COUNTING RESULTS FOR SPARSE PSEUDORANDOM HYPERGRAPHS II

YOSHIHARU KOHAYAKAWA, GUILHERME OLIVEIRA MOTA, MATHIAS SCHACHT, AND ANUSCH TARAZ

ABSTRACT. We present a variant of a universality result of Rödl [On universality of graphs with uniformly distributed edges, Discrete Math. 59 (1986), no. 1-2, 125–134] for sparse, 3-uniform hypergraphs contained in strongly jumbled hypergraphs. One of the ingredients of our proof is a counting lemma for fixed hypergraphs in sparse "pseudorandom" uniform hypergraphs, which is proved in the companion paper [Counting results for sparse pseudorandom hypergraphs I].

§1. INTRODUCTION

We say that a graph $G = (V, E)$ satisfies property $\mathcal{Q}(\eta, \delta, \alpha)$ if, for every subgraph $G[S]$ induced by $S \subset V$ with $|S| \geqslant \eta|V|$, we have $(\alpha - \delta)\binom{|S|}{2} < |E(G[S])| < (\alpha + \delta)\binom{|S|}{2}$. In [7, 13], answering affirmatively a question posed by Erdős (see, e.g.,[5] and [1, p. 363]; see also [10]), Rödl proved the following result.

Theorem 1.1. *For all $k \geqslant 1$ and $0 < \alpha$, $\eta < 1$, there exist δ, $n_0 > 0$ such that the following holds for all integer $n \geqslant n_0$.*

Every n-vertex graph G that satisfies $\mathcal{Q}(\eta, \delta, \alpha)$ contains all graphs with k vertices as induced subgraphs.

The quantification in Theorem 1.1 is what makes it unexpected. Indeed, note that η is not required to be small, it is allowed to be any constant less than 1.

We prove a variant of this result, which allows one to count the number of copies (not necessarily induced) of certain fixed 3-uniform linear hypergraphs in spanning subgraphs of sparse "jumbled" 3-uniform hypergraphs.

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The concept of jumbledness [14, 15] is well-known for graphs (see also [2–4, 9]). Let $\Gamma = (V, E)$ be a 3-uniform hypergraph and let $X \subset \binom{V}{2}$ and $Y \subset V$ be given. Denote by $E_\Gamma(X, Y)$ the set of triples in Γ containing a pair in X and a vertex in Y. Write $e_\Gamma(X, Y)$ for $|E_\Gamma(X, Y)|$. We say that Γ is (p, β)-*jumbled* if, for all subsets $X \subset \binom{V}{2}$ and $Y \subset V$, we have $\big|e_\Gamma(X, Y) - p|X||Y|\big| \leqslant \beta\sqrt{|X||Y|}$. A hypergraph H is called *linear* if every pair of edges shares at most one vertex. An edge e of a linear ℓ-uniform hypergraph H is a *connector* if there exist $v \in V(H) \smallsetminus e$ and ℓ edges $e_1, \ldots, e_\ell$ containing v such that $|e \cap e_i| = 1$ for $1 \leqslant i \leqslant \ell$. Note that, for $\ell = 2$, a connector is an edge that is contained in a triangle.

We prove a result that allows us to count the number of copies of small linear, connector-free 3-uniform hypergraphs H contained in certain n-vertex 3-uniform spanning subhypergraphs G_n of $(p, o(p^2 n^{3/2}))$-jumbled hypergraphs, for sufficiently large p and n. We remark that, if $p \gg n^{-1/4}$, then the random 3-uniform hypergraph, where each possible edge exists with probability p independently of all other edges, is $(p, \gamma p^2 n^{3/2})$-jumbled with high probability for all $\gamma > 0$. Therefore, our result applies to dense enough random 3-uniform hypergraphs.

This paper is organized as follows. In Section 2 we state the main result of this paper (Theorem 2.1) and we discuss the structure of its proof. Section 3 contains the statements and the proofs of the lemmas involved in the proof of Theorem 2.1. Section 4 contains the proof of Theorem 2.1. The final section contains some concluding remarks.

§2. Main result

We start by generalizing property $\mathcal{Q}(\eta, \delta, \alpha)$ to 3-uniform hypergraphs. We say that a 3-uniform hypergraph $G = (V, E)$ satisfies property $\mathcal{Q}'(\eta, \delta, q)$ if, for all $X \subset \binom{V}{2}$ and $Y \subset V$ with $|X| \geqslant \eta\binom{|V|}{2}$ and $|Y| \geqslant \eta|V|$, we have $(1 - \delta)q|X||Y| \leqslant |E_G(X, Y)| \leqslant (1 + \delta)q|X||Y|$. Considering the cardinality of $E_G(X, Y)$ for certain $X \subset \binom{V}{2}$ and $Y \subset V$ to obtain information on the subhypergraphs of G has recently been shown to be fruitful (see [11, 12]).

Given a pair $\{v_1, v_2\} \in \binom{V}{2}$, define $N_G(\{v_1, v_2\}) = \big\{v_3 \in V : \{v_1, v_2, v_3\} \in E\big\}$. We say that a 3-graph $G = (V, E)$ satisfies property $\mathrm{BDD}(k, C, q)$ if, for all $1 \leqslant r \leqslant k$ and for all distinct $S_1, \ldots, S_r \in \binom{V}{2}$, we have $|N_G(S_1) \cap \ldots \cap N_G(S_r)| \leqslant Cnq^r$.

An *embedding* of a hypergraph H into another hypergraph G is an injective mapping $\phi\colon V(H) \to V(G)$ such that $\{\phi(v_1), \ldots, \phi(v_k)\} \in E(G)$ whenever $\{v_1, \ldots, v_k\} \in E(H)$. We denote by $\mathcal{E}(H, G)$ the family of embeddings from H into G. The following variant of Theorem 1.1 for 3-uniform hypergraphs is our main result.

Theorem 2.1. *For all $0 < \varepsilon,\ \alpha,\ \eta < 1$, $C > 1$, and integer $k \geqslant 4$, there exist $\delta, \gamma > 0$ such that if $p = p(n) \gg n^{-1/k}$ and $p = p(n) = o(1)$ and n is sufficiently large, then the following holds for every $\alpha p \leqslant q \leqslant p$ and every $\beta \leqslant \gamma p^2 n^{3/2}$. Suppose that*

(*i*) $\Gamma = (V, E_\Gamma)$ *is an n-vertex (p, β)-jumbled 3-uniform hypergraph;*

(*ii*) $G = (V, E_G)$ *is a spanning subhypergraph of Γ with $|E_G| = q\binom{n}{3}$ and G satisfies $\mathcal{Q}'(\eta, \delta, q)$ and $\mathrm{BDD}(k, C, q)$.*

Then for every linear 3-uniform connector-free hypergraph H on k vertices we have

$$\big||\mathcal{E}(H, G)| - n^k q^{|E(H)|}\big| < \varepsilon n^k q^{|E(H)|}.$$

The proof of Theorem 2.1 requires several techniques. First, we shall prove that, under the conditions of the theorem, G satisfies a strong property involving degrees and co-degrees (see Lemmas 2.5, 3.3 and 3.4). After that we use an embedding result (Lemma 3.1) proved in [6]. Before we discuss the scheme of the proof, let us define some hypergraph properties, called *Discrepancy*, *Pair*, and *Tuple*.

Property 2.2 (DISC – Discrepancy property)**.** *Let $G = (V, E)$ be a 3-uniform hypergraph and let $X,\ Y \subset V$ be given. We say that the pair (X, Y) satisfies $\mathrm{DISC}(q, p, \varepsilon)$ in G (or $(X, Y)_G$ satisfies $\mathrm{DISC}(q, p, \varepsilon)$) if for all $X' \subset \binom{X}{2}$ and $Y' \subset Y$ we have*

$$\big|e_G(X', Y') - q|X'||Y'|\big| \leqslant \varepsilon p\binom{|X|}{2}|Y|.$$

Furthermore, if (V, V) satisfies $\mathrm{DISC}(q, p, \varepsilon)$ in G, then we say that the hypergraph G satisfies $\mathrm{DISC}(q, p, \varepsilon)$.

