Elliptic operators

$E \to M$ v.b. over (cpt, or) mf, ∇ conn on E

Def: The algebra $\mathcal{D}(E)$ of lin diff. op. on E is the algebra generated by $\Gamma(\mathrm{End}(E))$ and $\{\nabla_X \,/\, X \in \mathfrak{X}(M)\}$ acting on $\Gamma(E)$

$\mathcal{D}(E) = \bigcup_k \mathcal{D}^k(E)$ natural filtration $\quad \mathcal{D}^k$ generated by $\Gamma(\mathrm{End}(E))$ and $\nabla_{X_1} \circ \dots \circ \nabla_{X_l}$, $l \le k$

Def: $D \in \mathcal{D}(E,F) :\Longleftrightarrow \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix} \in \mathcal{D}(E \oplus F)$

Ex:
- $\nabla \in \mathcal{D}^1(E, T^*M \otimes E)$ as $\nabla = \sum_k \rho_k \sum \omega_i^k \otimes \nabla_{X_i}$ for $1 = \sum_k \rho_k$, X_i^k ω_i^k dual frames on U_k of TM & T^*M
- $\not{D} \in \mathcal{D}^1(S)$ as $\not{D} = \gamma \circ \# \circ \nabla$ $\quad \# \eta \otimes s := \eta^\sharp \otimes s$ with $\eta(X) = g(\eta^\sharp, X)$
- $d \in \mathcal{D}^1(\Lambda^* T^*M)$ because $d = \wedge \circ \nabla$ for a torsion-free conn ∇ & $\wedge : T^*M \otimes \Lambda^k T^*M \to \Lambda^{k+1} T^*M$

Symbol: $A = \bigcup_{i \ge 0} A_i \qquad A_i A_j \subseteq A_{i+j}$

$\mathrm{gr}\, A = \bigoplus_i A_i / A_{i-1}$ associated algebra

Thm: $\mathrm{gr}\, \mathcal{D}(E) \cong \Gamma(STM \otimes \mathrm{End}(E))$

via $\mathcal{D}^k / \mathcal{D}^{k-1} \underset{\sigma^k}{\cong} \Gamma(S^k TM \otimes \mathrm{End}(E))$

via $\sigma_k(D)(p,\eta) := \lim_{t \to \infty} t^{-k} \left(e^{-itf} D e^{itf}\right)(p) \in \mathrm{End}(E_p)$ where $d_pf = \eta$

$= \frac{(-i)^k}{k!} (\mathrm{ad}\, f)^k D \qquad \mathrm{ad}(f) D = Df - fD$

For $D \in \mathcal{D}^k(E,F)$

$\sigma^k \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix} =: \begin{pmatrix} 0 & 0 \\ \sigma^k(D) & 0 \end{pmatrix}$ generalizes symbol to $\mathcal{D}(E,F)$

$\boxed{\sigma^{k+l}(D \circ \tilde{D}) = \sigma^k(D) \cdot \sigma^l(\tilde{D})}$

Ex:
- $\sigma(d)(p, d_pf) = -i\, \underset{\in \mathrm{Hom}(\Lambda^k T_p^*M,\, \Lambda^{k+1} T_p^*M)}{d_pf \wedge \cdot}$ as $d f\omega - f d\omega = df \wedge \omega$
- $\sigma(\nabla_X)(df) = -i\, d_pf(X)$ i.e. $\sigma(\nabla_X) = X$
- $\sigma(\not{D})(df) = -i\, \gamma(\mathrm{grad}\, f)$

Thm: $\forall D \in \mathcal{D}^k(E,F)$ $\exists!$ $D^* \in \mathcal{D}^k(F^*, E^*)$ $\qquad$ M cpt, or., Riem metric g, vol

s.th. $\int_M f^*(De)\, \mathrm{vol} = \int_M (D^* f^*)(e)\, \mathrm{vol}$ $\quad \forall\, e \in \Gamma(E)$, $f^* \in \Gamma(F^*)$

$\sigma^k(D)^*(\eta) = \sigma^k(D^*)(\eta)$

if E, F hermitian, then $D^* \in \mathcal{D}^k(F,E)$

pf: $\nabla_X^* = -\nabla_X + \mathrm{div}\, X$ & $(D \circ \tilde{D})^* = \tilde{D}^* \circ D^*$

Analysis: **Def**: $D \in \mathcal{D}^k(E,F)$ is called elliptic iff
$\forall p \in M \quad df \neq 0 : \quad \sigma_k(D)(p, df) \in Hom(E_p, F_p)$ is invertible

Idea: takes $\underline{\underline{all}}$ highest order derivatives

Ex: • $\not{D}$ is elliptic (for M Riemannian)

• $\nabla^* \nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) \to \Gamma(E)$ is elliptic

pf: $\sigma(\nabla)(df) = -i\, grad\, f \otimes$

$\sigma(\nabla^*)\sigma(\nabla)(df) = (grad\, f \otimes)^*(grad\, f \otimes \cdot) = -i^2 \|grad\, f\|^2$

Big example: $* : \Lambda^k \to \Lambda^{n-k}$

st $\omega \wedge * \eta = \langle \omega, \eta \rangle\, vol$

we have $*^2 = \pm id$ e.g. $n=4$, $k=2$: $+1$; $n=2$, $k=1$: -1

Then $d^* =: \delta = (\pm 1)^? * d *$

pf: $\int_M \langle d\omega, \eta \rangle\, vol = \int_M d\omega \wedge * \eta = \pm \int \omega \wedge d * \eta = \pm \int \langle \omega, * d * \eta \rangle\, vol$

