Fachbereich Mathematik
Lecturer: Prof. Mathias Schacht
Assistant: Dr. Christian Reiher

Universität Hamburg
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Graph Theory 2

8th problem set

due January 26th, 10am

https://bit.ly/2M7QSLD

Exercise 1 [1 point]

Recall that Turán's theorem asserts that any n-vertex graph with more than $\frac{\ell-2}{\ell-1}\frac{n^2}{2}$ edges contains a copy of K_ℓ. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for sufficiently large n every n-vertex graph with more than $\left(\frac{\ell-2}{\ell-1}+\varepsilon\right)\frac{n^2}{2}$ edges contains not only one, but even at least δn^ℓ copies of K_ℓ.

Exercise 2 [1 point]

Deduce the Erdős–Stone theorem (Theorem 7.1.2) from Turán's theorem as outlined in § 7.5.

Exercise 3 [1 point]

Prove the asymptotically optimal lower bound on the Ramsey-Turán function $\mathrm{RT}(n; 5, o(n))$ (see Section 5 in the handout on the webpage) by showing that for every $\alpha > 0$ there is some n_0 such that for every $n \geqslant n_0$ we have

$$\mathrm{RT}(n; 5, \alpha n) > \frac{1}{2}\binom{n}{2}.$$

Hint: The graphs with high girth and high chromatic number from Theorem 11.2.2 from Graph Theory 1 might be useful. Recall that the proof indeed shows that for every g, $k \geqslant 3$ and sufficiently large n there exists an n-vertex graph G with girth at least g and $\alpha(G) < n/k$.

Exercise 4 [1 point]

Prove the asymptotically optimal upper bound on the Ramsey-Turán function $\mathrm{RT}(n; 5, o(n))$ (see Section 5 in the handout on the webpage) by showing that for every $\eta > 0$ there exist $\alpha > 0$ and n_0 such that for every $n \geqslant n_0$

$$\mathrm{RT}(n; 5, \alpha n) \leqslant \left(\frac{1}{2}+\eta\right)\binom{n}{2}.$$

Hint: Apply the regularity lemma and show that the reduced graph contains no triangle with a pair of density $> 1/2$.