For a 3-uniform hypergraph $G = (V, E)$, a set of vertices $Y \subset V$, and pairs $S_1, S_2 \in \binom{V}{2}$ we denote $N_G(S_1) \cap Y$ by $N_G(S_1; Y)$ and $N_G(S_1) \cap N_G(S_2) \cap Y$ by $N_G(S_1, S_2; Y)$.

Property 2.3 (PAIR – Pair property)**.** *Let $G = (V, E)$ be a 3-uniform hypergraph and let $X,\ Y \subset V$ be given. We say that the pair (X, Y) satisfies $\mathrm{PAIR}(q, p, \delta)$ in G (or $(X, Y)_G$ satisfies $\mathrm{PAIR}(q, p, \delta)$) if the following conditions hold:*

$$\sum_{S_1 \in \binom{X}{2}} \big||N_G(S_1; Y)| - q|Y|\big| \leqslant \delta p\binom{|X|}{2}|Y|,$$

$$\sum_{S_1 \in \binom{X}{2}} \sum_{S_2 \in \binom{X}{2}} \big||N_G(S_1, S_2; Y)| - q^2|Y|\big| \leqslant \delta p^2\binom{|X|}{2}^2|Y|.$$

Furthermore, if (V, V) satisfies $\mathrm{PAIR}(q, p, \delta)$ in G, then we say that the hypergraph G satisfies $\mathrm{PAIR}(q, p, \delta)$.

Property 2.4 (TUPLE – Tuple property). *We define* TUPLE(δ, q) *as the family of* n*-vertex* 3*-uniform hypergraphs* $G = (V, E)$ *such that the following two conditions hold:*

(*i*) $\big||N_G(S_1)| - nq\big| < \delta nq$ *for all but at most* $\delta\binom{n}{2}$ *sets* $S_1 \in \binom{V}{2}$*;*

(*ii*) $\big||N_G(S_1) \cap N_G(S_2)| - nq^2\big| < \delta nq^2$ *for all but at most* $\delta\binom{\binom{n}{2}}{2}$ *pairs* $\{S_1, S_2\}$ *of distinct sets in* $\binom{V}{2}$.

The next result allows us to obtain property TUPLE from PAIR. Since its proof is simple we will omit it.

Lemma 2.5. *For all* $0 < \alpha \leqslant 1$ *and* $0 < \delta < 1$ *there exists* $\delta' > 0$ *such that if a* 3*-uniform hypergraph* G *satisfies* PAIR(q, p, δ') *for* $\alpha p \leqslant q \leqslant p$*, then* G *satisfies* TUPLE(δ, q)*.*

In what follows we explain the organization of the proof. Consider the setup of Theorem 2.1. In order to obtain the conclusion of the theorem, we will use a counting result (Lemma 3.1), which requires that G satisfies properties BDD and TUPLE for the appropriate parameters. Since G satisfies BDD by hypothesis, it suffices to prove that G satisfies TUPLE. Using Lemma 3.3 it is possible to obtain DISC from property $\mathcal{Q}'$. Then, using that G satisfies DISC one can show that G satisfies PAIR using Lemma 3.4, which implies TUPLE by Lemma 2.5. The quantification used in these implications is carefully analyzed in Section 4.

§3. Main lemmas

We start by stating the counting lemma needed in the proof of Theorem 2.1. In order to apply it to a 3-uniform n-vertex hypergraph G, we shall prove that G satisfies TUPLE(δ, q) for a sufficiently small δ and sufficiently large $0 < q = q(n) \leqslant 1$. Since Lemma 2.5 allows us to obtain TUPLE from PAIR, we need to proof that G satisfies PAIR(q, p, δ') for a sufficiently small δ' and appropriate functions p and q. This is done using Lemmas 3.3 and 3.4, which are proved, respectively, in the Subsections 3.1 and 3.2

Given a 3-uniform hypergraph H, we define parameters $d_H = \max\{\delta(J)\colon J \subset H\}$ and $D_H = \min\{3d_H, \Delta(H)\}$. The following result, proved in [6], is our counting lemma.

Lemma 3.1. *Let* $k \geqslant 4$ *be an integer and let* $\varepsilon > 0$*,* $C > 1$ *and an integer* $d \geqslant 2$ *be fixed. Let* H *be a linear* 3*-uniform connector-free hypergraph on* k *vertices such that* $D_H \leqslant d$*. Then, there exists* $\delta > 0$ *for which the following holds for any* $q = q(n)$ *with* $q \gg n^{-1/d}$ *and* $q = o(1)$ *and for sufficiently large* n*.*

If G *is an* n*-vertex* 3*-uniform hypergraph with* $|E(G)| = q\binom{n}{3}$ *hyperedges and* G *satisfies* BDD(D_H, C, q) *and* TUPLE(δ, q)*, then*

$$\big||\mathcal{E}(H, G)| - n^k q^{|E(H)|}\big| < \varepsilon n^k q^{|E(H)|}.$$

3.1. $\mathcal{Q}'$ **implies** DISC**.** Given a 3-uniform hypergraph $G = (V, E)$ and subsets $A \subset \binom{V}{2}$ and non-empty $B \subset V$, the q-*density* between A and B is defined as

$$d_q(A, B) = \frac{|E_G(A, B)|}{q|A||B|}.$$

Before we state the main result of this subsection, Lemma 3.3, we shall prove the following result, which is inspired by a result in [13] for graphs.

Lemma 3.2. *For all* $0 < \eta < 1$ *and* $0 < \varepsilon^* < (1-\eta)/3$, *there exists* $\delta > 0$ *such that, if* $G = (V, E)$ *is an* n*-vertex* 3*-uniform hypergraph that satisfies* $\mathcal{Q}'(\eta, \delta, q)$, *then the following holds.*

For every $C \subset \binom{V}{2}$ *and* $D \subset V$ *such that* $|C|$ *is a multiple of* $\lceil \varepsilon^* \binom{n}{2} \rceil$ *and* $|D|$ *is a multiple of* $\lceil \varepsilon^* n \rceil$, *we have*

$$1 - \varepsilon^* < d_q(C, D) < 1 + \varepsilon^*.$$

Proof. Fix $\eta > 0$ and $0 < \varepsilon^* < (1-\eta)/3$. Let $\delta = \varepsilon^{*3}/24$ and put $t = 1/\varepsilon^*$. Suppose $G = (V, E)$ is an n-vertex 3-uniform hypergraph that satisfies $\mathcal{Q}'(\eta, \delta, q)$. Now, fix $C \subset \binom{V}{2}$ and $D \subset V$ such that $|C| = k_1 \lceil \varepsilon^* \binom{n}{2} \rceil$ and $|D| = k_2 \lceil \varepsilon^* n \rceil$ for some positive integers k_1 and k_2. Let $C_1, \ldots, C_{k_1}$ and $D_1, \ldots, D_{k_2}$ be, respectively, partitions of C and D such that $|C_1| = \ldots = |C_{k_1}| = \lceil \varepsilon^* \binom{n}{2} \rceil$ and $|D_1| = \ldots = |D_{k_2}| = \lceil \varepsilon^* n \rceil$. Now we partition the sets $\binom{V}{2} \setminus C$ and $V \setminus D$, respectively, in sets $C_{k_1+1}, \ldots, C_t$ and $D_{k_2+1}, \ldots, D_t$ such that $|C_{k_1+1}| = \ldots = |C_t| = \lceil \varepsilon^* \binom{n}{2} \rceil$ and $|D_{k_2+1}| = \ldots = |D_t| = \lceil \varepsilon^* n \rceil$. Note that $|C_t| \leqslant \varepsilon^* \binom{n}{2}$ and $|D_t| \leqslant \varepsilon^* n$.

