Background from Differential Topology

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1 (Co)homology, Orientability and Poincaré Duality

Recall that, using (e.g. singular) homology, we can associate to any topological space M its homology and cohomology groups with coefficient in any Abelian group G; most commonly, we will use the group $\mathbb{Z}$.

In the following I'll try to very briefly recall some fundamental results regarding the structure of the homology of my favorite class of topological spaces, namely connected (topological) manifolds (second countable Hausdorff spaces which are locally homeomorphic to $\mathbb{R}^n$). A first result is that $H_{k>n}(M^n; G) = H^{k>n}(M^n; G) = 0$.

The top degree homology is also of particular interest, especially in the case where M^n is furthermore assumed to be closed (i.e. compact without boundary). In this case, one may prove that $H_n(M^n; G)$ is either isomorphic to G or vanishes, and M^n is accordingly called G-orientable or G-non-orientable. If M^n is $\mathbb{Z}$-orientable, we call it orientable (since it follows that M^n is G-orientable for every G).

Remark 1. If $\pi_1(M^n)$ contains no subgroup of index two, M^n is always orientable. In particular, simply connected, closed manifolds are always orientable.

For an orientable, closed manifold M^n, there are precisely two orientations corresponding to the classes generating $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$. Choosing one of them, we call it the *fundamental class* of M and denote it by $[M]$. From now on, we will only discuss orientable manifolds, and presume that an orientation has been chosen.

Next, we turn to Poincaré duality, arguably the most important theorem regarding the (co)homology of closed manifolds. To introduce this, recall that there is a pairing between (singular) chains and cochains called the *cap product*, which we can define whenever the coefficient group G is even a commutative ring with unity.

A chain is a formal linear combination of maps $\sigma : \Delta^k \to M$; where we think of $\Delta^k = [e_1 : \cdots : e_{k+1}]$ as the convex hull of the standard unit vectors in $\mathbb{R}^{k+1}$. Now given a cochain $\varphi \in C^l(M; G)$ with $l \leq k$, we define the cap product $\sigma \frown \varphi \in C_{k-l}(M; G)$ by

$$\sigma \frown \varphi = \varphi\big(\sigma([e_1 : \cdots : e_{l+1}])\big) \cdot \sigma([e_{l+1}, \ldots, e_{k+1}])$$

where the dot denotes the product in G. This amounts to evaluating φ on the "front l-face" of σ. The cap product descends to a pairing between homology and cohomology classes.

Since there is no cohomology in degrees above the dimension, we can always pair a cohomology class with the fundamental class $[M]$. The statement of Poincaré duality is that the resulting map

$$\begin{aligned} H^k(M;G) &\longrightarrow H_{n-k}(M;G) \\ \alpha &\longmapsto [M] \frown \alpha \end{aligned}$$

is an isomorphism for every k. Corresponding classes in (co)homology are called (Poincaré) dual. Introducing the Betti numbers $\beta_k(M) = \dim_{\mathbb{Q}} H_k(M;\mathbb{Q})$ and the torsion subgroups $T_k \subset H_k(M;\mathbb{Z})$, we then have $H^k(M;\mathbb{Z}) \cong \mathbb{Z}^{\beta_{n-k}(M)} \oplus T_{n-k}$.

Another constraint is imposed by the *universal coefficients theorem*, which asserts the existence of a split short exact sequence

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(M;\mathbb{Z}),G) \longrightarrow H^n(M;\mathbb{Z}) \longrightarrow \operatorname{Hom}(H_n(M;\mathbb{Z}),G) \longrightarrow 0$$

which furthermore splits, i.e. $H^n(M;\mathbb{Z}) \cong \mathbb{Z}^{\beta_k(M)} \oplus T_{k-1}$ (where we are using the fact that $\operatorname{Ext}(H_{n-1}(M;\mathbb{Z}),\mathbb{Z}) \cong T_{n-1}$).

Put together, this shows that there is a *symmetry* in the homology of any compact, oriented manifold: $\beta_k(M) = \beta_{n-k}(M)$ and $T_{k-1} \cong T_{n-k}$.

Example 2. A simply connected, closed four-manifold has no torsion in its (co)homology. $H_0(M;\mathbb{Z})$ is always free Abelian of rank the number of connected components, hence $T_0 = 0 = T_3$. Since M is oriented, $T_4 = 0$ as well. Finally, since $H_1(M;\mathbb{Z}) \cong \pi_1(M)^{\mathrm{Ab}}$, it vanishes. Thus $T_1 = 0 = T_2$.

