PhD Seminar WS 2018/2019
Transversality and compactness

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Our goal for this session is to establish the following two facts about the SW moduli space $\mathscr{M}_\omega$ and its irreducible part $\mathscr{M}_\omega^*$:

- $\mathscr{M}_\omega^*$ is a smooth manifold of dimension $c_2(V_+)$ for ω generic.
- $\mathscr{M}_\omega$ is sequentially compact, meaning that any sequence of points in $\mathscr{M}_\omega$ has a convergent subsequence.

We take up the task of proving these one by one.

1 Transversality

We have previously seen in Danu's talk that if we assume that the SW map f_ω is transversal to $(0,0)$ in the codomain i.e. the cokernel $H^2_{(A,\Phi)}$ of its differential $\mathscr{T}_{(A,\Phi)} f_\omega$ vanishes whenever (A,Φ) is an SW solution, then $\mathscr{M}_\omega^*$ is a smooth manifold of dimension $c_2(V_+)$. This assumption isn't always true. In particular, it may fail to hold when ω identically vanishes. This is in fact the whole point of introducing an unphysical parameter like ω into the SW equations, because what is true is that Danu's assumption holds for generic choices of ω, so even if the honest-to-goodness physical SW equations don't have a smooth moduli space, a slight perturbation of the same does. This is essentially what we would like to show in this section.

Let's try to motivate the precise statements that we'll be proving. Since we want to prove something about genericity with respect to ω, it makes sense to redefine the SW map so that it takes in ω as part of its input in addition to A and Φ.

Definition 1 (Parametrised SW map)**.** The parametrised SW map

$$f : \mathscr{C}_{\mathfrak{s}} \times \mathrm{i}\Omega^2_+(X) \to \mathrm{i}\Omega^2_+(X) \times \Gamma(V_-)$$

is given by:

$$(A,\Phi,\omega) \mapsto f_\omega(A,\Phi) = (F_A^+ - \sigma(\Phi,\Phi) - \omega, D_A^+\Phi). \tag{1}$$

Gauge transformations $u \in \mathscr{G}$ on $\mathscr{C}_{\mathfrak{s}}$ lift to gauge transformations $u \times \mathrm{id}$ on $\mathscr{C}_{\mathfrak{s}} \times \mathrm{i}\Omega^2_+(X)$. Since there is a one-to-one correspondence between gauge transformations u and their lifts $u \times \mathrm{id}$, we will henceforth identify them in an abuse of notation. In particular, it makes sense to take the quotient $(\mathscr{C}_{\mathfrak{s}} \times \mathrm{i}\Omega^2_+(X))/\mathscr{G}$.

Note that we have $f \circ u = f$ for any gauge transformation $u \in \mathscr{G}$, so f descends to a map

$$\tilde{f} : (\mathscr{C}_{\mathfrak{s}} \times \mathrm{i}\Omega^2_+(X))/\mathscr{G} \to \mathrm{i}\Omega^2_+(X) \times \Gamma(V_-),$$

which we shall refer to as the descendant parametrised SW map.

Gauge transformations not only preserve the images of points under f but also the set of reducible solutions (although, not pointwise). This means that it also makes sense to take the quotient

$$(\mathscr{C}^*_{\mathfrak{s}} \times \mathrm{i}\Omega^2_+(X))/\mathscr{G} \subseteq (\mathscr{C}_{\mathfrak{s}} \times \mathrm{i}\Omega^2_+(X))/\mathscr{G},$$

and define the restriction $\tilde{f}^*$ of $\tilde{f}$ to $(\mathscr{C}^*_{\mathfrak{s}} \times \mathrm{i}\Omega^2_+(X))/\mathscr{G}$.

Definition 2 (Parametrised SW moduli space)**.** The parametrised SW moduli space $\mathscr{M}$ is defined to be the preimage $\tilde{f}^{-1}(0,0)$. In addition, its irreducible part $\mathscr{M}^*$ is defined to be $\tilde{f}^{*-1}(0,0)$.

There is a canonical projection $\pi : \mathscr{M}^* \to \mathrm{i}\Omega^2_+(X)$ given by $[A, \Phi, \omega] \mapsto \omega$. In order to retrieve $\mathscr{M}^*_\omega$, one simply has to take the preimage $\pi^{-1}(\omega)$.

