Fachbereich Mathematik — Universität Hamburg
Lecturer: Prof. Mathias Schacht — Fall 2017/18
Assistant: Oliver Ebsen — December 31st, 2017

Graph Theory 2

Exercise Sheet 9

due on January 11, 1pm

http://bit.ly/2z9dqPI

Exercise 1 [1 point]

Show that for every $\varepsilon > 0$ there exists some $\delta > 0$ such that if a bipartite graph $G = (X \cup Y, E)$ with density $d = \frac{|E|}{|X||Y|} > 0$ contains at most $(d^4 + \delta)|X|^2|Y|^2$ closed walks of length four starting in X, then

(*i*) all but at most $\varepsilon|Y|^2$ pairs y, $y' \in Y$ satisfy $\big||N(y) \cap N(y') \cap X| - d^2|X|\big| \leqslant \varepsilon|X|$ and

(*ii*) all but at most $\varepsilon|X|$ vertices $x \in X$ satisfy $\big||N(x) - d|Y|\big| \leqslant \varepsilon|Y|$.

Hint: The following consequence of the Cauchy-Schwarz inequality might be useful: if $\frac{1}{n}\sum_{i=1}^{n} x_i \geqslant \alpha$ and $\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leqslant \alpha^2 + \nu$, then $|\frac{1}{m}\sum_{i=1}^{m} x_i - \alpha|^2 \leqslant \nu\frac{n}{m}$ for every $m \in [n]$ (see next page for a proof).

Exercise 2 [1 point]

Under the same assumption as in Exercise 1, show that for sufficiently small $\delta > 0$ for every $Y' \subseteq Y$ all but at most $\varepsilon|X|$ vertices $x \in X$ satisfy $\big||N(x) \cap Y'| - d|Y'|\big| \leqslant \varepsilon|Y|$.

Exercise 3 [1 point]

Under the same assumption as in Exercise 1, show that for sufficiently small $\delta > 0$ the pair (X, Y) is (ε, d)-regular.

Exercise 4 [1 point]

Prove the asymptotically optimal lower bound on the Ramsey-Turán function $\mathrm{RT}(n; 5, o(n))$ (see Section 5 in the handout on the webpage) by showing that for every $\alpha > 0$ there is some n_0 such that for every $n \geqslant n_0$ we have

$$\mathrm{RT}(n; 5, \alpha n) > \frac{1}{2}\binom{n}{2}.$$

Hint: The graphs with high girth and high chromatic number from Theorem 11.2.2 from Graph Theory 1 might be useful. Recall that the proof indeed shows that for every g, $k \geqslant 3$ and sufficiently large n there exists an n-vertex graph G with girth at least g and $\alpha(G) < n/k$.

Written Exercise

Prove the asymptotically optimal upper bound on the Ramsey-Turán function $\mathrm{RT}(n; 5, o(n))$ (see Section 5 in the handout on the webpage) by showing that for every $\eta > 0$ there exist $\alpha > 0$ and n_0 such that for every $n \geqslant n_0$

$$\mathrm{RT}(n; 5, \alpha n) \leqslant \left(\frac{1}{2} + \eta\right)\binom{n}{2}.$$

Hint: Apply the regularity lemma and show that the reduced graph contains no triangle with a pair of density $> 1/2$.

Proof of the assertion in the hint of Exercise 1. The statement is obvious for $n = m$ and we may consider arbitrary integers $n > m \geqslant 1$ and real numbers $x_1, \dots, x_n$, $\alpha \in \mathbb{R}$, and $\nu \geqslant 0$. We observe

$$\frac{1}{m}\Big(\sum_{i=1}^{m} x_i - \alpha m\Big)^2 \leqslant \frac{1}{m}\Big(\sum_{i=1}^{m} x_i - \alpha m\Big)^2 + \frac{1}{n-m}\Big(\sum_{i=m+1}^{n} x_i - \alpha(n-m)\Big)^2$$
$$= \frac{1}{m}\Big(\sum_{i=1}^{m} x_i\Big)^2 + \frac{1}{n-m}\Big(\sum_{i=m+1}^{n} x_i\Big)^2 - 2\alpha\sum_{i=1}^{n} x_i + \alpha^2 n\,.$$

Applying the Cauchy-Schwarz inequality to each of the two squared sums yields

$$\frac{1}{m}\Big(\sum_{i=1}^{m} x_i - \alpha m\Big)^2 \leqslant \sum_{i=1}^{m} x_i^2 + \sum_{i=m+1}^{n} x_i^2 - 2\alpha\sum_{i=1}^{n} x_i + \alpha^2 n = \sum_{i=1}^{n} x_i^2 - 2\alpha\sum_{i=1}^{n} x_i + \alpha^2 n$$

and applying the assumptions $\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leqslant \alpha^2 + \nu$ and $\frac{1}{n}\sum_{i=1}^{n} x_i \geqslant \alpha$ leads to

$$\frac{1}{m}\Big(\sum_{i=1}^{m} x_i - \alpha m\Big)^2 \leqslant \alpha^2 n + \nu n - 2\alpha^2 n + \alpha^2 n = \nu n\,.$$

Dividing both sides by m gives the desired inequality. □