We divide the rest of the proof into two parts. First, we prove that for any triple $i, j, j' \in [t-1]$, $|e(C_i, D_j) - e(C_i, D_{j'})| \leqslant 6\delta \binom{n}{2} nq$, and for any triple $i, i', j \in [t-1]$, $|e(C_i, D_j) - e(C_{i'}, D_j)| \leqslant 6\delta \binom{n}{2} nq$. To finish the proof we put these estimates together to show that $1 - \varepsilon^* < d_q(C, D) < 1 + \varepsilon^*$.

Put $X = C_2 \cup \ldots \cup C_t$ and $Y = D_3 \cup \ldots \cup D_t$. Since $\varepsilon^* < (1-\eta)/3$, we have $|X| = (t-2)\lceil \varepsilon^* \binom{n}{2} \rceil + |C_t| \geqslant (t-2)\varepsilon^* \binom{n}{2} \geqslant \eta \binom{n}{2}$ and $|Y| = (t-3)\lceil \varepsilon^* n \rceil + |D_t| \geqslant (t-3)\varepsilon^* n \geqslant \eta n$. Therefore, using $\mathcal{Q}'(\eta, \delta, q)$, the following two inequalities hold.

$$\Big| e(X, D_1 \cup Y) - e(X, D_2 \cup Y) \Big| \leqslant 2\delta |X|(|D_1| + |Y|)q, \tag{1}$$

$$\left| \frac{e(C_1 \cup X, Y)}{(|C_1| + |X|)|Y|q} - \frac{e(C_1 \cup X, D_j \cup Y)}{(|C_1| + |X|)(|D_j| + |Y|)q} \right| \leqslant 2\delta, \text{ for } j \in \{1, 2\}. \tag{2}$$

Now we define the following for $j \in \{1, 2\}$

$$p_{1j} = \frac{e(C_1 \cup X, Y)}{(|C_1| + |X|)|Y|q} - \frac{e(C_1 \cup X, Y) + e(X, D_j)}{(|C_1| + |X|)(|D_j| + |Y|)q}.$$

By (2), the following holds for $j \in \{1,2\}$

$$p_{1j} - 2\delta \leqslant \frac{e(C_1, D_j)}{(|C_1| + |X|)(|D_j| + |Y|)q} \leqslant p_{1j} + 2\delta. \tag{3}$$

Note that $\big|e(X, D_1) - e(X, D_2)\big| = \big|e(X, D_1 \cup Y) - e(X, D_2 \cup Y)\big|$. Thus, using (1), we obtain the following inequality.

$$|p_{11} - p_{12}| = \left|\frac{e(X, D_1) - e(X, D_2)}{(|C_1| + |X|)(|D_1| + |Y|)q}\right| \leqslant \left(\frac{|X|}{|C_1| + |X|}\right) 2\delta < 2\delta. \tag{4}$$

Putting (3) and (4) together, we obtain the following inequality.

$$|e(C_1, D_1) - e(C_1, D_2)| < 6\delta(|C_1| + |X|)(|D_1| + |Y|)q < 6\delta\binom{n}{2}nq\,.$$

Applying the same strategy one can prove that, for any triple $i, j, j' \in [t-1]$,

$$|e(C_i, D_j) - e(C_i, D_{j'})| < 6\delta\binom{n}{2}nq. \tag{5}$$

Analogously, we obtain the following equation for any triple $i, i', j \in [t-1]$.

$$|e(C_i, D_j) - e(C_{i'}, D_j)| < 6\delta\binom{n}{2}nq. \tag{6}$$

By (5) and (6), we have $|e(C_i, D_j) - e(C_{i'}, D_{j'})| < 12\delta\binom{n}{2}nq$ for any $i, i', j, j' \in [t-1]$. Therefore,

$$|d_q(C_i, D_j) - d_q(C_{i'}, D_{j'})| < \frac{12\delta\binom{n}{2}nq}{|C_i||D_j|q} < \frac{12\delta}{(\varepsilon^*)^2} = \frac{\varepsilon^*}{2} \tag{7}$$

holds for any $i, i', j, j' \in [t-1]$. Put $W_C = C_1 \cup \ldots \cup C_{t-1}$ and $W_D = D_1 \cup \ldots \cup D_{t-1}$. Since $|W_C| \geqslant \eta\binom{n}{2}$ and $|W_D| \geqslant \eta n$, we know, by property $\mathcal{Q}'(\eta, \delta, q)$, that

$$1 - \delta < d_q(W_C, W_D) < 1 + \delta. \tag{8}$$

Suppose for a contradiction that there exist indexes $i_0, j_0 \in [t-1]$ such that either $d_q(C_{i_0}, D_{j_0}) > 1 + \varepsilon^*$ or $d_q(C_{i_0}, D_{j_0}) < 1 - \varepsilon^*$. Then, by (7), either for all $i, j \in [t-1]$ we have $d_q(C_i, D_j) > 1 + \varepsilon^*/2$ or for all $i, j \in [t-1]$ we have $d_q(C_i, D_j) < 1 - \varepsilon^*/2$. But note that

$$d_q(W_C, W_D) = \frac{\sum_{i,j\in[t-1]} d_q(C_i, D_j)|C_i||D_j|q}{|W_C||W_D|q}.$$

Then, either

$$d_q(W_C, W_D) < \frac{(t-1)^2(1 - \varepsilon^*/2)\left\lceil \varepsilon^*\binom{n}{2}\right\rceil \lceil \varepsilon^* n\rceil q}{|W_C||W_D|q} = (1 - \varepsilon^*/2) < 1 - \delta,$$

or

$$d_q(W_C, W_D) > \frac{(t-1)^2(1 + \varepsilon^*/2)\left\lceil \varepsilon^*\binom{n}{2}\right\rceil \lceil \varepsilon^* n\rceil q}{|W_C||W_D|q} = (1 + \varepsilon^*/2) > 1 + \delta,$$

a contradiction with (8). Therefore, for all $i, j \in [t-1]$,

$$1 - \varepsilon^* < d_q(C_i, D_j) < 1 + \varepsilon^*. \tag{9}$$

It remains to estimate the densities $d_q(C_{k_1}, D_j)$ and $d_q(C_i, D_{k_2})$ with $k_1 = t$ and $k_2 = t$ for all $1 \leqslant i \leqslant k_1$ and $1 \leqslant j \leqslant k_2$. Note that $k_1 = t$ ($k_2 = t$) if and only if $\lceil \varepsilon^* \binom{n}{2} \rceil = \varepsilon^* \binom{n}{2}$ ($\lceil \varepsilon^* n \rceil = \varepsilon^* n$), but in these cases one can prove in the same way we proved (9). Therefore, putting all these estimates together, we obtain $1 - \varepsilon^* < d_q(C, D) < 1 + \varepsilon^*$. □

The next lemma shows how one can obtain discrepancy properties from $\mathcal{Q}'$ in spanning subhypergraphs of sufficiently jumbled 3-uniform hypergraphs.

Lemma 3.3. *For all $0 < \varepsilon', \eta, \sigma < 1$ there exists $\delta > 0$ such that for every $\alpha > 0$ there exists $\gamma > 0$ such that the following holds.*

Let $\Gamma = (V, E_\Gamma)$ be an n-vertex (p, β)-jumbled 3-uniform hypergraph for $0 < p = p(n) \leqslant 1$ such that $\alpha p \leqslant q \leqslant p$ and $\beta \leqslant \gamma p n^{3/2}$. Let $G = (V, E_G)$ be a spanning subhypergraph of Γ. If G satisfies $\mathcal{Q}'(\eta, \delta, q)$, then every pair $(X, Y)_G$ with $X, Y \subset V$ such that $|X|, |Y| \geqslant \sigma n$ satisfies $\mathrm{DISC}(q, p, \varepsilon')$.

Proof. Fix $\varepsilon', \eta, \sigma > 0$ and let $\varepsilon^* = \min\left\{\varepsilon'^2 \sigma^2/24, (1-\eta)/4\right\}$. Let δ' be the constant given by Lemma 3.2 applied with η and ε^*. Put $\delta = \min\{\delta', \varepsilon'\}$, $\alpha > 0$ and $\gamma = \sigma^{3/2} \alpha \varepsilon'/2$.