Since $\mathrm{i}\Omega^2_+(X)$ is not finite-dimensional, $\mathscr{M}^*$ can't be either, given that it admits a projection onto $\mathrm{i}\Omega^2_+(X)$. However, we know from the implicit function theorem for Banach manifolds that if a Banach map between two Banach manifolds is Fredholm, then the preimage of any regular value is a manifold with dimension equal to the (finite) index of the Fredholm map. We aleady know that $\mathrm{i}\Omega^2_+(X)$ is a Banach manifold and that the preimage of a point $\omega \in \mathrm{i}\Omega^2_+(X)$ is the SW moduli space $\mathscr{M}^*_\omega$. So all that remains to show is that $\mathscr{M}^*$ is Banach and π is Fredholm.

Proposition 3. *The parametrised SW moduli space $\mathscr{M}^*$ is a Banach manifold.*

Proof. We use the implicit function theorem for Banach manifolds, according to which given a Banach map between two Banach manifolds, the preimage of regular point, that is a point in the codomain to which the map is transversal, is either empty or a Banach manifold. The Banach map in this case is the following:

$$\tilde{f}^* : (\mathscr{C}^*_{\mathfrak{s}} \times \mathrm{i}\Omega^2_+(X))/\mathscr{G} \to \mathrm{i}\Omega^2_+(X) \times \Gamma(V_-).$$

What we need to show is that $(0, 0)$ is a regular value, that is $\mathscr{T}_{[A,\Phi,\omega]}\tilde{f}^*$ is surjective for all $\tilde{[}A, \Phi, \omega] \in \tilde{f}^{*-1}(0, 0)$. Now, given a representative (A, Φ, ω) of the gauge equivalence class $[A, \Phi, \omega] \in (\mathscr{C}^*_{\mathfrak{s}} \times \mathrm{i}\Omega^2_+(X))/\mathscr{G}$, $\mathscr{T}_{[A,\Phi,\omega]}\tilde{f}^*$ is surjective if and only if $\mathscr{T}_{(A,\Phi,\omega)}f$ is surjective. In fact, we shall see that $\mathscr{T}_{(A,\Phi,\omega)}f$ is surjective for all irreducible (A, Φ, ω), not just when $f(A, \Phi, \omega) = 0$.

The way we'll go about this is to let $(\eta, \psi) \in \mathrm{i}\Omega^2_+(X) \times \Gamma(V_-)$ be orthogonal to the image of the differential $\mathscr{T}_{(A,\Phi,\omega)}f$ with respect to the inner product induced by the metric g on X, and then show that this necessarily means that (η, ψ) identically vanishes. Recall that given (a, φ, τ) in $\mathscr{T}_{(A,\Phi,\omega)}(\mathscr{C}^*_{\mathfrak{s}} \times \mathrm{i}\Omega^2_+(X))$, the differential $\mathscr{T}_{(A,\Phi,\omega)}f$ acts on it via:

$$\mathscr{T}_{(A,\Phi,\omega)}f(a, \varphi, \tau) = (2\mathrm{d}^+a - \sigma(\Phi, \varphi) - \sigma(\varphi, \Phi) - \tau, D^+_A\varphi + \gamma(a)\Phi). \tag{2}$$

The inner product of the above with (η, ψ) is then given by:

$$\begin{aligned}\langle \mathscr{T}_{(A,\Phi,\omega)}f(a, \varphi, \tau), (\eta, \psi)\rangle = \int_X \big(&\langle 2\mathrm{d}^+a - \sigma(\Phi, \varphi) - \sigma(\varphi, \Phi), \eta\rangle \\ &- \langle \tau, \eta\rangle + \langle D^+_A\varphi + \gamma(a)\Phi, \psi\rangle\big) \operatorname{vol}_g.\end{aligned} \tag{3}$$

The statement that (η, ψ) is orthogonal to the image of $\mathscr{T}_{(A,\Phi,\omega)}f$ then amounts to saying that the above vanishes for all choices of (a, φ, τ). Since τ is arbitrary and occurs only in the term $\langle \tau, \eta\rangle$, η must identically vanish. This implies that the entire first line on the right vanishes, leaving us with the following equation:

$$\int_X \langle D^+_A\varphi + \gamma(a)\Phi, \psi\rangle \operatorname{vol}_g = 0. \tag{4}$$

We may use the fact that D^+_A and D^-_A are adjoints to rewrite the above as:

$$\int_X \big(\langle \varphi, D^-_A\psi\rangle + \langle \gamma(a)\Phi, \psi\rangle\big) \operatorname{vol}_g = 0. \tag{5}$$

