} denotes Q traversed in reverse order, are closed walks in G whose lengths are at most $2n - 2$. Moreover, the parities of their lengths are different and, therefore, one of them is odd. Thereby we reach a contradiction to $\text{og}(G) \geq 2n - 1$. \square

Since $\text{og}(H) \geq 2n + 1$ implies $C \not\subseteq H$, it follows that one of the paths P_1, \dots, P_s has length at least n . This path needs to be unique and the other paths are contained in H . For these reasons we can rewrite (6.1) as $C = xPyQ$, where $x, y \in V(C) \cap V(H)$, the x - y -path P has no inner vertices in H , and the y - x -path $Q \subseteq H$ has length at most $n - 1$. Clearly, P is completely contained in a single standard copy $(\Pi_\circ, \mathcal{P}_B^\circ, \mathcal{P}_F^\circ)$. As the constituent of this picture over F_\bullet is distance preserving in H , it contains an x - y -path R whose length is at most the length of Q . Now

$$xRyQ \quad \text{and} \quad xPyR^{-1}$$

are closed walks in H and Π_\circ , respectively, whose lengths are at most $2n - 1$. Moreover, their parities are different, so that one of them is odd. Thus we get a contradiction either to $\text{og}(H) \geq 2n + 1$ or to $\text{og}(\Pi_\circ) \geq 2n + 1$. \square

Keeping a promise made in Remark 5.7 we can now easily deduce a further property of the clean partite lemma.

Corollary 6.3. *Let (B, \mathcal{B}_F) be a partite F -system with clean intersections. If for some natural number n we have $\mathcal{B}_F \subseteq \binom{B}{F}_n$, then for every number of colours r the clean conglomerate $\text{CPL}_r(B, \mathcal{B}_F) = (H, \mathcal{H}_B, \mathcal{H}_F)$ satisfies $\mathcal{H}_F \subseteq \binom{H}{F}_n$.*

Proof. This is clear for $n = 1$, so we can assume $n \geq 2$ from now on. Let us recall that $(H, \mathcal{H}_B, \mathcal{H}_F)$ is obtained by means of a partite construction. We keep using the notation from the explanation of this construction given immediately before Proposition 5.6.

Owing to the items (e) and (f) of Proposition 4.4 we know $\mathcal{G}_F \subseteq \binom{G}{F}_{n-1}$ and that the copies in \mathcal{G}_F have clean intersections. We want to establish $\mathcal{P}_{\alpha, F} \subseteq \binom{\Pi_\alpha}{F}_n$ for every non-negative integer $\alpha \leq N$ by induction on α . The case $\alpha = N$ of this assertion will then prove our corollary.

The base case $\alpha = 0$ is clear, because picture zero is just a disjoint union of copies of (B, \mathcal{B}_F) and $\mathcal{B}_F \subseteq \binom{B}{F}_n$ is assumed. Suppose now that we already know $\mathcal{P}_{\alpha-1, F} \subseteq \binom{\Pi_{\alpha-1}}{F}_n$ for some $\alpha \in [N]$. Utilising Proposition 4.4 one easily checks that all assumptions of Lemma 6.1(a) are satisfied for the partite amalgamation

$$(\Pi_\alpha, \mathcal{P}_{\alpha, B}, \mathcal{P}_{\alpha, F}) = (\Pi_{\alpha-1}, \mathcal{P}_{\alpha-1, B}, \mathcal{P}_{\alpha-1, F}) * (H_\alpha, \mathcal{H}_{\Pi_{\alpha-1}^{F(\alpha)}}, \mathcal{H}_{F(\alpha)})$$

and thus we have indeed $\mathcal{P}_{\alpha, F} \subseteq \binom{\Pi_\alpha}{F}_n$. \square

We conclude by establishing the following strengthening of Proposition 2.6.