Suppose that $\alpha p \leqslant q \leqslant p$ and $\beta \leqslant p \gamma n^{3/2}$. Let $\Gamma = (V, E_\Gamma)$ be an n-vertex (p, β)-jumbled 3-uniform hypergraph and let $G = (V, E_G)$ be a spanning subhypergraph of Γ such that G satisfies $\mathcal{Q}'(\eta, \delta, q)$. Let $(X, Y)_G$ be a pair with $X, Y \subset V$ such that $|X|, |Y| \geqslant \sigma n$. We want to prove that $(X, Y)_G$ satisfies $\mathrm{DISC}(q, p, \varepsilon')$. For this, fix arbitrary subsets $X' \subset \binom{X}{2}$ and $Y' \subset Y$. We will prove that $\left| e_G(X', Y') - q|X'||Y'| \right| \leqslant \varepsilon' p \binom{|X|}{2} |Y|$.

Upper bound. First, consider the case where $|X'| \leqslant \varepsilon' \binom{|X|}{2}$ or $|Y'| \leqslant \varepsilon' |Y|$. Note that, from the choice of γ and β, since $|X|, |Y| \geqslant \sigma n$, we have

$$\beta \sqrt{|X'||Y'|} \leqslant \alpha \varepsilon' p \binom{|X|}{2} |Y|. \tag{10}$$

Therefore,

$$\begin{aligned} e_G(X', Y') &\leqslant p|X'||Y'| + \beta \sqrt{|X'||Y'|} \\ &\leqslant q|X'||Y'| + (1-\alpha) p |X'||Y'| + \beta \sqrt{|X'||Y'|} \\ &\leqslant q|X'||Y'| + (1-\alpha) p \varepsilon' \binom{|X|}{2} |Y| + \beta \sqrt{|X'||Y'|} \\ &\leqslant q|X'||Y'| + \varepsilon' p \binom{|X|}{2} |Y|, \end{aligned} \tag{11}$$

where the first inequality follows from the jumbledness of Γ and the fact that G is a sub-hypergraph of Γ, the second one follows from the value of q, the third one follows from the fact that $|X'| \leqslant \varepsilon'\binom{|X|}{2}$ or $|Y'| \leqslant \varepsilon'|Y|$, and the last one is a consequence of (10). Thus, we may assume $|X'| > \varepsilon'\binom{|X|}{2}$ and $|Y'| > \varepsilon'|Y|$. We consider four cases, depending on the size of $|X'|$ and $|Y'|$.

Case 1: ($|X'| \geqslant (1-\varepsilon^*)\binom{n}{2}$ and $|Y'| \geqslant (1-\varepsilon^*)n$). By the choice of ε^*, we have $|X'| \geqslant \eta\binom{n}{2}$ and $|Y'| \geqslant \eta n$. By $\mathcal{Q}(\eta,\delta,q)$ we conclude that

$$e_G(X',Y') \leqslant (1+\delta)q|X'||Y'| \leqslant q|X'||Y'| + \varepsilon' p\binom{|X|}{2}|Y|.$$

Case 2: ($|X'| < (1-\varepsilon^*)\binom{n}{2}$ and $|Y'| < (1-\varepsilon^*)n$). Note that, since $|X'| < (1-\varepsilon^*)\binom{n}{2}$ and $|Y'| < (1-\varepsilon^*)n$, there exist subsets $X^* \subset \binom{V}{2}$ and $Y^* \subset V$ such that $X^* = X' \cup X''$ and $Y^* = Y' \cup Y''$, with $X' \cap X'' = \varnothing$ and $Y' \cap Y'' = \varnothing$, where $|X''| \leqslant \varepsilon^*\binom{n}{2}$ and $|X^*|$ is multiple of $\lceil \varepsilon^*\binom{n}{2}\rceil$, and $|Y''| \leqslant \varepsilon^* n$ and $|Y^*|$ is a multiple of $\lceil \varepsilon^* n\rceil$. Then, we can use Lemma 3.2 to obtain the following inequality.

$$\begin{aligned} e_G(X',Y') &\leqslant e_G(X^*,Y^*) \leqslant (1+\varepsilon^*)|X^*||Y^*|q \\ &\leqslant (1+\varepsilon^*)q|X'||Y'| + 2q\big(|X'||Y''| + |X''||Y'| + |X''||Y''|\big). \end{aligned}$$

Since $\varepsilon^* \leqslant \varepsilon'^2\sigma^2/16$, we have $|X''| \leqslant \varepsilon^*\binom{n}{2} \leqslant (\varepsilon'/8)|X'|$ and $|Y''| \leqslant \varepsilon^* n \leqslant (\varepsilon'/8)|Y'|$. Therefore,

$$\begin{aligned} e_G(X',Y') &\leqslant (1+\varepsilon^*)q|X'||Y'| + 2q\big(3(\varepsilon'/8)|X'||Y'|\big) \\ &\leqslant q|X'||Y'| + \frac{\varepsilon'}{4}q|X'||Y'| + \frac{3\varepsilon'}{4}q|X'||Y'| \leqslant q|X'||Y'| + \varepsilon' p\binom{|X|}{2}|Y|. \end{aligned}$$

Case 3: ($|X'| \geqslant (1-\varepsilon^*)\binom{n}{2}$ and $|Y'| < (1-\varepsilon^*)n$). As noticed before, since $|Y'| < (1-\varepsilon^*)n$, there exist subsets $Y^*, Y'' \subset V$ such that $Y^* = Y' \cup Y''$ with $Y' \cap Y'' = \varnothing$, where $|Y''| \leqslant \varepsilon^* n$ and $|Y^*|$ is a multiple of $\lceil \varepsilon^* n\rceil$. Note that there exist subsets $\tilde{X}, X'' \subset \binom{V}{2}$ such that $X' = \tilde{X} \cup X''$ with $\tilde{X} \cap X'' = \varnothing$, where $|X''| \leqslant \varepsilon^*\binom{n}{2}$ and $|\tilde{X}|$ is a multiple of $\lceil \varepsilon^*\binom{n}{2}\rceil$.

If X'' is empty, then put $W'' = \varnothing$. If X'' is not empty, then we "complete" X'' with elements of $\binom{V}{2}$ to obtain W'' such that $X'' \subset W''$ and $|W''| = \lceil \varepsilon^*\binom{n}{2}\rceil$ (note that possibly

$W'' \cap \tilde{X} \neq \varnothing$). Thus, $|\tilde{X}| + |W''| \leqslant |X'| + \varepsilon^*\binom{n}{2}$. By using Lemma 3.2, we have

$$\begin{aligned}
e_G(X', Y') &\leqslant e_G(W'', Y^*) + e_G(\tilde{X}, Y^*) \\
&\leqslant (1+\varepsilon^*)q\left(|Y^*||W''| + |Y^*||\tilde{X}|\right) \\
&= (1+\varepsilon^*)q\left(|Y'||W''| + |Y''||W''| + |Y'||\tilde{X}| + |Y''||\tilde{X}|\right) \\
&\leqslant (1+\varepsilon^*)q\left(|Y'|\left(|X'| + \varepsilon^*\binom{n}{2}\right) + |Y''|\left(|X'| + \varepsilon^*\binom{n}{2}\right)\right) \\
&\leqslant (1+\varepsilon^*)q|X'||Y'| + 2q\left(\varepsilon^*\binom{n}{2}|Y'| + |X'|\varepsilon^* n + \varepsilon^*\binom{n}{2}\varepsilon^* n\right).
\end{aligned}$$

Since $\varepsilon^* \leqslant \varepsilon'^2\sigma^2/16$, we have $\varepsilon^*\binom{n}{2} \leqslant (\varepsilon'/8)|X'|$ and $\varepsilon^* n \leqslant (\varepsilon'/8)|Y'|$. Therefore,

$$\begin{aligned}
e_G(X', Y') &\leqslant q|X'||Y'| + \frac{\varepsilon'}{4}q|X'||Y'| + 2q\left(\frac{3\varepsilon'}{8}|X'||Y'|\right) \\
&\leqslant q|X'||Y'| + \varepsilon'\binom{|X|}{2}|Y|p.
\end{aligned}$$

Case 4: ($|X'| < (1-\varepsilon^*)\binom{n}{2}$ and $|Y'| \geqslant (1-\varepsilon^*)n$). This case is analogous to Case 3.

