Áron Szabó: Linearization of the Seiberg-Witten equations

Some proofs are sketchy, and there might be errors in this note, please consult the script on the seminar's web page.

1 Warm-up: The half-de Rham complex [cf. Chapter 3.4 in the script]

1.1. A review of Hodge theory

Given an oriented Euclidean vector space (V, g) we can define the following objects

1. pointwise scalar product of k-forms: $\alpha, \beta \in \Lambda^k(V)$ $\quad \langle\alpha,\beta\rangle := \sum \alpha(e_1,\dots,e_k)\,\beta(e_1,\dots,e_k)$ where $\{e_i\}$ is a g-onb. This definition is independent of the choice of g-onb.
2. The Euclidean volume form $\quad \mathrm{vol}_g = \sqrt{\det g}\cdot e_1\wedge\dots\wedge e_n$
3. Hodge star operation $\quad * = *_g : \Lambda^k(V^*) \longrightarrow \Lambda^{\dim V - k}(V^*)$

$$\alpha \longmapsto *\alpha$$

via $\quad \alpha \wedge *_g\beta = \langle\alpha,\beta\rangle\,\mathrm{vol}_g$. For example we have $*1 = \mathrm{vol}_g$ and $*\mathrm{vol}_g = 1$.

(In fact, the metric g can be recovered from the corresponding Hodge star.)

We have $*^2 = \pm\,\mathrm{id}_V$ where the sign depends on $\dim X$ and k. $\quad *^2 = (-1)^{k(n-k)}\,\mathrm{id}_{\Lambda^k V^*}$

For $\dim V = 4$ and $k = 2$ we have $*: \Lambda^2 V^* \longrightarrow \Lambda^2 V^*$ and in fact $*^2 = +\,\mathrm{id}_V$.

Therefore the eigenvalues of $*$ are ± 1 and $\Lambda^2 V^*$ can be split accordingly:

<u>Definition</u> <u>Self-dual (antiselfdual) 2-forms on a 4 dimensional oriented Euclidean vector space</u> (V, g) are

$$\Lambda^2_\pm(V^*) = \{\, \alpha \in \Lambda^2(V^*) \mid *\alpha = \pm\alpha \,\}$$

With these, we have $\Lambda^2(V^*) = \Lambda^2_+(V^*) \oplus \Lambda^2_-(V^*)$. Moreover, this decomposition is orthogonal:

for $\alpha_\pm \in \Lambda^2_\pm(V^*)$, we have $\alpha_-\wedge\alpha_+ = \alpha_-\wedge *\alpha_+ = \langle\alpha_-,\alpha_+\rangle\mathrm{vol}_g = \langle\alpha_+,\alpha_-\rangle\mathrm{vol}_g = \alpha_+\wedge *\alpha_- = -\alpha_+\wedge\alpha_- = -\alpha_-\wedge\alpha_+$.

If (X, g) is a ccos Riemannian mfd, then these constructions can be carried out pointwise. In particular, we obtain self-dual and anti-self-dual 2 form fields $\Omega^2_\pm(X)$, and we have $\Omega^2(X) = \Omega^2_+(X) \oplus \Omega^2_-(X)$. If we introduce the L^2-scalar product on $\Omega^k(X)$ via $\quad (\alpha,\beta) := \int_X \langle\alpha,\beta\rangle\,\mathrm{vol}_g$, then it is easy to see that this decomposition is L^2-orthogonal.

We have the following important differential operators on a CCOS mfd

$d : \Omega^k(X) \longrightarrow \Omega^{k+1}(X)$ exterior derivative (smooth structure)

$\delta^g := d^*_g : \Omega^{k+1}(X) \longrightarrow \Omega^k(X)$ codifferential (divergence) (Riemannian str.)

where δ^g is defined via $(f_1, df_2)_{L^2} = (\delta^g f_1, f_2)_{L^2}$. We have $\delta^g = \pm * d *$

Definition The Hodge Laplacian of a CCOS mfd (X,g) is $\Delta^g := d\delta^g + \delta^g d : \Omega^\bullet(X) \longrightarrow \Omega^\bullet(X)$

Definition A form $\alpha \in \Omega^k(M)$ is a g-harmonic form if $\Delta^g \alpha = 0$. We use $\mathcal{H}^k(X,g) := \{\alpha \in \Omega^k(M) \mid \Delta^g \alpha = 0\}$.

