

MONOCHROMATIC TREES IN RANDOM GRAPHS

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ABSTRACT. Bal and DeBiasio [*Partitioning random graphs into monochromatic components*, Electron. J. Combin. **24** (2017), Paper 1.18] put forward a conjecture concerning the threshold for the following Ramsey-type property for graphs G : every k -colouring of the edge set of G yields k pairwise vertex disjoint monochromatic trees that partition the whole vertex set of G . We determine the threshold for this property for two colours.

§1. INTRODUCTION

For a graph $G = (V, E)$ we write $G \longrightarrow \Pi_2$ if for every 2-colouring of E , say with colours red and blue, there exist two monochromatic trees $T_1, T_2 \subseteq G$ such that

$$V(T_1) \cup V(T_2) = V,$$

i.e., V can be split into two sets each inducing a spanning monochromatic component. Here we allow one of the trees to be empty and we also allow both trees to be monochromatic of the same colour. In [1, Conjecture 8.1] Bal and DeBiasio conjectured that if

$$p = p(n) > (1 + \varepsilon) \left(\frac{2 \ln n}{n} \right)^{1/2}$$

for some $\varepsilon > 0$, then *asymptotically almost surely* (a.a.s.) the binomial random graph $G(n, p)$ satisfies $G(n, p) \longrightarrow \Pi_2$, i.e.,

$$\lim_{n \rightarrow \infty} \mathbb{P}(G(n, p) \longrightarrow \Pi_2) = 1.$$

One can observe that the conjectured condition on p would be best possible. In fact, if $p < (1 - \varepsilon) \left(\frac{2 \ln n}{n} \right)^{1/2}$ for some $\varepsilon > 0$, then a.a.s. $G(n, p)$ has diameter at least three (see, e.g., [3, Chapter 10]) and, hence, it contains two non-adjacent vertices u and v with disjoint neighbourhoods. Colouring all edges incident to u or v red and all other edges blue

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produces a colouring that requires at least three monochromatic trees in any decomposition of $V(G(n, p))$, since u and v cannot be in the same red tree.

Bal and DeBiasio showed that a.a.s. $G(n, p) \rightarrow \Pi_2$ provided that $p > C \left(\frac{\ln n}{n}\right)^{1/3}$ for some suitable constant $C > 1$. We improve on that result by showing that $\left(\frac{\ln n}{n}\right)^{1/2}$ is the threshold for that property.

Theorem 1.1. *If $p = p(n) \gg \left(\frac{\ln n}{n}\right)^{1/2}$, then a.a.s. $G(n, p) \rightarrow \Pi_2$.*

Combined with the discussion above, Theorem 1.1 implies that $\left(\frac{\ln n}{n}\right)^{1/2}$ is the threshold for the property $G \rightarrow \Pi_2$. We remark that our proof also yields a semi-sharp threshold, since with not much additional effort we could replace the assumption $p \gg \left(\frac{\ln n}{n}\right)^{1/2}$ by $p > C \left(\frac{\ln n}{n}\right)^{1/2}$ for some suitable constant $C > 1$. However, for a simpler presentation we chose to avoid these calculations and we will only consider the case stated in Theorem 1.1. In fact, since Theorem 1.1 implies that the threshold function for the monotone graph property $G \rightarrow \Pi_2$ is not of the form $n^{-\alpha}$ for some rational $\alpha \in \mathbb{Q}_{>0}$ it follows from Friedgut's criterion [7, Theorem 1.4] that $G \rightarrow \Pi_2$ has indeed a sharp threshold, i.e., there exist constants $c_1 > c_0 > 0$ and a function $c: \mathbb{N} \rightarrow \mathbb{R}$ with $c_0 < c(n) < c_1$ for every $n \in \mathbb{N}$ such that for every $\varepsilon > 0$ we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(G(n, p) \rightarrow \Pi_2) = \begin{cases} 0, & \text{if } p < (1 - \varepsilon)c(n) \left(\frac{\ln n}{n}\right)^{1/2} \\ 1, & \text{if } p > (1 + \varepsilon)c(n) \left(\frac{\ln n}{n}\right)^{1/2}. \end{cases}$$

In view of the question of Bal and DeBiasio [1] it remains to show that $c(n)$ is a constant independent of n and that we have $c(n) \equiv \sqrt{2}$.

Finally, we remark that Bal and DeBiasio [1] also considered multicoloured extensions of this problem and several other interesting variants. Among other they proposed an extension of Theorem 1.1 for r -colourings of the edges of $G(n, p)$. More precisely, Bal and DeBiasio conjectured that if $p = p(n) > (1 + \varepsilon) \left(\frac{r \ln n}{n}\right)^{1/r}$ for some $\varepsilon > 0$, then a.a.s. every r -colouring of the edges of $G(n, p)$ admits a partition of $V(G)$ into at most r sets each inducing a spanning monochromatic component. It was noted by Ebsen, Mota, and Schnitzer [6] that this conjecture fails to be true and that for $r \geq 3$ the threshold for the partition property is at least $\left(\frac{\ln n}{n}\right)^{\frac{1}{r+1}}$. We present their example in Proposition 4.1 in Section 4.

Roughly speaking, the proof of Theorem 1.1, given in Section 3, splits into two parts. We shall define what we mean by an *extremal* colouring of the edges of a graph, and we shall consider the extremal and the non-extremal cases separately. We shall first consider the somewhat simpler case of non-extremal colourings in Section 3.1. Extremal colourings will be harder to handle and such colourings will be analysed in Section 3.2. Before the

discussion of these two cases we collect a few observations concerning random graphs in Section 2.

§2. PRELIMINARIES

We consider finite simple graphs and follow standard notation and terminology (see [2, 4, 5] and [3, 8]). We shall make use of the following simple lemma on random graphs.

Lemma 2.1. *If $p = p(n) \gg ((\ln n)/n)^{1/2}$, then for every $\varepsilon > 0$ a.a.s. $G \in G(n, p)$ satisfies the following properties.*

- (i) *Every vertex $v \in V(G)$ has degree $d_G(v) = (1 \pm \varepsilon)pn$ and every pair of distinct vertices $u, w \in V(G)$ has $|N_G(u) \cap N_G(w)| = (1 \pm \varepsilon)p^2n$ joint neighbours.*
- (ii) *For every vertex $v \in V(G)$ and all disjoint subsets $U \subseteq V$ and $W \subseteq N_G(v)$ with $|U| \geq 100/p$ and $|W| \geq pn/100$ the number $e_G(U, W)$ of edges in the induced bipartite graph $G[U, W]$ satisfies $e_G(U, W) > p|U||W|/2$.*
- (iii) *For every vertex $v \in V(G)$ and $J \subseteq N_G(v)$ with $|J| \geq pn/100$, we have that all but at most $100/p$ vertices $x \in V(G) \setminus J$ satisfy $|N_G(x) \cap J| > p^2n/200$.*
- (iv) *For every vertex $y \in V(G)$ and $A \cup B = U \subseteq N_G(y)$ with $|U| \geq |N_G(y)| - p^2n/100$ and $|A|, |B| \geq p^2n/2$, the induced bipartite graph $G[A, B]$ contains at least $p^2n/100$ vertices of degree at least $p^2n/100$.*
- (v) *Every subgraph $H \subseteq G$ with minimum degree $\delta(H) \geq (1/2 + \varepsilon)pn$ is connected.*
- (vi) *Every subgraph $H \subseteq G$ on at most $100/p$ vertices is $10 \ln n$ -degenerate.*

Proof. Properties (i)–(vi) in Lemma 2.1 follow from the concentration of the binomial distribution. In fact, property (i) is a direct consequence of Chernoff's inequality.

Property (ii) also follows from that inequality by the following argument. For disjoint subsets $U, W \subseteq V$ Chernoff's inequality (see, e.g., [8, Theorem 2.1]) yields

$$\mathbb{P}(e_G(U, W) \leq \frac{1}{2}p|U||W|) \leq \exp(-p|U||W|/8).$$

Summing over all possible choices of $v \in V$ and all subsets $U \subseteq V$ and $W \subseteq N_G(v)$ considered in the property, we arrive at

$$\begin{aligned} \mathbb{P}(\text{property (ii) fails}) &\leq n \sum_{u \geq 100/p} \sum_{w \geq pn/100} \binom{n}{u} \binom{n}{w} p^w \exp(-puw/8) \\ &\leq n \sum_{u \geq 100/p} \sum_{w \geq pn/100} \exp(u \ln n) \left(\frac{enp}{w}\right)^w \exp(-puw/8) \\ &\leq n \sum_{u \geq 100/p} \sum_{w \geq pn/100} \exp(u \ln n + 6w - puw/8). \end{aligned}$$

Since $puw/16 - 6w \geq w/4$ for $u \geq 100/p$ and, since $puw/16 \geq up^2n/1600 \gg u \ln n$ for $w \geq pn/100$ and $p \gg ((\ln n)/n)^{1/2}$, it follows that

$$\mathbb{P}(\text{property (ii) fails}) \leq n \sum_{100/p \leq u \leq n} \sum_{w \geq pn/100} \exp(-w/4) = o(1),$$

which concludes the proof of Lemma 2.1 (ii).