Lower bound. If $|X'| \leqslant \varepsilon'\binom{|X|}{2}$ or $|Y'| \leqslant \varepsilon'|Y|$, then there is nothing to prove, because $\varepsilon'\binom{|X|}{2}|Y|p > q|X'||Y'|$. Therefore, assume that $|X'| > \varepsilon'\binom{|X|}{2}$ and $|Y'| > \varepsilon'|Y|$. Clearly, there exist subsets $\tilde{X} \subset \binom{V}{2}$ and $\tilde{Y} \subset V$ such that $X' = \tilde{X} \cup X''$ and $Y' = \tilde{Y} \cup Y''$, with $\tilde{X} \cap X'' = \varnothing$ and $\tilde{Y} \cap Y'' = \varnothing$, where $|X''| \leqslant \varepsilon^*\binom{n}{2}$ and $|\tilde{X}|$ is a multiple of $\lceil \varepsilon^*\binom{n}{2} \rceil$ and $|Y''| \leqslant \varepsilon^* n$ and $|\tilde{Y}|$ is a multiple of $\lceil \varepsilon^* n \rceil$.

Since $\varepsilon^* \leqslant \varepsilon'^2\sigma^2/8$, we have

$$|X''| \leqslant \varepsilon^*\binom{n}{2} \leqslant (\varepsilon'/4)|X'| \leqslant \big(\varepsilon'/4(1-\varepsilon^*)\big)|X'|$$

and $|Y''| \leqslant \varepsilon^* n \leqslant (\varepsilon'/4(1-\varepsilon^*))|Y'|$. Then, by Lemma 3.2, since $e_G(X', Y') \geqslant e_G(\tilde{X}, \tilde{Y})$, we have

$$\begin{aligned}
e_G(X', Y') &\geqslant (1-\varepsilon^*)|\tilde{X}||\tilde{Y}|q \\
&= (1-\varepsilon^*)q\big(|X'||Y'| - |X'||Y''| - |X''||Y'| + |X''||Y''|\big) \\
&\geqslant (1-\varepsilon^*)q|X'||Y'| - (1-\varepsilon^*)q\big(|X'||Y''| + |X''||Y'|\big) \\
&\geqslant q|X'||Y'| - \varepsilon^* q|X'||Y'| - (1-\varepsilon^*)q\big((\varepsilon'/2(1-\varepsilon^*))|X'||Y'|\big) \\
&\geqslant q|X'||Y'| - \frac{\varepsilon'}{2}q|X'||Y'| - \frac{\varepsilon'}{2}q|X'||Y'| \\
&\geqslant q|X'||Y'| - \varepsilon'\binom{|X|}{2}|Y|p.
\end{aligned}$$

□

3.2. DISC **implies** PAIR**.** The next lemma, which is a variation of Lemma 9 in [8], makes it possible to obtain PAIR from DISC in spanning subhypergraphs of sufficiently jumbled 3-uniform hypergraphs.

Lemma 3.4. *For all $0 < \alpha \leqslant 1$ and $\delta' > 0$ there exists $\varepsilon' > 0$ such that for all $\sigma > 0$ there exist $\gamma > 0$ such that the following holds for sufficiently large n.*

Suppose that

(*i*) $\Gamma = (V, E_\Gamma)$ *is an n-vertex 3-uniform (p, β)-jumbled hypergraph with $p \geqslant 1/\sqrt{n}$,*
(*ii*) $G = (V, E_G)$ *is a spanning subhypergraph of Γ, and*
(*iii*) $X, Y \subset V$ *with $|X|, |Y| \geqslant \sigma n$.*

Then, the following holds. If $\beta \leqslant \gamma p^2 n^{3/2}$ and $(X, Y)_G$ satisfies $\mathrm{DISC}(q, p, \varepsilon')$ *for some q with $\alpha p \leqslant q \leqslant p$, then $(X, Y)_G$ satisfies* $\mathrm{PAIR}(q, p, \delta')$.

We need the following results in order to prove Lemma 3.4. First, consider the following fact, which is similar to [8, Fact 13].

Fact 3.5. *Let Γ be a 3-uniform (p, β)-jumbled hypergraph. Let $U \subset \binom{V}{2}$ and $W \subset V$ and $\xi > 0$. If we have $|N_\Gamma(\{x, y\}, W)| \geqslant (1 + \xi)p|W|$ for every $\{x, y\} \in U$ or we have $|N_\Gamma(\{x, y\}, W)| \leqslant (1 - \xi)p|W|$ for every $\{x, y\} \in U$, then*

$$|U||W| \leqslant \frac{\beta^2}{\xi^2 p^2}.$$

Proof. Let Γ, U, W and ξ be as in the statement and suppose that for every $\{x, y\} \in U$ we have $\big|N_\Gamma(\{x, y\}, W)\big| \geqslant (1+\xi)p|W|$. Suppose for a contradiction that $|U||W| > \frac{\beta^2}{\xi^2 p^2}$. Then, $e_\Gamma(U, W) \geqslant |U|(1 + \xi)p|W| > p|U||W| + \beta\sqrt{|U||W|}$, a contradiction to the jumbledness of Γ. The case where $|N_\Gamma(\{x, y\}, W)| \leqslant (1 - \xi)p|W|$ for every $\{x, y\} \in U$ is analogous. □

Our next result, Lemma 3.8 below, is very similar to [8, Lemma 21], but in Lemma 3.8 we consider bipartite graphs $\Gamma = \big(\binom{V}{2}, V; E_\Gamma\big)$ instead of $\Gamma = (U, V; E_\Gamma)$ in [8], and we consider subsets X_1, X_2 of $\binom{V}{2}$ with $|X_1|, |X_2| \geqslant \eta\binom{n}{2}$ instead of subsets X_1, X_2 of V with $|X_1|, |X_2| \geqslant \eta n$. Due to this fact, the value of β in Lemma 3.8 is $\gamma p^2 n^{3/2}$, while in [8, Lemma 21] we have $\beta = \gamma p n$. The proof of Lemma 3.8 is identical to the proof of [8, Lemma 21] and we omit it here.

Let $\Gamma = (V, E_\Gamma)$ be a graph and let X, $Y \subset V$. As usual, we denote by $e_\Gamma(X, Y)$ the number of edges of Γ with one end-vertex in X and one end-vertex in Y, where edges contained in $X \cap Y$ are counted twice. We need to define jumbledness and discrepancy for graphs.

Definition 3.6 (Jumbledness for graphs). *We say that* $\Gamma = (V, E_\Gamma)$ *is a* (p, β)-jumbled graph *if, for all subsets* X, $Y \subset V$, *we have* $\big|e_\Gamma(X, Y) - p|X||Y|\big| \leqslant \beta\sqrt{|X||Y|}$. *Furthermore, a bipartite graph* $\Gamma_B = (U, V; E)$ *is called* (p, β)-jumbled *if, for all* $X \subset U$ *and* $Y \subset V$, *we have* $\big|e_\Gamma(X, Y) - p|X||Y|\big| \leqslant \beta\sqrt{|X||Y|}$.