In fact, in this case the groups $H_0 \cong H^4 \cong H_4 \cong H^0$ are always $\mathbb{Z}$ and $H_1 \cong H^3 \cong H_3 \cong H^1$ is zero, hence $H^2(M;\mathbb{Z})$ and its multiplicative structure determines the full cohomology ring.

2 The Intersection Form

Now assume that M^n is even-dimensional: $n = 2m$. Then we have a bilinear pairing

$$\begin{aligned} H^m(M;\mathbb{Z}) \times H^m(M;\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ (\alpha,\beta) &\longrightarrow [M] \frown (\alpha \smile \beta) \end{aligned}$$

which we can reformulate as the composition

$$H^m(M;\mathbb{Z}) \xrightarrow{h} \operatorname{Hom}(H_m(M;\mathbb{Z}),\mathbb{Z}) \xrightarrow{\cong} \operatorname{Hom}(H^m(M;\mathbb{Z}),\mathbb{Z})$$

using the universal coefficients theorem and Poincaré duality. In the absence of torsion (e.g. for a simply connected four-manifold), h is an isomorphism, hence $H^m(M;\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(H^m(M;\mathbb{Z}),\mathbb{Z})$. This shows that the above bilinear form is unimodular, i.e. represented by a matrix with integer entries and determinant ± 1. This bilinear form Q_M is called the *intersection form*, for reasons which I will explain soon.

There are two qualitatively completely different cases, depending on the parity of m. If m is odd, the intersection form is skew-symmetric. Any skew-symmetric, unimodular bilinear form is equivalent to the bilinear form represented by

$$\begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix}$$

This is a direct sum of copies of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Thus, the only invariant in this case is the rank, which is always a multiple of two.

The case $m = 2k$ (i.e. M is a $4k$-fold) is completely different: The intersection form is now symmetric, and we will see soon that there are many possibilities.

3 Transversality and Intersections

It remains to explain the peculiar name of the intersection form. As it turns out, this explanation also makes it very easy to compute the intersection form in many examples. To do this, we will need the notion of *transversality*.

Recall that for smooth manifolds M, N and a smooth map $f : M \to N$, $y \in N$ is called a *regular value* if for every $x \in f^{-1}(y)$, $D_x f$ is surjective. It is a basic result from differential geometry that the preimage is then a submanifold of M, with codimension the codimension of a point in N, i.e. the dimension of N.

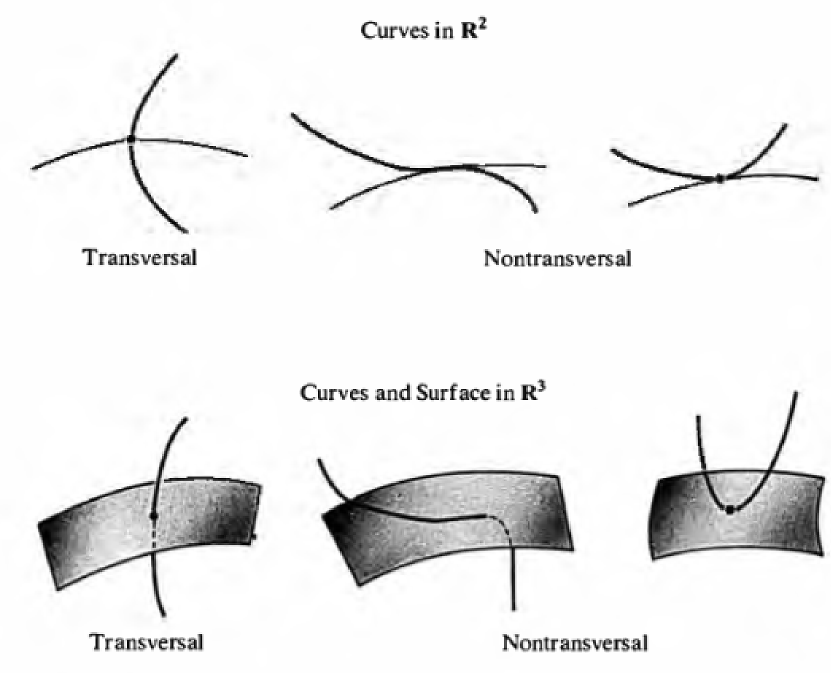
Instead of considering a point, we can consider a smooth submanifold $Y \subset N$, containing a point $y \in Y$ with preimage $X = f^{-1}(Y) \subset M$ containing a point x. Then the analog of surjectivity of $D_x f$ is that the image of $D_x f$ fills out (at least) the complement of $T_y Y$, or

$$D_x f(T_x M) + T_y Y = T_y N$$

where it is important to note that the sum doesn't have to be direct. If this condition is fulfilled for every $x \in X$, we say that the map f is *transversal* to Y, and write $f \pitchfork Y$. Transversality guarantees that X is indeed a submanifold of M, and $\operatorname{codim}_M X = \operatorname{codim}_N Y$.