Proposition 6.4. *For every ordered graph F and every integer $n \geq 1$ there exists a construction $\Upsilon^{(n)}$ which, given an ordered F -system (B, \mathcal{B}_F) with clean intersections satisfying $\mathcal{B}_F \subseteq \binom{B}{F}_n$ and a number of colours r , yields an ordered (B, \mathcal{B}_F) -conglomerate $\Upsilon_r^{(n)}(B, \mathcal{B}_F) = (H, \mathcal{H}_B, \mathcal{H}_F)$ such that*

- (a) $\mathcal{H}_B \longrightarrow (B, \mathcal{B}_F)_r^F$,
- (b) (1) $\mathcal{H}_F \subseteq \binom{H}{F}_n$,
- (2) $\omega(H) = \omega(B)$,
- (3) $\text{og}(H) \geq \min\{2n + 1, \text{og}(B)\}$,
- (c) and the (B, \mathcal{B}_F) -conglomerate $(H, \mathcal{H}_B, \mathcal{H}_F)$ is clean.

Before proving Proposition 6.4 we note that it implies Proposition 2.6. First we observe that the assumptions of Proposition 6.4 are less restrictive. Indeed, given an n -conform ordered F -system (B, \mathcal{B}_F) , as assumed in Proposition 2.6, conditions (i) and (iv) of Definition 2.3 ensure that (B, \mathcal{B}_F) has clean intersections and $\mathcal{B}_F \subseteq \binom{B}{F}_n$.

Concerning the conclusion of Proposition 6.4, we note that the ordered (B, \mathcal{B}_F) -conglomerate $\Upsilon_r^{(n)}(B, \mathcal{B}_F) = (H, \mathcal{H}_B, \mathcal{H}_F)$ clearly satisfies the Ramsey property (a) of Proposition 2.6. Moreover, part (c) of Proposition 6.4 implies that \mathcal{H}_F has clean intersections. The other three requirements needed for the n -conformity of (H, \mathcal{H}_F) , asserted

in part (b) of Proposition 2.6, follow directly from the three properties in part (b) of Proposition 6.4 combined with the assumption that (B, \mathcal{B}_F) is n -conform itself.

Proof. We argue by induction on n , using the construction $\Upsilon^{(1)}$ provided by Proposition 5.2 as a base case. Assuming now that for some $n \geq 2$ we have the construction $\Upsilon^{(n-1)}$ at our disposal and that an ordered F -system (B, \mathcal{B}_F) with clean intersections such that $\mathcal{B}_F \subseteq \binom{B}{F}_n$ as well as a number of colours r are given, we perform the same partite construction as in the proof of Proposition 5.2, the only difference being that vertically we use the conglomerate $(G, \mathcal{G}_B, \mathcal{G}_F) = \Upsilon_r^{(n-1)}(B, \mathcal{B}_F)$. Again this yields a sequence $(\Pi_\alpha, \mathcal{P}_{\alpha,B}, \mathcal{P}_{\alpha,F})_{\alpha \leq N}$ of clean pictures and the last of them can be converted into a clean ordered (B, \mathcal{B}_F) -conglomerate, which we shall denote by $\Upsilon_r^{(n)}(B, \mathcal{B}_F) = (H, \mathcal{H}_B, \mathcal{H}_F)$ from now on. In particular, $(H, \mathcal{H}_B, \mathcal{H}_F)$ satisfies property (c) of Proposition 6.4.

The partition relation (a) holds for the usual reason. Assertion (2) of part (b) follows from arguments already presented in the proof of Proposition 5.2, which we do not repeat.

Next, a straightforward induction on α based on Lemma 6.1(a), Proposition 5.6, Corollary 6.3, and our induction hypothesis on $\Upsilon^{(n-1)}$ reveals $\mathcal{P}_{\alpha,F} \subseteq \binom{\Pi_\alpha}{F}_n$ for every $\alpha \leq N$. The case $\alpha = N$ of this assertion proves (1) of part (b).

