The Linearized SW Equations (continued) and Structure of the Gauge Group

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1 Preliminaries

We start by recalling some of the background information that we need in this talk.

The input data for Seiberg-Witten (SW) theory is a CCOS Riemannian 4-manifold M, equipped with a fixed spin^c structure $\mathfrak{s}$ and a self-dual, imaginary two-form ω. The SW equations are then equations for (i) a section Φ of the positive spinor bundle V_+ and (ii) a spin^c connection A or equivalently a connection $\hat{A}$ on the characteristic line bundle $L_\mathfrak{s}$. The space of such variables of the SW equations is called the configuration space, and is denoted by $\mathcal{C}_\mathfrak{s}$.

Solutions of the SW equations are the same as zeros of the SW map $f_\omega : \mathcal{C}_\mathfrak{s} \to i\Omega^2_+(M) \times \Gamma(V_-)$. Fixing a point $(A, \Phi) \in \mathcal{C}_\mathfrak{s}$, $f_\omega(A, \Phi) = (F^+_{\hat{A}} - \sigma(\Phi, \Phi) - \omega, D^+_A \Phi)$. Its differential was computed in the last talk:

$$\begin{aligned} \mathcal{T}_{(A,\Phi)} f_\omega : i\Omega^1(M) \times \Gamma(V_+) &\longrightarrow i\Omega^2_+(M) \times \Gamma(V_-) \\ (a, \varphi) &\longmapsto (2\mathrm{d}^+ a - \sigma(\Phi, \varphi) - \sigma(\varphi, \Phi), D^+_A \varphi + \gamma(a)\Phi) \end{aligned}$$

There is a natural right action of the gauge group $\mathcal{G} = L^2_6(M, S^1)$ on $\mathcal{C}_\mathfrak{s}$, and the point (A, Φ) also defines the *orbit map* $o_{(A,\Phi)} : \mathcal{G} \to \mathcal{C}_\mathfrak{s}$ which sends $u \mapsto (A, \Phi) \cdot u$. We can also compute the differential in this case, and obtain the map $L_{(A,\Phi)} : \mathfrak{g} \cong i\Omega^0(M) \to T_{(A,\Phi)}\mathcal{C}_\mathfrak{s} = i\Omega^1(M) \times \Gamma(V_+)$, $\xi \mapsto (-\mathrm{d}\xi, \xi\Phi)$. The codomain of this map is the domain of $\mathcal{T}_{(A,\Phi)} f_\omega$, so we can compose the maps. If (A, Φ) is a solution to the SW equations—actually we only need $D^+_A \Phi = 0$—the result always vanishes, so the composition defines a complex, which was discussed in the previous talk:

Theorem 1. *For $(A, \Phi) \in \mathcal{Z}_\omega$, the composition*

$$i\Omega^0(M) \xrightarrow{L_{(A,\Phi)}} i\Omega^1(M) \times \Gamma(V_+) \xrightarrow{\mathcal{T}_{(A,\Phi)} f_\omega} i\Omega^2_+(M) \times \Gamma(V_-)$$

defines an elliptic complex, with Euler characteristic (also known as the index, since it is the index of $L_{(A,\Phi)} \oplus \mathcal{T}_{(A,\Phi)} f_\omega$) $\frac{1}{4}(2\chi(M) + 3\sigma(M) - c_1^2(L_\mathfrak{s}) = -c_2(V_+)$

Remark 2. The last equality can be traced back to the Clifford multiplication isomorphism $\Lambda^2_+ \otimes \mathbb{C} \cong \mathrm{End}_0(V_+)$.

I will spend the first part of this talk continuing this study of the linearized SW equations.

2 The Linearized SW Equations (Continued)

2.1 The Cohomology Groups

A first, rather striking, observation is that the index of the elliptic complex is in fact independent of (A, Φ); it only depends on the spin^c structure. This is interesting because it can be expressed as the alternating sum $H^0_{\mathrm{el}} - H^1_{\mathrm{el}} + H^2_{\mathrm{el}}$; all summands may a priori depend on (A, Φ). Let us investigate the individual cohomology groups in some more detail:

Lemma 3. *Let $(A, \Phi) \in \mathcal{Z}_\omega$. Then*

$$H^0_{el} \cong \begin{cases} 0 & \Phi \not\equiv 0 \\ i\mathbb{R} & \Phi \equiv 0 \end{cases}$$

Proof. $H^0_{\mathrm{el}} = \ker L_{(A,\Phi)}$, so its elements must satisfy $\mathrm{d}\xi = 0$ and $\xi\Phi = 0$. This clearly implies the claim. □

