ON THE RAMSEY NUMBER OF SPARSE 3-GRAPHS

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Abstract. We consider a hypergraph generalization of a conjecture of Burr and Erdős concerning the Ramsey number of graphs with bounded degree. It was shown by Chvátal, Rödl, Trotter, and Szemerédi [*The Ramsey number of a graph with bounded maximum degree*, J. Combin. Theory Ser. B **34** (1983), no. 3, 239–243] that the Ramsey number $R(G)$ of a graph G of bounded maximum degree is linear in $|V(G)|$. We derive the analogous result for 3-uniform hypergraphs.

1. Introduction

For an r-graph (or r-uniform hypergraph) $\mathcal{F}^{(r)}$, the *Ramsey number* $R(\mathcal{F}^{(r)})$ is the smallest integer N so that every 2-coloring of the complete r-graph $K_N^{(r)}$ yields a copy of $\mathcal{F}^{(r)}$ as a monochromatic sub-hypergraph. When $r = 2$, Burr and Erdős [2] stated the conjecture that for each Δ, every graph $F = \mathcal{F}^{(2)}$ with maximum degree at most Δ satisfies $R(F) \le C|V(F)|$ where $C = C(\Delta)$ is a constant depending only on Δ. This conjecture was proven by Chvátal, Rödl, Szemerédi and Trotter [4]. Further results on more general linear Ramsey conjectures of Burr and Erdős were later considered in [1, 3, 10, 11, 13, 18].

While there are several results on Ramsey numbers on various classes of graphs (see, e.g., [7] and the references therein), not much is known about the related problem for hypergraphs. Basically the only hypergraphs known to have linear Ramsey numbers are paths and cycles (see Haxell et al. [8]). A result concerning arbitrary hypergraphs of bounded maximum degree was recently obtained by Kostochka and Rödl. In [12] these authors proved that for each Δ, every r-graph with maximum degree at most Δ satisfies

$$R(\mathcal{F}^{(r)}) \le |V(\mathcal{F}^{(r)})|^{1+o(1)}, \tag{1}$$

where $o(1) \to 0$ as $|V(\mathcal{F}^{(r)})| \to \infty$. In this paper, we sharpen (1) to a linear bound for 3-graphs.

Theorem 1. *For all integers Δ, there exists $C = C(\Delta)$ so that, for every 3-graph $\mathcal{F}$ with maximum degree at most Δ, $R(\mathcal{F}) \le C|V(\mathcal{F})|$.*

Our proof of Theorem 1 is similar, in spirit, to the proof given in [4], where the analogous graph result is proved. The proof in [4] is based on *Szemerédi's regularity lemma* for graphs. Here, we use a 3-graph regularity lemma (Theorem 5) from [15]. In addition, we use a so-called *embedding lemma* (Lemma 8), which is, in fact, the main work of this paper.

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In the course of writing this paper, we learned that Cooley, Fountoulakis, Kühn and Osthus [5] independently obtained a proof of Theorem 1. Their proof is also based on a 3-graph regularity lemma and an embedding lemma. However, the proof of the respective embedding lemma is different.

We would also like to mention that, in an earlier version of our paper, we employed a proof of the embedding lemma different from our current proof (which was also different from that of [5]). This proof, which is conceptually similar to the original approach used for graphs in [4], can be found in the Master's thesis [16] of the second author.

2. The regular approximation lemma and the embedding lemma

The objective of this section is to state the two main tools in our proof of Theorem 1; the *regular approximation lemma* (Theorem 5) and the *embedding lemma* (Lemma 8). The regular approximation lemma (from [15]) is a variant of the original 3-graph regularity lemma from [6]. The embedding lemma shall be proved in this paper. To proceed, we require some definitions.

2.1. Notation.

In this paper, the notion of ε*-regularity* (for graphs) plays an important role. Let G be a graph and $X\dot\cup Y$ be two disjoint, non-empty sets of vertices. Write $e_G(X,Y) = |\{\{x,y\}\in G : x\in X, y\in Y\}|$ for the number of edges of G between X and Y and define $d_G(X,Y) = e_G(X,Y)/(|X||Y|)$ for the *density* of G. For $d,\varepsilon>0$, we say that the induced bipartite subgraph $G[X,Y]$ is (d,ε)*-regular* if for all $X'\subseteq X$, $|X'|>\varepsilon|X|$, and $Y'\subseteq Y$, $|Y'|>\varepsilon|Y|$, we have $|d(X',Y')-d|<\varepsilon$. We say that $G[X,Y]$ is ε*-regular* if it is (d,ε)-regular for some $0\le d\le 1$.

The definitions which follow (hypergraph density, regularity and partitions) emanate from the paper of Frankl and Rödl [6]. For a graph P with vertex set V, let $\mathcal{K}_3(P)$ denote the family of *triangles* K_3 of P, i.e.,

$$\mathcal{K}_3(P) = \Big\{\{x,y,z\}\in\tbinom{V}{3}\colon\ \{x,y\},\,\{x,z\},\{y,z\}\in E(P)\Big\}.$$

For a 3-graph $\mathcal{H}$ with vertex set V, we define the *(relative) density* $d_{\mathcal{H}}(P)$ *of* $\mathcal{H}$ *w.r.t.* P as

$$d_{\mathcal{H}}(P) = \begin{cases}\frac{|\mathcal{H}\cap\mathcal{K}_3(P)|}{|\mathcal{K}_3(P)|} & \text{if } |\mathcal{K}_3(P)|>0\\ 0 & \text{otherwise.}\end{cases}$$

Note that we identify the 3-graph $\mathcal{H}$ with its edge set. The following definition generalizes the notion of ε-regularity from graphs to 3-graphs.

Definition 2. *Let constants d, $\delta>0$ be given as well as a 3-partite 3-graph $\mathcal{H}$ and a 3-partite graph P on common vertex partition $V_1\dot\cup V_2\dot\cup V_3 = V$. We say that $\mathcal{H}$ is (d,δ)-*regular w.r.t. P *if for any subgraph $Q\subseteq P$,*

$$|\mathcal{K}_3(Q)| > \delta|\mathcal{K}_3(P)| \quad\Longrightarrow\quad |d_{\mathcal{H}}(Q)-d|<\delta.$$

*We say that $\mathcal{H}$ is δ-*regular w.r.t. P *if it is (d,δ)-regular for some constant d.*

Note that if $\mathcal{H}$ is (d,δ)-regular w.r.t. P, then for *every* subgraph $Q\subseteq P$ we have

$$\Big||\mathcal{H}\cap\mathcal{K}_3(Q)| - d|\mathcal{K}_3(Q)|\Big| < \max\Big\{\delta\big|\mathcal{K}_3(P)\big|,\,\delta\big|\mathcal{K}_3(Q)\big|\Big\} \le \delta|\mathcal{K}_3(P)|\,. \tag{2}$$

We now consider a specific type of partition.

Definition 3 ((ℓ, t, δ)-partition)**.** *Let integers ℓ and t and constant $\delta > 0$ be given. For a vertex set V, an (ℓ, t, δ)-*partition *$\boldsymbol{P}$ of V consists of a vertex partition $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_t = V$ and, for each $1 \le i < j \le t$, a partition of pairs*

$$\bigcup_{1 \le \alpha \le \ell} P_\alpha^{ij} = K(V_i, V_j)\,,$$

of the complete bipartite graph $K(V_i, V_j)$ with vertex classes V_i and V_j so that the following conditions hold:

(1) $|V_1| = \dots = |V_t|$ *and* $|V_0| < t$*;*
(2) for each $1 \le i < j \le t$ and each $1 \le \alpha \le \ell$, the bipartite graph P_α^{ij} is $(1/\ell, \delta)$-regular.

For an (ℓ, t, δ)-partition $\boldsymbol{P}$ of V and indices $1 \le i < j < k \le t$ and $1 \le \alpha, \beta, \gamma \le \ell$, we shall refer to the 3-partite graph $P_\alpha^{ij} \cup P_\beta^{jk} \cup P_\gamma^{ik}$ (which has vertex 3-partition $V_i \dot\cup V_j \dot\cup V_k$) as a *triad* of $\boldsymbol{P}$. Let $\mathrm{Triad}(\boldsymbol{P})$ denote the collection of triads of $\boldsymbol{P}$, i.e.,

$$\mathrm{Triad}(\boldsymbol{P}) = \left\{ P_\alpha^{ij} \cup P_\beta^{jk} \cup P_\gamma^{ik} : 1 \le i < j < k \le t,\ 1 \le \alpha, \beta, \gamma \le \ell \right\}.$$

We close this section with the definition of a regular partition.

Definition 4. *Let 3-graph $\mathcal{H}$ have vertex set V with (ℓ, t, δ)-partition $\boldsymbol{P}$. For $\delta > 0$, we say that $\mathcal{H}$ is δ-*regular w.r.t. *$\boldsymbol{P}$ if for each triad $P \in \mathrm{Triad}(\boldsymbol{P})$, $\mathcal{H}$ is δ-regular w.r.t. P.*

2.2. **The regular approximation lemma.** We now present the so-called *regular approximation lemma*. This lemma appeared for 3-graphs in [15] and was generalized to k-graphs in [17].

Theorem 5 (Regular approximation lemma)**.** *For all $\mu > 0$, integers t_0 and functions $\delta \colon \mathbb{N} \to (0, 1]$, there exist integers L_0, T_0 and N_0 so that for every 3-graph $\mathcal{G}$ with vertex set V of size $|V| = N > N_0$, there exist an $(\ell, t, \delta(\ell))$-partition $\boldsymbol{P}$ of V, where $1 \le \ell \le L_0$ and $t_0 \le t \le T_0$, and a 3-graph $\mathcal{H}$ with vertex set V so that the following conditions hold:*

(1) $\mathcal{H}$ is $\delta(\ell)$-regular w.r.t. $\boldsymbol{P}$;
(2) for all but at most $\mu\binom{t}{3}\ell^3$ triads $P \in \mathrm{Triad}(\boldsymbol{P})$, we have $|(\mathcal{G} \triangle \mathcal{H}) \cap \mathcal{K}_3(P)| < \mu|\mathcal{K}_3(P)|$.

For our purposes in this paper, we shall need a version of Theorem 5 suited for 2-colorings. For a 2-coloring $K_V^{(3)} = \mathcal{G}_r \dot\cup \mathcal{G}_b$ we will find $\mathcal{H}_r \dot\cup \mathcal{H}_b = K_V^{(3)}$ such that *(1)* and *(2)* of Theorem 5 hold for $\mathcal{H}_r$ and $\mathcal{G}_r$ and $\mathcal{H}_b$ and $\mathcal{G}_b$, respectively.

Corollary 6. *For all $\mu > 0$, integers t_0 and functions $\delta \colon \mathbb{N} \to (0, 1]$, there exist integers L_0, T_0 and N_0 so that for every 2-coloring $K_V^{(3)} = \mathcal{G}_r \dot\cup \mathcal{G}_b$ of the 3-uniform clique on vertex set V, $|V| = N > N_0$, there exist an $(\ell, t, \delta(\ell))$-partition $\boldsymbol{P}$ of V, where $1 \le \ell \le L_0$ and $t_0 \le t \le T_0$, and a 2-coloring $K_V^{(3)} = \mathcal{H}_r \dot\cup \mathcal{H}_b$ so that the following conditions hold:*

(1) *both $\mathcal{H}_r$ and $\mathcal{H}_b$ are $\delta(\ell)$-regular w.r.t. $\boldsymbol{P}$;*
(2) *for all but at most $\mu\binom{t}{3}\ell^3$ triads $P \in \mathrm{Triad}(\boldsymbol{P})$,*

$$|(\mathcal{G}_r \triangle \mathcal{H}_r) \cap \mathcal{K}_3(P)| = |(\mathcal{G}_b \triangle \mathcal{H}_b) \cap \mathcal{K}_3(P)| < \mu|\mathcal{K}_3(P)|\,.$$

The proof of Corollary 6 is straightforward, and we sketch it for completeness.