Property (iii) follows from (ii). Given a vertex v and a subset $J \subseteq N_G(v)$ of size at least $pn/100$ we consider the set

$$U = \{x \in V(G) \setminus J : |N_G(x) \cap J| \leq p^2n/200\}.$$

Assuming for a contradiction that $|U| > 100/p$ we infer from (ii) that

$$e_G(U, J) > p|U||J|/2 \geq p|U| \cdot pn/200 = p^2n|U|/200,$$

which contradicts the definition of the set U . Consequently, $|U| \leq 100/p$ and property (iii) is established.

The proof of property (iv) makes use of the fact that a.a.s. for every $y \in V$ and every subset $A \subseteq N_G(y)$ with $p^2n/2 \leq |A| \leq |N_G(y) \setminus A|$ we have

$$e_G(A, N_G(y) \setminus A) > \frac{4}{25}p^2n|A|. \quad (2.1)$$

In fact, property (iv) follows from (2.1) and we prove this implication first. Let a vertex y and sets A , B and U be as in the statement of (iv). Without loss of generality, we may suppose $|A| \leq |B| \leq |N_G(y) \setminus A|$, and hence we can apply (2.1). Removing all vertices from A that have less than $p^2n/50$ neighbours in $N_G(y) \setminus A$ and using the bound $|N_G(y) \cap N_G(a)| \leq 2p^2n$ for all $a \in A$, which is given by (i), we deduce from (2.1) that at least

$$\frac{4p^2n|A|/25 - |A|p^2n/50}{2p^2n} = \frac{7|A|}{100} > \frac{p^2n}{100}$$

vertices in A have at least $p^2n/50$ neighbours in $N_G(y) \setminus A$. Since $B = (N_G(y) \setminus A) \setminus B'$ for some $|B'| \leq p^2n/100$, property (iv) follows and it is left to verify (2.1).

For the proof of (2.1) we may assume that $|A| \leq |N_G(y) \setminus A|$ and we consider two cases depending on the size of A . If $|A| \geq 100/p$ inequality (2.1) is a consequence of property (ii) applied with $v = y$ and the disjoint sets A and $N_G(y) \setminus A$ combined with the first part of (i), which leads to

$$e_G(A, N_G(y) \setminus A) \stackrel{(ii)}{\geq} \frac{1}{2}p|A||N_G(y) \setminus A| \stackrel{(i)}{\geq} \frac{1}{2}p|A| \cdot \frac{1}{3}pn > \frac{4}{25}p^2n|A|.$$

For the case $|A| \leq 100/p$ we have $p^2n|A| \gg p|A|^2$. Hence, we may use the concentration inequality $\mathbb{P}(X > t) \leq \exp(-t)$ for binomially distributed random variables X satisfying $\mathbb{E}[X] \leq t/7$ (see, e.g., [8, Corollary 2.4]) to derive that, for every fixed set A , we

have

$$\mathbb{P}(2e_G(A) > p^2n|A|/4) \leq \exp(-p^2n|A|/4).$$

Summing over all sets A of size at most $100/p$ yields

$$\begin{aligned} \mathbb{P}(\exists A \subseteq V \text{ with } |A| \leq 100/p \text{ such that } 2e_G(A) > p^2n|A|/4) \\ \leq \sum_{a=p^2n/4}^{100/p} n^a \exp(-p^2na/4) = o(1), \end{aligned} \quad (2.2)$$

where the last inequality follows from our assumption on p . We infer (2.1) from (2.2). Given $y \in V(G)$ and $A \subseteq N_G(y)$ with $p^2n/2 \leq |A| \leq 100/p$ we appeal to the second assertion of property (i) with $\varepsilon = 1/2$ for all pairs of the form y, a with $a \in A$. Summing $|N_G(y) \cap N_G(a)|$ over all $a \in A$ yields

$$e_G(A, N_G(y) \setminus A) > \frac{1}{2}p^2n|A| - 2e_G(A) \stackrel{(2.2)}{>} \frac{1}{6}p^2n|A|$$

and (2.1) follows. This concludes the proof of property (iv).

For property (v) we observe that for $p \gg (\ln n)/n$ and every fixed $\delta > 0$, again Chernoff's inequality implies that a.a.s., for every subset $U \subseteq V$, we have

$$2e_G(U) < p|U|^2 + \delta pn|U|. \quad (2.3)$$

To prove (2.3), one can analyse the cases in which $\delta n/|U| \leq 3/2$, $3/2 < \delta n/|U| < 7$ and $\delta n/|U| \geq 7$ separately. For the first two cases, one can use one of the standard forms of Chernoff's inequality, as given in, e.g., [8, Corollary 2.3]. For the third case, one can again use [8, Corollary 2.4].

Next we consider an arbitrary component C of the subgraph $H \subseteq G$ and let $U = V(C)$. Combining (2.3) for $\delta = \varepsilon$ with the minimum degree assumption tells us that

$$|U| \cdot (1/2 + \varepsilon)pn \leq 2e_G(U) < p|U|^2 + \varepsilon pn|U|,$$

which implies $|U| > n/2$. Consequently, every component of H spans more than $n/2$ vertices, which implies that H is connected.

For the proof of (vi) it suffices to show that every subset $U \subseteq V$ of size at most $100/p$ contains a vertex of degree at most $10 \ln n$. However, this follows from the observation that for every such set U we have

$$e_G(U) \leq |U| \cdot 5 \ln n,$$

which again can be deduced from the concentration inequality given in [8, Corollary 2.4]. \square

§3. PROOF OF THE MAIN RESULT

We introduce some further notation and classify the two-colourings into two classes (see Definition 3.1 below). For a colouring $\varphi: E \rightarrow \{\text{red}, \text{blue}\}$ of the edges of a graph $G = (V, E)$ we write $\varphi \rightarrow \Pi_2$ to indicate that there exist two monochromatic trees $T_1, T_2 \subseteq G$ such that

$$V(T_1) \cup V(T_2) = V.$$

In particular, $G \rightarrow \Pi_2$ if $\varphi \rightarrow \Pi_2$ holds for all 2-colourings φ of E . We denote the two edge disjoint spanning monochromatic subgraphs induced by φ by G_{red}^φ and G_{blue}^φ , i.e.,

$$G_{\text{red}}^\varphi = (V, \varphi^{-1}(\text{red})) \quad \text{and} \quad G_{\text{blue}}^\varphi = (V, \varphi^{-1}(\text{blue})).$$

For a vertex $v \in V$ we consider its red- and blue-*neighbourhood*

$$N_{\text{red}}^\varphi(v) = \{u \in N(v) : \varphi(\{v, u\}) = \text{red}\} \quad \text{and} \quad N_{\text{blue}}^\varphi(v) = \{u \in N(v) : \varphi(\{v, u\}) = \text{blue}\}$$

and the corresponding degrees $d_{\text{red}}^\varphi(v) = |N_{\text{red}}^\varphi(v)|$ and $d_{\text{blue}}^\varphi(v) = |N_{\text{blue}}^\varphi(v)|$. We roughly classify the vertices depending on these degrees by defining the following sets

$$R^\varphi = \{v \in V : d_{\text{red}}^\varphi(v) > \frac{1}{3}d(v)\} \quad \text{and} \quad B^\varphi = \{v \in V : d_{\text{blue}}^\varphi(v) > \frac{1}{3}d(v)\}. \quad (3.1)$$

These sets might not be disjoint, but every vertex is a member of at least one of them and vertices v in the symmetric difference of these sets have at least $2d(v)/3$ neighbours in one colour. In the proof of Theorem 1.1 we consider two cases depending, whether there is a monochromatic path between some vertex in R^φ and a different vertex in B^φ .