Property 3.7 (Discrepancy for graphs). *Let* $G = (V, E)$ *be a graph and let* X, $Y \subset V$ *be disjoint. We say that* (X, Y) *satisfies* $\mathrm{DISC}(q, p, \varepsilon)$ *in* G *(or* $(X, Y)_G$ *satisfies* $\mathrm{DISC}(q, p, \varepsilon)$*) if for all* $X' \subset X$ *and* $Y' \subset Y$ *we have*

$$\big|e_G(X', Y') - q|X'||Y'|\big| \leqslant \varepsilon p|X||Y|.$$

Lemma 3.8. *For all positive real* ϱ_0 *and* ν, *there exists a positive real* μ *such that, for all* $\sigma' > 0$, *there exist* $\gamma > 0$ *and* $n_0 > 0$ *such that for all* $n \geqslant n_0$, *the following holds.*

Suppose

(*i*) $\Gamma = \big(\binom{V}{2}, V; E_\Gamma\big)$ *is a bipartite* (p, β)*-jumbled graph with* $|V| \geqslant n$, $p \geqslant 1/\sqrt{n}$ *and* $\beta \leqslant \gamma p^2 n^{3/2}$,

(*ii*) $X_1, X_2 \subset \binom{V}{2}$ *and* $Y \subset V$ *with* $|X_1|, |X_2| \geqslant \sigma'\binom{n}{2}$, $|Y| \geqslant \sigma' n$,

(*iii*) $B = (X_1, X_2; E_B)$ *is an arbitrary bipartite graph.*

Then, if $(X_1, X_2)_B$ *satisfies* $\mathrm{DISC}(\varrho, 1, \mu)$ *for some* ϱ *with* $\varrho_0 \leqslant \varrho \leqslant 1$, *then for all but at most* $\nu|Y|$ *vertices* $y \in Y$, *the pair* $\big(N_\Gamma(y, X_1), N_\Gamma(y, X_2)\big)_B$ *satisfies* $\mathrm{DISC}(\varrho, 1, \nu)$.

We need two facts before proving of Lemma 3.4.

Fact 3.9 ([8, Fact 22]). *Suppose* $\varrho_0 > 0$, $\mu > 0$ *and* $B = (X, E_B)$ *is a graph with* $|E_B| \geqslant \varrho_0\binom{|X|}{2}$. *Then there exist disjoint subsets* $X_1, X_2 \subset X$ *such that*

(*i*) $(X_1, X_2)_B$ *satisfies* $\mathrm{DISC}(\varrho, 1, \mu)$ *for some* $\varrho \geqslant \varrho_0$,

(*ii*) $|X_1|, |X_2| \geqslant \zeta|X|$ *for* $\zeta = {\varrho_0}^{100/\mu^2}/4$.

Fact 3.10. *Let* $\Gamma = (V, E)$ *be a 3-uniform hypergraph and let* $\Gamma' = \big(\binom{V}{2}, V; E'\big)$ *be a bipartite graph, where* $E' = \big\{\{\{v_1, v_2\}, v\} \colon \{v_1, v_2\} \in \binom{V}{2}, v \in V \text{ and } \{v_1, v_2, v\} \in E\big\}$. *Then,* Γ *is* (p, β)*-jumbled if and only if* Γ' *is* (p, β)*-jumbled.*

We have stated all the tools needed in the proof of Lemma 3.4. This proof is very similar to the proof of [8, Lemma 9].

Proof of Lemma 3.4. Let $0 < \alpha \leqslant 1$ and $0 < \delta' < 1$ be given. Put $\xi = \delta'/6$, $\varrho_0 = \delta'/50$ and $\nu = \alpha^2\xi\varrho_0/64$. Let μ be obtained by an application of Lemma 3.8 with parameters ϱ_0 and ν. Without loss of generality, assume $\mu < \xi\varrho_0/4$. Let $\zeta = \varrho_0^{100/\mu^2}/4$ be given and put $\varepsilon' = \min\big\{\alpha\delta'^2/36, (\alpha^3\xi\varrho_0\zeta/64)^2\big\}$. Now fix $\sigma > 0$ and let $\sigma' = \zeta\sigma^2/2$. Following the quantification of Lemma 3.8 applied with parameter σ' we obtain γ' and n_0. Then, put

$$\gamma = \min\left\{\gamma', \sqrt{\sigma^3\delta'/12}, (\alpha/2)\sqrt{\xi\varrho_0\sigma\sigma'/24}\right\}.$$

Finally, consider n sufficiently large and suppose $p \geqslant 1/\sqrt{n}$.

Fix $\beta \leqslant \gamma p^2 n^{3/2}$ and consider a 3-uniform (p,β)-jumbled hypergraph $\Gamma = (V, E_\Gamma)$ such that $|V| = n$ and let $G = (V, E_G)$ be a spanning subhypergraph of Γ. Let X, Y be subsets of V such that $|X|, |Y| \geqslant \sigma n$. Suppose that $(X,Y)_G$ satisfies $\mathrm{DISC}(q,p,\varepsilon')$ for some q with $\alpha p \leqslant q \leqslant p$, i.e., for all $X' \subset \binom{X}{2}$ and $Y' \subset Y$ the following holds.

$$\left|e_G(X',Y') - q|X'||Y'|\right| \leqslant \varepsilon' p\binom{|X|}{2}|Y|. \tag{12}$$

We want to prove that the following inequalities hold:

$$\sum_{S_1 \in \binom{X}{2}} \left||N_G(S_1;Y)| - q|Y|\right| \leqslant \delta' p\binom{|X|}{2}|Y|, \tag{13}$$

$$\sum_{S_1 \in \binom{X}{2}} \sum_{S_2 \in \binom{X}{2}} \left||N_G(S_1,S_2;Y)| - q^2|Y|\right| \leqslant \delta' p^2\binom{|X|}{2}^2|Y|. \tag{14}$$

We start by verifying (13). For at most $\delta'\binom{|X|}{2}/6$ pairs $S \in \binom{X}{2}$, we have

$$\left||N_G(S,Y)| - q|Y|\right| > (\delta'/3)q|Y|\,.$$

Indeed, otherwise there would be a set $B_X \subset \binom{X}{2}$ with at least $\delta'\binom{|X|}{2}/12$ elements such that, for all $\{x,x'\} \in B_X$, either $|N_G(\{x,x'\},Y)| > (1+\delta'/3)q|Y|$ or for all of them we have $|N_G(\{x,x'\},Y)| < (1-\delta'/3)q|Y|$. In either case, we would have

$$\left|e_G(B_X,Y) - q|B_X||Y|\right| > \frac{\delta'^2}{36}q\binom{|X|}{2}|Y| \geqslant \frac{\delta'^2\alpha}{36}p\binom{|X|}{2}|Y| \geqslant \varepsilon' p\binom{|X|}{2}|Y|,$$

where the last inequality follows from the choice of ε'. But this contradicts (12) when we put $X' = B_X$ and $Y' = Y$.

Let W be the set of pairs $S \in \binom{X}{2}$ such that $|N_\Gamma(S,Y)| \geqslant 2p|Y|$. By Fact 3.5 applied to W and Y with $\xi = 1$, we know that there exist at most $\beta^2/p^2|Y|$ elements $S \in W$ such that $|N_\Gamma(S,Y)| \geqslant 2p|Y|$. Therefore,

$$\begin{aligned}\sum_{S \in \binom{X}{2}} \left||N_G(S,Y)| - q|Y|\right| &\leqslant \binom{|X|}{2}\frac{\delta'}{3}q|Y| + \left(\frac{\delta'}{6}\binom{|X|}{2}\right)2p|Y| + (\beta^2/p^2|Y|)|Y| \\ &\leqslant p\binom{|X|}{2}|Y|\left(\frac{2\delta'}{3}\right) + (\beta/p)^2 \leqslant \delta' p\binom{|X|}{2}|Y|,\end{aligned}$$

where the last inequality follows from the facts that $\beta \leqslant \gamma p^2 n^{3/2}$ and $\gamma \leqslant \sqrt{\sigma^3\delta'/12}$. We just proved that (13) holds.