Proof (Sketch). Given $K_V^{(3)} = \mathcal{G}_r \dot\cup \mathcal{G}_b$ and given input parameters, apply Theorem 5 to $\mathcal{G}_r$ to obtain $\mathcal{H}_r$ and a regular partition $\boldsymbol{P}$ so that (*1*) and (*2*) of Theorem 5 hold.

It follows immediately from the definitions that $\boldsymbol{P}$ is also a regular partition for $\mathcal{H}_b = K_V^{(3)} \setminus \mathcal{H}_r$ and, consequently (*1*) of Corollary 6 holds. Moreover, note that

$$\mathcal{G}_b \triangle \mathcal{H}_b = (K_V^{(3)} \setminus \mathcal{G}_b) \triangle (K_V^{(3)} \setminus \mathcal{H}_b) = \mathcal{G}_r \triangle \mathcal{H}_r \,.$$

Consequently, for any triad $P \in \mathrm{Triad}(\boldsymbol{P})$ we have

$$(\mathcal{G}_b \triangle \mathcal{H}_b) \cap \mathcal{K}_3(P) = (\mathcal{G}_r \triangle \mathcal{H}_r) \cap \mathcal{K}_3(P)\,,$$

and so (*2*) of Corollary 6 follows from (*2*) of Theorem 5. □

2.3. **The embedding lemma.** We now state the *embedding lemma*, the second main tool in our proof of Theorem 1. The embedding lemma will take place in the following environment. (For a brief motivation of this environment, see Remark 9 below.)

Setup 7. *Let integers k and n and constants d_3, μ, d_2, $\delta > 0$ be given. Suppose graph P and 3-graph $\mathcal{H} = \mathcal{G} \dot\cup \mathcal{B}$ have common vertex set V where the following conditions are satisfied:*

(*1*) *$V = V_1 \dot\cup \ldots \dot\cup V_k$ is a k-partition, where $|V_1| = \cdots = |V_k| = n$;*
(*2*) *$P = \bigcup_{1 \le i < j \le k} P^{ij}$ is a k-partite graph with k-partition above, where each induced bipartite subgraph $P^{ij} = P[V_i, V_j]$, $1 \le i < j \le k$, is (d_2, δ)-regular;*
(*3*) *$\mathcal{H} = \bigcup_{1 \le h < i < j \le k} \mathcal{H}^{hij} \subseteq \mathcal{K}_3(P)$ is a k-partite 3-graph with k-partition above, where each induced 3-partite sub-hypergraph $\mathcal{H}^{hij} = \mathcal{H}[V_h, V_i, V_j]$, $1 \le h < i < j \le k$, is (d_3, δ)-regular w.r.t. $P^{hij} = P^{hi} \cup P^{hj} \cup P^{ij}$;*
(*4*) *$\mathcal{B} = \bigcup_{1 \le h < i < j \le k} \mathcal{B}^{hij} \subseteq \mathcal{H}$ is a sub-hypergraph of $\mathcal{H}$ where each $\mathcal{B}^{hij} = \mathcal{B}[V_h, V_i, V_j]$, $1 \le h < i < j \le k$, satisfies $|\mathcal{B}^{hij}| \le \mu |\mathcal{K}_3(P^{hij})|$.*

Lemma 8 (Embedding lemma)**.** *For all integers $\Delta \ge 1$ and all $d_3 > 0$, there exists $\mu > 0$ so that for all $d_2 > 0$, there exist $\delta > 0$, $c > 0$ and n_0 so that for all $n \ge n_0$, the following holds. Suppose*

(*i*) *V, P and $\mathcal{H} = \mathcal{G} \dot\cup \mathcal{B}$ are as in Setup 7 with the constants $k \ge (2\Delta)^2$, d_3, μ d_2, δ, and n;*
(*ii*) *$\mathcal{F}_0$ is a 3-graph on $m \le cn$ vertices which has maximum degree $\Delta(\mathcal{F}_0) \le \Delta$.*

Then, there exists a copy of $\mathcal{F}_0$ appearing as a sub-hypergraph of $\mathcal{G} = \mathcal{H} \setminus \mathcal{B}$.

We prove Lemma 8 in Section 4. (In fact, we prove a stronger version of Lemma 8 (see Proposition 13).)

We mention that the role of the hypergraph $\mathcal{B}$ (seen above) may not become clear until a few pages into the proof of Theorem 1. As such, we make the following remark on how, in general, Theorem 5 (the regular approximation lemma) and Lemma 8 (the embedding lemma) may be jointly applied. (Joint applications of Corollary 6 and Lemma 8 are analogous.)

Remark 9. *In context, the hypergraph $\mathcal{G}'$ of Lemma 8 will be a sub-hypergraph of the input hypergraph $\mathcal{G}$ of Theorem 5. The hypergraph $\mathcal{H}'$ of Lemma 8 will be a sub-hypergraph of the output hypergraph $\mathcal{H}$ of Theorem 5. Theorem 5 creates the 'very regular' $\mathcal{H}'$ by combining $\mathcal{G}'$ ('good' triples) with a small number of artificial triples $\mathcal{B}$ ('bad' triples). Lemma 8 finds a copy of $\mathcal{F}_0$ which lies within the very regular $\mathcal{H}'$ and which misses all unwanted (but rare) triples $\mathcal{B}$. This places $\mathcal{F}_0$ within $\mathcal{G}' \subseteq \mathcal{G}$.*

3. Proof of main result

Let integer Δ be given. To prove Theorem 1, we must first define the constant $C = C(\Delta)$ promised by Theorem 1. We do so now.

3.1. Constants.

We define C in terms of constants given by the regular approximation lemma (Corollary 6) and the embedding lemma (Lemma 8). Our definition of $C = C(\Delta)$ will appear in (12) below.

We first define the auxiliary constants

$$k = (2\Delta)^2 \qquad \text{and} \qquad d_3 = \frac{1}{2}\,. \tag{3}$$

Let

$$R = R(K_k^{(3)}) \tag{4}$$

be the Ramsey number for the 3-uniform clique $K_k^{(3)}$ on k vertices. We continue by defining auxiliary constants in terms of the embedding lemma. Recall the quantification of Lemma 8: $\forall \Delta, d_3\, \exists \mu\, \forall d_2\, \exists \delta, c, n_0$. With Δ given and d_3 already fixed in (3), we now let

$$\mu' = \mu(\Delta, d_3) > 0 \tag{5}$$

be the constant guaranteed by Lemma 8. Let ℓ be an integer variable and consider functions $\delta\colon \mathbb{N} \to (0,1]$, $c\colon \mathbb{N} \to (0,1]$, and $n_0\colon \mathbb{N} \to \mathbb{N}$ defined by

$$\delta(\ell) = \delta(\Delta, d_3, \mu', d_2 = 1/\ell) \tag{6}$$

$$c(\ell) = c(\Delta, d_3, \mu', d_2 = 1/\ell) \tag{7}$$

$$n_0(\ell) = n_0(\Delta, d_3, \mu', d_2 = 1/\ell) \tag{8}$$

guaranteed by Lemma 8 (for the choice of density $d_2 = 1/\ell$).

We continue by defining auxiliary constants in terms of the regular approximation lemma. Corollary 6 is quantified: $\forall \mu, t_0, \delta\colon \mathbb{N} \to (0,1]\,, \exists L_0, T_0, N_0$. We first fix μ and t_0 by setting

$$\mu = \min\left\{\tfrac{1}{2}\tbinom{R}{3}^{-1}, \mu'\right\} \qquad \text{and} \qquad t_0 = R\,. \tag{9}$$

With μ and t_0 fixed above and $\delta\colon \mathbb{N} \to (0,1]$ given in (6), let

$$L_0 = L_0(\mu, t_0, \delta(\cdot)), \quad T_0 = T_0(\mu, t_0, \delta(\cdot)), \quad N_0 = N_0(\mu, t_0, \delta(\cdot)) \tag{10}$$

be the constants given by Corollary 6 for constants μ and t_0 and function $\delta(\cdot)$.

We now use the constants above to define the promised constant $C = C(\Delta)$. To this end, set

$$c_{\min} = \min_{\ell=1,\dots,L_0} c(\ell)\,,\ n_{0,\max} = \max_{\ell=1,\dots,L_0} n_0(\ell)\,,\ \text{and } N_1 = \max\{n_{0,\max}, N_0\}\,, \tag{11}$$

where $c(\ell)$ and $n_0(\ell)$ are the functions defined in (7) and (8) and L_0 and N_0 are the constants given in (10). Finally, we define

$$C = \frac{2}{c} T_0 N_1\,. \tag{12}$$

Note that all constants above were defined after displaying Δ only, hence, the constant $C = C(\Delta)$ depends only on Δ. We now prove Theorem 1 for this choice of C, i.e., we verify that $R(\mathcal{F}_0) \le C|V(\mathcal{F}_0)|$ for any 3-graph $\mathcal{F}_0$ of maximum degree Δ.

3.2. **Proof of Theorem 1.** Let $\mathcal{F}_0$ be a 3-graph on m vertices with maximum degree $\Delta(\mathcal{F}_0) \leq \Delta$. We have to show $R(\mathcal{F}_0) \leq Cm$ where $C = C(\Delta)$ is the constant defined in (12). For that, let

$$N = Cm \tag{13}$$

and let V be an arbitrary set of N vertices. We will show that for every 2-coloring $K_V^{(3)} = \mathcal{G}_r \dot\cup \mathcal{G}_b$ of the 3-uniform clique $K_V^{(3)}$ on vertex set V,

$$\mathcal{F}_0 \subseteq \mathcal{G}_r \quad \text{or} \quad \mathcal{F}_0 \subseteq \mathcal{G}_b. \tag{14}$$

To prove (14), our first step is to apply Corollary 6 to the 2-coloring $K_V^{(3)} = \mathcal{G}_r \dot\cup \mathcal{G}_b$. To that end, let constants μ and t_0 be given in (9) and let function $\delta : \mathbb{N} \to (0,1]$ be given in (6). With these parameters, apply Corollary 6 to the 2-coloring $K_V^{(3)} = \mathcal{G}_r \dot\cup \mathcal{G}_b$ to obtain $(\ell, t, \delta(\ell))$-partition $\boldsymbol{P}$ and 2-coloring $K_V^{(3)} = \mathcal{H}_r \dot\cup \mathcal{H}_b$ (as described by Corollary 6) where $1 \leq \ell \leq L_0$ and $t_0 \leq t \leq T_0$, for the constants L_0 and T_0 given in (10). (Note that we may apply Corollary 6 to $K_V^{(3)} = \mathcal{G}_r \dot\cup \mathcal{G}_b$ since, by (12) and (13), $|V| = N = Cm = (2/c)T_0 N_1 m > N_0$.) Let partition $\boldsymbol{P}$ have vertex partition $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_t = V$ and system of bipartite graphs P_α^{ij}, $1 \leq i < j \leq t$, $1 \leq \alpha \leq \ell$, as described by Corollary 6.

We are going to use the partition $\boldsymbol{P}$ obtained above, together with Lemma 8, to prove (14). For that, we need to first locate an appropriate region of $\boldsymbol{P}$ to which to apply Lemma 8 (which will satisfy the assumptions of Lemma 8). This will be done in the following three steps.

Recall that, for every $1 \leq i < j \leq t$, the partition $\boldsymbol{P}$ admits ℓ graphs P_α^{ij} with $1 \leq \alpha \leq \ell$, while in Lemma 8, we need just a single such graph (see Setup 7). In the first step, we will select, for every $1 \leq i < j \leq t$, a graph P^{ij} in such a way that in most of the resulting triads the hypergraph $\mathcal{H}_r$ and $\mathcal{G}_r$ are "essentially the same." We will make this more precise now.