Definition 3.1. *Let $G = (V, E)$ be a graph and $\varphi: E \rightarrow \{\text{red}, \text{blue}\}$. We say φ is extremal if there is a pair of distinct vertices $r \in R^\varphi$ and $b \in B^\varphi$ for which no monochromatic r - b -path exists. If no such pair of vertices exists, then we say φ is non-extremal.*

For the proof of Theorem 1.1 we consider non-extremal and extremal colourings φ separately. Before we proceed, let us remark that the property $G \rightarrow \Pi_2$ is an increasing property, that is, if G is a spanning subgraph of G' and $G \rightarrow \Pi_2$ holds, then $G' \rightarrow \Pi_2$ also holds. This implies that it suffices to prove Theorem 1.1 under the additional hypothesis that $p = o(1)$.

3.1. Non-extremal colourings. The following proposition addresses the case when φ is non-extremal.

Proposition 3.2 (Non-extremal case). *If $p = p(n) \gg ((\ln n)/n)^{1/2}$ and $p = o(1)$, then a.a.s. $G \in G(n, p)$ satisfies $\varphi \rightarrow \Pi_2$ for every non-extremal colouring $\varphi: E(G) \rightarrow \{\text{red}, \text{blue}\}$.*

In the proof of Proposition 3.2 we shall make use of the following simple observation, which is closely related to the fact that every 2-colouring of the edges of the complete graph yields a monochromatic spanning tree.

Lemma 3.3. *Let $G = (V, E)$ be a graph and $\varphi: E \rightarrow \{\text{red}, \text{blue}\}$. If for a subset $U \subseteq V$ all pairs of vertices $u, u' \in U$ are connected by a monochromatic path, then there exists a monochromatic tree T with $V(T) \supseteq U$.*

Proof. Let T be a monochromatic tree containing the maximum number of vertices from U . We may assume that T is colored red. If there is some vertex $u \in U \setminus V(T)$, then it must be connected to every vertex $u' \in U \cap V(T)$ by a blue u - u' -path, which results in a monochromatic tree containing at least one more vertex from U than T . \square

With this observation at hand we can now establish the proof of the proposition.

Proof of Proposition 3.2. Owing to $p \gg \left(\frac{\ln n}{n}\right)^{1/2}$ we may and shall assume that for $\varepsilon = 1/10$ the graph $G = (V, E) \in G(n, p)$ satisfies properties (i)–(vi) given in Lemma 2.1. Moreover, let $\varphi: E \rightarrow \{\text{red}, \text{blue}\}$ be a non-extremal colouring, which is fixed throughout the proof. For simpler notation, we suppress the superscript φ in terms like G_{red}^φ , $N_{\text{red}}^\varphi(v)$, $d_{\text{red}}^\varphi(v)$, R^φ , and their blue counterparts.

If one of the sets R or B , say R , is empty, then it follows from property (i) that every vertex in G satisfies $d_{\text{blue}}(v) \geq (2/3 - \varepsilon)pn$. Hence, by property (v) there exists a blue spanning tree of G and $\varphi \rightarrow \Pi_2$.

Since φ is non-extremal, between every vertex $r \in R$ and every $b \in B$ there exists a monochromatic r - b -path. In particular, vertices contained in the intersection $R \cap B$ are connected to every other vertex by a monochromatic path.

Below we show that there exist monochromatic components $C_{\text{red}} \subseteq G_{\text{red}}$ and $C_{\text{blue}} \subseteq G_{\text{blue}}$ covering V , i.e.,

$$V(C_{\text{blue}}) \cup V(C_{\text{red}}) = V. \quad (3.2)$$

Consider a monochromatic component C containing the most number of vertices. In particular, any pair of vertices in C can be connected by a monochromatic path. If C would be completely contained in R or B , say without loss of generality in R , then we can fix an arbitrary vertex $v \in B$ and Lemma 3.3 would show that there exists a monochromatic component containing C and v , which violates the maximal choice of C . Therefore, C intersects each set R and B in at least one vertex, say $v_r \in R$ and $v_b \in B$ and without loss of generality we may assume C is coloured red.

Then for every vertex $u \in R \setminus V(C)$ the monochromatic v_b - u -path must be blue and, hence, all pairs of vertices in $R \setminus V(C)$ are connected by a blue path. Consequently, all

pairs of vertices in

$$(V(C) \cap B) \cup (R \setminus V(C)) \quad (3.3)$$

are connected by monochromatic paths and another application of Lemma 3.3 yields a monochromatic component C' containing the vertices from (3.3). Similarly, there exists a monochromatic component C'' containing all vertices from

$$(V(C) \cap R) \cup (B \setminus V(C)).$$

In particular, C' and C'' cover all vertices of G . If both these components have the same colour then we either found two disjoint monochromatic trees covering V or one such tree, i.e., $\varphi \longrightarrow \Pi_2$. If C' and C'' are of different colours then (3.2) follows.

It is left to deduce the proposition from (3.2). Let $C_{\text{red}} \subseteq G_{\text{red}}$ and $C_{\text{blue}} \subseteq G_{\text{blue}}$ satisfy (3.2). We may assume that both components are maximal, i.e., every vertex in the complement of C_{red} has only blue neighbours in C_{red} and, analogously, every vertex in the complement of C_{blue} has only red neighbours in C_{blue} . We consider the symmetric difference of C_{red} and C_{blue} and let

$$O_{\text{red}} = V(C_{\text{red}}) \setminus V(C_{\text{blue}}) \quad \text{and} \quad O_{\text{blue}} = V(C_{\text{blue}}) \setminus V(C_{\text{red}})$$

be the two parts of the symmetric difference, where vertices in O_{red} are only contained in C_{red} and those from O_{blue} are only contained in C_{blue} . Note that the maximal choice of C_{red} and C_{blue} implies that there is no edge between O_{red} and O_{blue} . In fact, there is not even a monochromatic path between O_{red} and O_{blue} , since every edge leaving O_{red} is blue and every edge entering O_{blue} is red. Owing to the assumption that every vertex in R is connected by a monochromatic path with every vertex in B we arrive at one of the following two cases

- (I) $O_{\text{red}} = \emptyset$ or $O_{\text{blue}} = \emptyset$,
- (II) $O_{\text{red}} \cup O_{\text{blue}} \subseteq R \setminus B$ or $O_{\text{red}} \cup O_{\text{blue}} \subseteq B \setminus R$.

To see that one of the cases must occur, let us assume case (I) does not hold and let $v \in O_{\text{red}}$ and $u \in O_{\text{blue}}$. As noted above it is not possible that one of the vertices is contained in R , while the other one is a member of B . Consequently, both of them must be contained in $R \setminus B$ or in $B \setminus R$. Repeating the same argument for pairs (v, u') with $u' \in O_{\text{blue}}$ and pairs (v', u) with $v' \in O_{\text{red}}$ yields case (II).

Next we note that case (I) asserts that one of the parts of the symmetric difference of C_{red} and C_{blue} is empty, which combined with (3.2) implies the existence of a monochromatic spanning tree in G .

For case (II) we can assume without loss of generality that $O_{\text{red}} \cup O_{\text{blue}} \subseteq R \setminus B$. We infer from the maximality of C_{red} that no vertex in O_{blue} has a red neighbour in C_{red} , and,

therefore,

$$N_{\text{red}}(v) \subseteq O_{\text{blue}}$$

for every $v \in O_{\text{blue}}$. Since $O_{\text{blue}} \subseteq R \setminus B$ it follows from property (i) that G_{red} induced on O_{blue} has minimum degree $(2/3 - \varepsilon)pn$. Consequently, property (v) yields a red spanning tree on O_{blue} and combined with a red spanning tree on C_{red} we found two vertex disjoint red trees covering G , which concludes the proof of Proposition 3.2. \square

3.2. Extremal colourings. In this section we consider extremal colourings φ and establish an analogous proposition as in the non-extremal case. Together Propositions 3.2 and 3.4 establish Theorem 1.1.

Proposition 3.4 (Extremal case). *If $p = p(n) \gg ((\ln n)/n)^{1/2}$ and $p = o(1)$, then a.a.s. $G \in G(n, p)$ satisfies $\varphi \rightarrow \Pi_2$ for every extremal colouring $\varphi: E(G) \rightarrow \{\text{red}, \text{blue}\}$.*

Proof. As in the proof of Proposition 3.2 we may and shall assume that $G = (V, E) \in G(n, p)$ satisfies properties (i)–(vi) for $\varepsilon = 1/100$ given in Lemma 2.1. Let $\varphi: E \rightarrow \{\text{red}, \text{blue}\}$ be a fixed extremal colouring and again, for simpler notation, in what follows we suppress the superscript φ in terms like G_{red}^φ , $N_{\text{red}}^\varphi(v)$, $d_{\text{red}}^\varphi(v)$, R^φ , and their blue counterparts.