Finally, (3) only requires attention if $\text{og}(B) \geq 5$. Now $\bar{n} = \min\{n, \frac{1}{2}(\text{og}(B) - 1)\}$ is at least 2 and we can apply Lemma 6.1(b) with \bar{n} instead of n . This allows us to show $\text{og}(\Pi_\alpha) \geq 2\bar{n} + 1$ for every $\alpha \leq N$ by induction on α , and thus we have indeed $\text{og}(H) = \text{og}(\Pi_N) \geq 2\bar{n} + 1$. \square

§7 CONCLUDING REMARKS

7.1. Generalisations of the main result. The first generalisation concerns the girth of the canonical Ramsey graph H in Theorem 1.1. It seems likely that, as long as F is not a forest, Theorem 2.4 remains valid when we add the demand that H and F should have the same girth. In the non-canonical setting, such a result was obtained recently by Reiher and Rödl. In fact, the proof of their result provides a system \mathcal{H}_F of copies of F in H , which itself contains no short cycles (see [29, §5]), and we believe that those results extend to the canonical setting.

The second generalisation concerns hypergraphs. The proof of Theorem 1.1 presented here can be partly extended to linear hypergraphs. For hypergraphs the clique number is replaced by the size of the largest subset X of vertices such that any two vertices in X are contained in a hyperedge. However, it is perhaps less clear what the analogous concepts of odd girth, n -inducedness, and distance preservation for such hypergraphs are. It seems plausible that versions of Theorems 1.1 and 2.4 hold for general hypergraphs.

The last generalisation we briefly mention concerns a common generalisation of our result and Theorem 5.8 in the context of Ramsey classes, when we colour arbitrary subgraphs.

In that context an induced canonical Ramsey theorem with maintained clique number was provided by Prömel and Voigt [27] and it would be of some interest to establish a result subsuming both their work and Theorem 2.4.

7.2. Classical versus canonical Ramsey theory. As it turned out, many results in Ramsey theory can be paralleled in the canonical context. For example, it follows from the work of Erdős, Hajnal, and Rado [10, §16.4] (see also reference [8, (4.2)]) and of Shelah [34] that in terms of the number of exponentiations the canonical Ramsey number and the non-canonical Ramsey number (for at least four colours) for cliques in hypergraphs display the same behaviour. Another example, in the somewhat different context of random graphs, concerns the threshold for the corresponding Ramsey properties. For graph cliques with at least four vertices those thresholds coincide, which follows from the work of Rödl and Ruciński [32, 33] and of Kamčev and Schacht [18]. A far-reaching generalisation of these parallel developments would be an affirmative answer to the question of whether having the Ramsey property for sufficiently many colours yields the canonical Ramsey property as well. For graphs this naïve question reads as follows.

Question. Given an ordered graph F , does there exist an integer r such that for every ordered graph H that satisfies the relation $H \longrightarrow (F)_r$ we also have $H \xrightarrow{*} (F)$?

Note that for graphs F with at most two edges this question is trivial, since any colouring of such small graphs is canonical.

More interestingly, by applying the machinery developed in [29] we can answer this question in the negative when we replace H by a system \mathcal{H}_F of copies of an ordered graph F with at least three edges. For such graphs there are at least five equivalence relations on $E(F)$ and, hence, at least one of them corresponds to a non-canonical colouring. We fix such a non-canonical equivalence relation, say \equiv^F , and let $r \in \mathbb{N}$ be given.

With the notation developed in [29] the pair (F, \equiv^F) is a pretrain and [29, Proposition 9.1] applied to (F, \equiv^F) and r yields a pretrain system $(H, \equiv^H, \mathcal{H}_F)$ with the Ramsey property $\mathcal{H}_F \longrightarrow (F)_r$. Moreover, the restriction of \equiv^H to copies of F in \mathcal{H}_F is identical to \equiv^F . So, we can colour $E(H)$ in such a way that no copy of F from \mathcal{H}_F is canonically coloured.

Consequently, the version of the question for the system \mathcal{H}_F has the expected negative answer. Moreover, if $F = K_k$ for some $k \geq 3$ (and for certain other graphs F), the constructions involved in the proof of [29, Proposition 9.1] yield $\mathcal{H}_{K_k} = \binom{H}{K_k}$. This answers the question posed above negatively for such instances.

Nevertheless, in light of the discussion above one may wonder whether these parallel developments in canonical and non-canonical Ramsey theory are just anecdotal or whether there is some profound connection.

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