The most important case for us will be when f is the inclusion map of a submanifold $X \hookrightarrow M$, and Y is some other submanifold of M. If such an f is transversal, i.e. $T_p X + T_p Y = T_p M$ for every point in the intersection, we say that X and Y are transversal and write $X \pitchfork Y$. In that case we see that the intersection is a manifold and $\operatorname{codim}_X(X \cap Y) = \operatorname{codim}_M Y$ or in other words $\operatorname{codim}_M(X \cap Y) = \operatorname{codim}_M X + \operatorname{codim}_M Y$.

Let's look at some simple examples of transversality to get a feel for it:



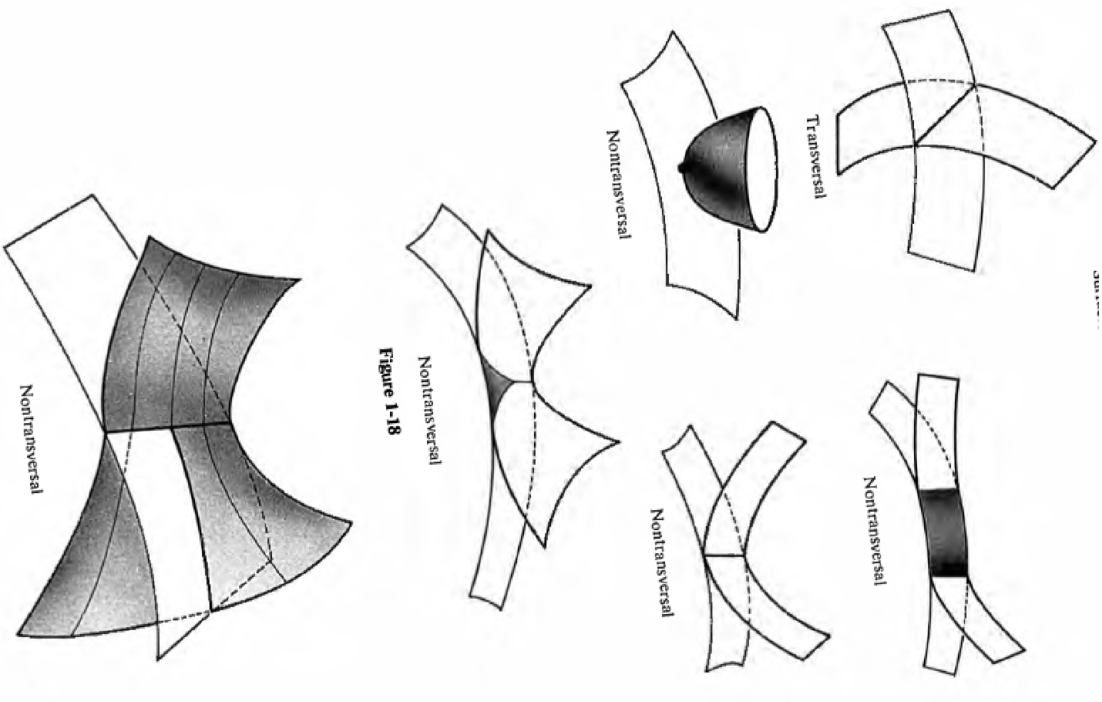


Figure 1-18

From the examples, it should be plausible that if we wiggle transversal submanifolds, they remain transversal, while if we wiggle non-transversal submanifolds, we can easily make them transversal. The former is made precise by the statement that transversality is an open condition, i.e. stable under small perturbations (at least if the domain of f is compact). The latter can be formulated as a corollary to the following theorem:

Theorem 3 (Transversality Theorem). *Suppose $f : M \to N$ is a smooth map, and $Y \subset N$ a submanifold. Then there is an open unit ball B^k (k large enough, at least $2 \dim N$) and a map $F : M \times S \to N$ such that $f = F(-, 0)$ and $f_s := F(-, s)$ is transversal to Y for almost every $s \in B^k$.*

Corollary 4. *In the above set-up, f is always homotopic to a map transversal to Y, and we may choose the transversal map to be as close to f as we like.*

In this sense, transversality is a *generic*, as well as stable, property.