Let $r \in R$ and $b \in B$ be two distinct vertices for which no monochromatic r - b -path exists. We shall build a red and a blue tree with roots r and b . We sometimes refer to r as the *red root* and to b as the *blue root*. The trees will be built in two stages. In the first stage every vertex $v \in V \setminus \{r, b\}$ will be assigned a *preferred colour* $\varrho(v)$, which indicates its ‘preference’. In fact, the preferred colour $\varrho(v)$ will be chosen in such a way that v can be connected in the ‘right colour’ to r or b in a robust way, that is, there will be ‘many’ $\varrho(v)$ -coloured paths from v to the root of colour $\varrho(v)$. The preferred colours will be assigned vertex by vertex and earlier choices may influence those chosen later. However, in this process it might turn out that a later vertex v needs to be connected to the blue tree through an earlier vertex u with $\varrho(u) = \text{red}$ (thus u would in principle belong to the red tree that we are building). To resolve such conflicts, we finalise the choices in a second round after every vertex has chosen its preferred colour and, in fact, here some vertices may get connected to the tree opposite to its preferred colour (e.g., because of v above we may decide to override u ’s preference ($\varrho(u) = \text{red}$) and connect u to the blue tree). Below we give the details of this approach.

First stage: choosing preferred colours. We begin with the neighbours of r and b which are connected by an edge of the ‘right colour’ to the respective root. For those

vertices v , we set the preferred colour to the obvious choice:

$$\varrho(v) = \begin{cases} \text{red,} & \text{if } v \in N_{\text{red}}(r) \setminus N_{\text{blue}}(b) \\ \text{blue,} & \text{if } v \in N_{\text{blue}}(b) \setminus N_{\text{red}}(r). \end{cases} \quad (3.4)$$

For symmetry reasons we defer the assignment of $\varrho(v)$ to the vertices v in $N_{\text{red}}(r) \cap N_{\text{blue}}(b)$ for a moment. Next we consider the edges between $N_{\text{red}}(r)$ and $N_{\text{blue}}(b)$. Recall that we assume that properties (i)–(vi) in Lemma 2.1 hold for G . Recall also that we suppose that $p = o(1)$. Both assertions in property (i), combined with the definition of the sets R and B , allow us to invoke property (ii) to obtain that

$$e_G(N_{\text{red}}(r) \setminus N_{\text{blue}}(b), N_{\text{blue}}(b) \setminus N_{\text{red}}(r)) \geq \frac{p}{2} |N_{\text{red}}(r) \setminus N_{\text{blue}}(b)| |N_{\text{blue}}(b) \setminus N_{\text{red}}(r)|.$$

At least half of these edges have the same colour and, by symmetry, *we may assume that they are red*. We continue with the following claim.

Claim 3.5. *At least $pn/100$ vertices $v \in N_{\text{blue}}(b) \setminus N_{\text{red}}(r)$ satisfy*

$$|N_{\text{red}}(v) \cap (N_{\text{red}}(r) \setminus N_{\text{blue}}(b))| > \frac{p^2 n}{25}. \quad (3.5)$$

Proof. The vertices $v \in N_{\text{blue}}(b) \setminus N_{\text{red}}(r)$ with

$$|N_{\text{red}}(v) \cap (N_{\text{red}}(r) \setminus N_{\text{blue}}(b))| \leq \frac{p}{8} |N_{\text{red}}(r) \setminus N_{\text{blue}}(b)| \quad (3.6)$$

can account for at most $(p/8)|N_{\text{red}}(r) \setminus N_{\text{blue}}(b)||N_{\text{blue}}(b) \setminus N_{\text{red}}(r)|$ red edges between the sets $N_{\text{red}}(r) \setminus N_{\text{blue}}(b)$ and $N_{\text{blue}}(b) \setminus N_{\text{red}}(r)$, of which there are at least

$$\frac{1}{4} p |N_{\text{red}}(r) \setminus N_{\text{blue}}(b)| |N_{\text{blue}}(b) \setminus N_{\text{red}}(r)|.$$

Therefore, in view of property (i), there must be at least

$$\frac{\frac{p}{8} |N_{\text{red}}(r) \setminus N_{\text{blue}}(b)| |N_{\text{blue}}(b) \setminus N_{\text{red}}(r)|}{(1 + \varepsilon)p^2 n} > \frac{1}{25} |N_{\text{blue}}(b) \setminus N_{\text{red}}(r)| > \frac{pn}{100} \quad (3.7)$$

vertices $v \in N_{\text{blue}}(b) \setminus N_{\text{red}}(r)$ with

$$|N_{\text{red}}(v) \cap (N_{\text{red}}(r) \setminus N_{\text{blue}}(b))| > \frac{p}{8} |N_{\text{red}}(r) \setminus N_{\text{blue}}(b)| > \frac{p^2 n}{25}, \quad (3.8)$$

as required. \square

The vertices v satisfying (3.5) play a special rôle in the proof, since they can be used to connect other vertices to both roots, as they are blue neighbours of b and connect (robustly) by red paths of length two to r . Furthermore, the vertices in $N_{\text{red}}(r) \cap N_{\text{blue}}(b)$ are even direct neighbours of both roots in the right colour. We will refer to the vertices in

$$J = \{v \in N_{\text{blue}}(b) \setminus N_{\text{red}}(r) : v \text{ satisfies (3.5)}\} \cup (N_{\text{red}}(r) \cap N_{\text{blue}}(b)) \quad (3.9)$$

as the *joker vertices*. Note that Claim 3.5 implies

$$|J| > \frac{pn}{100}. \quad (3.10)$$

For the presentation, it will also be simpler to give all joker vertices the same preferred colour and, hence, we set

$$\varrho(v) = \text{blue}$$

for all $v \in N_{\text{red}}(r) \cap N_{\text{blue}}(b)$. This way we have defined $\varrho(v)$ for every $v \in N_{\text{red}}(r) \cup N_{\text{blue}}(b)$.

Among the vertices not considered so far we turn first to those with a decent number of joker vertices as neighbours. More precisely, we set

$$X = \left\{ x \in V \setminus (N_{\text{red}}(r) \cup N_{\text{blue}}(b) \cup \{r, b\}) : |N(x) \cap J| > \frac{p^2n}{200} \right\}. \quad (3.11)$$

In particular, every vertex $x \in X$ has more than $p^2n/400$ jokers as neighbours in one colour and this will be its preferred colour, i.e., for every $x \in X$ we set

$$\varrho(x) = \begin{cases} \text{red,} & \text{if } |N_{\text{red}}(x) \cap J| > \frac{p^2n}{400} \\ \text{blue,} & \text{if } |N_{\text{blue}}(x) \cap J| > \frac{p^2n}{400}, \end{cases} \quad (3.12)$$

for vertices x satisfying both conditions in (3.12), we pick the value of $\varrho(x)$ arbitrarily. Note that, for every vertex v which has been assigned a preferred colour $\varrho(v)$ already,

$$\text{there exists a } \varrho(v)\text{-coloured path from } v \text{ to the root of colour } \varrho(v). \quad (3.13)$$

We shall keep this invariant in the assignment of the preferred colours to the remaining vertices.

Before we continue, we make the following remark, which partly explains some of the underlying ideas in our approach.

Remark 3.6. If we have reached every vertex of G at this point (that is, if $V = \{r, b\} \cup N_{\text{red}}(r) \cup N_{\text{blue}}(b) \cup X$), then we can finish the proof as follows. For every vertex in J we decide independently with probability $1/2$ whether we attach it to the red tree or to the blue tree and every other vertex will be attached to the tree matching its preferred colour. This clearly works for the vertices in $N_{\text{red}}(r) \cup N_{\text{blue}}(b)$. Moreover, since every vertex $x \in X$ connects to at least $\frac{p^2n}{400} \gg \ln n$ neighbours in J in its preferred colour, at least one of those neighbours will obtain that colour in the random assignment (with high probability) and this would conclude the proof. Note that, for this argument to work, it would suffice if the joker vertices in $N_{\text{blue}}(b) \setminus N_{\text{red}}(r)$ had just one red neighbour in $N_{\text{red}}(r) \setminus N_{\text{blue}}(b)$. ■

Unfortunately, some vertices may have only a few neighbours in J , and therefore we could have that $V \neq \{r, b\} \cup N_{\text{red}}(r) \cup N_{\text{blue}}(b) \cup X$. Let

$$Y = V \setminus (N_{\text{red}}(r) \cup N_{\text{blue}}(b) \cup \{r, b\} \cup X).$$

