Fachbereich Mathematik — Universität Hamburg
Lecturer: Prof. Mathias Schacht — Fall 2020/21
Assistant: Dr. Christian Reiher — November 17th, 2020

Graph Theory 2

2nd problem set

due November 24th, 10am

https://bit.ly/3lFwcaK

Exercise 1 [1 point]

Let G be a k-linked graph. Show that

(i) G is $(2k-1)$-connected;

(ii) if $s_1, \dots, s_k, t_1, \dots, t_k$ are not necessarily distinct vertices of G such that $s_i \neq t_i$ for all i, then G contains independent paths $P_i = s_i \dots t_i$ for $i = 1, \dots, k$, i.e., no two of these paths share an internal vertex.

Exercise 2 [1 point]

A central open problem for linkages in graphs is to determine for every $k \in \mathbb{N}$ the minimal integer $f(k)$ such that every $f(k)$-connected graph is k-linked. In the lecture we have seen that $f(k) \leqslant 2^{\binom{k}{2}}$ and currently the best published upper bound due to Thomas and Wollan asserts $f(k) \leqslant 10k$.

In the other direction, Watkins found an infinite sequence of 5-connected graphs $(G_n)_{n\in\mathbb{N}}$ with $|V(G_n)| \longrightarrow \infty$ with none of them being 2-linked. In particular, this shows that $f(2) \geqslant 6$ and a simple modification of these graphs can be used to derive $f(k) \geqslant 2k+2$ for every $k \geqslant 2$, which was conjectured to be optimal by Thomassen. However, this was disproved by Jørgensen, who showed that for every $k \geqslant 1$ there is a $(3k-3)$-connected graph that is not k-linked. All known constructions that disprove Thomassen's conjecture are realised by small graphs only, i.e., the number of vertices is bounded by a function of k. In fact, for large graphs (number of vertices unbounded in terms of k) it is believed that Thomassen's conjecture holds.

(i) For every $k \in \mathbb{N}$ find a lower bound for $f(k)$ as close as possible to Jørgensen's bound.

(ii) Show $f(2) \geqslant 6$. Can you find an infinite sequence like Watkins?
Hint: Consider nearly maximal planar graphs.

Exercise 3 [1 point]

Show directly that every k^2-linked graph contains a topological minor of K_k.

Exercise 4 [1 point]

Prove that the edge set of any graph G can be written as a disjoint union $E(G) = C \mathbin{\dot\cup} B$ with C and B being elements from the cycle space $\mathcal{C}(G)$ and the cut space $\mathcal{B}(G)$, respectively.