On the local density problem for graphs of given odd-girth

Wiebke Bedenknecht [1,2]

Fachbereich Mathematik, Universität Hamburg
20146 Hamburg, Germany

Guilherme Oliveira Mota [3]

Instituto de Mathemática e Estatística
Universidade de São Paulo, Rua do Matão 1010
05508-090, São Paul, Brazil

Christian Reiher and Mathias Schacht [4]

Fachbereich Mathematik, Universität Hamburg
20146 Hamburg, Germany

Abstract

Erdős conjectured that every n-vertex triangle-free graph contains a subset of $\lfloor n/2 \rfloor$ vertices that spans at most $n^2/50$ edges. Extending a recent result of Norin and Yepremyan, we confirm this for graphs homomorphic to so-called Andrásfai graphs. As a consequence, Erdős' conjecture holds for every triangle-free graph G with minimum degree $\delta(G) > 10n/29$ and if $\chi(G) \leq 3$ the degree condition can be relaxed to $\delta(G) > n/3$. In fact, we obtain a more general result for graphs of higher odd-girth.

Keywords: Andrásfai graphs, Erdős $(1/2, 1/50)$ - conjecture, sparse halves

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1 Introduction

We say an n-vertex graph G is (α, β)-*dense* if every subset of $\lfloor \alpha n \rfloor$ vertices spans more than βn^2 edges. Given $\alpha \in (0, 1]$ Erdős, Faudree, Rousseau, and Schelp [5] asked for the minimum $\beta = \beta(\alpha)$ such that every (α, β)-dense graph contains a triangle. For example, Mantel's theorem asserts that $\beta(1) = 1/4$. More generally, Erdős et al. conjectured that for $\alpha \geq 17/30$ the balanced complete bipartite graph gives the best lower bound for the function $\beta(\alpha)$, which leads to

$$\beta(\alpha) = \frac{1}{4}(2\alpha - 1)\,. \tag{1}$$

The same authors verified this conjecture for $\alpha \geq 0.648$ and the best result in this direction is due to Krivelevich [9], who verified it for every $\alpha \geq 3/5$. We say a graph G is a *blow-up* of some graph F, if there exists a partition $V(G) = \dot{\bigcup}_{x \in V(F)} V_x$ such that

$$\forall x, y \in V(F)\ \forall a \in V_x\ \forall b \in V_y : ab \in E(G) \Leftrightarrow xy \in E(F)\,.$$

For $\alpha < 17/30$ balanced blow-ups of the 5-cycle yield a better lower bound for $\beta(\alpha)$ and Erdős et al. conjectured

$$\beta(\alpha) = \frac{1}{25}(5\alpha - 2) \tag{2}$$

for $\alpha \in [53/120, 17/30]$. For $\alpha < 53/120$ it is known that balanced blow-ups of the Andrásfai graph F_3 (see Figure 1) lead to a better bound. The special case $\beta(1/2) = 1/50$ was considered before by Erdős (see, e.g., [4] for a monetary bounty for this problem).

Conjecture 1.1 (Erdős) *Every* $(1/2, 1/50)$*-dense graph contains a triangle.*

Besides the balanced blow-up of the 5-cycle Simonovits (see, e.g., [4]) noted that balanced blow-ups of the Petersen graph yield the same lower bound for Conjecture 1.1 and, more generally, for (2) in the corresponding range.

Conjecture 1.1 asserts that every triangle-free n-vertex graph G contains a subset of $\lfloor n/2 \rfloor$ vertices that spans at most $n^2/50$ edges. Our first result

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[2] Email: Wiebke.Bedenknecht@uni-hamburg.de
[3] Email: mota@ime.usp.br
[4] Email: Christian.Reiher@uni-hamburg.de; schacht@math.uni-hamburg.de

(see Theorem 1.2 below) verifies this for graphs G that are homomorpic to a triangle-free graph from a special class.