For a function $\lambda\colon \binom{\{1,\dots,t\}}{2} \to \{1,\dots,\ell\}$ and indices $1 \leq i < j \leq t$, we shall write $P^{ij}(\lambda) = P^{ij}_{\lambda(\{i,j\})}$ for the bipartite graph P_α^{ij} (from the partition $\boldsymbol{P}$) satisfying $\alpha = \lambda(\{i,j\})$. We then write

$$\mathrm{Triad}_\lambda(\boldsymbol{P}) = \big\{ P^{hi}(\lambda) \cup P^{ij}(\lambda) \cup P^{hj}(\lambda)\colon\ 1 \leq h < i < j \leq t \big\} \subseteq \mathrm{Triad}(\boldsymbol{P}).$$

Let

$$\mathrm{Triad}_{\mathrm{bad}}(\boldsymbol{P}) = \big\{ P \in \mathrm{Triad}(\boldsymbol{P})\colon\ |(\mathcal{G}_r \triangle \mathcal{H}_r) \cap \mathcal{K}_3(P)| \geq \mu |\mathcal{K}_3(P)| \big\}.$$

Due to property *(2)* of Corollary 6, we have $|\mathrm{Triad}_{\mathrm{bad}}(\boldsymbol{P})| \leq \mu \binom{t}{3} \ell^3$. Consequently, a simple averaging argument yields that there exists a function $\lambda_0\colon \binom{\{1,\dots,t\}}{2} \to \{1,\dots,\ell\}$ such that

$$|\mathrm{Triad}_{\lambda_0}(\boldsymbol{P}) \cap \mathrm{Triad}_{\mathrm{bad}}(\boldsymbol{P})| \leq \mu \tbinom{t}{3}. \tag{15}$$

(See, e.g., [14, Fact 4.1] for a similar argument.) In the remainder of our proof, we shall only work with bipartite graphs $P^{ij}(\lambda_0)$, $1 \leq i < j \leq t$. For simplicity of presentation, we shall now set $P^{ij} = P^{ij}(\lambda_0)$ (although, we will *not* drop the λ_0 from $\mathrm{Triad}_{\lambda_0}(\boldsymbol{P})$).

We now come to the second step. To each $1 \leq h < i < j \leq t$, we have associated precisely one triad in $\mathrm{Triad}_{\lambda_0}(\boldsymbol{P})$, and by the choice of μ in (9) and the inequality in (15), at most $\mu\binom{t}{3} < \binom{t}{3}/\binom{R}{3}$ of these triads are in $\mathrm{Triad}_{\mathrm{bad}}(\boldsymbol{P})$. Consequently, there exists an R-tuple of vertex classes V_i, w.l.o.g. $V_1, \dots, V_R$, so that for *every* $1 \le h < i < j \le R$,

$$P^{hij} = P^{hi} \cup P^{ij} \cup P^{hj} \text{ satisfies } P^{hij} \notin \mathrm{Triad}_{\mathrm{bad}}(\boldsymbol{P})\,. \tag{16}$$

(Here, we used $\mathrm{ex}(t, K_R^{(3)}) \le (1 - 1/\binom{R}{3})\binom{t}{3}$, where $\mathrm{ex}(t, K_R^{(3)})$ is the Turán number of $K_R^{(3)}$.)

In the third and last step, we consider the densities of $\mathcal{H}_r$ and $\mathcal{H}_b$ for each triad selected in the previous two steps. Since $K_V^{(3)} = \mathcal{H}_r \dot\cup \mathcal{H}_b$, we have, for every $1 \le h < i < j \le R$, either

$$d_{\mathcal{H}_r}(P^{hij}) \ge \tfrac{1}{2} \qquad \text{or} \qquad d_{\mathcal{H}_b}(P^{hij}) \ge \tfrac{1}{2}\,.$$

Consequently, by the choice of R in (4), there exists a k-tuple of vertex classes V_i, say w.l.o.g. $V_1, \dots, V_k$, and a color from $\{r, b\}$, say r, so that for every $1 \le h < i < j \le k$

$$d_{\mathcal{H}_r}(P^{hij}) \ge \tfrac{1}{2} \tag{17}$$

holds.

This defines the location of the region of $\boldsymbol{P}$ to which we want to apply Lemma 8. Set

$$\mathcal{H} = \mathcal{H}_r \cap \bigcup_{1 \le h < i < j \le k} \mathcal{K}_3(P^{hij})\,, \quad \mathcal{G} = \mathcal{G}_r \cap \bigcup_{1 \le h < i < j \le k} \mathcal{K}_3(P^{hij})\,, \quad \text{and} \quad \mathcal{B} = \mathcal{H}_r \setminus \mathcal{G}_r.$$

Note that, due to (16) and (17), $\mathcal{H} = \mathcal{G} \dot\cup \mathcal{B}$ defined above and $V_1 \dot\cup \dots \dot\cup V_k$ satisfy Setup 7 for constants k, d_3, μ' (chosen in (3) and (9)), $d_2 = 1/\ell$ and $\delta(\ell)$ (chosen in (6)) and $n = \lfloor N/t \rfloor$. Moreover, it is easy to check that all constants were appropriately chosen so that we can apply Lemma 8. Lemma 8 then yields a copy of $\mathcal{F}_0$ in $\mathcal{G} \subseteq \mathcal{G}_r$, which concludes the proof of Theorem 1. □

4. Proof of the 3-graph embedding lemma

In this section we reduce the proof of the 3-graph embedding lemma (Lemma 8) to a statement for embeddings of graphs (see Proposition 11 below). The proof of this reduction is similar to the proof of the *counting lemma* in [19, Lemma 3.4]. We first state the graph embedding lemma.

4.1. **Statement of the graph embedding lemma.**

Let P be a k-partite graph with vertex partition $V_1 \dot\cup \dots \dot\cup V_k = V = V(P)$ and let J_0 be k-partite graph with vertex partition $U_1 \dot\cup \dots \dot\cup U_k = U = V(J_0)$. We are interested in labeled embeddings (or copies) of J_0 in P. For a copy J of J_0 in P and a vertex $u \in U$ we denote by $J(u)$ the vertex $v \in V(J) \subseteq V$ which corresponds to the vertex u in J_0. Alternatively, we may view J as a injective mapping from $V(J_0)$ to $V(P)$ which preserves edges. We say a copy J of J_0 is a *partite* embedding if for every $i \in [k]$ and every $u \in U_i$ we have $J(u) \in V_i$. For the rest of the paper we restrict our attention to labeled, partite embeddings of J_0 and denote by $\mathscr{J} = \mathscr{J}(J_0, P)$ a family of (not necessarily all) partite embeddings of J_0 in P.

We also consider partial partite embeddings of induced subgraphs of J_0 in P. Let $X \subseteq U$ be fixed and write $J_0[X]$ for the subgraph of J_0 induced on the set X. Denote by $J|_X$ the partial embedding of $J_0[X]$. We define the family of all partial embeddings of $J_0[X]$ in $\mathscr{J}$ by

$$\mathscr{J}[X] = \big\{ J|_X \colon\ J \in \mathscr{J} \big\}\,.$$

For a given partial embedding $\tilde{J}$ of $J_0[X]$, we want to control the number of extensions of that particular copy to a full copy of J_0 in $\mathscr{J}$. With this in mind, we define

$$\operatorname{ext}_{\mathscr{J}}(\tilde{J}) = \left|\{J \in \mathscr{J} :\ J|_X = \tilde{J}\}\right|.$$

Note, that $\operatorname{ext}_{\mathscr{J}}(\tilde{J}) = 0$ if $\tilde{J} \notin \mathscr{J}[X]$, i.e., if there is no extension of $\tilde{J}$ to a full copy $J \in \mathscr{J}$. We now arrive at a crucial definition we use for the remainder of the paper.

Definition 10. *Let $d > 0$ and $\varepsilon > 0$ and let P be a k-partite graph with vertex partition $V_1 \dot\cup \ldots \dot\cup V_k = V$ and $|V_1| = \cdots = |V_k| = n$. Let J_0 be a k-partite graph on $m = |V(J_0)|$ vertices and let $\mathscr{J}$ be a family of labeled, partite embeddings of J_0 in P.*

We say $\mathscr{J}$ is (d,ε)-extendable *if for every $X \subseteq V(J_0)$ and every $\tilde{J} \in \mathscr{J}[X]$ we have*

$$\operatorname{ext}_{\mathscr{J}}(\tilde{J}) = (d \pm \varepsilon)^{e(J_0) - e(J_0[X])} n^{m - |X|}.$$

Note that a (d,ε)-extendable family must be "large". Indeed, applying the definition for $X = \emptyset$ gives

$$|\mathscr{J}| = |\mathscr{J}[\emptyset]| = (d \pm \varepsilon)^{e(J_0)} n^m.$$

The next lemma ensures the existence of (d,ε)-extendable families if P is sufficiently regular and J_0 is a graph of bounded maximum degree and size $c|V(P)|$ for sufficiently small $c > 0$. For the statement of the lemma, we also need the following concept. For a vertex $u \in U = V(J_0)$, let $N^2(u)$ be the set of all vertices of distance at most 2 from u, other than u itself, i.e.,

$$N^2(u) = \left(N(u) \cup \bigcup_{u' \in N(u)} N(u')\right) \setminus \{u\}. \tag{18}$$

Proposition 11. *For every integer $k \geq 2$ and all $d > 0$ and $\varepsilon > 0$ there exist $\delta > 0$, $c > 0$ and n_0 so that the following holds for every $n \geq n_0$.*

Let $P = \bigcup_{1 \leq i < j \leq k} P^{ij}$ be a k-partite graph with vertex partition $V_1 \dot\cup \ldots \dot\cup V_k = V = V(P)$, $|V_1| = \cdots = |V_k| = n \geq n_0$, where P^{ij} is (d,δ)-regular for every $1 \leq i < j \leq k$. Moreover, let J_0 be a k-partite graph on $m \leq cn$ vertices with vertex partition $U_1 \dot\cup \ldots \dot\cup U_k = U = V(J_0)$ such that for every $u \in U$ and every $i \in [k]$ we have $|(N^2(u) \cup \{u\}) \cap U_i| \leq 1$.

Then there exists a (d,ε)-extendable family $\mathscr{J} = \mathscr{J}(J_0, P)$ of labeled, partite embeddings of J_0 in P. In particular, $|\mathscr{J}| = (d \pm \varepsilon)^{e(J_0)} n^m$.

Proposition 11 can be considered a strengthened version of the graph embedding lemma of Chvátal et al. [4] (which found at least one embedding into P of any graph J_0 with maximum degree Δ). Indeed, for a graph J_0 of maximum degree Δ, one obtains the partition $U = U(J_0) = U_1 \cup \cdots \cup U_k$ satisfying the hypothesis of Proposition 11 by coloring the square J_0^2 of J_0 using $k = \Delta(\Delta - 1) + 1$ colors. We give the proof of Proposition 11 in Section 6 and first deduce Lemma 8 from Proposition 11.

4.2. **Proof of Lemma 8.** We are going to prove Lemma 8 by induction on the number of hyperedges in $\mathcal{F}_0$. For inductive purposes we will generalize Lemma 8 and prove a stronger statement (see Proposition 13 below). We will work with the following setup.

Setup 12. *Let integers k and m be given. Suppose a k-partite graph J_0 on vertex set U and 3-graph $\mathcal{F}_0$ with $V(\mathcal{F}_0) \subseteq U$ satisfy the following:*

(*a*) $U = U_1 \dot\cup \ldots \dot\cup U_k$ *is a k-partition and* $|U| = m$*;*
(*b*) *for every* $u \in U$ *and every* $i \in [k]$ *we have* $|(N^2_{J_0}(u) \cup \{u\}) \cap U_i| \leq 1$*;*
(*c*) $\mathcal{F}_0 \subseteq \mathcal{K}_3(J_0)$*.*

In context, the graph J_0 will be the 'shadow' $\partial\mathcal{F}_0$ of $\mathcal{F}_0$, which consists of all pairs from U covered by a triple of $\mathcal{F}_0$ (cf. (19) below).