Now, since φ is arbitrary and occurs only in the term $\langle \varphi, D^-_A\psi\rangle$, $D^-_A\psi$ must identically vanish. This leaves us with:

$$\int_X \langle \gamma(a)\Phi, \psi\rangle \operatorname{vol}_g = 0. \tag{6}$$

Our hypothesis is that (A, Φ, ω) is irreducible, which just means that Φ is not identically zero. So, there is at least one point, say $p \in X$, such that $\Phi(p) \neq 0$. Using any trivialisation of the bundle V_- over a sufficiently small neighbourhood $U \ni p$ and any identification thereof with the standard Clifford module over $\mathbb{R}^4$, one may see that the map $a(p') \mapsto \gamma(a(p'))\Phi(p')$ is a surjection from $\mathscr{T}^*_{p'}(X)$ onto the fibre $V_{-,p'}$ for all $p' \in U$. In particular one can always choose some a such that $\gamma(a(p'))\Phi(p') = \psi(p')$ for all $p' \in U$. If we assume that ψ doesn't vanish everywhere in U, we may then ensure that $\langle \gamma(a)\Phi, \psi\rangle$ is nonnegative throughout U and positive on some open subset U' of U. It follows that if h is a smooth function with nonempty support contained in U', then the integrand

$$\int_X \langle \gamma(ha)\Phi, \psi\rangle \operatorname{vol}_g$$

is strictly positive. This is a contradiction and so our assumption that ψ doesn't vanish everywhere in U must have been wrong. An standard property of elliptic operators such as D_A^- tells us that any section in their kernel which identically vanishes in some open neighbourhood U of X must identically vanish all over X. We have already shown that ψ is in the kernel of D_A^-, so ψ must identically vanish over X. □

Proposition 4. *The canonical projection $\pi : \mathscr{M}^* \to \mathrm{i}\Omega^2_+(X)$ is a Fredholm map of index $c_2(V_+)$.*

Proof. The way we shall prove this is by showing that the kernel and cokernel of $\mathscr{T}_{[A,\Phi,\omega]}\pi$ may be respectively identified in a canonical way with the kernel and cokernel of the restriction of the differential of the descendant SW map

$$\mathscr{T}_{[A,\Phi]}\tilde{f}^*_\omega : \mathscr{T}_{[A,\Phi]}(\mathscr{C}^*_{\mathfrak{s}}/\mathscr{G}) \to \mathrm{i}\Omega^2_+(X) \times \Gamma(V_-).$$

Danu's already done the hard work of showing that *this* is Fredholm with index $c_2(V_+)$. Since that just means that the dimensions of the kernel and cokernel are finite and their difference is $c_2(V_+)$, it would immediately follow that π is Fredholm with index $c_2(V_+)$ as well.

The tangent space $\mathscr{T}_{[A,\Phi,\omega]}\mathscr{M}^*$ consists equivalence classes $[a,\varphi,\tau]$ of triples

$$(a,\varphi,\tau) \in \mathscr{T}_{[A,\Phi,\omega]}(\mathscr{C}^*_{\mathfrak{s}} \times \mathrm{i}\Omega^2_+(X))$$

modulo tangent vectors to the orbits of $\mathscr{G}$ i.e. triples of the form $(-\mathrm{i}\,\mathrm{d}\xi, \mathrm{i}\xi\Phi, 0)$ with $\xi \in C^\infty(X)$, such that (a,φ,τ) lies in the kernel of the differential $\mathscr{T}_{(A,\Phi,\omega)}f^*$. In other words, the following has to hold:

$$2\mathrm{d}^+a - \sigma(\Phi,\varphi) - \sigma(\varphi,\Phi) - \tau = 0, \quad D_A^+\varphi + \gamma(a)\Phi = 0. \tag{7}$$

The differential $\mathscr{T}_{[A,\Phi,\omega]}\pi$ is given by $[a,\varphi,\tau] \mapsto \tau$, so an element $[a,\varphi,\tau] \in \mathscr{T}_{[A,\Phi,\omega]}\mathscr{M}^*$ lies in its kernel if and only if $\tau = 0$. Elements $[a,\varphi,0]$ kernel of $\mathscr{T}_{[A,\Phi,\omega]}\pi$ therefore satisfy:

$$2\mathrm{d}^+a - \sigma(\Phi,\varphi) - \sigma(\varphi,\Phi) = 0, \quad D_A^+\varphi + \gamma(a)\Phi = 0. \tag{8}$$

