An expected polynomial time algorithm for coloring 2-colorable 3-graphs

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Abstract

We present an algorithm that colors a random 2-colorable 3-uniform hypergraph optimally in expected running time $O(n^5 \log^2 n)$.

Keywords: coloring, hypergraphs, average case analysis

1 Introduction

One of the classical problems in complexity theory is to decide whether a given k-uniform hypergraph is 2-colorable (or *bipartite*). While for bipartite graphs a 2-coloring can be found in linear time, it was shown by Lovász [10] that the problem to decide whether a given k-uniform hypergraph is bipartite is NP-complete for all $k \geq 3$. Moreover, Guruswami et al. [6] proved that it is NP-hard to color bipartite, k-uniform hypergraphs with a constant number of colors for $k \geq 4$. It was also shown by Dinur et al. [3] that this problem remains inapproximable by a constant for 3-uniform hypergraphs. On the other hand, recently, Krivelevich et al. [9] gave a polynomial time algorithm which colors 3-uniform bipartite hypergraphs using $O(n^{1/5} \log^c n)$ colors. Another positive result is due to Chen and Frieze [1]. Those authors studied colorings of so-called α-dense bipartite 3-uniform hypergraphs, where a 3-uniform hypergraph is α-dense if the collective degree of any two vertices is at least αn. They

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found a randomized algorithm that colors an α-dense 3-uniform hypergraph H in $n^{O(1/\alpha)}$ time.

The purpose of this note is to present an algorithm that colors a hypergraph chosen uniformly at random from the family of all labeled, 3-uniform, bipartite hypergraphs on n vertices in $O(n^5 \log^2 n)$ expected time. Indeed, we prove a slightly more general result for the class of Fano-free hypergraphs, see Theorem 1.1. Before we state it precisely we review related results for graphs.

In 1984 Wilf [15] noted, using a simple counting argument, that one can decide in constant expected time, whether a graph is p-colorable. Few years later Turner [14] found an $O(|V| + |E| \log p)$ algorithm for optimally coloring almost all p-colorable graphs. This result was further expanded by Dyer and Frieze [4] who developed an algorithm which colored every p-colorable graph on n vertices properly (with p colors) in $O(n^2)$ expected time.

Another line of research concerns the study of monotone properties of the type $\mathrm{Forb}(n, L)$ for a fixed graph L, i.e., the family of all labeled graphs on n vertices, which contain no copy of L as a (not necessarily induced) subgraph. Prömel and Steger [13] discovered an algorithm that colors properly (regardless of its value χ) a randomly chosen member from $\mathrm{Forb}(n, K_{p+1})$, i.e., the class of all labeled K_{p+1}-free graphs, in $O(n^2)$ expected time. This is clearly a generalization of the result of Dyer and Frieze in the light of the well known result of Kolaitis et al. [8] that almost all K_{p+1}-free graphs are p-colorable.

In [12] we studied $\mathrm{Forb}(n, F)$, where F is the 3-uniform hypergraph of the Fano plane, which is the unique triple system with 7 hyperedges on 7 vertices where every pair of vertices is contained in precisely one hyperedge. It was shown independently by Füredi and Simonovits [5] and Keevash and Sudakov [7], that for large n the unique extremal Fano-free hypergraph is the balanced, complete, bipartite hypergraph $B_n = (U \dot\cup W, E_{B_n})$, where $|U| = \lfloor n/2 \rfloor$, $|W| = \lceil n/2 \rceil$ and E_{B_n} consists of all hyperedges with at least one vertex in U and one vertex in W. The hypergraph of the Fano plane F is not bipartite, i.e., for every vertex partition $X \dot\cup Y = V(F)$ into two classes there exists a hyperedge of F which is either contained in X or in Y. Consequently, $\mathrm{Forb}(n, F)$ contains any bipartite 3-uniform hypergraph on n vertices. However, deleting any hyperedge from F results in a bipartite hypergraph.

Let $\mathcal{B}_n$ be the class of all labeled bipartite hypergraphs on n vertices. It was shown in [12] that

$$|\mathrm{Forb}(n, F)| \leq (1 + 2^{-\Omega(n^2)})|\mathcal{B}_n|\,. \tag{1}$$

Our main result here states that one can color a 3-uniform hypergraph chosen uniformly at random from $\mathrm{Forb}(n, F)$ in polynomial expected time.

Theorem 1.1 *There is an algorithm with average running time $O(n^5 \log^2 n)$ which colors every member from $\mathrm{Forb}(n, F)$ optimally.*

Together with (1) we immediately derive in a similar manner to Steger and Prömel [13] that one can color a 3-uniform hypergraph chosen uniformly at random from $\mathcal{B}_n$ in polynomial expected time.