We now proceed to define $\varrho(y)$ for every $y \in Y$. Since $J \subseteq N_{\text{blue}}(b)$ we can apply property (iii) to obtain that

$$m = |Y| \leq \frac{100}{p}. \quad (3.14)$$

Consequently, we infer from property (vi) that we can order the vertices in Y as y_1, \dots, y_m in such a way that for every $i \in [m]$ we have

$$|N(y_i) \cap Y_{i+1}| \leq 10 \ln n \quad \text{for } Y_{i+1} = \{y_{i+1}, \dots, y_m\}. \quad (3.15)$$

We shall assign the preferred colours to the vertices in Y in this order. Let $i \in [m]$ and suppose the preferred colours $\varrho(y_j)$ for $j \in [i-1]$ were already fixed. We consider two cases depending on the preferred colours appearing in the neighbourhood of y_i . We split $N(y_i)$ according to the preferred colours of the vertices, i.e., we consider the partition

$$N(y_i) = (N(y_i) \cap \varrho^{-1}(\text{red})) \cup (N(y_i) \cap \varrho^{-1}(\text{blue})) \cup (N(y_i) \cap Y_{i+1}).$$

We say y_i is *canonically connected in red* (resp. blue) if y_i connects in red (resp. blue) to many vertices with preferred colour red (resp. blue), i.e.,

$$|N_{\text{red}}(y_i) \cap \varrho^{-1}(\text{red})| \geq \frac{p^2 n}{400} \quad (3.16)$$

(resp. $|N_{\text{blue}}(y_i) \cap \varrho^{-1}(\text{blue})| \geq p^2 n/400$). If y_i fails to be canonically connected in either colour, then we say it is *non-canonically connected*.

We set $\varrho(y_i) = \text{red}$ (resp. $\varrho(y_i) = \text{blue}$) if y_i is canonically connected in red (resp. blue). Clearly, by induction, with this choice of $\varrho(y_i)$ we also ensure property (3.13).

It is left to consider vertices y_i that are non-canonically connected. Since

$$\begin{aligned} & (N_{\text{blue}}(y_i) \cap \varrho^{-1}(\text{red})) \cup (N_{\text{red}}(y_i) \cap \varrho^{-1}(\text{blue})) \\ &= N(y_i) \setminus \left((N_{\text{red}}(y_i) \cap \varrho^{-1}(\text{red})) \cup (N_{\text{blue}}(y_i) \cap \varrho^{-1}(\text{blue})) \cup (N(y_i) \cap Y_{i+1}) \right), \end{aligned}$$

in this case we have

$$\begin{aligned} |(N_{\text{blue}}(y_i) \cap \varrho^{-1}(\text{red})) \cup (N_{\text{red}}(y_i) \cap \varrho^{-1}(\text{blue}))| &> |N(y_i)| - \frac{p^2 n}{200} - 10 \ln n \\ &> |N(y_i)| - \frac{p^2 n}{100}. \end{aligned} \quad (3.17)$$

In other words, the preferred colour $\varrho(v)$ of almost all neighbours v of y_i mismatches the colour of the edge $\{y_i, v\}$, i.e., $\varphi(\{y_i, v\}) \neq \varrho(v)$. Next we show that both mismatching sets are large enough to ensure quite a few edges crossing these sets. More precisely, we will show that the induced bipartite subgraph

$$\begin{aligned} G_{\text{mis}}(y_i) &= G[N_{\text{blue}}(y_i) \cap \varrho^{-1}(\text{red}), N_{\text{red}}(y_i) \cap \varrho^{-1}(\text{blue})] \\ &\text{contains } p^2 n/100 \text{ vertices of degree at least } p^2 n/100. \end{aligned} \quad (3.18)$$

Note that the existence of any edge $\{u, v\}$ in the graph $G_{\text{mis}}(y_i)$ allows us to connect y_i in colour $\varphi(\{u, v\})$ to the root of colour $\varphi(\{u, v\})$. More precisely, if $u \in N_{\text{blue}}(y_i) \cap \varrho^{-1}(\text{red})$ and $v \in N_{\text{red}}(y_i) \cap \varrho^{-1}(\text{blue})$ and $\varphi(\{u, v\}) = \text{red}$, then there exists a red y_i - r -path using the red u - r -path guaranteed by (3.13) and the red edges $\{y_i, v\}$ and $\{v, u\}$. This then would allow us to assign preferred colour red to y_i . However, for a path as above we use v for a red path, even though v 's preferred colour is blue ($\varrho(v) = \text{blue}$). Such ‘‘conflicts’’ will be resolved in the second stage and for that we need a more ‘‘robust’’ way to connect y_i to the root of its preferred colour. We prepare for that by proving (3.18). We also remark that the proof of (3.18) is the only place in the proof where it will be essential that there is no monochromatic path between r and b and that $p \gg \left(\frac{\ln n}{n}\right)^{1/2}$.

Proof of (3.18). As it turns out, it suffices to establish a suitable lower bound on the cardinality of the two types of mismatching neighbourhoods of y_i ; namely, it is enough to prove that

$$|N_{\text{blue}}(y_i) \cap \varrho^{-1}(\text{red})| \geq \frac{1}{2}p^2n \quad \text{and} \quad |N_{\text{red}}(y_i) \cap \varrho^{-1}(\text{blue})| \geq \frac{1}{2}p^2n. \quad (3.19)$$

Indeed, property (iv) tells us that (3.19) combined with (3.17) yields (3.18).

For the proof of (3.19) we first observe that

$$\begin{aligned} N_{\text{blue}}(y_i) \cap \varrho^{-1}(\text{red}) &= (N(y_i) \cap \varrho^{-1}(\text{red})) \setminus (N_{\text{red}}(y_i) \cap \varrho^{-1}(\text{red})) \\ &\supseteq (N(y_i) \cap N(r) \cap \varrho^{-1}(\text{red})) \setminus (N_{\text{red}}(y_i) \cap \varrho^{-1}(\text{red})). \end{aligned} \quad (3.20)$$

We shall next consider the joint neighbourhood of y_i and r . Note that no $v \in N_{\text{blue}}(r)$ can have preferred colour blue. In fact, if $\varrho(v) = \text{blue}$, then there exists a blue v - b -path in G (see (3.13)) and combined with $\varphi(\{r, v\}) = \text{blue}$ this leads to a blue path between r and b , which was excluded by the choice of r and b . Moreover, every red neighbour v of r outside $N_{\text{red}}(r) \cap N_{\text{blue}}(b) \subseteq J$ (i.e., every $v \in N_{\text{red}}(r) \setminus (N_{\text{red}}(r) \cap N_{\text{blue}}(b))$) was assigned preferred colour red in (3.4). Therefore,

$$N(r) \subseteq \varrho^{-1}(\text{red}) \cup J \cup Y_i,$$

whence we deduce that

$$N_{\text{blue}}(y_i) \cap \varrho^{-1}(\text{red}) \stackrel{(3.20)}{\supseteq} (N(y_i) \cap N(r)) \setminus (Y_{i+1} \cup J \cup (N_{\text{red}}(y_i) \cap \varrho^{-1}(\text{red}))).$$

From (3.15), the fact that $y_i \notin X$ (see (3.11)), and the fact that y_i is not canonically connected in red (see (3.16)), we infer that

$$|N_{\text{blue}}(y_i) \cap \varrho^{-1}(\text{red})| \geq |N(y_i) \cap N(r)| - 10 \ln n - \frac{p^2n}{200} - \frac{p^2n}{400}.$$

Therefore, the first inequality in (3.19) follows from property (i) and $p^2n \gg \ln n$. The second inequality in (3.19) follows by the symmetric argument with colours exchanged. As observed above, this establishes (3.18) as well. \square

Finally, we define the preferred colour of y_i by

$$\varrho(y_i) = \begin{cases} \text{red,} & \text{if } E(G_{\text{mis}}(y_i)) \cap \varphi^{-1}(\text{red}) \text{ induces } \frac{p^2n}{200} \text{ vertices of degree } \geq \frac{p^2n}{200} \\ \text{blue,} & \text{otherwise.} \end{cases} \quad (3.21)$$

Recalling the discussion following (3.18) we note that also in this case we ensure property (3.13) for the vertex y_i . Note that in view of property (iv), if $\varrho(y_i)$ is blue, then $E(G_{\text{mis}}(y_i)) \cap \varphi^{-1}(\text{blue})$ induces $\frac{p^2n}{200}$ vertices of degree $\geq \frac{p^2n}{200}$.

This concludes the discussion of the first stage and we assigned preferred colours $\varrho(v)$ to every vertex $v \in V \setminus \{r, b\}$. For that we considered the vertices in $(N_{\text{red}}(r) \cup N_{\text{blue}}(b)) \setminus J$, in the joker set J , in the set X connected ‘‘robustly’’ to the joker set, and in the remaining set Y differently. Moreover, the vertices in Y were treated differently depending on whether they are canonically connected or not.