Note that if a graph J_0 and a 3-graph $\mathcal{F}_0$ satisfy Setup 12, then from (*b*) and (*c*) we infer $\Delta(\mathcal{F}_0) \leq \Delta(J_0) \leq \max_{u\in U} |N^2_{J_0}(u)| < k$. In other words, $\mathcal{F}_0$ and J_0 have bounded maximum degree.

Consider V, P and $\mathcal{H} = \mathcal{G} \dot\cup \mathcal{B}$ as in Setup 7 and U, J_0, and $\mathcal{F}_0$ as in Setup 12. Moreover, let $\mathscr{J}$ be a family of labeled, partite embeddings of J_0 in P. This will be the environment of upcoming Proposition 13 (see (*i*)–(*iii*) in Proposition 13). For the statement of Proposition 13 we need some further definitions.

In Proposition 13, we are interested in labeled, partite embeddings of $\mathcal{F}_0$ in $\mathcal{G}$. We will also consider embeddings of $\mathcal{F}_0$ in $\mathcal{H}$. As before, for a copy $\mathcal{F}$ of $\mathcal{F}_0$ in $\mathcal{H}$ and $u \in V(\mathcal{F}_0)$, we denote by $\mathcal{F}(u)$ the vertex $v \in V$ that corresponds to u (in the copy $\mathcal{F}$). We restrict our attention to copies $\mathcal{F}$ of $\mathcal{F}_0$ with the additional properties that there exists a copy $J \in \mathscr{J}$ of J_0 in P such that $\mathcal{F} \subseteq \mathcal{K}_3(J)$ and that $\mathcal{F}(u) = J(u)$ for every $u \in V(\mathcal{F}_0)$. Note that since J is a labeled, partite copy of J_0, there exists at most one copy $\mathcal{F}$ of $\mathcal{F}_0$ for every $J \in \mathscr{J}$ and each such $\mathcal{F}$ must be a partite copy of $\mathcal{F}_0$. We denote the family of all those partite embeddings by

$$(P,\mathcal{H})^{(J_0,\mathcal{F}_0)}_{\mathscr{J}} = \big\{(J,\mathcal{F})\colon\ J \in \mathscr{J} \text{ and } \mathcal{F} \text{ is a copy of } \mathcal{F}_0 \text{ such that } \mathcal{F} \subseteq \mathcal{K}_3(J) \cap \mathcal{H} \text{ and } \mathcal{F}(u) = J(u) \text{ for every } u \in V(\mathcal{F}_0)\big\}\,.$$

More generally, for a set $X \subseteq U = V(J_0)$, the induced subgraph $\tilde{J}_0 = J_0[X]$ and a 3-graph $\tilde{\mathcal{F}}_0 \subseteq \mathcal{K}_3(\tilde{J}_0)$ we write

$$(P,\mathcal{H})^{(\tilde{J}_0,\tilde{\mathcal{F}}_0)}_{\mathscr{J}} = \Big\{(\tilde{J},\tilde{\mathcal{F}})\colon\ \tilde{J} \in \mathscr{J}[X] \text{ and } \tilde{\mathcal{F}} \text{ is a copy of } \tilde{\mathcal{F}}_0 \text{ such that } \tilde{\mathcal{F}} \subseteq \mathcal{K}_3(\tilde{J}) \cap \mathcal{H} \text{ and } \tilde{\mathcal{F}}(u) = \tilde{J}(u) \text{ for every } u \in V(\tilde{\mathcal{F}}_0)\Big\}\,.$$

The following lemma can be viewed as the induction statement for the proof of Lemma 8 on the number of edges of $\mathcal{F}_0$.

Proposition 13. *For all integers $k \geq 1$ and all $d_3 > 0$ and $\gamma \in (0, d_3)$, there exists $\mu > 0$ so that for all $d_2 > 0$ and $\varepsilon \in (0, d_2/2]$, there exist $\delta > 0$, $c > 0$ and n_0 so that for all $n \geq n_0$ the following holds. Suppose*

(*i*) *V, P and $\mathcal{H} = \mathcal{G} \dot\cup \mathcal{B}$ are as in Setup 7 for constants k, d_3, μ, d_2, δ, and n;*
(*ii*) *U, J_0, and $\mathcal{F}_0$ are as in Setup 12 for k and $m \leq cn$;*
(*iii*) *$\mathscr{J}$ is a (d_2, ε)-extendable family of labeled, partite embeddings of J_0 in P.*

Then, for every $e_0 \in \mathcal{F}_0$ we have

$$\Big|(P,\mathcal{G})^{(J_0,\mathcal{F}_0)}_{\mathscr{J}}\Big| = (d_3 \pm \gamma)\Big|(P,\mathcal{G})^{(J_0,\mathcal{F}_0\setminus e_0)}_{\mathscr{J}}\Big|\,.$$

We defer the proof of Proposition 13 to Section 5 in favor of first deducing Lemma 8 from Proposition 11 and Proposition 13. We first make the following remark regarding notation.

Remark 14. *In the remainder of this paper we will use the following conventions. Recall that J_0 (resp. $\mathcal{F}_0$) is a graph (3-graph) which we want to embed. We will denote by J and $\mathcal{F}$ embedded copies of J_0 and $\mathcal{F}_0$. We also consider embeddings of various sub-(hyper)graphs $\tilde{J}_0, J_0^* \subseteq J_0$ and $\tilde{\mathcal{F}}_0, \mathcal{F}_0^*, \mathcal{F}_0^- \subseteq \mathcal{F}_0$. For embedded copies of those sub-(hyper)graphs we will use the same notation and drop the zero in the subscript.*

Proof of Lemma 8. First, we determine the constants involved in the proof. For that, recall the quantification of Lemma 8, Proposition 11, and Proposition 13. As in Lemma 8, let $\Delta \geq 1$ and $d_3 > 0$ be given. We set $k = (2\Delta)^2$ and $\gamma = d_3/2$. For the fixed k, d_3, and γ, Proposition 13 yields $\mu > 0$. This is the μ we define for Lemma 8. Now, let $d_2 > 0$ be given. We then apply Proposition 11 with k, d_2, and $\varepsilon = d_2/2$ and obtain δ', $c' > 0$ and n_0'. Moreover, we apply Proposition 13 for d_2 and $\varepsilon = d_2/2$ and get δ'', $c'' > 0$ and n_0''. Finally, we fix the promised constants for Lemma 8 by setting $\delta = \min\{\delta', \delta''\}$, $c = \min\{c', c''\}$ and $n_0 = \max\{n_0', n_0''\}$.

Now, let V, P, $\mathcal{H} = \mathcal{G} \dot\cup \mathcal{B}$ and $\mathcal{F}_0$ satisfying (i) and (ii) of Lemma 8 be given. Set $J_0 = \partial \mathcal{F}_0$, the shadow of $\mathcal{F}_0$, defined as the graph with the same vertex set $U = V(J_0) = V(\mathcal{F}_0)$ and with edge set

$$E(J_0) = \{\{u, v\} \subseteq U\colon \text{ there is } w \in U \text{ such that } \{u, v, w, \} \in \mathcal{F}_0\}\,. \qquad (19)$$

Since $\Delta(\mathcal{F}_0) \leq \Delta$, the maximum degree in J_0 is bounded by 2Δ. Therefore, $\max_{u \in U} |N^2_{J_0}(u)| \leq 2\Delta(2\Delta - 1) < (2\Delta)^2$ and hence, for $k \geq (2\Delta)^2$, there exists a k-partition of $U_1 \dot\cup \dots \dot\cup U_k = U$ such that property (b) of Setup 12 holds (see the discussion after Proposition 11). Therefore, by (i) of Lemma 8, V, P, and J_0 satisfy the assumptions of Proposition 11 and we infer that there exists a (d_2, ε)-extendable family $\mathscr{J}$ of labeled, partite embeddings of J_0 in P where

$$|\mathscr{J}| = (d_2 \pm \varepsilon)^{e(J_0)} n^m \geq (d_2/2)^{e(J_0)} n^m\,. \qquad (20)$$

We now appeal to Proposition 13, and first check that its hypothesis is met. Clearly, by the definition of J_0, we have $\mathcal{F}_0 \subseteq \mathcal{K}_3(J_0)$ and J_0 and $\mathcal{F}_0$ satisfy part (c) of Setup 12, and hence, assumption (ii) of Proposition 13. Assumption (i) is satisfied since V, P, $\mathcal{H} = \mathcal{G} \dot\cup \mathcal{B}$ and $\mathcal{F}_0$ are given by Lemma 8, and the family $\mathscr{J}$ yields assumption (iii). Therefore, we can apply Proposition 13 for any $e_0 \in \mathcal{F}_0$. Moreover, we can apply Proposition 13 with $\mathcal{F}_0$ replaced by $\mathcal{F}_0 \setminus e_0$ and some $e_0' \in \mathcal{F}_0 \setminus e_0$. Hence, after $|\mathcal{F}_0|$ applications of Proposition 13, we have

$$\left|(P, \mathcal{G})^{(J_0, \mathcal{F}_0)}_{\mathscr{J}}\right| = (d_3 \pm \gamma)^{|\mathcal{F}_0|} \left|(P, \mathcal{G})^{(J_0, \emptyset)}_{\mathscr{J}}\right| \geq (d_3/2)^{|\mathcal{F}_0|} \left|(P, \mathcal{G})^{(J_0, \emptyset)}_{\mathscr{J}}\right|.$$

Noting that $\left|(P, \mathcal{G})^{(J_0, \emptyset)}_{\mathscr{J}}\right| = |\mathscr{J}|$, we obtain from (20)

$$\left|(P, \mathcal{G})^{(J_0, \mathcal{F}_0)}_{\mathscr{J}}\right| \geq (d_3/2)^{|\mathcal{F}_0|} (d_2/2)^{e(J_0)} n^m\,.$$

Since $\left|(P, \mathcal{G})^{(J_0, \mathcal{F}_0)}_{\mathscr{J}}\right|$ is a lower bound on the number of copies of $\mathcal{F}_0$ in $\mathcal{G} = \mathcal{H} \setminus \mathcal{B}$, this concludes the proof of Lemma 8 based on Propositions 11 and 13. □

5. Proof of Proposition 13

Recall the quantification of Proposition 13:

$$\forall\, k, d_3 > 0, \gamma \in (0, d_3),\ \exists\, \mu > 0:\ \forall\, d_2 > 0, \varepsilon \in (0, d_2/2],\ \exists\, \delta > 0, c > 0, n_0\,.$$

In order to simplify the presentation, we will not calculate estimates on the constants μ, δ, c, and n_0. Instead, we will verify Proposition 13 under the assumption that those constants were chosen in such a way that

$$\max\{\tfrac{1}{n_0}, c, \delta\} \ll \min\{\varepsilon, d_2, \mu\} \le \mu \ll \min\{\gamma, d_3, \tfrac{1}{k}\}\,, \tag{21}$$

where $a \ll b$ means that a was chosen sufficiently smaller than a function of b (and possibly other variables to the "right" of b).