Since (co)homology and many other constructions in algebraic topology are invariant under homotopy, we may always assume maps are transversal in these settings. In particular, if we have two closed submanifolds $X, Y \subset M$, we may homotope them to be transversal (also called *in general position*) and then take the homology class of the intersection (also a closed submanifold) $[X \cap Y]$. This then defines a notion of intersection on homology classes representable by submanifolds. It is an important theorem that this intersection is dual to the cup product via Poincaré duality: Denoting Poincaré duality by a bar, we have

$$\overline{M \cdot N} = \bar{M} \smile \bar{N}$$

An interesting situation occurs when considering $X^m, Y^m \subset M^{2m}$. Then $\operatorname{codim}_M(X \cap Y) = \operatorname{codim}_M X + \operatorname{codim}_M Y = \dim M$ and therefore $X \cap Y$ is a finite set of points. We assume that X, Y, M are all oriented, and compare the induced orientations in the intersection point: If they agree, we assign $+1$ and otherwise -1. Adding these up yields the *intersection number* of M and N. The above then says that this intersection number coincides with the result we get by first dualizing M and N and taking the cup product, then evaluating on the fundamental class. Now, if we can prove that any class in $H_2(M; \mathbb{Z})$ can be represented by a surface, then we have truly justified the name *intersection form*. This is not very hard, using transversality:

Lemma 5. *Let M^4 be closed, connected, oriented, smooth and simply connected. Then every class in $H_2(M; \mathbb{Z})$ is realized by an smooth immersed sphere, as well as by a smooth embedded surface.*

Remark 6. The simply connected assumption is actually superfluous, but simplifies the proof.

Proof. Since $\pi_1(M) = 1$, Hurewicz' theorem tells us that $\pi_2(M) \cong H_2(M; \mathbb{Z})$ is surjective, hence there exists a continuous map $f : S^2 \to M$ such that $f_*(S^2) = \alpha$ (for any $\alpha \in H_2(M; \mathbb{Z})$). Any continuous map between manifolds is homotopic to a smooth map, so we may assume f to be smooth. Using transversality, we may also assume that the image of the sphere only self-intersects in so-called *transverse double-points*. Thus, the sphere is immersed.

Now we may resolve double-points to obtain an immersion, at the cost of raising the genus. In charts, such an intersection looks like the set $xy = 0$ in $\mathbb{C}^2$, i.e. $\mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}$. We can perturb this to $xy = \epsilon$ for small ϵ, which simply amounts to cutting out a disk around each point and gluing the two ends together i.e. a "self-sum". This clearly raises the genus by one. □

Thus, we may work in terms of intersecting embedded submanifolds or cohomology classes, whichever suits us best.

Now, we examine some important examples of intersection forms:

Example 7.

(i) $H^2(S^4;\mathbb{Z}) = 0$ so the intersection form is empty.

(ii) The standard "projective line" $\mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^2$ represents the generator of $H_2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ and satisfies $\mathbb{C}\mathrm{P}^1 \cdot \mathbb{C}\mathrm{P}^1 = +1$, i.e. (1) is the intersection form of $\mathbb{C}\mathrm{P}^2$.

(iii) The second cohomology of $S^2 \times S^2$ is generated by $S^2 \times \{q\}$ and $\{p\} \times S^2$, which intersect in $\{(p,q)\}$ but do not self-intersect transversely ($S^2 \times \{q'\}$ doesn't touch $S^2 \times \{q\}$), hence $Q_{S^2\times S^2} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We say that they form a "hyperbolic pair", and denote this matrix by H.

(iv) Another famous example of a 4-manifold is called the E_8-manifold, because its intersection form is the E_8 Cartan matrix

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}$$

It follows from Donaldson's theorem that E_8 is nonsmoothable.

(v) Reversing orientation means we evaluate on $-[M]$ instead of $[M]$, hence $Q_{\bar{M}} = -Q_M$, e.g. $Q_{\overline{\mathbb{C}\mathrm{P}^2}} = (-1)$. But for instance $H \cong_{\mathbb{Z}} -H$, corresponding to the fact that there is an orientation-reversing diffeomorphism (send $(x,y) \mapsto (x,-y)$).

(vi) Given two 4-manifolds M, N we can take out a small ball from each and the complements together using a tube $S^3 \times \mathbb{R}$. The resulting manifold is called the *connected sum* $M\#N$ and satisfies $Q_{M\#N} = Q_M \oplus Q_N$. Thus, $\#_p\mathbb{C}\mathrm{P}^2\#\#_q\overline{\mathbb{C}\mathrm{P}^2}$ has diagonal intersection from $\oplus_p(1) \bigoplus \oplus_q(-1)$.