For later reference we note the following properties in addition to (3.13) for every vertex from the set $(J \setminus (N_{\text{red}}(r) \cap N_{\text{blue}}(b))) \cup X \cup Y$.

(a) If $v \in J \setminus (N_{\text{red}}(r) \cap N_{\text{blue}}(b))$, then it follows from the definition (3.9) of J that

$$|N_{\text{red}}(v) \cap (N_{\text{red}}(r) \setminus N_{\text{blue}}(b))| \geq \frac{p^2n}{25}.$$

(b) If $x \in X$, then it follows from (3.12) that

$$|N_{\varrho(x)}(x) \cap J| \geq \frac{p^2n}{400}.$$

(c) If $y_i \in Y$ is canonically connected in colour $\varrho(y_i)$, then it follows from (3.16) that

$$|(N_{\varrho(y_i)}(y_i) \setminus Y_i) \cap \varrho^{-1}(\varrho(y_i))| \geq \frac{p^2n}{400}.$$

(d) If $y_i \in Y$ is not canonically connected in either colour, then by (3.21) the bipartite subgraph of G with edges of colour $\varrho(y_i)$ induced across the two types of mismatched vertices in $N(y_i) \setminus Y_i$, which we denote by

$$G_{\varrho(y_i)} \left[(N_{\text{blue}}(y_i) \cap \varrho^{-1}(\text{red})) \setminus Y_i, (N_{\text{red}}(y_i) \cap \varrho^{-1}(\text{blue})) \setminus Y_i \right],$$

contains at least $p^2n/200$ vertices of degree at least $p^2n/200$.

Second stage: finalising the choices. We shall now assign *final* colours to the vertices of G to establish $\varphi \longrightarrow \Pi_2$. More precisely, we shall define a function $f: V \rightarrow \{\text{red}, \text{blue}\}$ with $f(r) = \text{red}$ and $f(b) = \text{blue}$ so that

$$G_{\text{red}}[f^{-1}(\text{red})] \quad \text{and} \quad G_{\text{blue}}[f^{-1}(\text{blue})] \quad \text{are connected.} \quad (3.22)$$

Since our process for defining f is somewhat lengthy, we first give a rough outline. The assignment of the colours $f(v)$ for $v \in V$ will be achieved in two rounds.

The function f will start as a partial function with domain $\text{dom } f$ close to half of V . At this stage, on most of $\text{dom } f$, we shall have $f \equiv \varrho$, but for about half of the joker vertices v we shall ‘switch’ and pick as v ’s final colour the colour opposite to its preferred colour: $f(v) = \bar{\varrho}(v)$, where $\bar{\varrho}(v) = \text{red}$ if $\varrho(v) = \text{blue}$ and $\bar{\varrho}(v) = \text{blue}$ if $\varrho(v) = \text{red}$. At this point, we shall have that

$$G_{\text{red}}[f^{-1}(\text{red}) \setminus Y] \quad \text{and} \quad G_{\text{blue}}[f^{-1}(\text{blue}) \setminus Y] \quad \text{are connected.} \quad (3.23)$$

(The comment above is somewhat similar to Remark 3.6.) From this point in the proof onwards, we shall increase $\text{dom } f$ in smaller steps. It will be convenient to say that, once $f(v)$ has been defined for a vertex v , the vertex v has been *finalised*. Also, we remark that, once we choose the value of $f(v)$ for some v , we shall not change it afterwards.

What we discussed above corresponds to most of the first round. However, still in the first round, we shall have to finalise some other vertices $z \notin \text{dom } f$, setting $f(z) = \bar{\varrho}(z)$ so that we can improve (3.23) by replacing Y by some substantially smaller subset Y' (in fact, $|Y'|$ will roughly be $|Y|/2$). This final stage of the first round is encapsulated in Claim 3.8 below.

In the second round of our procedure defining f , we pick the colour of the remaining vertices $v \in V \setminus \text{dom } f$. This process will be guided by the vertices in Y' . This concludes our outline of what comes next, and we proceed to define f precisely.

Consider a random bipartition $Z_0 \cup Z_1 = V \setminus \{r, b\}$ where every vertex $v \in V \setminus \{r, b\}$ is included independently with probability $1/2$ into Z_0 or Z_1 . Since $p^2 n \gg \ln n$ we deduce from (a)–(d) that with positive probability there exists a partition $Z_0 \cup Z_1 = V \setminus \{r, b\}$ such that for every vertex in $(J \setminus N_{\text{red}}(r) \cap N_{\text{blue}}(b)) \cup X \cup Y$ the following holds:

- (a') If $v \in J \setminus (N_{\text{red}}(r) \cap N_{\text{blue}}(b))$, then $N_{\text{red}}(v) \cap (N_{\text{red}}(r) \setminus N_{\text{blue}}(b)) \cap Z_0 \neq \emptyset$.
- (b') If $x \in X$, then $N_{\varrho(x)}(x) \cap J \cap Z_\xi \neq \emptyset$ for both $\xi \in \{0, 1\}$.
- (c') If $y_i \in Y$ is canonically connected in colour $\varrho(y_i)$, then

$$(N_{\varrho(y_i)}(y_i) \setminus Y_i) \cap \varrho^{-1}(\varrho(y_i)) \cap Z_0 \neq \emptyset.$$

(d') If $y_i \in Y$ is non-canonically connected, then there exists an edge $\{u, v\} \in E(G_{\varrho(y_i)})$ such that

$$u \in (N_{\bar{\varrho}(y_i)}(y_i) \cap \varrho^{-1}(\varrho(y_i)) \cap Z_0) \setminus Y_i$$

and

$$v \in (N_{\varrho(y_i)}(y_i) \cap \varrho^{-1}(\bar{\varrho}(y_i)) \cap Z_1) \setminus Y_i,$$

where, we recall, $\bar{\varrho}(y_i)$ denotes the colour different from $\varrho(y_i)$.

Note that we considered at most n such sets of size $\Omega(p^2n)$ in (a)–(c) and $O(n \cdot p^2n) = O(n^2)$ stars of size $\Omega(p^2n)$ in (d). Consequently, the existence of a partition $Z_0 \cup Z_1 = V \setminus \{r, b\}$ satisfying (a')–(d') indeed follows from $p^2n \gg \ln n$ and a standard application of Chernoff's inequality. We fix such a partition for the remainder of the proof.

After this preparatory random splitting we start defining the final colours $f(v)$ for $v \in V$. We start with r and b in the obvious manner:

$$f(r) = \text{red} \quad \text{and} \quad f(b) = \text{blue}.$$

Moreover, every $v \in Z_0$ will be assigned its preferred colour and every joker vertex in Z_1 will be assigned the opposite of its preferred colour:

$$f(v) = \begin{cases} \varrho(v), & \text{if } v \in Z_0 \\ \bar{\varrho}(v), & \text{if } v \in J \cap Z_1. \end{cases} \quad (3.24)$$

Note that we now have $\text{dom } f = Z_0 \cup J$. We have thus committed ourselves in which of the two monochromatic subgraphs in (3.22) the vertices in $Z_0 \cup J$ are. We mention that, owing to the definition of ϱ , our *tendency* is to set $f(v) = \varrho(v)$ for the remaining vertices $v \in Z_1 \setminus J = V \setminus \text{dom } f$. However, if we do this blindly, assertion (3.22) will not hold. In what follows, we shall “switch” the colour of *some* vertices $v \in Z_1 \setminus J$ and we shall set $f(v) = \bar{\varrho}(v)$ (in the same way we did for the vertices in $Z_1 \cap J$). Such switchings will basically be forced on us as we proceed to increase $\text{dom } f$ in our proof.

Before we continue, we make the following remark, which is closely related to the discussion in Remark 3.6.

Remark 3.7. Suppose every vertex of Y is canonically connected in some colour. Then properties (a')–(c') and an inductive argument would show that (3.22) holds for our current function f . ■

Remark 3.7 above deals with the lucky case in which every vertex of Y is canonically connected in some colour. In general, there will be vertices y in Y that are non-canonically

connected. Such vertices y will force us to set $f(z) = \bar{\varrho}(z)$ for some $z \in Z_1 \setminus J$ also. This is made precise in the following claim.

Claim 3.8. *There exists a subset $Z'_1 \subseteq Z_1 \setminus J$ for which the following holds. If we set*

$$f(z) = \bar{\varrho}(z) \tag{3.25}$$

for every $z \in Z'_1$, then $\text{dom } f = Z_0 \cup J \cup Z'_1 \cup \{r, b\}$ and (3.22) holds.