Let V, P and $\mathcal{H} = \mathcal{G}\dot\cup\mathcal{B}$ satisfying (i) of Proposition 13 and U, J_0, and $\mathcal{F}_0$ satisfying (ii) be given. Moreover, let $\mathscr{J}$ be a (d_2, ε)-extendable family of embeddings of J_0 in P, i.e., part (iii) of Proposition 13 holds. We prove Proposition 13 by induction on $|\mathcal{F}_0|$. If $\mathcal{F}_0$ contains no edges then there is nothing to show. So let $\mathcal{F}_0$ with at least one edge be given and assume Proposition 13 holds for 3-graphs $\mathcal{F}_0'$ which have at most $|\mathcal{F}_0| - 1$ edges and which satisfy property (ii) of the proposition for the chosen constants. Note that since J_0 and $\mathcal{F}_0$ satisfy assumption (ii) of Proposition 13 we have

$$\Delta(\mathcal{F}_0) \le \Delta(J_0) \le k - 1\,. \tag{22}$$

Fix an edge $e_0 \in \mathcal{F}_0$. We denote by $\mathcal{F}_0^- = \mathcal{F}_0 \setminus e_0$ the sub-hypergraph of $\mathcal{F}_0$ on the same vertex set which we obtain by removing e_0 from $\mathcal{F}_0$. For a labeled copy $\mathcal{F}^-$ of $\mathcal{F}_0^-$ in $\mathcal{H}$, we denote by $\eta_{\mathcal{F}^-}$ the set of those three vertices which correspond to e_0 in $\mathcal{F}_0$. In other words, if $e_0 = \{x_0, y_0, z_0\}$ then

$$\eta_{\mathcal{F}^-} = \big\{\mathcal{F}^-(x_0),\, \mathcal{F}^-(y_0),\, \mathcal{F}^-(z_0)\big\}\,.$$

Note that $\eta_{\mathcal{F}^-}$ is not necessarily an edge in $\mathcal{H}$. Let $1_{\mathcal{H}}\colon \binom{V}{3} \to \{0,1\}$ be the indicator function for edges in $\mathcal{H}$.

Our first step will be to prove the following estimate for the number of copies $\mathcal{F}$ of $\mathcal{F}_0$ in $\mathcal{H}$ for which $\mathcal{F}^-$ is contained in $\mathcal{G}$ (in other words, only the edge corresponding to $e_0 \in \mathcal{F}_0$ is allowed to be in $\mathcal{B} = \mathcal{H} \setminus \mathcal{G}$):

$$\Big|\big\{(J,\mathcal{F}) \in (P,\mathcal{H})^{(J_0,\mathcal{F}_0)}_{\mathscr{J}}\colon\ \mathcal{F}^- \subseteq \mathcal{G}\big\}\Big| = (d_3 \pm \sqrt{\delta})\Big|(P,\mathcal{G})^{(J_0,\mathcal{F}_0^-)}_{\mathscr{J}}\Big|\,. \tag{23}$$

Before we prove (23), note that it yields the upper bound in the statement of Proposition 13, since $\mathcal{G} \subseteq \mathcal{H}$ and $\delta \ll \gamma$ (see (21)) (in fact, since $\delta \ll \gamma$, this upper bound is stronger than what we promised). Shortly, we will also use (23) to prove the lower bound in the statement of Proposition 13, but first proceed to prove (23).

Proof of (23). Observe

$$\begin{aligned}
\Big|\big\{(J,\mathcal{F}) \in (P,\mathcal{H})^{(J_0,\mathcal{F}_0)}_{\mathscr{J}}\colon\ \mathcal{F}^- \subseteq \mathcal{G}\big\}\Big| &= \sum\Big\{1_{\mathcal{H}}(\eta_{\mathcal{F}^-})\colon\ (J,\mathcal{F}^-) \in (P,\mathcal{G})^{(J_0,\mathcal{F}_0^-)}_{\mathscr{J}}\Big\}\\
&= \sum\Big\{d_3 + 1_{\mathcal{G}}(\eta_{\mathcal{F}^-}) - d_3\colon\ (J,\mathcal{F}^-) \in (P,\mathcal{G})^{(J_0,\mathcal{F}_0^-)}_{\mathscr{J}}\Big\} \qquad (24)\\
&= d_3\Big|(P,\mathcal{G})^{(J_0,\mathcal{F}_0^-)}_{\mathscr{J}}\Big| \pm \Big|\sum\Big\{1_{\mathcal{H}}(\eta_{\mathcal{F}^-}) - d_3\colon\ (J,\mathcal{F}^-) \in (P,\mathcal{G})^{(J_0,\mathcal{F}_0^-)}_{\mathscr{J}}\Big\}\Big|.
\end{aligned}$$

We will bound the contribution of the "±-term" in the last inequality by using the regularity of the hypergraph $\mathcal{H}$ (cf. (3) in Setup 7). For that, consider the induced sub-hypergraphs $J_0^* = J_0[U \setminus e_0]$ and $\mathcal{F}_0^* = \mathcal{F}_0[U \setminus e_0]$ of J_0 and $\mathcal{F}_0$ on $U \setminus e_0$. In other words, we obtain J_0^* from J_0 (resp. $\mathcal{F}_0^*$ from $\mathcal{F}_0$) by removing the vertices of e_0. For a copy $(J^*,\mathcal{F}^*) \in (P,\mathcal{G})^{(J_0^*,\mathcal{F}_0^*)}_{\mathscr{J}}$, let $\mathrm{EXTR}(J^*,\mathcal{F}^*)$ be the set of all triples $\eta \in \binom{V}{3}$ such that $V(J^*) \cup \eta = V(\mathcal{F}^*) \cup \eta$ extends J^* (resp. $\mathcal{F}^*$) to a copy $J \in \mathscr{J}$ (resp. copy $\mathcal{F}^- \subseteq \mathcal{G}$ of $\mathcal{F}_0^-$) such that $(J, \mathcal{F}^-) \in (P, \mathcal{G})_{\mathscr{J}}^{(J_0, \mathcal{F}_0^-)}$. Since $\mathcal{F}_0 \subseteq \mathcal{K}_3(J_0)$, e_0 induces a triangle in J_0 and hence $\mathrm{EXTR}(J^*, \mathcal{F}^*) \subseteq \mathcal{K}_3(P^{hij})$ for some $1 \leq h < i < j \leq k$. Moreover, as we now show, there exists a subgraph $Q(J^*, \mathcal{F}^*) \subseteq P^{hij}$ for which $\mathcal{K}_3(Q(J^*, \mathcal{F}^*)) = \mathrm{EXTR}(J^*, \mathcal{F}^*)$.

Indeed, let $(J^*, \mathcal{F}^*)$ be fixed and suppose $e_0 = \{x_0, y_0, z_0\}$ with $x_0 \in U_h$, $y_0 \in U_i$, and $z_0 \in U_j$. For a vertex $v \in V(\mathcal{H})$ let H_v be the *link-graph* of v, i.e., $V(H_v) = V(\mathcal{H}) \setminus \{v\}$ and $E(H_v) = \{\{v', v''\} \colon \{v, v', v''\} \in \mathcal{H}\}$. We also write $N_{\mathcal{H}}(v', v'')$ for the set of vertices $v \in V(\mathcal{H})$ which form an edge with v' and v'' in $\mathcal{H}$, i.e., $N_{\mathcal{H}}(v', v'') = \{v \in V(\mathcal{H}) \colon \{v, v', v''\} \in \mathcal{H}\}$. Before we define $Q(J^*, \mathcal{F}^*)$, we consider an auxiliary 3-partite graph $R = R(J^*, \mathcal{F}^*)$ defined as follows

$$V(R) = V_h \dot\cup V_i \dot\cup V_j \quad \text{and} \quad E(R) = E(R^{hi}) \dot\cup E(R^{hj}) \dot\cup E(R^{ij})$$

where

$$E(R^{hi}) = \bigcap \Big\{ E(H_{\mathcal{F}^*(u)}[V_h, V_i]) \colon u \in U \setminus e_0 \text{ and } \{x_0, y_0, u\} \in \mathcal{F}_0 \Big\},$$
$$E(R^{hj}) = \bigcap \Big\{ E(H_{\mathcal{F}^*(u)}[V_h, V_j]) \colon u \in U \setminus e_0 \text{ and } \{x_0, u, z_0\} \in \mathcal{F}_0 \Big\},$$

and

$$E(R^{ij}) = \bigcap \Big\{ E(H_{\mathcal{F}^*(u)}[V_i, V_j]) \colon u \in U \setminus e_0 \text{ and } \{u, y_0, z_0\} \in \mathcal{F}_0 \Big\}.$$

Since $\mathcal{H} \subseteq \mathcal{K}_3(P)$, we have $R \subseteq P^{hij}$. We then set $Q(J^*, \mathcal{F}^*)$ to be the induced subgraph of R defined by

$$Q(J^*, \mathcal{F}^*) = R[X, Y, Z],$$

where

$$\begin{aligned} X = (V_h \setminus V(J^*)) &\cap \bigcap \Big\{ N_P(J^*(u)) \colon \{x_0, u\} \in E(J_0) \text{ and } u \in U \setminus e_0 \Big\} \\ &\cap \bigcap \Big\{ N_{\mathcal{H}}(\mathcal{F}^*(u), \mathcal{F}^*(u')) \colon \{x_0, u, u'\} \in \mathcal{F}_0 \text{ and } u, u' \in U \setminus e_0 \Big\}, \\ Y = (V_i \setminus V(J^*)) &\cap \bigcap \Big\{ N_P(J^*(u)) \colon \{y_0, u\} \in E(J_0) \text{ and } u \in U \setminus e_0 \Big\} \\ &\cap \bigcap \Big\{ N_{\mathcal{H}}(\mathcal{F}^*(u), \mathcal{F}^*(u')) \colon \{y_0, u, u'\} \in \mathcal{F}_0 \text{ and } u, u' \in U \setminus e_0 \Big\}, \end{aligned}$$

and

$$\begin{aligned} Z = (V_j \setminus V(J^*)) &\cap \bigcap \Big\{ N_P(J^*(u)) \colon \{z_0, u\} \in E(J_0) \text{ and } u \in U \setminus e_0 \Big\} \\ &\cap \bigcap \Big\{ N_{\mathcal{H}}(\mathcal{F}^*(u), \mathcal{F}^*(u')) \colon \{z_0, u, u'\} \in \mathcal{F}_0 \text{ and } u, u' \in U \setminus e_0 \Big\}. \end{aligned}$$

Clearly $Q(J^*, \mathcal{F}^*) \subseteq P^{hij}$ and, by construction, $\mathcal{K}_3(Q(J^*, \mathcal{F}^*)) = \mathrm{EXTR}(J^*, \mathcal{F}^*)$.