But this is precisely the necessary and sufficient condition for $[a,\varphi]$ being in the kernel of $\mathscr{T}_{_{[A,\Phi]}\tilde{f}^*_\omega}$. So, there is a canonical identification:

$$\ker\, \mathscr{T}_{[A,\Phi,\omega]}\pi \cong \ker\, \mathscr{T}_{[A,\Phi]}f^*_\omega, \quad \text{via} \quad [a,\varphi,0] \leftrightarrow [a,\varphi]. \tag{9}$$

Next, we consider the cokernel. The image of $\mathscr{T}_{[A,\Phi,\omega]}\pi$ just consists of 2-forms $\tau \in \mathrm{i}\Omega^2_+(X)$ which can be written as:

$$\tau = 2\mathrm{d}^+a - \sigma(\Phi,\varphi) - \sigma(\varphi,\Phi). \tag{10}$$

Meanwhile, the image of $\mathscr{T}_{[A,\Phi]}f^*_\omega$ consists of elements of the form

$$(2\mathrm{d}^+a - \sigma(\Phi,\varphi) - \sigma(\varphi,\Phi), D_A^+\varphi + \gamma(a)\Phi).$$

In the proof of Proposition 3, we saw that the differential $\mathscr{T}_{(A,\Phi,\omega)}f$ is surjecive whenever Φ is not identically zero. In particular, this means that given any section η of V_-, we can always find a and φ such that:

$$\psi = D_A^+\varphi + \gamma(a)\Phi. \tag{11}$$

So, the image of $\mathscr{T}_{[A,\Phi]}f_\omega^*$ is really just im $\mathscr{T}_{[A,\Phi,\omega]}\pi \times \Gamma(V_-)$, giving us:

$$\begin{aligned}\text{coker } \mathscr{T}_{[A,\Phi]}f_\omega^* &:= (\mathrm{i}\Omega_+^2(X) \times \Gamma(V_-))/\text{im } \mathscr{T}_{[A,\Phi]}f_\omega^* \\ &= (\mathrm{i}\Omega_+^2(X) \times \Gamma(V_-))/(\text{im } \mathscr{T}_{[A,\Phi,\omega]}\pi \times \Gamma(V_-)) \\ &\cong \mathrm{i}\Omega_+^2(X)/\text{im } \mathscr{T}_{[A,\Phi,\omega]}\pi =: \text{coker } \mathscr{T}_{[A,\Phi,\omega]}\pi.\end{aligned} \tag{12}$$

□

As a straightforward corollary of the above two results, we have:

Theorem 5 (Transversality). *$\mathscr{M}_\omega^*$ is a smooth manifold of dimension $c_2(V_+)$ for ω a generic element of $\mathrm{i}\Omega_+^2(X)$.*

Proof. Now that we know $\pi : \mathscr{M}^* \to \mathrm{i}\Omega_+^2(X)$ is Fredholm with index $c_2(V_-)$, it follows by the implicit function theorem for Banach manifolds that the preimage $\mathscr{M}_\omega^* = \pi^{-1}(\omega)$ is a smooth manifold of finite dimension $c_2(V_-)$ whenever $\omega \in \mathrm{i}\Omega_+^2(X)$ is a regular value. The theorem then follows from ageneralisation of Sard's theorem to Banach manifolds which says that the regular values of any Banach map is generic. □

What we have really shown is that even if $(0,0)$ is not a regular value of the map $\tilde{f}_\omega^*$, $(\tau, 0)$ is, for a generic choice of τ in $\mathrm{i}\Omega_+^2(X)$. The reason we couldn't directly use Sard's theorem to argue this is that although τ is generic, $(\tau, 0)$ isn't generic in $\mathrm{i}\Omega_+^2(X) \times \Gamma(V_-)$.

2 Compactness

Next we turn to the question of compactness of the SW moduli space $\mathscr{M}$. Since points in $\mathscr{M}_\omega$ are gauge equivalence classes of SW solutions $(A, \Phi) \in \mathscr{Z}_\omega$, this boils down to the following statement:

Theorem 6 (Compactness). *Let (A_i, Φ_i) be a sequence of C^∞ SW solutions. Then there exists a sequence of C^∞ gauge transformations u_i such that there is a subsequence of $u_i(A_i, \Phi_i)$ that converges in the C^∞ topology to a C^∞ Seiberg–Witten solution (A, Φ).*

Note that (A_i, Φ_i) and u_i don't need to be C^∞; all we want are that $u_i(A_i, \Phi_i)$ are C^∞ solutions.