Proof. We first consider our current function f with $\text{dom } f = Z_0 \cup J$ and verify the following fact.

Fact 3.9. *Assertion (3.23) holds for f .*

Proof. We consider the different types of vertices encountered in the first stage separately. First we recall that vertices $v \in (N_{\text{red}}(r) \cup N_{\text{blue}}(b)) \setminus J$ are directly connected to their respective roots in colour $\varrho(v)$. Consequently, all vertices

$$v \in Z_0 \cap ((N_{\text{red}}(r) \cup N_{\text{blue}}(b)) \setminus J)$$

are in the same component in $G_{f(v)} = G_{\varrho(v)}$ as the respective root.

Secondly, we consider the joker vertices. Note that nothing needs to be shown for the vertices $v \in N_{\text{red}}(r) \cap N_{\text{blue}}(b)$ as they are directly connected to both roots in the appropriate colour and, hence, for these vertices it does not matter which final colour $f(v)$ is assigned to them. Moreover, for every joker vertex $v \in J \cap Z_0$ we have $f(v) = \varrho(v) = \text{blue}$ and since $J \subseteq N_{\text{blue}}(b)$, these vertices are also directly connected to b in $G_{f(v)}$. For the remaining joker vertices $v \in (J \setminus (N_{\text{red}}(r) \cap N_{\text{blue}}(b))) \cap Z_1$ we appeal to (a'). Owing to (3.24) the final colour $f(v)$ of v is red and, by (a'), every such v has at least one red neighbour u in $Z_0 \cap (N_{\text{red}}(r) \setminus N_{\text{blue}}(b)) \subseteq \text{dom } f$. Since we have $f(u) = \varrho(u) = \text{red}$, the vertex v is also connected to r in $G_{\text{red}}[f^{-1}(\text{red})]$.

Next we move to the vertices x in $X \cap Z_0$ and for those vertices we appeal to (b'). If $f(x) = \varrho(x) = \text{red}$, then (b') applied with $\xi = 1$ tells us that x has at least one red neighbour $v \in J \cap Z_1 \subseteq \text{dom } f$ (i.e., there is $v \in N_{\text{red}}(x) \cap J \cap Z_1 \subseteq \text{dom } f$). Since $\varrho(v) = \text{blue}$ and, therefore, $f(v) = \text{red}$ (see (3.24)), we infer from the discussion above that x is connected by a red path to r in $G_{\text{red}}[f^{-1}(\text{red})]$. If $f(x) = \varrho(x) = \text{blue}$, then the same argument with (b') applied with $\xi = 0$ yields that x is connected by a blue path to b in $G_{\text{blue}}[f^{-1}(\text{blue})]$. \square

We shall now improve Fact 3.9: we shall prove that (3.22) holds for f , as long as we enlarge the domain of f suitably. Roughly speaking, what we have to do is to ‘attach’ the vertices in $Y \cap Z_0$ to $G_{\text{red}}[f^{-1}(\text{red})]$ or to $G_{\text{blue}}[f^{-1}(\text{blue})]$, with edges (or paths) of the

correct colour. We shall proceed vertex by vertex following the order y_1, \dots, y_m (ignoring vertices outside Z_0). For certain vertices $y_i \in Y \cap Z_0$, this will be a matter of realizing that a suitable edge is there; for other vertices $y_i \in Y \cap Z_0$, we may have to finalise a vertex $v \in Z_1 \setminus J$: every time we do this, we add v to Z'_1 and Z'_1 increases (we start with $Z'_1 = \emptyset$). Let us remark that, when we put a vertex v in Z'_1 and finalise it, we shall set $f(v) = \bar{\varrho}(v)$. At the end of this process, assertion (3.22) will hold for our f . We now go into the details of this process.

We proceed inductively and use the fixed ordering of the vertices in Y . At first we have $\text{dom } f = Z_0 \cup J$ and $Z'_1 = \emptyset$. Suppose now that $1 \leq i \leq m$, $y_i \in Y \cap Z_0$, and the vertices in some set $Z'_1 \subseteq Z_1 \setminus J$ have been finalised with $f(z') = \bar{\varrho}(z')$ for every $z' \in Z'_1$. Suppose further that

$$G_{\text{red}}[f^{-1}(\text{red}) \setminus Y_i] \quad \text{and} \quad G_{\text{blue}}[f^{-1}(\text{blue}) \setminus Y_i] \quad \text{are connected.} \quad (3.26)$$

We now finalise y_i analysing two cases.

Case 1. If y_i is canonically connected in colour $\varrho(y_i)$, then we proceed in a similar manner as for the vertices in $X \cap Z_0$. In fact, it follows from (c') that in this case y_i has a neighbour $v \in N_{\varrho(y_i)}(y_i) \setminus Y_i$ such that

$$f(y_i) = \varrho(y_i) = \varrho(v) = f(v),$$

where the first and last identities follow from the fact that $y_i \in Z_0$ and $v \in Z_0$. Since $v \in (\text{dom } f) \setminus Y_i$, the inductive assumption (3.26) and the edge $\{y_i, v\}$ of colour $\varrho(y_i) = f(y_i)$ tells us that $G_{f(y_i)}[f^{-1}(f(y_i)) \setminus Y_{i+1}]$ is connected, completing the induction step in this case.

Case 2. We now consider the case in which $y_i \in Y \cap Z_0$ is non-canonically connected. In this case we may have to enlarge the set Z'_1 by adding some vertex v , but we will ensure the monochromatic connection for v as well. By symmetry we may assume that the preferred colour of y_i is red and, since $y_i \in Z_0$, we have

$$\varrho(y_i) = f(y_i) = \text{red}.$$

Let $\{u, v\}$ be the edge given by (d') of colour $\varrho(y_i) = \text{red}$. In particular,

$$u \in (\varrho^{-1}(\text{red}) \cap Z_0) \setminus Y_i.$$

Therefore, we already finalised u and $f(u) = \text{red}$. Furthermore, by the induction assumption (3.26), we already know that u is connected to r by a red path in $G_{\text{red}}[f^{-1}(\text{red}) \setminus Y_i]$. Furthermore,

$$v \in (\varrho^{-1}(\text{blue}) \cap Z_1) \setminus Y_i.$$

In case v has already been put into Z'_1 in this inductive process, then we already “switched” its colour and finalised it to be red. If not, then we add v to Z'_1 at this point and finalise it with $f(v) = \text{red}$. In any case we may use the red edges $\{u, v\}$ and $\{v, y_i\}$ to connect the vertices v and y_i to r by a red path in $G_{\text{red}}[f^{-1}(\text{red}) \setminus Y_{i+1}]$. This concludes our induction step in this case and completes the proof of Claim 3.8. \square

It is left to finalise the colours of the vertices in $Z_1 \setminus (J \cup Z'_1)$. Again we consider the vertices separately, according to their membership in the sets $N_{\text{red}}(r) \cup N_{\text{blue}}(b)$, X or Y . This time we reverse the order in which we deal with the vertices and begin with the vertices in Y .

We iterate over the vertices in $Y \cap (Z_1 \setminus (J \cup Z'_1))$ in *reverse order*: y_m, \dots, y_1 . In this process, we shall finalise the vertices $y \notin \text{dom } f$ that we encounter one by one. For some y , it may happen that some other vertex $v \notin \text{dom } f$ has to be finalised also. When this does happen, we shall say that v has been *pulled forward* and we shall always let $f(v) = \bar{\varrho}(v)$, that is, we shall switch the colour of v . We now describe this inductive process precisely.

Let $i \in [m]$ be the largest index such that y_i has not been finalised yet. We proceed as in the proof of Claim 3.8. If y_i is canonically connected in colour $\varrho(y_i)$, then we set $f(y_i) = \varrho(y_i)$. Owing to (c') there exists a neighbour in $v \in N_{\varrho(y_i)}(y_i) \cap Z_0$ with preferred colour $\varrho(v) = \varrho(y_i)$. Since $v \in Z_0$, in fact, we already have $f(v) = \varrho(v)$ and, in view of Claim 3.8, the vertex v is connected to the root of the corresponding colour with an $f(v)$ -coloured path. Extending this path with the edge $\{v, y_i\}$ of colour $f(y_i) = f(v)$ to y_i concludes this case.