Hence, we can now rewrite the "±-term" of (24) by considering copies $(J^*, \mathcal{F}^*) \in (P, \mathcal{G})^{(J_0^*, \mathcal{F}_0^*)}_{\mathscr{J}}$:

$$\begin{aligned}\Big|\sum\Big\{1_{\mathcal{H}}(\eta_{\mathcal{F}^-}) - d_3\colon (J, \mathcal{F}^-) \in (P, \mathcal{G})^{(J_0, \mathcal{F}_0^-)}_{\mathscr{J}}\Big\}\Big| \\ = \sum\Big\{\Big|\sum_{\eta \in \mathrm{EXTR}(J^*, \mathcal{F}^*)} (1_{\mathcal{H}}(\eta) - d_3)\Big|\colon (J^*, \mathcal{F}^*) \in (P, \mathcal{G})^{(J_0^*, \mathcal{F}_0^*)}_{\mathscr{J}}\Big\} \\ = \sum\Big\{\Big|\sum_{\eta \in \mathcal{K}_3(Q(J^*, \mathcal{F}^*))} (1_{\mathcal{H}}(\eta) - d_3)\Big|\colon (J^*, \mathcal{F}^*) \in (P, \mathcal{G})^{(J_0^*, \mathcal{F}_0^*)}_{\mathscr{J}}\Big\}.\end{aligned}$$

Since $Q(J^*, \mathcal{F}^*) \subseteq \mathcal{K}_3(P^{hij})$ for every $(J^*, \mathcal{F}^*) \in (P, \mathcal{G})^{(J_0^*, \mathcal{F}_0^*)}_{\mathscr{J}}$, we can use the (d_3, δ)-regularity of $\mathcal{H}$ w.r.t. P^{hij} to bound the inner sum by $\delta|\mathcal{K}_3(P^{hij})| \le \delta n^3$ (see (2)) and obtain

$$\Big|\sum\Big\{(1_{\mathcal{H}}(\eta_{\mathcal{F}^-}) - d_3)\colon (J, \mathcal{F}^-) \in (P, \mathcal{G})^{(J_0, \mathcal{F}_0^-)}_{\mathscr{J}}\Big\}\Big| \le \delta n^3 \times \Big|(P, \mathcal{G})^{(J_0^*, \mathcal{F}_0^*)}_{\mathscr{J}}\Big|. \tag{25}$$

Since $\mathscr{J}$ is (d_2, ε)-extendable, *every* $J^* \in \mathscr{J}[U \setminus e_0]$ extends to at least $(d_2 - \varepsilon)^{e(J_0) - e(J_0^*)} n^3$ copies J of J_0 in $\mathscr{J}$; this holds, in particular, for any copy J^* contained in a pair $(J^*, \mathcal{F}^*) \in (P, \mathcal{G})^{(J_0^*, \mathcal{F}_0^*)}_{\mathscr{J}}$. Consequently,

$$\delta n^3 \times \Big|(P, \mathcal{G})^{(J_0^*, \mathcal{F}_0^*)}_{\mathscr{J}}\Big| \le \frac{\delta \times \Big|(P, \mathcal{G})^{(J_0, \mathcal{F}_0^*)}_{\mathscr{J}}\Big|}{(d_2 - \varepsilon)^{e(J_0) - e(J_0^*)}}. \tag{26}$$

Now, we apply the induction assumption $|\mathcal{F}_0^-| - |\mathcal{F}_0^*|$ times and obtain

$$\frac{\delta \times \Big|(P, \mathcal{G})^{(J_0, \mathcal{F}_0^*)}_{\mathscr{J}}\Big|}{(d_2 - \varepsilon)^{e(J_0) - e(J_0^*)}} \le \frac{\delta \times \Big|(P, \mathcal{G})^{(J_0, \mathcal{F}_0^-)}_{\mathscr{J}}\Big|}{(d_2 - \varepsilon)^{e(J_0) - e(J_0^*)} (d_3 - \gamma)^{|\mathcal{F}_0^-| - |\mathcal{F}_0^*|}} \overset{(21)}{\le} \sqrt{\delta}\Big|(P, \mathcal{G})^{(J_0, \mathcal{F}_0^-)}_{\mathscr{J}}\Big|,$$

where we used (22) (which combined with $|V(J_0^*)| = |V(\mathcal{F}_0^*)| = m - 3$ yields $\max\{e(J_0) - e(J_0^*), |\mathcal{F}_0^-| - |\mathcal{F}_0^*|\} < 3k$) in the last inequality. Combining, the last estimate with (24)–(26) renders (23). □

We will now derive the lower bound on $\big|(P, \mathcal{G})^{(J_0, \mathcal{F}_0)}_{\mathscr{J}}\big|$ from (23). For that, we will estimate the contribution of edges in $\mathcal{H}$ (in fact, edges in $\mathcal{B} = \mathcal{H} \setminus \mathcal{G}$) contained in copies of $\mathcal{F}_0$ in $(P, \mathcal{G})^{(J_0, \mathcal{F}_0)}_{\mathscr{J}}$. More precisely, for any $\eta \in \mathcal{K}_3(P^{hij})$, we consider

$$\deg(\eta) = \Big|\Big\{(J, \mathcal{F}^-) \in (P, \mathcal{G})^{(J_0, \mathcal{F}_0^-)}_{\mathscr{J}}\colon \eta = \eta_{\mathcal{F}^-}\Big\}\Big|,$$

and shall derive an upper bound on $\deg(\eta)$ for $\eta \in \mathcal{K}_3(P^{hij})$. In particular, we will prove that for each $\eta \in \mathcal{K}_3(P^{hij})$,

$$\deg(\eta) \le \frac{1}{\sqrt{\mu}} \times \frac{\Big|(P, \mathcal{G})^{(J_0, \mathcal{F}_0^-)}_{\mathscr{J}}\Big|}{|\mathcal{K}_3(P^{hij})|}. \tag{27}$$

This allows us to derive the required lower bound on $\big|(P,\mathcal{G})^{(J_0,\mathcal{F}_0)}_{\mathscr{J}}\big|$. Indeed, we have

$$\begin{aligned}\Big|(P,\mathcal{G})^{(J_0,\mathcal{F}_0)}_{\mathscr{J}}\Big| &= \Big|\big\{(J,\mathcal{F})\in(P,\mathcal{H})^{(J_0,\mathcal{F}_0)}_{\mathscr{J}}\colon\ \mathcal{F}^-\subseteq\mathcal{G}\big\}\Big| - \sum_{\eta\in\mathcal{K}_3(P^{hij})\cap\mathcal{B}}\deg(\eta)\\ &\overset{(23)}{\geq} (d_3-\sqrt{\delta})\Big|(P,\mathcal{G})^{(J_0,\mathcal{F}_0^-)}_{\mathscr{J}}\Big| - \sum_{\eta\in\mathcal{K}_3(P^{hij})\cap\mathcal{B}}\deg(\eta)\\ &\overset{(27)}{\geq} (d_3-\sqrt{\delta})\Big|(P,\mathcal{G})^{(J_0,\mathcal{F}_0^-)}_{\mathscr{J}}\Big| - \big|\mathcal{K}_3(P^{hij})\cap\mathcal{B}\big|\times\frac{1}{\sqrt{\mu}}\frac{\Big|(P,\mathcal{G})^{(J_0,\mathcal{F}_0^-)}_{\mathscr{J}}\Big|}{|\mathcal{K}_3(P^{hij})|}\\ &\geq (d_3-\sqrt{\delta}-\sqrt{\mu})\Big|(P,\mathcal{G})^{(J_0,\mathcal{F}_0^-)}_{\mathscr{J}}\Big|,\end{aligned}$$

where the last inequality follows from (i) of Proposition 13, i.e., that part (4) of Setup 7 holds. The required lower bound on $\big|(P,\mathcal{G})^{(J_0,\mathcal{F}_0)}_{\mathscr{J}}\big|$ now follows from $\delta\ll\mu\ll\gamma$.

All that remains is to prove (27), for which we use the following notation. Consider the graph J_0 and suppose $e_0=\{x_0,y_0,z_0\}\subset U$. Let

$$N_{J_0}(e_0)=\big(N_{J_0}(x_0)\cup N_{J_0}(y_0)\cup N_{J_0}(z_0)\big)\setminus e_0$$

be the neighbors of the vertices in e_0 in J_0. Let $\tilde{J}_0=J_0[U\setminus(e_0\cup N_{J_0}(e_0))]$ be the induced subgraph of J_0, which we obtain after removing all vertices contained in $e_0\cup N_{J_0}(e_0)$. Similarly, we define $\tilde{\mathcal{F}}_0=\mathcal{F}_0[U\setminus(e_0\cup N_{J_0}(e_0))]$ as the induced sub-hypergraph after removing the same vertices.

We now prove (27). For a copy $\tilde{J}$ of $\tilde{J}_0$ and a set of three vertices η which is disjoint from $V(\tilde{J})$ and which spans a triangle in P, we denote by $\tilde{J}+\eta$ the union of the graph $\tilde{J}$ with that triangle. In particular, $\tilde{J}+\eta$ is then a copy of $J_0[U\setminus N_{J_0}(e_0)]$. With this notation, we have, for every $\eta\in\mathcal{K}_3(P^{hij})$,

$$\deg(\eta)\leq\sum\Big\{\operatorname{ext}_{\mathscr{J}}(\tilde{J}+\eta)\colon\ (\tilde{J},\tilde{\mathcal{F}})\in(P,\mathcal{G})^{(\tilde{J}_0,\tilde{\mathcal{F}}_0)}_{\mathscr{J}}\Big\}.$$

Since $\mathscr{J}$ is a (d_2,ε)-extendable family, we can bound

$$\operatorname{ext}_{\mathscr{J}}(\tilde{J}+\eta)\leq(d_2+\varepsilon)^{e(J_0)-e(\tilde{J}_0)-3}n^{|N_{J_0}(e_0)|}$$

and, therefore,

$$\deg(\eta)\leq(d_2+\varepsilon)^{e(J_0)-e(\tilde{J}_0)-3}n^{|N_{J_0}(e_0)|}\times\Big|(P,\mathcal{G})^{(\tilde{J}_0,\tilde{\mathcal{F}}_0)}_{\mathscr{J}}\Big|. \tag{28}$$

Repeating similar arguments as for (26), we can again use the property that $\mathscr{J}$ is (d_2,ε)-extendable to show

$$\Big|(P,\mathcal{G})^{(\tilde{J}_0,\tilde{\mathcal{F}}_0)}_{\mathscr{J}}\Big|\leq\frac{\Big|(P,\mathcal{G})^{(J_0,\tilde{\mathcal{F}}_0)}_{\mathscr{J}}\Big|}{(d_2-\varepsilon)^{e(J_0)-e(\tilde{J}_0)}n^{|N_{J_0}(e_0)|+3}} \tag{29}$$

Continuing similarly as before (see paragraph after (26)), i.e., applying the induction assumption $|\mathcal{F}_0^-|-|\tilde{\mathcal{F}}_0|$ times, we obtain

$$\Big|(P,\mathcal{G})^{(J_0,\tilde{\mathcal{F}}_0)}_{\mathscr{J}}\Big|\leq\frac{\Big|(P,\mathcal{G})^{(J_0,\mathcal{F}_0^-)}_{\mathscr{J}}\Big|}{(d_3-\gamma)^{|\mathcal{F}_0^-|-|\tilde{\mathcal{F}}_0|}}. \tag{30}$$

Combining (28)–(30) with $\varepsilon \le d_2/2$ (which yields $(d_2+\varepsilon)/(d_2-\varepsilon) \le 3$), we get for every $\eta \in \mathcal{K}_3(P^{hij})$

$$\deg(\eta) \le \frac{3^{e(J_0)-e(\tilde{J}_0)}}{(d_3-\gamma)^{|\mathcal{F}_0^-|-|\tilde{\mathcal{F}}_0|}} \times \frac{\left|(P,\mathcal{G})_{\mathscr{J}}^{(J_0,\mathcal{F}_0^-)}\right|}{(d_2+\varepsilon)^3 n^3} \overset{(21)}{\le} \frac{1}{\sqrt{\mu}} \times \frac{\left|(P,\mathcal{G})_{\mathscr{J}}^{(J_0,\mathcal{F}_0^-)}\right|}{(d_2+\varepsilon)^3 n^3},$$

where we used (22) (which yields $|N_{J_0}(e)| \le 3k-3$ and hence $\max\{e(J_0)-e(\tilde{J}_0), |\mathcal{F}_0^-|-|\tilde{\mathcal{F}}_0|\} < 3k^2$) for the last inequality, in addition to (21). Moreover, we infer from the folklore "triangle counting lemma" for (d_2,δ)-regular graphs (see, e.g., [6, Fact A]), that $|\mathcal{K}_3(P^{hij})| \le (d_2+\varepsilon)^3 n^3$ for sufficiently small $\delta \ll \min\{\varepsilon, d_2\}$. Therefore, we can rewrite the last estimate in the form

$$\deg(\eta) \le \frac{1}{\sqrt{\mu}} \times \frac{\left|(P,\mathcal{G})_{\mathscr{J}}^{(J_0,\mathcal{F}_0^-)}\right|}{|\mathcal{K}_3(P^{hij})|},$$

which is (27). □

6. Proof of the graph embedding lemma

The proof of the graph embedding lemma, Proposition 11, presented in this section, is an adaptation of the proof from [4]. In Section 6.1 we introduce the concepts used in the proof and review a few auxiliary facts concerning (d,δ)-regular graphs. The proof of Proposition 11 then follows in Section 6.2.