1.1 Andrásfai graphs

A well studied family of triangle-free graphs, which appear in the lower bound constructions for the function $\beta(\alpha)$ above, are the so-called *Andrásfai graphs* (see also Woodall [12]). For an integer $d \geq 1$ the Andrásfai graph F_d is the d-regular graph with vertex set

$$V(F_d) = \{v_1, \ldots, v_{3d-1}\},$$

where $\{v_i, v_j\}$ forms an edge if

$$d \leq |i - j| \leq 2d - 1. \tag{3}$$

Note that $F_1 = K_2$ and $F_2 = C_5$ (see Figure 1). It is easy to check that Andrásfai graphs are triangle-free and balanced blow-ups of these graphs play a prominent role in connection with extremal problems for triangle-free graphs (see, e.g., [1],[6],[7],[3]).

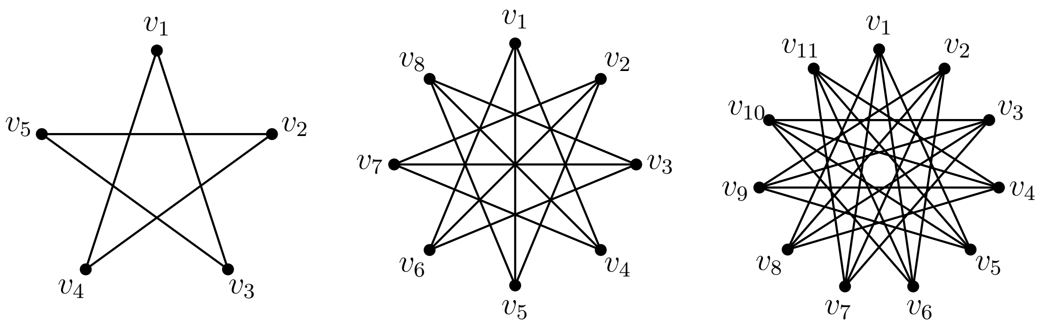

Fig. 1. Andrásfai graphs F_2, F_3, and F_4.

Our first result validates Conjecture 1.1 (stated in the contrapositive) for graphs homomorphic to some Andrásfai graph.

Theorem 1.2 *If a graph G is homomorphic to an Andrásfai graph F_d for some integer $d \geq 1$, then G is not $(1/2, 1/50)$-dense.*

Since F_d is homomorphic to $F_{d'}$ if and only if $d' \geq d$, Theorem 1.2 extends recent work of Norin and Yepremyan [11], who obtained such a result for n-vertex graphs G homomorphic to F_5 with the additional minimum degree assumption $\delta(G) \geq 5n/14$.

Owing to the work of Chen, Jin, and Koh [3], which asserts that every triangle-free 3-chromatic n-vertex graph G with minimum degree $\delta(G) > n/3$ is homomorphic to some Andrásfai graph, we deduce from Theorem 1.2 that Conjecture 1.1 holds for all such graphs G.
Similarly, combining Theorem 1.2 with a result of Jin [7], which asserts that triangle-free graphs G with $\delta(G) > 10n/29$ are homomorphic to F_9, implies Conjecture 1.1 for those graphs as well. We summarise these direct consequences of Theorem 1.2 in the following corollary.

Corollary 1.3 *Let G be a triangle-free graph on n vertices.*

(i) *If $\delta(G) > 10n/29$, then G is not $(1/2, 1/50)$-dense.*

(ii) *If $\delta(G) > n/3$ and $\chi(G) \leq 3$, then G is not $(1/2, 1/50)$-dense.*

We remark that part (i) slightly improves earlier results of Krivelevich [9] and of Norin and Yepremyan [11] (see also [8] where an average degree condition was considered).

1.2 Generalised Andrásfai graphs of higher odd-girth

We consider the following straightforward variation of Andrásfai graphs of *odd-girth* at least $2k+1$, i.e., graphs without odd cycles of length at most $2k-1$. For integers $k \geq 2$ and $d \geq 1$ let F_d^k be the d-regular graph with vertex set

$$V(F_d^k) = \{v_1, \dots, v_{(2k-1)(d-1)+2}\},$$

where $\{v_i, v_j\}$ forms an edge if

$$(k-1)(d-1)+1 \leq |i-j| \leq k(d-1)+1. \tag{4}$$

In particular, for $k = 2$ we recover the definition of the Andrásfai graphs from (3) and for general $k \geq 2$ we have $F_1^k = K_2$, $F_2^k = C_{2k+1}$ and for every $d \geq 2$ the graph F_d^k has odd-girth $2k+1$ (see Figure 2).