Proposition $\alpha \in \Omega^2(X) \Rightarrow \left(\Delta^g \alpha = 0 \iff (d\alpha = 0 \text{ and } \delta^g \alpha = 0) \right)$

proof $(\Delta^g \alpha, \alpha)_{L^2} = (d\delta^g \alpha, \alpha)_{L^2} + (\delta^g d\alpha, \alpha)_{L^2} = (\delta^g \alpha, \delta^g \alpha)_{L^2} + (d\alpha, d\alpha)_{L^2} = \|\delta^g \alpha\|^2_{L^2} + \|d\alpha\|^2_{L^2}$ □

Theorem (Hodge) (X,g): CCOS mfd

$\Rightarrow$ ① Every de Rham cohomology class contains a unique harmonic representative.

② We have the L^2-orthogonal decomposition

$$\Omega^k(X) = d(\Omega^{k-1}(X)) \oplus \mathcal{H}^k(X,g) \oplus \delta^g(\Omega^{k+1}(X))$$

1.2 The half-de Rham complex

Consider the following sequence for a CCOS 4-mfd (X,g)

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d^+} \Omega^2_+(X) \longrightarrow 0$$

This is indeed a complex since $d^+ = \pi_+ \circ d$.

Proposition The index of the half-de Rham complex is $\frac{1}{2}(\chi(X) + \sigma(X))$

proof ① $H^0 = H^0_{dR}(X)$: Clear

② $H^1 = H^1_{dR}(X)$:

For $\alpha \in \ker d^+ \subset \Omega^1(X)$, we have

$$0 \overset{\text{Stokes}}{=} \int_X d(\alpha \wedge d\alpha) = \int_X d\alpha \wedge d\alpha = \int (\cancel{d^+\alpha} + d^-\alpha) \wedge (\cancel{d^+\alpha} + d^-\alpha)$$

$$= \int_X d^-\alpha \wedge d^-\alpha$$

$$= \int_X \langle d^-\alpha, * \underbrace{d^-\alpha}_{\in \Omega^2_-(X)} \rangle \mathrm{vol}_g = -\int \langle d^-\alpha, d^-\alpha \rangle \mathrm{vol}_g = -\|d^-\alpha\|^2_{L^2}$$

This means $d^+\alpha = 0$ already implies $d^-\alpha = 0$, hence $d\alpha = 0$ too. $(d\alpha = 0 \iff d^+\alpha = 0)$

③ $\underline{H^2 \simeq \mathcal{H}^2_+(X)}$:

We need coker d^+. Let $h \in \mathcal{H}^2_+(X)$ and $\alpha \in \Omega^1(X)$. Then

$$0 \overset{\text{Stokes}}{=} \int d(h \wedge \alpha) = \int dh \wedge \alpha + \int h \wedge d\alpha = \int \cancel{dh} \wedge \alpha + \int h \wedge d^+\alpha + \int h \wedge d^-\alpha$$

$$= \int_X \langle h, * d^+\alpha \rangle \, vol_g + \int \langle h, * d^-\alpha \rangle \, vol_g$$

$$= \int_X \langle h, d^+\alpha \rangle \, vol_g - \cancel{\int_X \langle h, d^-\alpha \rangle \, vol_g} + \int \langle dh, *\alpha \rangle vol_g$$

$$= (h, d^+\alpha) \implies \operatorname{im} d^+ \perp \mathcal{H}^2_+$$

Now by the Hodge decomposition for any $\omega \in \mathcal{H}^2_+$, we have α, β, γ

$$\omega = d\alpha + \delta^g \beta + h \qquad \alpha \in \Omega^1(X), \ \beta \in \Omega^3(X), \ h \in \mathcal{H}^2(X)$$