Recall that this was the key deficiency with Donaldson theory: the curvature of the nonabelian gauge field could blow up at a point, and so the moduli space failed to be compact. To prove that this is not the case with Seiberg–Witten theory, we run the argument in reverse. That is, very roughly speaking we first argue that the only way the SW moduli space could fail to be compact, is if k-th order derivatives of a sequence of solutions blew up at some point, for some value of k, and then we show that this can't in fact happen (up to gauge transformations, that is).

In order to make this precise, we have first clarify what we mean when we say that the k-th order derivative blows up. The formalism of Sobolev norms and Sobolev spaces is the natural setting for formulating this idea. Recall that the L^p_k Sobolev norm on some vector bundle $E \to X$ equipped with a connection ∇ is given by:

$$\|s\|_{L^p_k} = \left(\int_X (|s|^p + |\nabla s|^p + \cdots + |\nabla^k s|^p) \,\mathrm{vol}_g \right)^{1/p}. \tag{13}$$

The norm depends on the choice of connection but the Banach space completion of $\Gamma(E)$ with respect to it does not. This is the Sobolev space $L^p_k(E)$.

What we mean when we say that the k-th derivative blows up is that the L^p_k norm is unbounded. The relationship to compactness is given by the following deep theorem:

Theorem 7 (Sobolev embedding)**.** *Let X be a compact n-manifold. Then:*

- *There exists an embedding $L^p_{j+m}(E) \subset C^j(E)$ if $mp \geq n$.*
- *The embedding is compact if $mp > n$.*

In fact, we will only be needing the following corollary obtained by setting $n = 4$, $p = 2$, $m = 3$, and $j = k - 3$ for some $k \geq 3$:

Corollary 8. *Let X be a compact 4-manifold and $k \geq 3$. Then there exists a compact embedding $L^2_k(E) \subset C^{k-3}(E)$. Hence, every bounded sequence in $L^2_k(E)$ has a convergent subsequence in $C^{k-3}(E)$.*

In particular this means that if $\mathscr{X} \subset C^\infty(E)$ is closed subset satisfying a (possibly k-dependent) bound with respect to an L^2_k norm for every integer $k \geq 3$, then $\mathscr{X}$ is sequentially compact.

In order to apply this to the SW story, we need to first address what we mean by L^2_k norms in this case. The issue is that the connections A_i don't form sections of a vector bundle but instead an affine space modeled on the space of sections of a vector bundle. This matter is easily resolved by fixing some connection A_0 and considering the differences $a_i := A_i - A_0$ which are indeed sections of a vector bundle. The SW equations may be rewritten in terms of $a := A - A_0$ as follows:

$$D^+_{A_0}\Phi = -\gamma(a)\Phi, \quad 2\mathrm{d}^+ a = \sigma(\Phi, \Phi) + \omega_0, \tag{14}$$

where the self-dual part is as usual and $F^+_{A_0}$ has been absorbed into ω_0. Gauge transformations, parametrised by $\xi \in C^\infty(M)$, are meanwhile given by:

$$(a, \Phi) \mapsto (a - \mathrm{i}\,\mathrm{d}\xi, e^{\mathrm{i}\xi}\Phi). \tag{15}$$

In particular, a changes by an exact form. Since any differential form can be uniquely written as the sum of an exact and a coclosed form, we can always fix the gauge so that a is coclosed. Thus we have the gauge-fixed SW equations:

$$D^+_{A_0}\Phi = -\gamma(a)\Phi, \quad 2\mathrm{d}^+ a = \sigma(\Phi,\Phi) + \omega_0, \quad \mathrm{d}^* a = 0. \tag{16}$$

Thus, in order to prove Theorem 6, that is, $\mathscr{M}_\omega$ is sequentially compact, it is sufficient to show the following:

Proposition 9. *There exist constants c_k, c'_k, depending only on (X,g) and an integer $k \geq 3$, such that solutions (a,Φ) to the gauge-fixed SW equations* (22) *admit the following bounds for any integer $k \geq 3$:*

$$\|a\|_{L^2_k} \leq c_k, \quad \|\Phi\|_{L^2_k} \leq c'_k. \tag{17}$$

In order to show this, we will break up this result into the following more manageable chunks:

- Given that a and Φ admit L^2_3 bounds depending on (X,g), they also admit L^2_k bounds depending on (X,g) and k for any integer $k \geq 3$.
- Given that a and Φ admit L^2 and L^∞ (i.e. pointwise) bounds depending on (X,g), they also admit L^2_3 bounds depending on (X,g).
- Any a and Φ satisfying the gauge-fixed SW equations admit L^2 and L^∞ (i.e. pointwise) bounds depending on (X,g).

