Graph Theory 2

7th problem set

due January 19th, 10am

https://bit.ly/3siKemc

Exercise 1 [1 point]

Formulate an *induced version* of the counting lemma for partite copies and deduce it from the counting lemma for cliques (Proposition 2.1).

Exercise 2 [1 point]

The general counting lemma implies that every (ε, d)-regular pair (X, Y) contains at most $(d^4 + 4\varepsilon)|X|^2|Y|^2$ partite homomorphism of C_4, i.e., there are at most $(d^4 + 4\varepsilon)|X|^2|Y|^2$ closed walks $xyx'y'x$ in G such that $x, x' \in X$ and $y, y' \in Y$.

Provide a matching lower bound for all (not necessarily ε-regular) bipartite graphs $G = (X \dot\cup Y, E)$ with $|E| = d|X||Y|$, i.e., show that G contains at least $d^4|X|^2|Y|^2$ partite homomorphism of C_4.

Exercise 3 [1 point]

Show that the exceptional class V_0 in *Szemerédi's regularity lemma* can be avoided by redistributing its vertices. In particular, show that the assertion

(i) $V = \dot\bigcup_{i=0}^{t} V_i$ with $|V_0| \leqslant \varepsilon|V|$ and $|V_1| = \cdots = |V_t|$

in the regularity lemma can be replaced by

(i') $V = \dot\bigcup_{i=1}^{t} V_i$ and $|V_1| \leqslant \cdots \leqslant |V_t| \leqslant |V_1| + 1$.

Exercise 4 [1 point]

Let (X, Y) be an (ε, d)-regular pair with $d^3 > \varepsilon > 0$. What can you say about the size of a largest matching in such a pair? What can you say if in addition we assume that $|X| = |Y| =: m$ and minimum degree is at least dm?