Suppose for a contradiction that (14) does not holds. Then,

$$\sum_{S_1 \in \binom{X}{2}} \sum_{S_2 \in \binom{X}{2}} \big| |N_G(S_1, S_2; Y)| - q^2|Y| \big| > \delta' p^2 \binom{|X|}{2}^2 |Y|. \tag{15}$$

Define the following sets of "bad" pairs.

$$\mathcal{B}_1 = \left\{ (S_1, S_2) \in \binom{X}{2} \times \binom{X}{2} : |N_\Gamma(S_1, Y)| > 2p|Y| \right\},$$
$$\mathcal{B}_2 = \left\{ (S_1, S_2) \in \binom{X}{2} \times \binom{X}{2} \smallsetminus \mathcal{B}_1 : |N_\Gamma(S_1, S_2, Y)| > 4p^2|Y| \right\}.$$

Since Γ is (p, β)-jumbled, it follows that

$$|\mathcal{B}_1| \leqslant \frac{\beta^2}{p^2|Y|} \binom{|X|}{2} \leqslant \frac{\gamma^2 n^3 p^2}{|Y|} \binom{|X|}{2} \leqslant \frac{\gamma^2 n^2 p^2}{\sigma} \binom{|X|}{2} \leqslant \frac{\delta'}{3} p^2 \binom{|X|}{2}^2.$$

where the first inequality follows from Fact 3.5 applied to the sets

$$W = \{ S_1 \in \binom{X}{2} : |N_\Gamma(S_1, Y)| \geqslant 2p|Y| \}$$

and Y with $\xi = 1$. The second inequality follows from the choice of β, the third one follows from $|Y| \geqslant \sigma n$, and the last one holds because $|X| \geqslant \sigma n$ and $\gamma \leqslant \sqrt{\sigma^3 \delta'/12}$.

We want to bound $|\mathcal{B}_2|$ from above. By definition, if a pair of vertices belongs to $\mathcal{B}_2$, then it does not belong to $\mathcal{B}_1$. Then, consider a pair of vertices $S_1 \in \binom{X}{2}$ such that $|N_\Gamma(S_1, Y)| \leqslant 2p|Y|$. Consider a set $Y' \subset Y$ of size exactly $2p|Y|$ that contains $N_\Gamma(S_1, Y)$. Applying Fact 3.5 to the sets $\{S_2 \in \binom{X}{2} : |N_\Gamma(S_2, Y')| \geqslant 2p|Y'|\}$ and Y' with $\xi = 1$, we conclude that there are at most $\beta^2/p^2|Y'|$ pairs $S_2 \in \binom{X}{2}$ such that $|N_\Gamma(S_1, S_2, Y)| > 4p^2|Y|$. Therefore,

$$|\mathcal{B}_2| \leqslant \binom{|X|}{2} \frac{\beta^2}{p^2 2p|Y|} \leqslant \binom{|X|}{2} \frac{\gamma^2 n^2 p}{2\sigma} \leqslant \frac{\delta'}{6} p \binom{|X|}{2}^2,$$

The summation below is over the pairs $(S_1, S_2) \in \binom{X}{2} \times \binom{X}{2} \smallsetminus \mathcal{B}_1 \cup \mathcal{B}_2$. By (15) and the upper bounds on $\mathcal{B}_1$ and $\mathcal{B}_2$ we conclude that

$$\begin{aligned}
\sum \big| |N_G(S_1, S_2; Y)| - q^2|Y| \big| &> \delta' p^2 \binom{|X|}{2}^2 |Y| - |\mathcal{B}_1||Y| - |\mathcal{B}_2| 2p|Y| \\
&\geqslant \delta' p^2 \binom{|X|}{2}^2 |Y| - \frac{2\delta'}{3} p^2 \binom{|X|}{2}^2 |Y| \\
&= \frac{\delta'}{3} p^2 \binom{|X|}{2}^2 |Y|.
\end{aligned} \tag{16}$$

The contribution of the pairs $(S_1, S_2) \notin (\mathcal{B}_1 \cup \mathcal{B}_2)$ with $\big||N_G(S_1, S_2; Y)| - q^2|Y|\big| \leqslant \delta' q^2|Y|/6$ to the sum in (16) is at most

$$\frac{\delta'}{6} p^2 \binom{|X|}{2}^2 |Y|. \tag{17}$$

Note that, by the definition of $\mathcal{B}_2$, for all $(S_1, S_2) \notin \mathcal{B}_1 \cup \mathcal{B}_2$, the following holds.

$$\big||N_G(S_1, S_2; Y)| - q^2|Y|\big| \leqslant \max\big\{q^2|Y|, (4p^2 - q^2)|Y|\big\} \leqslant 4p^2|Y|.$$

Hence, by (16) and (17), there exist at least $\delta'\binom{|X|}{2}^2/24$ pairs $(S_1, S_2) \in \binom{X}{2} \times \binom{X}{2} \smallsetminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ such that

$$\big||N_G(S_1, S_2; Y)| - q^2|Y|\big| > \frac{\delta'}{6} q^2|Y| = \xi q^2|Y|. \tag{18}$$

Now let us define two auxiliary graphs B^+ and B^- with vertex-set $\binom{X}{2}$ and edge-sets as follows.

$$E(B^+) = \left\{\{S_1, S_2\} \in \binom{\binom{X}{2}}{2} : (1 + \xi)q^2|Y| < |N_G(S_1, S_2; Y)| \leqslant 4p^2|Y|\right\}$$
$$E(B^-) = \left\{\{S_1, S_2\} \in \binom{\binom{X}{2}}{2} : |N_G(S_1, S_2; Y)| < (1 - \xi)q^2|Y|\right\}.$$

Since there are at least $\delta'\binom{|X|}{2}^2/24$ pairs $(S_1, S_2) \in \binom{X}{2} \times \binom{X}{2}$ such that (18) holds, we have

$$\max\{e(B^+), e(B^-)\} \geqslant \frac{\binom{|X|}{2}^2 \delta'/24}{4} - \binom{|X|}{2} \geqslant \varrho_0 \binom{\binom{|X|}{2}}{2},$$

where in the first inequality the term "4" in the denominator comes from the fact that now we are counting unordered pairs and the edges belongs either to $E(B^+)$ or $E(B^-)$. Furthermore, we discount the pairs $\{S_1, S_1\}$.

Suppose without lost of generality that $e(B^+) \geqslant \varrho_0\binom{\binom{|X|}{2}}{2}$. Then, Fact 3.9 implies that there exist subsets $X_1, X_2 \subset \binom{X}{2}$ with $|X_1|, |X_2| \geqslant \zeta\binom{|X|}{2}$ such that $(X_1, X_2)_{B^+}$ satisfies $\mathrm{DISC}(\varrho, 1, \mu)$ for some $\varrho \geqslant \varrho_0$.

Recall that $\Gamma = (V, E_\Gamma)$ is a 3-uniform (p, β)-jumbled hypergraph with n vertices. By Fact 3.10, the bipartite graph $\Gamma' = (\binom{V}{2}, V; E_{\Gamma'})$, where

$$E_{\Gamma'} = \left\{\{\{v_1, v_2\}, v\} : \{v_1, v_2\} \in \tbinom{V}{2}, v \in V \text{ and } \{v_1, v_2, v\} \in E_\Gamma\right\}$$

is a (p, β)-jumbled graph. Note that $X_1, X_2 \subset \binom{X}{2} \subset \binom{V}{2}$ with $|X_1|, |X_2| \geqslant \zeta\binom{|X|}{2} \geqslant \zeta\binom{\sigma n}{2} \geqslant (\zeta\sigma^2/2)\binom{n}{2} \geqslant \sigma'\binom{n}{2}$. Therefore, the hypotheses of Lemma 3.8 are satisfied. By Lemma 3.8 we conclude that for all but at most $\nu|Y|$ vertices $y \in Y$, the pair $(N_{\Gamma'}(y, X_1), N_{\Gamma'}(y, X_2))_{B^+}$ satisfies $\mathrm{DISC}(\varrho, 1, \nu)$, which implies the following statement for all but at most $\nu|Y|$ vertices $y \in Y$.

$$(N_\Gamma(y, X_1), N_\Gamma(y, X_2))_{B^+} \text{ satisfies } \mathrm{DISC}(\varrho, 1, \nu). \tag{19}$$

Now let us estimate the number of triplets (S_1, S_2, y) in $X_1 \times X_2 \times Y$ such that $\{S_1, S_2\}$ is an edge of B^+ and the pairs S_1 and S_2 belong to the neighbourhood of y in G. Formally, we define such triplets as follows.

$$\mathcal{T} = \{(S_1, S_2, y) \in X_1 \times X_2 \times Y : S_1 \in N_G(y, X_1), S_2 \in N_G(y, X_2), \{S_1, S_2\} \in E(B^+)\}.$$

By the definition of B^+ we have

$$\begin{aligned} |\mathcal{T}| &> (1+\xi)q^2|Y|e_{B^+}(X_1, X_2) \geqslant (1+\xi)q^2|Y|(\varrho - \mu)|X_1||X_2| \\ &> \left(1 + \frac{\xi}{2}\right)\varrho q^2|X_1||X_2||Y|, \end{aligned} \tag{20}$$

where in the second inequality we used the fact that $(X_1, X_2)_{B^+} \in \mathrm{DISC}(\varrho, 1, \mu)$ and the last one follows from the choice of μ.

