Graph Theory 2

Exercise Sheet 8

due on December 14, 1pm

http://bit.ly/2AuJF09

Exercise 1 [1 point]

In class we defined a pair (X, Y) in a graph $G = (V, E)$ for nonempty, disjoint sets X, $Y \subseteq V$ to be (ε, d)-*regular pair*, if for all $X' \subseteq X$ and $Y' \subseteq Y$ we have

$$\left|e_G(X', Y') - d\,|X'||Y'|\right| \leqslant \varepsilon|X||Y|\,.$$

Show that this definition is equivalent to the one in the textbook [Diestel, *Graph Theory*, 5th ed., p. 187/188]. More precisely, show that for every $\varepsilon > 0$ there is some $\delta > 0$ such that every (δ, d)-regular pair satisfying the definition above is ε-regular following the definition from the textbook and in the opposite direction that every δ-regular pair satisfying the textbook definition is (ε, d)-regular as defined above with $d = d(X, Y) := \frac{e_G(X,Y)}{|X||Y|}$.

Exercise 2 [1 point]

Show that the exceptional class V_0 in *Szemerédi's regularity lemma* can be avoided by redistributing its vertices. In particular, show that the assertion

(i) $V = \dot\bigcup_{i=0}^{t} V_i$, $|V_0| \leqslant \varepsilon|V|$, and $|V_1| = \cdots = |V_t|$

in the regularity lemma can be replaced by

(i') $V = \dot\bigcup_{i=1}^{t} V_i$ and $|V_1| \leqslant \cdots \leqslant |V_t| \leqslant |V_1| + 1$.

Exercise 3 [1 point]

In the special case of triangles the *counting lemma* asserts that if all bipartite pairs (V_i, V_j) of a tripartite graph $G = (V_1 \dot\cup V_2 \dot\cup V_3, E)$ are (ε, d_{ij})-regular for some $\varepsilon > 0$ and $d_{ij} \in [0, 1]$ for $1 \leqslant i < j \leqslant 3$, then the number of triangles in G is "close" to $d_{12}d_{13}d_{23}|V_1||V_2||V_3|$.

Is a similar assertion true, if only two, or if only one of the three pairs is ε-regular? If so, what would be an appropriate generalisation for arbitrary fixed graphs F instead of K_3?

Exercise 4 [1 point]

Let (X, Y) be an (ε, d)-regular pair in some graph $G = (V, E)$ for some $d^3 > 4\varepsilon > 0$. What can you say about the size of a largest matching in the induced bipartite graph $G[X, Y]$? What can you say if we in addition assume that $|X| = |Y| =: m$ and minimum degree of $G[X, Y]$ is at least $dm/2$.

Written Exercise

Let $G = (X \dot\cup Y, E)$ be a nonempty, bipartite graph of density $d := \frac{|E|}{|X||Y|}>0$.

(i) Show that G contains at least $d^4|X|^2|Y|^2$ closed walks of of length 4 starting in X.
Hint: Cauchy-Schwarz inequality

(ii) Show that the lower bound is *approximately optimal* if (X, Y) is in addition (ε, d)-regular.
Hint: Define a notion of δ-*approximately optimal* and show for all $\delta > 0$, there is $\varepsilon > 0$ s.t. ...