By self-duality of ω, we have

$$*\omega = *h + *d\alpha + *\delta^g\beta$$
$$= *h + \delta^g(*\alpha) + d*\beta$$
$$= h + \delta^g\beta + d\alpha$$

By uniqueness of the Hodge decomposition, we must have

$*d\alpha = +\beta \qquad *h = h$

$d*\beta = +d\alpha$

$$\omega = h + d\alpha + \delta^g\beta = h + d\alpha + *d*\beta = h + d\alpha + *d\alpha = h + 2d^+\alpha$$

But then clearly $\operatorname{coker} d^+ = \Omega^2_+(X)/\operatorname{im} d^+ \simeq \mathcal{H}^2_+$.

④ The index is

$$b_0(X) - b_1(X) + b_2^+(X) = \frac{1}{2}\big(b_0(X) - b_1(X) + b_2(X)\big) + \frac{1}{2}\underset{\overset{\parallel}{b_4}}{b_0(X)} - \frac{1}{2}\underset{\overset{\parallel}{b_3}}{b_1(X)} + \frac{1}{2}b_2^+(X) - \frac{1}{2}b_2^-(X)$$

$$= \frac{1}{2}\big(b_0 - b_1 + b_2 - b_3 + b_4\big) + \frac{1}{2}\big(b_2^+ - b_2^-\big)$$

$$= \frac{1}{2}\big(\chi(X) + \sigma(X)\big)$$

□

$\sigma(X) = b_2^+(X) - b_2^-(X)$ Def 3.9

2 The linearized Seiberg-Witten equations [cf. Chapter 5.1 in the script]

Sometimes we need to work with nonsmooth sections.

Definition The kth Sobolev space of sections of a bundle $E \to M$ with connection ∇ is $\overline{\Gamma(E)}^{\|\cdot\|_{H^k}}$ where $\|\Psi\|^2_{H^k} = \sum_{l=0}^{k} \|\nabla^l \Psi\|_{L^2}$.

- Positive spinors $\Phi \in \Gamma(V_+)$ lie in $L^2_5(V_+)$. $H^5(V_+)$
- Sections $i\Lambda^2_+(X) \times V_-$ are elements of $L^2_4(i\Lambda^2_+(X) \times V_-)$. $H^4(i\,\Lambda^2_+(X) \times V_-)$
- $\mathcal{A}_\mathfrak{s} \in L^2_5(\mathcal{A})$, i.e. of the form $\hat{A}_0 + a$ for $\hat{A}_0$ a smooth connection on $L_\mathfrak{s}$ and $a \in iL^2_5(T^*X)$.
- $\mathcal{G}$ consists of maps in $L^2_6(X, S^1)$. $\mathcal{G} \subset H^6(X, S^1)$

Lemma 5.2 ① The H^6-gauge group is an infinite dimensional abelian Hilbert-Lie group

② The H^6-gauge group acts smoothly on the H^5-configuration space and on H^4-sections of $i\Lambda^2_+(X) \oplus V_-$

proof In the script. $\square$

Definition The base space of the Seiberg-Witten equations is $\mathcal{B} := \mathcal{C}_\mathfrak{s}/\mathcal{G}$.

Definition The moduli space of (solutions of) the Seiberg-Witten equations is $\mathcal{M}_w := Z_w/\mathcal{G}$.

Main interest: how nice an object is $\mathcal{M}_w$? (As nice as you can wish!)

Plan: ① get an "expected dimension" of the moduli space (this talk)

② show that $\mathcal{M}_w$ has a smooth structure (next talk)

Analogy: the tale of the topologist who looked at the n-sphere and found a cohomology

Consider $X := \mathbb{R}^{n+2}\setminus\{0\}$, the group $G := \mathbb{R}_+$ acting by scaling and the function

$$f: \mathbb{R}^{n+2} \to \mathbb{R} \quad x \mapsto x^{n+2} \text{ (last component)}.$$

Now $Z := f^{-1}(0) = \mathbb{R}^{n+1} \times \{0\} \simeq \mathbb{R}^{n+1}$, and the G-action on X descends to Z.

