Fachbereich Mathematik — Universität Hamburg
Lecturer: Prof. Mathias Schacht — Fall 2017/18
Assistant: Oliver Ebsen — Oktober 19th, 2017

Graph Theory 2

Exercise Sheet 1

due on Oktober 26, 1pm

http://bit.ly/2kYKM1V

Exercise 1 (§2.20) [1 Punkt]

Show that a graph $G = (V, E)$ contains a matching with k edges if and only if

$$q(G - S) \leqslant |S| + |V| - 2k$$

for all sets $S \subseteq V$, where $q(G - S)$ denotes the number of odd components in $G - S$.

Exercise 2 (§2.25) [1 Punkt]

For cubic graphs, Lemma 2.3.1 is considerably stronger than the Erdős-Pósa theorem. Extend the lemma to arbitrary multigraphs of minimum degree $\geqslant 3$, by finding a function $g\colon \mathbb{N} \to \mathbb{N}$ such that every multigraph of minimum degree $\geqslant 3$ and order at least $g(k)$ contains k disjoint cycles, for all $k \in \mathbb{N}$. Alternatively, show that no such function g exists.

Exercise 3 (§2.27) [1 Punkt]

Show that if G has two edge-disjoint spanning trees, it has a connected spanning subgraph all whose degrees are even.

Exercise 4 (§2.30) [1 Punkt]

A graph G is called *balanced* if $\varepsilon(H) \leqslant \varepsilon(G)$ for every subgraph $H \subseteq G$, where $\varepsilon(H) = \frac{|E(H)|}{|V(H)|}$.

(i) Find a few natural classes of balanced graphs.

(ii) Give an example of a graph which is not balanced.

(iii) Show that the arboricity of a balanced graph is bounded from above by its average degree. Is it even bounded by ε? Or by $\varepsilon + 1$?

(iv) Characterise the graphs G satisfying $\varepsilon(H) \geqslant \varepsilon(G)$ for every induced subgraph $H \subseteq G$ on at least two vertices.

Written Exercise (§2.22)

Derive Hall's marriage theorem from Tutte's theorem.