6.1. Preparations.

In Proposition 11 we consider a given a k-partite graph $P = \bigcup_{1\le i<j\le k} P^{ij}$ with vertex partition $V_1 \dot\cup \ldots \dot\cup V_k = V = V(P)$, $|V_1| = \cdots = |V_k| = n$, where P^{ij} is (d,δ)-regular for every $1 \le i < j \le k$. Later we will embed a given graph J_0 with bounded maximum degree into P. In order to obtain an extendable family of embeddings $\mathscr{J}$ (see Definition 10) we will try to avoid sets of vertices which have the "wrong" number of joint neighbors. For that we first define sets of "bad" vertices.

In what follows, we abuse cross-product notation and write, for a set $I \subseteq [k]$, $\prod_{i\in I} V_i = \{\{v_i\}_{i\in I} : v_i \in V_i \text{ for all } i \in I\}$ for the set of all unordered $|I|$-tuples of vertices which are transversal to $\bigcup_{i\in I} V_i$.

Definition 15. *For $\varepsilon > 0$ and P as above set $B_{[k]}(P,\varepsilon) = \emptyset$ and for every proper subset $\emptyset \ne I \subsetneq [k]$ define recursively*

$$B_I(P,\varepsilon) = \Bigg\{\{v_i\}_{i\in I} \in \prod_{i\in I} V_i \colon \exists \ell \in [k]\setminus I \text{ s.t. } \Big|\bigcap_{i\in I} N(v_i) \cap V_\ell\Big| \ne (d\pm\varepsilon)^{|I|} n \\ \text{or } \big|\{v_\ell \in V_\ell \colon \{v_i\}_{i\in I\cup\{\ell\}} \in B_{I\cup\{\ell\}}(P,\varepsilon)\}\big| \ge \varepsilon n\Bigg\}.$$

If P and ε are clear from context, we will simply write B_I for $B_I(P,\varepsilon)$. We also set $B = B(P,\varepsilon) = \bigcup_{\emptyset\ne I\subsetneq[k]} B_I(P,\varepsilon)$.

It is a well known fact that if $\delta \ll \min\{\varepsilon, d, 1/k\}$, then the (d,δ)-regularity of P implies that for every $\emptyset \ne I \subsetneq [k]$, all but at most $\varepsilon n^{|I|}$ $|I|$-tuples $\{v_i\}_{i\in I} \in \prod_{i\in I} V_i$ satisfy

$$\Big|\bigcap_{i\in I} N(v_i) \cap V_\ell\Big| = (d\pm\varepsilon)^{|I|} n \tag{31}$$

for every $\ell \in [k] \setminus I$ (see, e.g., [9, Fact 1.4]). The following fact is a simple consequence of (31).

Fact 16. *For all integers k and d, $\varepsilon > 0$ there exists $\delta > 0$ so that for every graph P as above the following holds. For every proper subset $\emptyset \neq I \subsetneq [k]$ we have $|B_I(P, \varepsilon)| \leq \varepsilon n^{|I|}$.*

Proof. The proof is by induction on $k - |I|$. If $|I| = k-1$ then the statement follows directly from (31), since $B_{[k]}(P, \varepsilon) = \emptyset$.

Let $\emptyset \neq I \subsetneq [k]$ be fixed. Fix $\varepsilon' > 0$ such that $\varepsilon' + k\sqrt{\varepsilon'} \leq \varepsilon$ and let δ be small enough so that (31) holds with ε replaced by ε' and so that $|B_J(P, \varepsilon')| \leq \varepsilon' n^{|J|}$ for all super-sets $J \supsetneq I$.

Note that (31) implies there are at most $\varepsilon' n^{|I|}$ $|I|$-tuples in $\prod_{i \in I} V_i$ which belong to $B_I(P, \varepsilon')$ due to the first reason, i.e., they fail to satisfy (31) with ε replaced by ε'. Moreover, by induction, $|B_{I \cup \ell}(P, \varepsilon')| \leq \varepsilon' n^{|I|+1}$ for every $\ell \in [k] \setminus I$. Hence, there are at most $(k - |I|)\sqrt{\varepsilon'} n^{|I|}$ $|I|$-tuples in $\{v_i\}_{i \in I} \in \prod_{i \in I} V_i$ for which there exist an $\ell \in [k] \setminus I$ so that

$$\left|\left\{v_\ell \in V_\ell \colon \{v_i\}_{i \in I \cup \{\ell\}} \in B_{I \cup \{\ell\}}(P, \varepsilon')\right\}\right| \geq \sqrt{\varepsilon'} n\,.$$

Consequently, since $\varepsilon' \leq \varepsilon$ we have

$$|B_I(P, \varepsilon)| \leq |B_I(P, \varepsilon')| \leq \varepsilon' n^{|I|} + (k - |I|)\sqrt{\varepsilon'} n^{|I|} \leq \varepsilon n^{|I|}$$

due to the choice of ε'. □

Next we consider $|I|$-tuples which contain no bad sub-tuple.

Definition 17. *For $\varepsilon' > 0$ and P as above, and for a proper subset $\emptyset \neq I \subsetneq [k]$, set*

$$Z_I = Z_I(P, \varepsilon') = \left\{\{v_i\}_{i \in I} \in \prod_{i \in I} V_i \colon \{v_i\}_{i \in I'} \notin B_{I'}(P, \varepsilon') \text{ for all } \emptyset \neq I' \subseteq I\right\}.$$

Set $Z = Z(P, \varepsilon') = \bigcup_{\emptyset \neq I \subsetneq [k]} Z_I(P, \varepsilon')$.

The following fact is an immediate consequence of Fact 16.

Fact 18. *For all integers k and d, $\varepsilon' > 0$ there exists $\delta > 0$ so that for every graph P as above, and for every proper subset $\emptyset \neq I \subsetneq [k]$, we have $|Z_I(P, \varepsilon')| \geq (1 - \varepsilon')n^{|I|}$.*

Proof. Let k and d, $\varepsilon' > 0$ be given. Set $\varepsilon'' = \varepsilon'/(2^{k-1} - 1)$ and let δ be sufficiently small so that Fact 16 holds for k, d, and ε''. Fix a proper subset $\emptyset \neq I \subsetneq [k]$. By the choice of δ, for every $\emptyset \neq I' \subseteq I$ we have $|B_{I'}(P, \varepsilon'')| \leq \varepsilon'' n^{|I'|}$. Therefore, there are at most $\varepsilon'' n^{|I|}$ $|I|$-tuples $\{v_i\}_{i \in I} \in \prod_{i \in I} V_i$ for which $\{v_i\}_{i \in I'} \in B_{I'}(P, \varepsilon'')$. Since $B_{I'}(P, \varepsilon') \subseteq B_{I'}(P, \varepsilon'')$, we see

$$\left|Z_I(P, \varepsilon')\right| \geq n^{|I|} - \sum_{\emptyset \neq I' \subseteq I} \varepsilon'' n^{|I|} \geq (1 - \varepsilon')n^{|I|},$$

where the last inequality holds due to the choice of ε''. □

We close this section with the following simple observation.

Fact 19. *For all integers k and $\varepsilon^* > 0$, there exists $\varepsilon' > 0$ so that for every d and $\delta > 0$ and every graph P as above*[1] *the following holds. For every proper subset $\emptyset \neq I \subsetneq [k]$, every $\{v_i\}_{i\in I} \in Z_I(P,\varepsilon')$ and every $\ell \in [k] \setminus I$, we have*

$$\Big|\big\{v_\ell \in V_\ell \colon \{v_i\}_{i\in I\cup\{\ell\}} \notin Z_{I\cup\{\ell\}}\big\}\Big| \leq \varepsilon^* n\,.$$

Proof. Let k and $\varepsilon^* > 0$ be given and set $\varepsilon' = \varepsilon^*/(2^{k-1}-1)$. Let $\emptyset \neq I \subsetneq [k]$, $\{v_i\}_{i\in I} \in Z_I(P,\varepsilon')$ and $\ell \in [k]\setminus I$ be given. By definition, we know $\{v_i\}_{i\in I'} \notin B_{I'}(P,\varepsilon')$ for every $\emptyset \neq I' \subseteq I$, and hence,

$$\Big|\big\{v_\ell \in V_\ell \colon \{v_i\}_{i\in I'\cup\{\ell\}} \in B_{i\in I'\cup\{\ell\}}(P,\varepsilon')\big\}\Big| \leq \varepsilon' n\,.$$

Applying this observation for all $\emptyset \neq I' \subseteq I$, we obtain

$$\Big|\big\{v_\ell \in V_\ell \colon \exists\, \emptyset \neq I' \subseteq I \text{ s.t. } \{v_i\}_{i\in I'\cup\{\ell\}} \in B_{i\in I'\cup\{\ell\}}(P,\varepsilon')\big\}\Big| \leq (2^{|I|}-1)\varepsilon' n \leq \varepsilon^* n$$

and the fact follows. □

6.2. **Proof of Proposition 11.** In this section, we prove Proposition 11. Our proof is somewhat similar to the proof of the graph embedding lemma in [4] and will be based on the concepts and observations from Section 6.1.

Proof of Proposition 11. First we recall the quantification of Proposition 11: $\forall\, k \geq 2$, $d > 0$, $\varepsilon > 0$ $\exists\, \delta > 0$, $c > 0$, and n_0. Again, instead of calculating somewhat tedious estimates on the promised constants $\delta > 0$, $c > 0$ and n_0, we will simply verify Proposition 11 under the assumption the constants were chosen to satisfy

$$\max\{\tfrac{1}{n_0}, c, \delta\} \ll \varepsilon' \ll \varepsilon^* \ll \min\{\varepsilon, d, \tfrac{1}{k}\}\,, \tag{32}$$

for auxiliary constant ε' given by Fact 19 applied with k and auxiliary constant ε^*. Moreover, we will assume that δ is sufficiently small so that the conclusion of Fact 18 holds for k, δ and ε'.

Let $P = \bigcup_{1\leq i<j\leq k} P^{ij}$ be a k-partite graph with vertex partition $V_1 \dot\cup \ldots \dot\cup V_k = V = V(P)$, $|V_1| = \cdots = |V_k| = n \geq n_0$, where P^{ij} is (d,δ)-regular for every $1 \leq i < j \leq k$. Moreover, let J_0 be a k-partite graph on $m \leq cn$ vertices with vertex partition $U_1 \dot\cup \ldots \dot\cup U_k = U = V(J_0)$ such that for every $u \in U$ and every $i \in [k]$ we have $|(N^2_{J_0}(u) \cup \{u\}) \cap U_i| \leq 1$, where $N^2_{J_0}(u)$ is the second neighborhood of u in J_0 (see (18)). We have to show that there exists a (d,ε)-extendable family of $\mathscr{J}$ of labeled, partite embeddings of J_0 in P (see Definition 10). We will now define a family of embeddings $\mathscr{J}$, and then show that this family is, in fact, a (d,ε)-extendable family.