Moreover, $Z/G = S^n$ and $TS^n \simeq TM/TG$, so

$\dim S^n = \dim(TM/TG) = \dim TM - \dim TG = n+1-1 = n$

On the other hand TM/TG looks a lot like a cohomology group if you look at it like a topologist, namely $0 \xrightarrow{\iota} G \xrightarrow{\text{action}} X \xrightarrow{f} \mathbb{R} \longrightarrow 0$

If we take the "derivative of this chain at a given point in Z", we'll get something nice

$$0 \longrightarrow TG \xrightarrow{T(\text{action})} TX \xrightarrow{Tf} T\mathbb{R} \longrightarrow 0$$

The expected dimension of $\mathcal{Z}/\mathcal{G}$ is $\dim_{exp} \mathcal{Z}/\mathcal{G} = h^2$, so if we knew that $h^1 = h^3 = 0$, then we'd be able to tell the expected dimension based on the index $\text{ind} = h^1 - h^2 - h^3 = -h^2 = \dim_{exp}$.

The linearized Seiberg-Witten equations

Fact $f_\omega : \mathcal{C}_\mathfrak{s} \longrightarrow i\Omega^2_+ \times \Gamma(V_-)$ $\quad (A,\phi) \mapsto \left(F^+_{\hat{A}} - \sigma(\phi,\phi) - \omega,\ D^+_A\phi\right)$ is a smooth map

Lemma 5.4 $\omega \in iH^4(\Lambda^2_+ T^*X)$, $(A,\phi) \in \mathcal{C}_\mathfrak{s}$ we have

$$T_{(A,\phi)} f_\omega : i\Omega^1(X) \times \Gamma(V_+) \longrightarrow i\Omega^2_+(X) \times \Gamma(V_-)$$

$$(a,\varphi) \longmapsto \left(2d^+a - \sigma(\phi,\varphi) - \sigma(\varphi,\phi),\ D^+_A\varphi + \gamma(a)\phi\right)$$

The principal part of $T_{(A,\phi)} f_\omega$ is $(a,\varphi) \longmapsto (2d^+a,\ D^+_A\varphi)$.
highest-order

proof ① Consider first the curve $(A+ta, \phi)$ through (A,ϕ)

Now $T_{(A,\phi)}(f_\omega)(a,0) = \left.\frac{d}{dt}\right|_{t=0} (f_\omega)(A+ta,\ \phi) = \left.\frac{d}{dt}\right|_{t=0} \left(F^+_{\widehat{A+ta}} - \sigma(\phi,\phi) - \omega,\ D^+_{A+ta}\phi\right)$

$$F^+_{\widehat{A+ta}} = F^+_{\hat{A}} + 2t\,d^+a$$

$$D^+_{A+ta} = \sum_i e_i \cdot \left(\nabla^A_{e_i} + ta(e_i)\right)\phi = D^+_A\phi + t\gamma(a)\phi$$

$$= (2d^+a,\ \gamma(a))$$

② Now consider the curve $(A, \phi+t\varphi)$ through (A,ϕ)

$$T_{(A,\phi)}(f_\omega)(0,\varphi) = \left.\frac{d}{dt}\right|_{t=0} (f_\omega)(A, \phi+t\varphi) = \left.\frac{d}{dt}\right|_{t=0} \left(F^+_{\hat{A}} - \sigma(\phi+t\varphi, \phi+t\varphi) - \omega,\ D^+_A\phi\right)$$

$$\overset{\sigma \text{ is bilinear}}{=} \left.\frac{d}{dt}\right|_{t=0} \left(F^+_{\hat{A}} - \sigma(\phi,\phi) - t\left(\sigma(\phi,\varphi) + \sigma(\varphi,\phi)\right) + t^2\sigma(\varphi,\varphi) - \omega,\ D^+_A\phi\right)$$

$$= \left(\sigma(\phi,\varphi) + \sigma(\varphi,\phi),\ 0\right)$$

Thus the claim. □

Definition Let $M \curvearrowleft G$ be a smooth action. The infinitesimal action of (fundamental vector field corresponding to $\xi \in \mathfrak{g}$ is $X_\xi \in \mathfrak{X}(M)$ defined via $(X_\xi)_p := \left.\frac{d}{dt}\right|_{t=0} (p \cdot \exp(t\xi))$ $\forall p \in M$