Now we will give an upper bound on $|\mathcal{T}|$ that contradicts (20). For that, we write $|\mathcal{T}| = \sum_{y \in Y} e_{B^+}\big(N_G(y, X_1), N_G(y, X_2)\big)$. Put

$$Y' = \big\{y \in Y : d_\Gamma(y, X_i) \leqslant 2p|X_i| \text{ for both } i = 1, 2\big\}.$$

By (19), for all but at most $\nu|Y|$ vertices $y \in Y'$ we have

$$\begin{aligned} e_{B^+}(N_G(y, X_1), N_G(y, X_2)) &\leqslant \varrho|N_G(y, X_1)||N_G(y, X_2)| \\ &\qquad + \nu|N_\Gamma(y, X_1)||N_\Gamma(y, X_2)| \\ &\leqslant \varrho\; d_G(y, X_1)d_G(y, X_2) + 4\nu p^2|X_1||X_2|. \end{aligned}$$

The last inequality follows from the fact that $y \in Y'$. Now we will bound the terms related to vertices in $Y \smallsetminus Y'$. By Fact 3.5, we have $|Y \smallsetminus Y'| \leqslant \beta^2\big(1/p^2|X_1| + 1/p^2|X_2|\big)$. Then,

$$\begin{aligned} |\mathcal{T}| \leqslant &\sum_{y \in Y'} \big(\varrho\; d_G(y, X_1)d_G(y, X_2) + 4\nu p^2|X_1||X_2|\big) + \nu|Y|4p^2|X_1||X_2| \\ &+ \left(\frac{\beta^2}{p^2|X_1|} + \frac{\beta^2}{p^2|X_2|}\right)|X_1||X_2|. \end{aligned}$$

The next inequality is obtained by putting the following facts together: $\nu = \alpha^2\xi\varrho/64$, $\gamma \leqslant (\alpha/2)\sqrt{\xi\varrho_0\sigma\sigma'/24}$, $q \geqslant \alpha p$, $|X_1|, |X_2| \geqslant \sigma'\binom{n}{2}$ and $|Y| \geqslant \sigma n$.

$$|\mathcal{T}| \leqslant \varrho \sum_{y \in Y'} (d_G(y, X_1)d_G(y, X_2)) + \frac{\xi}{4}\varrho q^2|X_1||X_2||Y|. \tag{21}$$

Define $Y_i'' = \big\{y \in Y : d_G(y, X_i) > (1 + \sqrt{\varepsilon'})q|X_i|\big\}$ for both $i = 1, 2$. Since $(X, Y)_G \in \mathrm{DISC}(q, p, \varepsilon')$, it is not hard to see that $|Y_i''| \leqslant \sqrt{\varepsilon'}p\binom{|X|}{2}|Y|/q|X_i|$ for both $i = 1, 2$. Since $|X_1|, |X_2| \geqslant \zeta\binom{|X|}{2}$, $q \geqslant \alpha p$ and $\varepsilon' \leqslant (\alpha^3\xi\varrho_0\zeta/64)^2$, the following holds for both $i = 1, 2$.

$$|Y_i''| \leqslant \frac{\xi\varrho\alpha^2}{64}|Y|.$$

Note that

$$\begin{aligned}\sum_{y\in Y'}(d_G(y,X_1)d_G(y,X_2)) &= \sum_{y\in Y'\smallsetminus(Y_1''\cup Y_2'')}(d_G(y,X_1)d_G(y,X_2))\\ &\quad+\sum_{y\in Y'\cap(Y_1''\cup Y_2'')}(d_G(y,X_1)d_G(y,X_2))\\ &\leqslant |Y|(1+\sqrt{\varepsilon'})^2q^2|X_1||X_2|+\frac{\xi\varrho\alpha^2}{32}|Y|\left(4p^2|X_1||X_2|\right).\end{aligned}$$

Therefore, since $\sqrt{\varepsilon'}\leqslant\xi/24$, the above inequality together with (21) implies

$$|\mathcal{T}|\leqslant\left(1+\frac{\xi}{2}\right)\varrho q^2|X_1||X_2||Y|,$$

a contradiction with (20). □

§4. Proof of the main result

In this section we show how to combine the lemmas presented in Section 3 in order to prove Theorem 2.1.

Proof of Theorem 2.1. Let ε, α, $\eta>0$, $C>1$ and $k\geqslant 4$ be given. Let $H_1,\dots,H_r$ be all the k-vertex 3-uniform hypergraphs which are linear and connector-free. Applying Lemma 3.1 with parameters k, C, ε and $d=k$ for $H_1,\dots,H_r$, we obtain, respectively, constants $\delta_1,\dots,\delta_r$. Now put $\delta_{\min}=\min\{\delta_1,\dots,\delta_r\}$. Let δ' be given by Lemma 2.5 applied with α and $\delta_{\min}$. Let ε' be given by Lemma 3.4 applied with α and δ'. Lemma 3.3 applied with ε', η and $\sigma=1$ gives δ. Following the quantification of Lemma 3.3 applied with α we obtain γ_1. Finally, following the quantification of Lemma 3.4 applied with $\sigma=1$ we obtain γ_2.

Put $\gamma=\min\{\gamma_1,\gamma_2\}$. Let $p=p(n)=o(1)$ with $p\gg n^{-1/k}$ and let $q=q(n)$ be such that $\alpha p\leqslant q\leqslant p$. In what follows we suppose that n is sufficiently large.

Let $\Gamma=(V,E_\Gamma)$ be an n-vertex (p,β)-jumbled 3-uniform hypergraph and let G be a spanning subhypergraph of Γ with $|E(G)|=q\binom{n}{3}$ such that G satisfies $\mathcal{Q}'(\eta,\delta,q)$ and $\mathrm{BDD}(k,C,q)$. Suppose that $\beta\leqslant\gamma p^2n^{3/2}$. We want to prove that G contains $(1\pm\varepsilon)n^kq^{|E(H)|}$ copies of all linear 3-uniform connector-free hypergraphs H with k vertices. By Lemma 3.3, our hypergraph G satisfies $\mathrm{DISC}(q,p,\varepsilon')$. Now apply Lemmas 3.4 and 2.5 in succession to deduce that G satisfies $\mathrm{PAIR}(q,p,\delta')$ and $\mathrm{TUPLE}(\delta,q)$. Now let H be any linear 3-uniform connector-free hypergraphs H with k vertices. Since G satisfies $\mathrm{TUPLE}(\delta,q)$ and $\mathrm{BDD}(k,C,q)$, by Lemma 3.1, we conclude that

$$\big||\mathcal{E}(H,G)|-n^kq^{|E(H)|}\big|<\varepsilon n^kq^{|E(H)|}.$$

□

§5. Concluding Remarks

Most of the definitions in this paper generalize naturally to k-uniform hypergraphs, for k larger than 3. Lemma 3.1 holds for k-uniform hypergraphs for every $k \geqslant 2$ (for details, see [6]). It would be interesting to obtain a version of Theorem 2.1 for k-uniform hypergraphs when $k > 3$, but unfortunately such a generalization presents new difficulties and will be considered elsewhere.

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Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brazil
E-mail address: {yoshi|mota}@ime.usp.br

Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany
E-mail address: schacht@math.uni-hamburg.de

Institut für Mathematik, Technische Universität Hamburg–Harburg, Hamburg, Germany
E-mail address: taraz@tuhh.de