(The rationale behind this split shall become evident in a moment.)

In addition, we shall be using without proof the following results from Sobolev and elliptic theory:

Lemma 10 (Sobolev multiplication above the borderline)**.** *Let X be a compact n-manifold, E_1, E_2, E be smooth inner product bundles on it. Then any $C^\infty(X)$-bilinear map $M : \Gamma(E_1) \times \Gamma(E_2) \to \Gamma(E)$ satisfies the following estimates for all $s_1 \in L^p_j(E_1)$ and $s_2 \in L^q_k(E_2)$ and some fixed constants $c^{p,q}_{j,k}$:*

$$\|M(s_1,s_2)\|_{L^q_k} \leq c^{p,q}_{j,k}\|s_1\|_{L^p_j}\|s_2\|_{L^q_k}, \tag{18}$$

whenever the following inequalities hold:

$$\begin{aligned} &n < pj, \quad 0 \leq k \leq j, \\ &\frac{1}{p} - \frac{1}{q} \leq \frac{j-k}{n}. \end{aligned} \tag{19}$$

Remark 11. A Sobolev norm L^p_j is said to be below, at, or above the borderline depending on whether $j/n - 1/p$ is negative, zero, or positive.

Lemma 12 (Elliptic estimate)**.** *Let X be a compact manifold and E, E' be smooth inner product bundles over it. Then any elliptic linear differential operator $P : \Gamma(E) \to \Gamma(E')$ of order l satisfies the following estimates for all $s \in L^2_{k+l}(E)$ and some fixed constants c^p_k:*

$$\|s\|_{L^p_{k+l}} \leq c^p_k(\|Ps\|_{L^p_k} + \|s\|_{L^p}). \tag{20}$$

When s is L^2-orthogonal to ker P*, the estimates can be improved to:*

$$\|s\|_{L^p_{k+l}} \leq c^p_k \|Ps\|_{L^p_k}. \tag{21}$$

Let's now turn to proving the results we have broken up Proposition 9 into.

Proposition 13. *Given that a and Φ admit L^2_3 bounds depending on (X, g), they also admit L^2_k bounds depending on (X, g) and k for any integer $k \geq 3$.*

Proof. The trick is to rewrite the gauge-fixed SW equations in the following form:

$$D^+_{A_0}\Phi = -\gamma(a)\Phi, \quad 2(\mathrm{d}^+ + \mathrm{d}^{*-})a = \sigma(\Phi, \Phi) + \omega_0, \quad \mathrm{d}^* a = 0. \tag{22}$$

Here, the second equation is to be understood as an equality on the exterior algebra of forms $\Omega^\bullet(X)$, which is just the direct sum of all the $\Omega^i(X)$. The point of this is that $D^+_{A_0}$ and $\mathrm{d}^+ + \mathrm{d}^{*-}$ are elliptic linear differential operators.

Setting

$$n = 4, \quad p = q = 2, \quad j = k \geq 3,$$

in the Sobolev multiplication lemma and applying it to the $C^\infty(X)$-bilinear map

$$\Omega^1(X) \times \Gamma(V_+) \to \Gamma(V_-), \quad (a, \Phi) \mapsto -\gamma(a)\Phi,$$

tells us that if a and Φ admit L^2_k bounds, then so does $-\gamma(a)\Phi$. Meanwhile, applying the lemma to the $C^\infty(X)$-bilinear map

$$\Gamma(V_+) \times \Gamma(V_+) \to \Omega^\bullet(X), \quad (\Psi, \Phi) \mapsto \sigma(\overline{\Psi}, \Phi),$$

tells us that if Φ (and hence, $\overline{\Phi}$) admits an L^2_k bound, then so does $\sigma(\Phi, \Phi)$. The gauge-fixed SW equations then imply the existence of L^2_k bounds on $D^+_{A_0}\Phi$ and $(\mathrm{d}^+ + \mathrm{d}^{*-})a$. Finally, we may use elliptic estimates on $D^+_{A_0}$ and $\mathrm{d}^+ + \mathrm{d}^{*-}$ to obtain L^2_{k+1} bounds on a and Φ. The result to be proved then follows by induction. □

Now we see the rationale behind splitting Proposition 9. The above argument works only for $k \geq 3$, since the case $k < 3$ is beyond the jurisdiction of Sobolev multiplication above the borderline.

[to be wrapped up]