Lemma 5.6 ① The Lie algebra of $\mathcal{G} = C^\infty(X, S^1)$ is $\mathfrak{g} = C^\infty(X, i\mathbb{R}) = i\Omega^0(X)$

② The fundamental vector field of $\xi \in \mathfrak{g}$ at $(A,\phi) \in \mathcal{C}_\mathfrak{s}$ is $(X_\xi)_{(A,\phi)} = (-d\xi, \xi\phi)$

③ The principal part of this operator is $\xi \longmapsto (-d\xi, 0)$.

proof $(X_\xi)_{(A,\phi)} = \frac{d}{dt}\Big|_{t=0} \left((A,\phi)\cdot \exp(t\xi) \right)$

$= \frac{d}{dt}\Big|_{t=0} \left(A + \exp(t\xi)\, d \exp(-t\xi),\ \exp(t\xi)\phi \right)$

$A.g = A + g d g^{-1}$

$= \frac{d}{dt}\Big|_{t=0} \left(A + \exp(t\xi)\exp(-t\xi)\cdot(-t) d\xi,\ \exp(t\xi)\phi \right)$

$= \frac{d}{dt}\Big|_{t=0} \left(A - t\, d\xi,\ \exp(t\xi)\phi \right)$

$= (-d\xi, \xi\phi)$, as claimed. $\square$

Now for the main theorem

<u>Proposition 5.6</u> Fix $(A,\phi) \in \mathcal{C}_\xi$. If $D_A^+\phi = 0$ (in particular on Z_ω), the sequence

$$i\Omega^0(X) \xrightarrow{(X_\cdot)_{(A,\phi)}} i\Omega^1 \times \Gamma(V_+) \xrightarrow{T_{(A,\phi)}(f_\omega)} i\Omega^2_+(X) \times \Gamma(V_-) \qquad (C)$$

is an elliptic complex with index $-\frac{1}{4}\left(c_1^2(L_\xi) - 2\chi(X) - 6\sigma(X)\right)$

proof ① The composition is zero.

② According to the "sledge-hammer" Atiyah-Singer index theorem, the index of an elliptic complex depends only on the (symbol of the operators, which in turn depend only on the) principal part of the operators.
Thus the complex (C') has the same index as (C)

$$i\Omega^0(X) \longrightarrow i\Omega^1(X) \times \Gamma(V_+) \longrightarrow \Omega^2_+(X) \times \Gamma(V_-) \qquad (C')$$

$$\xi \longmapsto (-d\xi, 0)$$

$$(a, \varphi) \longmapsto (2d^+a, D_A^+\varphi)$$

This is the direct sum of the complexes

$$i\Omega^0(X) \xrightarrow{d} i\Omega^1(X) \xrightarrow{d^+} i\Omega^2_+(X) \qquad (C_1) \qquad (\text{HdR})$$

and

$$0 \longrightarrow \Gamma(V_+) \xrightarrow{D_A^+} \Gamma(V_-) \qquad (C_2)$$

Now from the Atiyah-Singer index theorem $\operatorname{ind} C_2 = -\operatorname{ind}_{\mathbb{R}} D_A^+ = -2\operatorname{ind}_{\mathbb{C}} D_A^+ = -2\cdot\frac{1}{8}\left(c_1^2(L_\xi) - \sigma(X)\right)$ $\square$

Now $H^1(C)$ is the formal tangent space of the moduli space, so if we could calculate the dimensions of $H^0(C)$ and $H^2(C)$, then we'd have an "expected dimension" of the moduli space.
E.g. if $h^0(C) = h^2(C) = 0$ ("vanishing theorems"), then $\dim_{\exp} \mathcal{M}_\omega := h^1(C) = -\operatorname{ind}(C)$.