Roughly speaking, we let $\mathscr{J}$ be the family of all labeled, partite embeddings of J_0 in P for which the embedding of the second neighborhood $N^2(u)$ of any vertex $u \in U$ lies in $Z(P,\varepsilon')$. More precisely, let $Z(P,\varepsilon')$ be defined as in Definition 17. Set

$$\mathscr{J} = \Big\{J \subset P\colon\ J \text{ is a labeled, partite copy of } J_0 \text{ in } P \text{ s.t.}$$
$$\big\{J(u')\colon\ u' \in N^2_{J_0}(u)\big\} \in Z(P,\varepsilon') \text{ for all } u \in U\Big\}\,, \tag{33}$$

[1] In fact, for the statement of this fact we don't need the (d,δ)-regularity of P. However, we keep the assumption to be consistent with the other facts in this section.

where, as before, for a copy J of J_0 in P and a vertex $u' \in U = V(J_0)$, we denote by $J(u')$ the vertex in $V(J) \subseteq V(P)$ which corresponds to u' in J_0. In what follows, we verify that $\mathscr{J}$, defined above, is indeed a (d, ε)-extendable family of embeddings. Our proof is based on the following claim.

Claim 20. *Suppose $X \subseteq U = V(J_0)$ and $\tilde{J}$ is a labeled, partite embedding of $J_0[X]$ with the property*

$(*)$ $\{\tilde{J}(u')\colon u' \in N^2_{J_0}(u) \cap X\} \in Z(P, \varepsilon')$ *for every* $u \in U \setminus X$

Then $\operatorname{ext}_{\mathscr{J}}(\tilde{J}) = (d \pm \varepsilon)^{e(J_0) - e(J_0[X])} n^{m-|X|}$.

(Note, since $\tilde{J}$ satisfies both $(*)$ and, by Claim 20, $\operatorname{ext}_{\mathscr{J}}(\tilde{J}) \geq 1$, we have $\tilde{J} \in \mathscr{J}[X]$.)

Before we verify the claim, we observe that it immediately implies that $\mathscr{J}$ is (d, ε)-extendable. In fact, for any $X \subseteq U$ and $\tilde{J} \in \mathscr{J}[X]$ it follows from the definition of $\mathscr{J}$ in (33) and from the definition of $Z(P, \varepsilon')$ in Definition 17, that $\tilde{J}$ satisfies $(*)$ of Claim 20. Consequently, $\operatorname{ext}_{\mathscr{J}}(\tilde{J}) = (d \pm \varepsilon)^{e(J_0) - e(J_0[X])} n^{m-|X|}$. Since this holds for any $X \subseteq U$ and $\tilde{J} \in \mathscr{J}[X]$ this shows that $\mathscr{J}$ is (d, ε)-extendable and concludes the proof of Proposition 11, based on Claim 20. □

Proof of Claim 20. We prove Claim 20 by induction on $m - |X|$. If $|X| = m$, then $X = U$ and the conclusion of Claim 20 holds trivially for every copy J of $J_0 = J_0[U]$.

For an integer $t \geq 0$, suppose Claim 20 holds for all sets $X' \subseteq U$ for which $m - |X'| = t \geq 0$. Let $X \subseteq U$ be a set for which $m - |X| = t + 1 \geq 1$. Fix some copy $\tilde{J}$ of $J_0[X]$ satisfying $(*)$. Fix a vertex $y \in U \setminus X$ arbitrarily and let $\ell \in [k]$ be such that $y \in U_\ell$.

It follows from Definition 17 that since

$$\begin{aligned} \tilde{J}(N_{J_0}(y) \cap X) &:= \{\tilde{J}(u)\colon u \in N_{J_0}(y) \cap X\} \\ &\subseteq \{\tilde{J}(u)\colon u \in N^2_{J_0}(y) \cap X\} \in Z(P, \varepsilon'), \end{aligned}$$

we have $\tilde{J}(N_{J_0}(y) \cap X) \in Z(P, \varepsilon')$. Therefore, $\tilde{J}(N_{J_0}(y) \cap X) \notin B(P, \varepsilon')$, and consequently, setting

$$C(y) = V_\ell \cap \bigcap \Big\{ N_P(v)\colon v \in \tilde{J}\big(N_{J_0}(y) \cap X\big) \Big\}$$

we infer

$$|C(y)| = (d \pm \varepsilon')^{\deg(y,X)} n\,,$$

where $\deg(y, X) = |N_{J_0}(y) \cap X| = |\tilde{J}\big(N_{J_0}(y) \cap X\big)|$. We may think of $C(y)$ as the candidate set for y, i.e., the set of vertices which could extend $\tilde{J}$ from a labeled, partite copy of $J_0[X]$ to a labeled, partite copy of $J_0[X \cup \{y\}]$. In order to extend $\tilde{J}$ we have to ensure that we do not reuse any vertex from $\tilde{J}$. Hence, we set $C'(y) = C(y) \setminus V(\tilde{J})$ and note

$$(d + \varepsilon')^{\deg(y,X)} n \geq |C(y)| \geq |C'(y)| \geq (d - \varepsilon')^{\deg(y,X)} n - cn\,. \tag{34}$$

Note that by definition of $C'(y)$, every vertex $v \in C'(y)$ extends $\tilde{J}$ to a labeled, partite copy of $J_0[X \cup \{y\}]$. For $v \in C'(y)$ we denote the particular such copy of $J_0[X \cup \{y\}]$ by $\tilde{J} + v$. In order to later appeal to the induction assumption, we shall have to restrict our attention to those $v \in C'(y)$ for which $\tilde{J} + v$ satisfies $(*)$ with X replaced by

$$X' := X \cup \{y\}\,.$$

We will show that indeed $(*)$ holds for $\tilde{J}+v$ for "most" $v \in C'(y)$.

For that, let $u \in U \setminus X'$ be fixed and consider $N^2_{J_0}(u) \cap X'$. We say $v \in C'(y)$ is *Z-bad for u* if

$$\left\{(\tilde{J}+v)(u')\colon\ u' \in N^2_{J_0}(u) \cap X'\right\} \notin Z(P,\varepsilon')$$

and we denote by Z-bad(u) the set of vertices $v \in C'(y)$ which are Z-bad for u. Note that $\tilde{J}+v$ satisfies $(*)$ if and only if $v \in C'(y)$ and v is not Z-bad for any $u \in U \setminus X'$. Hence, we want to show that the number of Z-bad vertices is small. More precisely, we are going to show that for all but at most $k-1$ vertices $u \in U \setminus X'$, the set Z-bad(u) is empty, and for all other vertices $u \in U \setminus X'$, it is very small. We consider three cases of vertices $u \in U' \setminus X$ depending on the set $N^2_{J_0}(u) \cap X'$.

Case 1 (u satisfies that $y \notin N^2_{J_0}(u) \cap X'$)**.** By the assumption of this case, we have $N^2_{J_0}(u) \cap X' = N^2_{J_0}(u) \cap X$, and since $(*)$ holds for $\tilde{J}$, it is easy to check that for every $v \in C'(y)$,

$$\left\{(\tilde{J}+v)(u') = \tilde{J}(u')\colon\ u' \in N^2_{J_0}(u) \cap X'\right\} = \tilde{J}(N^2_{J_0}(u) \cap X) \in Z(P,\varepsilon')\,,$$

i.e., $(*)$ holds for u and X' and $\tilde{J}+v$ for any $v \in C'(y)$. In other words,

$$y \notin N^2_{J_0}(u) \cap X' \;\Rightarrow\; Z\text{-bad}(u) = \emptyset\,. \tag{35}$$

Before we continue with the next case, we note that for all but at most $k-1$ vertices $u \in U \setminus X'$, we are in Case 1. More precisely, let

$$W = \{w \in U \setminus X'\colon\ y \in N^2_{J_0}(w)\}\,.$$

Since, as is implied by the hypothesis (on J_0) in Proposition 10, we know $|N^2_{J_0}(y)| < k$, and since $y \in N^2_{J_0}(w)$ if and only if $w \in N^2_{J_0}(y)$, we infer

$$|W| < k\,. \tag{36}$$

Case 2 (u satisfies that $\{y\} = N^2_{J_0}(u) \cap X'$)**.** By Fact 18, applied with $I = \{\ell\}$, we have $|V_\ell \setminus Z_{\{\ell\}}| \le \varepsilon' n$. Therefore, $|C'(y) \setminus Z_{\{\ell\}}| \le \varepsilon' n$ and

$$\{y\} = N^2_{J_0}(u) \cap X' \;\Rightarrow\; |\,Z\text{-bad}(u)| \le \varepsilon' n\,. \tag{37}$$

In the final case, when $y \in N^2_{J_0}(u) \cap X'$, but $\{y\} \ne N^2_{J_0}(u) \cap X'$, we shall verify a similar estimate on $|\,Z$-bad$(u)|$.

Case 3 (u satisfies that $\{y\} \subsetneq N^2_{J_0}(u) \cap X'$)**.** By the assumption of this case, we have $N^2_{J_0}(u) \cap X' = (N^2_{J_0}(u) \cap X) \cup \{y\}$ where $N^2_{J_0}(u) \cap X \ne \emptyset$. Moreover, by the assumption of the claim, i.e., by $(*)$ for $\tilde{J}$, we have $\tilde{J}(N^2_{J_0}(u) \cap X) = \{\tilde{J}(u')\colon\ u' \in N^2_{J_0}(u) \cap X\} \in Z(P,\varepsilon')$. Consequently, Fact 19 implies

$$\left|\left\{v \in V_\ell\colon\ \tilde{J}(N^2_{J_0}(u) \cap X) \cup \{v\} \notin Z(P,\varepsilon')\right\}\right| \le \varepsilon^* n\,,$$

i.e.,

$$\{y\} \subsetneq N^2_{J_0}(u) \cap X' \;\Rightarrow\; |\,Z\text{-bad}(u)| \le \varepsilon^* n\,. \tag{38}$$

This concludes our cases.

Based on the observations (35)–(38), we now finish the proof of Claim 20. We first refine the set $C'(y)$ and define

$$C''(y) = C'(y) \setminus \bigcup_{u \in U \setminus X'} Z\text{-bad}(u)\,. \tag{39}$$

Due to (34), combined with (35)–(38), we obtain

$$(d+\varepsilon')^{\deg(y,X)}n \geq |C''(y)| \geq (d-\varepsilon')^{\deg(y,X)}n - cn - (k-1)\times\max\{\varepsilon' n, \varepsilon^* n\}\,.$$

Since $\deg(y,X) < k$, we infer from $c \ll \varepsilon' \ll \varepsilon^* \ll \min\{\varepsilon, d, 1/k\}$ that

$$|C''(y)| = (d\pm\varepsilon)^{\deg(y,X)}n\,. \tag{40}$$

Finally, we observe that, by definition of the family $\mathscr{J}$ in (33), every extension $J \in \mathscr{J}$ of $\tilde{J}$ must map y to a vertex v in $C''(y)$, since otherwise either $\tilde{J}+v$ would not be a labeled, partite copy of $J_0[X']$ (if $v \notin C'(y)$) or, for some $u \in U\setminus X'$, the second neighborhood would not be embedded in a set from $Z(P,\varepsilon')$ (if $v \in C'(y)\setminus C''(y)$). Consequently,

$$\mathrm{ext}_{\mathscr{J}}(\tilde{J}) = \sum_{v\in C''(y)} \mathrm{ext}_{\mathscr{J}}(\tilde{J}+v)\,. \tag{41}$$

For every $v \in C''(y)$, we can apply induction to $\tilde{J}+v$, since $m-|X'| = m-|X|-1$ and, by definition of the set $C''(y)$ in (39), the copy $\tilde{J}+v$ satisfies the assumption ($*$). Hence, we infer by induction, combined with (40) and (41), that

$$\begin{aligned}\mathrm{ext}_{\mathscr{J}}(\tilde{J}) &= \sum_{v\in C''(y)} \mathrm{ext}_{\mathscr{J}}(\tilde{J}+v)\\ &= (d\pm\varepsilon)^{\deg(y,X)}n \times (d\pm\varepsilon)^{e(J_0)-e(J_0[X'])}n^{m-|X'|}\\ &= (d\pm\varepsilon)^{e(J_0)-e(J_0[X])}n^{m-|X|}\,,\end{aligned}$$

which concludes the proof of Claim 20. □

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