Fachbereich Mathematik — Universität Hamburg
Lecturer: Prof. Mathias Schacht — Fall 2020/21
Assistant: Dr. Christian Reiher — February 2nd, 2021

# Graph Theory 2

***10th problem set***

**due February 9th, 10am**

https://bit.ly/3pNrt8W

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**Exercise 1** [1 point]

Let $m, n \in \mathbb{N}$, and assume that $m-1$ divides $n-1$. Show that for every tree $T$ with $m$ vertices the Ramsey number $R^{(2)}(T, K_{1,n})$ equals $m+n-1$.

**Exercise 2** [1 point]

We say an infinite family $\mathscr{G}$ of finite graphs has the *Ramsey property*, if for every integer $r \geqslant 2$ and every $H \in \mathscr{G}$ there is some $G \in \mathscr{G}$ satisfying $G \longrightarrow (H)_r$, i.e., every $r$-colouring of $E(G)$ yields a monochromatic copy of $H$.

(*i*) Which examples of families having the Ramsey property do you know from class?

(*ii*) Prove or disprove that the set of hypercubes has the Ramsey property.

(*iii*) Find a family with the Ramsey property and prove it directly (without using results from class).

**Exercise 3** [1 point]

We write $G \longrightarrow (H)_r^I$ for graphs $G$, $H$, $I$, and an integer $r \geqslant 2$, if every $r$-colouring of the copies of $I$ in $G$ yields a subgraph $H_0 \subseteq G$ isomorphic to $H$ such that all copies of $I$ in $H_0$ are monochromatic.

(*i*) For $I = K_2$, Ramsey's theorem fasserts that for every $H$ and $r$ there is some $G$ such that $G \longrightarrow (H)_r^{K_2}$. Does the same hold for some graph $I$ with at least three vertices?

(*ii*) Find graphs $I$ and $H$ such that $G \longrightarrow (H)_2^I$ fails for all graphs $G$.

**Exercise 4** [1 point]

Erdős established for every integer $r \geqslant 2$ the existence of a graph $G$ with chromatic number and girth at least $r$ by a probabilistic argument. Present a constructive proof of this result by partite construction as given by Nešetřil and Rödl in [*A short proof of the existence of highly chromatic graphs without short cycles*, J. Combinat. Theory B, 27 (1979), pp. 225-227].