Let's assume we're at an irreducible solution, so that $H^0_{\mathrm{el}} = 0$. To understand H^1_{el} and H^2_{el}, we introduce a *local slice* for the action $\mathcal{G} \curvearrowright \mathcal{C}_{\mathfrak{s}}$, near (A, Φ). This is a closed submanifold S such that a neighborhood of (A, Φ) is diffeomorphic to $S \times \mathcal{G}$: You can construct it by adding to (A, Φ) all sufficiently small tangent vectors orthogonal with respect to the L^2-inner product to $L_{(A,\Phi)}(\mathfrak{g})$. Then we may restrict f_ω to S and we see that

$$\begin{aligned} H^1_{\mathrm{el}} &= \ker \mathcal{T}_{(A,\Phi)} f_\omega / \operatorname{im} L_{(A,\Phi)} = \ker \mathcal{T}_{(A,\Phi)} \big(f_\omega\big|_S\big) \\ H^2_{\mathrm{el}} &= \operatorname{coker} \mathcal{T}_{(A,\Phi)} f_\omega = \operatorname{coker} \mathcal{T}_{(A,\Phi)} \big(f_\omega\big|_S\big) \end{aligned}$$

2.2 Implicit Function Theorem for Banach Manifolds

Now we want to use these observations to study the moduli space of solutions to the SW equations modulo gauge. In particular, we want to determine its dimension. To do this, we need an implicit function-type argument. Recall that, in finite-dimensional differential geometry, the level sets of a smooth map $f : M^{r+m} \to N^{r+n}$ of constant rank r (that is, its differential has constant rank) are submanifolds of codimension r. The proof proceeds by showing that near a point $x_0 \in M$ with $y_0 = f(x_0) \in N$, we can write f with respect to some charts (centered on x_0, y_0) as $\tilde{f}(x, y) = (x, \psi(x))$ for a smooth map $\psi : \mathbb{R}^r \to \mathbb{R}^n$ and in fact as $f(x, y) = (x, 0)$ so that with respect to these charts $f^{-1}(y_0) = \tilde{f}(0, 0)$ is locally simply given by setting the x-coordinates to zero. These are compatible charts, showing it's a submanifold. The case which is most frequently used is when f is a submersion, i.e. has trivial cokernel.

In the infinite-dimensional setting there is an analog, at least in the case where the infinite-dimensional spaces are locally modeled on a Banach space, and the map f is Fredholm, i.e. its differential has finite dimensional kernel and cokernel.

Again, one may write $f(x, y) = (x, \psi(x))$; the map ψ is then called a *Kuranishi map*. As in the finite-dimensional case, the situation where there is no cokernel (i.e. $\mathcal{T}f$ is surjective) leads to the level set being a submanifold of dimension $\ker \mathcal{T}f$.

Let us apply this in our situation. Recall that we had a local slice S for the $\mathcal{G}$-action on $\mathcal{C}_{\mathfrak{s}}$, and the restricted SW map $\big(f_\omega\big)\big|_S : S \to i\Omega^2_+(M) \times \Gamma(V_-)$. This map is Fredholm, and a neighborhood of $[A, \Phi] \in \mathcal{M}_\omega$ equals $(f_\omega\big|_S)^{-1}(0)$. Since $H^1_{\mathrm{el}} = \ker(f_\omega\big|_S)$ and $H^2_{\mathrm{el}} = \operatorname{coker}(f_\omega\big|_S)$, the Kuranishi map is $\psi : H^1_{\mathrm{el}} \to H^2_{\mathrm{el}}$. In particular, if $\mathcal{T}f$ is surjective, i.e. H^2_{el} vanishes, we see:

Proposition 4. *Assuming $H^0_{el} = 0$ and $H^2_{el} = 0$, a neighborhood of $[A, \Phi] \in \mathcal{M}_\omega$ is a smooth, manifold of dimension $H^1_{el} = \frac{1}{4}(c_1^2(L_{\mathfrak{s}} - (2\chi(M) + 3\sigma(M)) = \dim^{exp} \mathcal{M}_\omega$.*

This justifies the name “expected dimension”.

In the reducible case, things are a little more complicated:

Proposition 5. *If $H^0_{el} \cong \mathbb{R}$ and $H^2_{el} = 0$, a neighborhood of $[A, \Phi] \in \mathcal{M}_\omega$ is the quotient of a smooth manifold of dimension $\dim^{\mathrm{exp}} \mathcal{M}_\omega + 1$ by a $U(1)$-action.*

Proof. There is still the Kuranishi map $\psi : H^1_{\mathrm{el}} \to 0$. The constant gauge transformations act on $\mathcal{C}_{\mathfrak{s}}$ and $i\omega^2_+(M) \times \Gamma(V_-)$ and the former descends to H^1_{el}. We now have

$$1 - \dim H^1_{\mathrm{el}} = \frac{1}{4}((2\chi(M) + 3\sigma(M)) - c_1^2(L_{\mathfrak{s}}))$$

and thus $\dim H^1_{\mathrm{el}} = 1 + \dim^{\mathrm{exp}} \mathcal{M}_\omega$, which one still has to quotient by $U(1)$. □

3 Structure of the Gauge Group

We want to split the gauge group up into some simpler parts which we can treat more or less independently.