Next we consider the case in which y_i is non-canonically connected. In this case we also set $f(y_i) = \varrho(y_i)$, but we shall make use of the edge $\{u, v\}$ of colour $\varrho(y_i)$ guaranteed by (d'). Since $u \in \varrho^{-1}(\varrho(y_i)) \cap Z_0$, the colour $f(u)$ of u was chosen in the first round of the second stage already, and we have $f(u) = \varrho(u) = \varrho(y_i) = f(y_i)$. Claim 3.8 then tells us that there is a path from u to the root of colour $f(y_i)$ in $G_{f(y_i)}[f^{-1}(f(y_i))]$. On the other hand, the vertex v is contained in $Z_1 \setminus Y_i$ and $\varrho(v) = \bar{\varrho}(y_i)$. We now proceed differently depending on whether or not $v \in \text{dom } f$.

If $f(v)$ has not been set already, then we pull this vertex forward and finalise its colour opposite to its preferred colour, i.e., we treat the vertex v as the vertices $z \in Z'_1$ in (3.25). As a result we obtain $f(v) = f(y_i)$ and, since the edges $\{u, v\}$ and $\{v, y_i\}$ are coloured $f(y_i)$, we ensure the invariant that y_i and v are connected to the root of colour $f(y_i) = f(v)$ in $G_{f(y_i)}[f^{-1}(f(y_i))]$.

If $f(v)$ has already been set before, then either (a) $v \in (J \cap Z_1) \cup Z'_1$ and, by (3.24) and (3.25), the final colour of v was set opposite to its preferred colour, or else (b) v was

pulled forward because of some other vertex y_j with $j > i$. However, also in case (b), the colour of v was switched and we have $f(v) = \bar{\rho}(v) = \rho(y_i) = f(y_i)$. Consequently, in both cases (a) and (b), we already established a connection of v to the root of colour $f(v)$ in $G_{f(v)}[f^{-1}(f(v))]$. Extending this path with the edge $\{v, y_i\}$ of colour $f(v) = f(y_i)$ establishes the required connection for y_i . Here, we are using that $v \in Z_1 \setminus Y_i$ being in $(J \cap Z_1) \cup Z'_1$ or being pulled forward are the only reasons that could have led to the finalisation of v . This concludes the discussion of the vertices in Y .

Next we move to the vertices in X . Note that some of the vertices $x \in X \cap (Z_1 \setminus (Z'_1 \cup J))$ may have been pulled forward to attach some $y \in Y$ that is non-canonically connected. However, such a vertex x was finalised and the desired connection to the root of colour $f(x)$ was established on that occasion.

For every vertex $x \in X \setminus \text{dom } f$, we simply set

$$f(x) = \rho(x).$$

By (b') there exist vertices $u \in J \cap Z_0$ and $v \in J \cap Z_1$, both contained in $N_{f(x)}(x)$. Since all joker vertices were assigned preferred colour blue and $u \in Z_0$, we have $f(u) = \rho(u) = \text{blue}$. On the other hand, since $v \in J \cap Z_1$, we infer from (3.24) that $f(v) = \text{red}$. Hence, no matter what $f(x)$ is, there exists a path from x to the root of colour $f(x)$ in $G_{f(x)}[f^{-1}(f(x))]$.

It is left to finalise the remaining vertices $v \in (N_{\text{red}}(r) \cup N_{\text{blue}}(b)) \cap (Z_1 \setminus (Z'_1 \cup J))$ that have not been pulled forward. Obviously, setting $f(v) = \text{red}$ if $v \in N_{\text{red}}(r)$ and blue otherwise connects v to the root in the appropriate colour.

Summarising, we finalised every vertex $v \in V$ in such a way that v is connected to the root of colour $f(v)$ in $G_{f(v)}[f^{-1}(f(v))]$ (i.e., assertion (3.22) holds). Consequently, the partition

$$f^{-1}(\text{red}) \cup f^{-1}(\text{blue}) = V$$

shows that $\varphi \longrightarrow \Pi_2$, which concludes the proof of Proposition 3.4. \square

§4. EXTENSION FOR MORE COLOURS

In this section we show that Theorem 1.1 does not extend in the expected way to more than two colours. For $r \geq 2$ and a graph $G = (V, E)$ we write $G \longrightarrow \Pi_r$ if for every r -colouring of E there exist r monochromatic trees $T_1, \dots, T_r \subseteq G$ such that

$$V(T_1) \cup \dots \cup V(T_r) = V.$$

Since it is not hard to obtain a lower bound construction for the threshold $p = p(n)$ for $G(n, p) \longrightarrow \Pi_r$ as long as there are r vertices with no joint neighbour, one may wonder whether $p = p(n) = \left(\frac{r \ln n}{n}\right)^{1/r}$ is the sharp threshold for this property. Such a conjecture

was indeed put forward by Bal and DeBiasio [1, Conjecture 8.1]. However, it was noted by Ebsen, Mota, and Schnitzer [6] that for $r \geq 3$ the threshold is larger and we include their example below.

Proposition 4.1. *For any integer $r \geq 3$ and $p = p(n) \ll \left(\frac{\ln n}{n}\right)^{\frac{1}{r+1}}$ a.a.s. $G \in G(n, p)$ fails to satisfy $G \rightarrow \Pi_r$.*

Proof. For a simpler presentation we only prove the proposition for $r = 3$, since the adjustments for $r > 3$ are rather straightforward. Suppose $p = p(n) \ll \left(\frac{\ln n}{n}\right)^{1/4}$. We show that a.a.s. $G = (V, E) \in G(n, p)$ admits a 3-colouring of E with colours red, blue, and green such that there is no partition $V(G) = V(T_1) \cup V(T_2) \cup V(T_3)$ with monochromatic trees $T_1, T_2, T_3 \subseteq G$.

By our choice of p a.a.s. there are four vertices r, b, g , and z that are independent in G and that have no common neighbour, i.e.,

$$N(z) \cap N(r) \cap N(b) \cap N(g) = \emptyset.$$

Below we write $N(r, g, b)$ for the joined neighbourhood $N(r) \cap N(g) \cap N(b)$.

We now describe a colouring $\varphi: E \rightarrow \{\text{red, blue, green}\}$ with the desired property. The edges incident to r are coloured red, those incident to b are coloured blue, and those incident to g are coloured green. This choice ensures that we need at least three monochromatic trees to partition V and below we will ensure that z cannot be connected to any of these three trees.

Next we colour the edges induced in

$$X = N(r) \cup N(b) \cup N(g).$$

in such a way that for every vertex $x \in X \setminus N(r, b, g)$, the edges incident to x are coloured with at most two of the three colours and we fix one of the ‘‘missing colours’’ that do not appear on edges incident to x , which we denote by $\text{mc}(x)$. The following colourings have this property:

For every edge we list at least one *allowed* colour and if an edge is assigned to more than one allowed colour, then one may pick arbitrarily one of the allowed colours

- edges within $N(r)$ are allowed to be coloured red, within $N(b)$ are allowed to be coloured blue, and within $N(g)$ are allowed to be coloured green;
- edges between $N(r) \setminus N(b)$ and $N(b) \setminus (N(r) \cup N(g))$ are coloured red, between $N(b) \setminus N(g)$ and $N(g) \setminus (N(b) \cup N(r))$ are coloured blue, and between $N(g) \setminus N(r)$ and $N(r) \setminus (N(g) \cup N(b))$ are coloured green.

Then we colour the edges incident with z . Edges zx with $x \in X \setminus N(r, b, g)$ are coloured with colour $\text{mc}(x)$. Note that from the definition of $\text{mc}(x)$, if zx is coloured green, then there is no monochromatic green path between g and x , and similar for the symmetric cases.

Let Y be the set of vertices not considered so far, i.e., $Y = V(G) \setminus (X \cup \{r, b, g, z\})$. It remains to colour the edges incident to Y . We will prevent z to be connected by a monochromatic path to r , b , or g using vertices from Y . For that, we give colour blue to the edges zy with $y \in Y$, while edges between $N(r, b, g)$ and Y and within Y are coloured red. For the edges yx with $y \in Y$ and $x \in X \setminus N(r, b, g)$, the colours $\{\text{red}, \text{green}\} \setminus \{\text{mc}(x)\}$ are allowed. Since for every $x \in X \setminus N(r, b, g)$ and every $y \in Y$ the colours of the edges zx and yx are different, and the only edge incident to x that has colour $\text{mc}(x)$ is zx , there is no monochromatic path from x to r , b or g containing vertices from Y . Moreover, one can check that for any colouring φ as described, it is impossible to connect z by a monochromatic path with r , b , or g and, hence φ has the desired property. \square

It would be interesting to determine the threshold for $G(n, p) \rightarrow \Pi_r$ for $r \geq 3$ and to decide if the lower bound in Proposition 4.1 is optimal. We remark that the construction given in Proposition 4.1 also works for covering (instead of partitioning) the vertices of $G(n, p)$ with monochromatic trees.

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