Corollary 1.2 *There is an algorithm with average running time $O(n^5 \log^2 n)$ which finds a bipartition of every member from $\mathcal{B}_n$.*

2 Algorithm for coloring Fano-free hypergraphs

Below we first present the simple algorithm $\mathrm{Color}(H)$ which will be based on the subroutine $\mathrm{Partition}(H, \alpha)$:

Algorithm 1 Color (H)

Input: *H from $\mathrm{Forb}(n, F)$;*

Output: *Optimal coloring of H;*

(1) *choose "small" $\alpha > 0$ appropriately;*

(2) $(X, Y) \leftarrow$ *Partition* (H, α);

(3) **If** $e(X) + e(Y) = 0$

(4) **then** *output 2-coloring corresponding to (X, Y);*

(5) **else** *try all $n^n = 2^{n \log n}$ possible colorings and output the one that minimizes the number of colors used;*

Obviously, $\mathrm{Color}(H)$ finds an optimal coloring of H. For proving Theorem 1.1 we will show that there exists an $\alpha > 0$ such that Step 5 of the algorithm will be executed for at most $2^{-n \log n} |\mathrm{Forb}(n, F)|$ 3-uniform hypergraphs from $\mathrm{Forb}(n, F)$, while Step 2 has a running time of $O(n^5 \log^2 n)$ for all H.

The subroutine $\mathrm{Partition}(H, \alpha)$ finds a *locally minimal* partition $X_H \dot{\cup} Y_H = V(H)$, i.e., a partition for which $e(X_H) + e(Y_H)$ cannot be decreased by moving a single vertex from one class to another. Moreover, for "most" 3-uniform hypergraphs H from $\mathrm{Forb}(n, F)$ the algorithm $\mathrm{Partition}(H, \alpha)$ outputs a partition with the additional property $e(X_H) + e(Y_H) < \alpha n^3$.

Algorithm 2 Partition (H, α)

Input: *$H \in \mathrm{Forb}(n, F)$, $\alpha > 0$;*

Output: *locally minimal vertex partition of H: $V = X_H \dot{\cup} Y_H$;*

(1) *choose* $\varepsilon := \varepsilon(\alpha)$ *and* $\eta := \eta(\alpha)$ *appropriately;*

(2) *apply* Regularize$(H, \varepsilon, \lceil 1/\varepsilon \rceil)$ *and obtain an* ε*-regular partition* $V_1, \ldots, V_t$*;*

(3) *define cluster hypergraph* $H(\eta)$ *with densities at least* η*;*

(4) *find minimal vertex bipartition of* $H(\eta)$*;*

(5) *retrieve corresponding vertex bipartition of* $H : W_1 \dot\cup W_2 = V$*;*

(6) **while** $\exists w \in W_i$ *s.t.* $\deg_{W_i}(w) > \deg_{W_{[2]\setminus\{i\}}}(w)$ **do** *move* w *to* $W_{[2]\setminus\{i\}}$*;*

In Step 2 the algorithm Regularize(H, ε, t_0) was used. This algorithm, due to Czygrinow and Rödl [2], finds an ε-regular partition of a 3-uniform hypergraph H on n vertices and at least t_0 many clusters in time $O(n^5 \log^2 n)$. Since all of the steps can be implemented in $O(n^5 \log^2 n)$ time, it follows that Partition(H, α) requires $O(n^5 \log^2 n)$ steps.

Thus, it is still left to analyze the amount of the hypergraphs H from Forb(n, F) for which an exhaustive search in Color(H) is needed (Step 5). In the main part of the proof we show that there are at most $2^{-\Omega(n^2)}|\text{Forb}(n, F)|$ such hypergraphs in Forb(n, F). To prove this, we study structural properties of a typical H from Forb(n, F). Our analysis is based on the techniques from [12]. We introduce a chain of subsets of Forb(n, F) such that all members of them possess certain "typical" properties. The first subset of it will be $\mathcal{F}'_n(\alpha)$, which will consist of those members that admit a bipartition such that the number of hyperedges inside the bipartition classes is at most αn^3. Using the properties of the weak hypergraph regularity lemma (i.e. it partitions the vertex set of a hypergraph into constantly many equal-sized pieces), it can be shown that, firstly, most of the hypergraphs from Forb(n, F) lie in $\mathcal{F}'_n(\alpha)$ and, secondly, that for most of the members from $\mathcal{F}'_n(\alpha)$ the algorithm Partition(H, α) finds a locally minimal partition for given α.

The further analysis proceeds as follows. We introduce two more proper subsets of Forb(n, F), which describe two further "useful" properties of almost all Fano-free hypergraphs on n vertices. We then deduce that the last property implies in fact bipartiteness. As a seemingly surprising fact, we obtain, that for almost all members from Forb(n, F) *any* locally minimal partition for some appropriate α already satisfies $e(X_H) + e(Y_H) = 0$. The details can be found in the full version of the article [11].

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