Our main result generalises Theorem 1.2 for graphs of odd-girth at least $2k+1$. In fact, the constant $\frac{1}{2(2k+1)^2}$ appearing in Theorem 1.4 is best possible as balanced blow-ups of C_{2k+1} show.

Theorem 1.4 *If a graph G is homomorphic to a generalised Andrásfai graph F_d^k for some integers $k \geq 2$ and $d \geq 1$, then G is not $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense.*

Analogous to the relation between Conjecture 1.1 and Theorem 1.2 one may wonder if every $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense graph contains an odd cycle of length

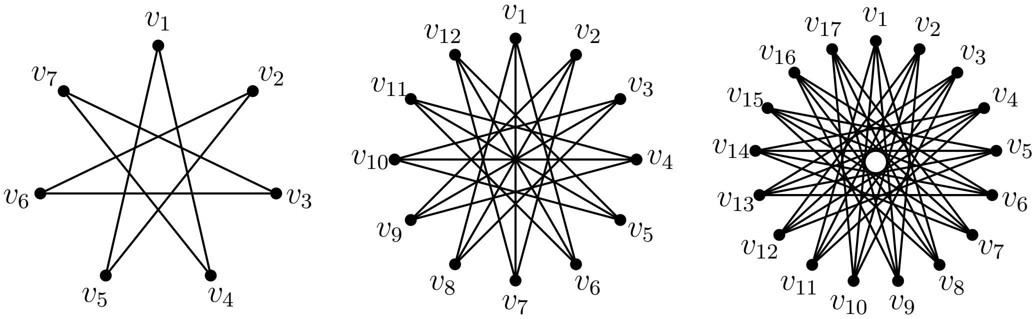

Fig. 2. Generalised Andrásfai graphs F_2^3, F_3^3, and F_4^3 of odd-girth 7.

at most $2k-1$. For n-vertex graphs G with $\delta(G) > \frac{3n}{4k}$ such a result follows from Theorem 1.4 combined with the work from [10].

Corollary 1.5 *Let G be a graph with odd-girth at least $2k+1$ on n vertices. If $\delta(G) > \frac{3n}{4k}$, then G is not $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense.*

For $k = 2$ Theorem 1.4 reduces to Theorem 1.2. For the proof of Theorem 1.4 it will be convenient to work with a geometric representation of such graphs G. In that representation we will arrange the vertices of G on the unit circle $\mathbb{R}/\mathbb{Z}$ and edges between two vertices x and y may only appear depending on their angle with respect to the centre of the circle.

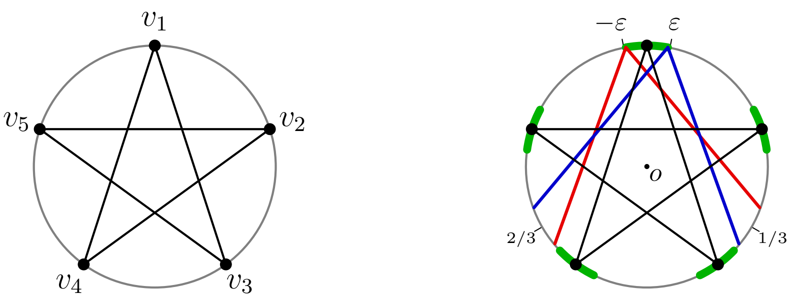

Fig. 3. A copy of $F_2 = C_5$ and a representation of a blow-up on the unit circle.

For example, let G be a blow-up of $F_2 = C_5$. One can distribute the vertices of F_2 equally spaced on the unit circle (see Figure 3). Then we place all vertices of G that correspond to the blow-up class of v_i into a small ε-ball around v_i on the unit circle (cf. green arcs in Figure 3). For a sufficiently small ε, all vertices in an ε-ball around v_i have the same neighbours and they can be characterised by having their smaller angle with respect to the centre bigger than 120° (cf. red and blue lines in Figure 3).

For the proof of Theorem 1.2 we distinguish two cases depending on the independence number $\alpha(G)$ and refer to [2].

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