It is clear that one can produce quite a few different forms, so let us introduce some invariants. Perhaps the most obvious is their rank, given by $b_2(M)$. We also define the parity by saying that Q_M is *even* if $Q_M(\alpha,\alpha) \in 2\mathbb{Z}$ for any α, and *odd* otherwise. Furthermore, we say that Q_M is *definite* if $Q_M(\alpha,\alpha)$ has the same sign for any α, and *indefinite* otherwise.

Besides these simple invariants, the most important invariant is the *signature*. While they are not always diagonalizable over $\mathbb{Z}$, every symmetric bilinear form is diagonalizable over $\mathbb{R}$, i.e. if we consider the intersection form on $H^2(M;\mathbb{R}) = H^2(M;\mathbb{Z}) \otimes \mathbb{R}$. Since the matrix is non-degenerate, we may bring it to a form where it has only ± 1-entries on the diagonal. If the form Q_M then has p times $+1$ and q times -1 on the diagonal, we define its signature as $\sigma(Q_M) = p - q$. We also set $b_2^+(M) = p$ and $b_2^-(M) = q$.

The classification of these symmetric bilinear and unimodular forms is a very deep and difficult problem. It turns out that the indefinite case is the more tractable one:

Theorem 8 (Hasse-Minkowski Classification). *Every integer-valued, unimodular, indefinite, symmetric bilinear form is equivalent over $\mathbb{Z}$ to one of the following $\mathbb{Z}$:*

(i) *If the form is odd, it is equivalent to $p(1) \oplus q(-1)$ (of course, $p, q \geq 1$).*

(ii) *If the form is even, it is equivalent to $aH \oplus bE_8$, where we may choose $a \geq 1$ since $H \cong_{\mathbb{Z}} -H$ ($a = 0$ is not possible since E_8 is negative-definite).*

Corollary 9. *If Q_M is even and indefinite, then $\sigma(M) = -8b \equiv 0 \mod 8$.*

The classification in the definite case is considered intractable. However, Donaldson's theorem tells us that almost none of the possible definite forms occur for smooth manifolds:

Theorem 10 (Donaldson). *If an integer-valued, unimodular, definite, symmetric bilinear form Q is realized as the intersection form of a CCOS manifold, then it is equivalent over $\mathbb{Z}$ to either $p(1)$ or $q(-1)$ for some $p, q \geq 1$.*

This yields a classification of simply connected CCOS four-manifolds up to homeomorphism, when combined with Freedman's celebrated theorem, whose statement we also recall:

Theorem 11 (Freedman). *Let Q be an integer-valued, unimodular, symmetric bilinear form. Then there exists a simply connected compact, connected, oriented topological 4-manifold realizing Q as its intersection form. If Q is even, this manifold is unique up to homeomorphism. If Q is odd, there are precisely two homeomorphism classes of manifolds realizing Q as the intersection form, which are distinguished by the so-called Kirby-Siebenmann invariant.*

Remark 12. For smooth 4-manifolds, the Kirby-Siebenmann invariant automatically vanishes, since its vanishing indicates the possibility of putting a piecewise-linear (PL) structure on a manifold, which is stricly weaker than a smooth structure.

Corollary 13. *Simply connected CCOS four-manifolds are classifed up to homeomorphism by the Euler characteristic, signature, and parity of the intersection form.*

Proof. Let X, Y be simply connected CCOS four-folds. Since $H_1 = H_3 = 0$, we have $\chi(X) = 2+b_2(X)$ and similarly for Y. Thus, we see that $b_2^+(X)+b_2^-(X) = b_2^+(Y)+b_2^-(Y)$. Since the signatures coincide, we see $b_2^{\pm}(X) = b_2^{\pm}(Y)$.

Now assume the parity of the intersection forms is even. Then Donaldson's theorem says that they are not definite, hence Hasse-Minkowski applies and shows that $Q_X = aH \oplus bE_8 = Q_Y$. In case the intersection forms are odd, the definite case yields $Q_X = p(1) = Q_Y$ or $Q_X = q(-1) = Q_Y$. Finally, the indefinite case leads to $Q_X = p(1) \oplus q(-1) = Q_Y$. In all cases, the intersection forms agree, hence X and Y are homeomorphic (the Kierby-Siebenmann invariant always vanishes). $\square$