Proposition 6. $[M, S^1] \cong H^1(M; \mathbb{Z})$

Proof. Homotopy classes of maps correspond bijectively to maps $\pi_1(M) \to \mathbb{Z} \cong \pi_1(S^1)$. Clearly $f \simeq g \implies f_* = g_*$. Conversely, if $f_* = g_*$, then using the multiplication in S^1 clearly the map $(f^{-1} \cdot g)_*$ has trivial image, hence $f^{-1} \cdot g$ lifts to $\mathbb{R}$ and is null-homotopic. This induces a homotopy $f \simeq g$. Now $\mathbb{Z}$ is Abelian, hence maps $\pi_1(M) \to \mathbb{Z}$ are elements of $\operatorname{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z}) \cong H^1(M; \mathbb{Z})$. □

Note that this isomorphism maps $[f] \rightsquigarrow f_* \rightsquigarrow f^*\mu$, where μ is a fixed generator of $H^1(S^1; \mathbb{Z})$. This suggests the following definition:

Definition 7. The *degree* of a map $M \to S^1$ is the map

$$\begin{aligned} \deg : \mathcal{G} &\longrightarrow H^1(M; \mathbb{Z}) \\ u &\longmapsto u^*\mu \end{aligned}$$

Corollary 8. *$u \in \mathcal{G}$ is nullhomotopic if and only if $\deg u = 0$, if and only if $u = e^{if}$ for some $f : M \to \mathbb{R}$. .*

Definition 9. We say $u \in \mathcal{G}_0$ if $\deg u = 0$.

Corollary 10. *$\mathcal{G}_0$ is the connected component of $1 \in \mathcal{G}$ and $\mathcal{G}/\mathcal{G}_0 \cong H^1(M; \mathbb{Z}) \cong \mathbb{Z}^{b_1(M)}$.*

Now we consider another subdivision of $\mathcal{G}_0$ into two parts. Consider the following subgroups:

(i) $U(1) = e^{ic}$ for c constant.

(ii) $\mathcal{G}^\perp := \{e^{if} \mid f \in L^2_6(M),\ \int_M f \operatorname{vol}_g = 0\}$.

Proposition 11. *$U(1) \times \mathcal{G}^\perp \cong \mathcal{G}_0$, the isomorphism being given by $(e^{ic}, e^{if}) \mapsto e^{i(c+f)}$.*

Proof. Let $h = e^{if} \in \mathcal{G}_0$, and set $\lambda_h = \exp\left(\frac{i}{\operatorname{vol} M} \int_M f \operatorname{vol}_g\right)$; it is well-defined since if $f' = f + 2\pi i k$ then the integrals over M differ by multiples of $2\pi i \operatorname{vol} M$. Then the inverse of our isomorphism is given by $h \mapsto (\lambda_h, \lambda_h^{-1} h)$. □

We now want to rewrite the quotient $\mathcal{G}/\mathcal{G}_0$ in a simpler way, using one more subgroup:

Definition 12. A map $u : M \to S^1$ is called harmonic if $\alpha = u\mathrm{d}u^{-1}$ is a harmonic form. We write $u \in \mathcal{G}^h$.

$\mathcal{G}^h$ is indeed a subgroup of $\mathcal{G}$, as is easily checked. Since $0 = \mathrm{d}^2(uu^{-1}) = 2(\mathrm{d}u)(\mathrm{d}u^{-1}) = 2\mathrm{d}\alpha$, $u \in \mathcal{G}^h$ if and only if $\mathrm{d}^*\alpha = 0$.

Proposition 13. *For any $u \in \mathcal{G}$ there is a unique, up to constant, $f_u : M \to \mathbb{R}$ such that ue^{-if_u} is harmonic.*

To prove it, we have to quote the following result:

Theorem 14. *There exists a "Green's operator" for Δ, given by $G : \Omega^k(M) \to (\mathcal{H}^k(M))^\perp$, which maps α to the unique $\omega \in (\mathcal{H}^k(M))^\perp$ with $\Delta\omega = \alpha - H(\alpha)$, where $H : \Omega^k(M) \to \mathcal{H}^k(M)$ is the orthogonal projection.*

Note that $\Delta G = \mathrm{id} - H = G\Delta$ and $HG = GH$.
Proof of Proposition. Set $\beta = u\mathrm{d}u^{-1}$ and $f = iG(\mathrm{d}^*\beta)$. □