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$d + \delta : \Lambda^* \to \Lambda^*$ is elliptic

idea of pf: $\sigma(\delta)(\eta) = (\eta \wedge)^* = i_{\eta^*} \lrcorner$, and

$v \longmapsto \omega \mapsto v \wedge \omega + i_v \omega$ is a Clifford action

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We want the following Theorem

Thm: $D \in \mathcal{D}^1(E,F)$ elliptic of Dirac type and $D^* = D$

then D has a discrete real spectrum of finite-dim eigenspace of smooth sections & eigenvalues have precise asymptotic to $\pm\infty$

Sobolev spaces: k-Norm $\quad \|s\|_k^2 = \int_M \|s\|^2 + \|\nabla s\|^2 + \dots + \|\nabla \dots \nabla s\|^2\, vol$

E with hermitian metric

$W^k(E) = \overline{(\Gamma(E), \|\cdot\|_k)}$ completion

Fundamental properties: • $W^k \to W^{k-1}$ is compact

• $\bigcap_k W^k = \Gamma(E)$

Idea of proof: • Def independent of ∇

• via partition of unity it reduces to trivial bundles on tori

• pf for functions on tori via standard Fourier computations

Thm: $D = \gamma \circ \nabla$ for unitary connection ∇ on E which is compatible with skew-herm. Clifford action γ: $\nabla(X \cdot e) = (\nabla \gamma) \cdot e + X \nabla e$
$\overset{\shortparallel}{\gamma(X) \cdot e}$

Then $D^2 = \nabla^* \nabla + C$
for some $C \in \Gamma(\mathrm{End}(E))$

Pf: $D^2 s|_p = \sum_i X_i \nabla_{X_i} \left(\sum_j X_j \nabla_{X_j} s \right)|_p = \left(\sum_{i,j} X_i \cdot X_j \nabla_{X_i} \nabla_{X_j} s \right)|_p$

$(\nabla X_i)|_p = 0$, ON-basis

$= \sum_i \underbrace{X_i \cdot X_i}_{=-1} \cdot \nabla_{X_i} \nabla_{X_i} s + \sum_{i<j} X_i X_j (\nabla_{X_i} \nabla_{X_j} - \nabla_{X_j} \nabla_{X_i} s)$

$= - \nabla^* \nabla s|_p + R^c s$

where $R^c \cdot s = \gamma(R) \cdot s$ curvature of ∇ on E

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locally $E = \Delta \otimes V$ where Δ is ~~irred.~~ or direct sum of spin rep.
Δ equipped with spin-connection LC
V equipped with some hermitian connection
and $\nabla^E = \nabla^\Delta \otimes \nabla^V$ so $R = R^\Delta \otimes R^V$
Then $\gamma(R^\Delta) = \frac{1}{4}$ scal
R^V commutes with Clifford multiplication

Corollary: $s \mapsto \sqrt{\|Ds\|^2 + \|s\|^2}$ defines a norm on W^1 equivalent to $\|\cdot\|_1$

Rmk: a bit more work shows: $\|s\|_{k+1} \leq C(\|Ds\|_k + \|s\|_k) \;\; \forall s \in \Gamma(E)$

Pf of theorem: • Define $G \subseteq L^2 \oplus L^2$ to be the graph of D
$\overset{\shortparallel}{\{(s, Ds) \,/\, s \in W^1\}}$

$J : L^2 \oplus L^2 \to L^2 \oplus L^2$ $\quad J \hat{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Claim: $L^2 \oplus L^2 = G \oplus JG$
pf: $JG \subseteq G^\perp$ if $(a,b) \perp G$ then $(a,s) + (b, Ds) = 0 \;\; \forall s$
i.e. $\langle a, s \rangle = \langle Db, s \rangle$
which means that $Db = -a$ weakly $\xrightarrow[\text{analysis}]{\text{some}}$ $b \in W^1$, $Db = -a$ $\leadsto (a,b) \in JG$

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Define $Q : L^2 \to L^2 \oplus L^2 \xrightarrow{\pi_G} G \xrightarrow{\pi_1} L^2$
$x \mapsto (x, 0)$

Claim: Q is a continuous s.a. operator | s.a. is a short computation
We have $\|x\|^2 \geq \|Qx\|^2 + \|DQx\|^2 \geq \|Qx\|_1^2$
$\to Q : L^2 \to W^1 \to L^2$
bounded, compact

Hence Q has discrete spectrum of finite-dim eigenspaces (Q is injective & positive)

Eigenvectors x of Q with ev $\varrho > 0$

correspond to pairs of eigenvectors $x \pm z$ of D with ev $\pm\sqrt{\frac{1-\varrho}{\varrho}}$ (note $\varrho < 1$)

where $z = -\frac{1}{\varrho\lambda} y$ for $(x,0) = (\varrho x, \varrho D x) + (-Dy, y)$

Smoothness: idea: $\|s\|_{H_{k+1}} \le C(\|s\|_{H_k} + \|Ds\|_{H_k})$ makes sense

heat-kernel: Consider $e^{-tD^2} : L^2 \to L^2$

via s with $Ds = \lambda s$

$s \mapsto e^{-t\lambda^2} \cdot s$

for $t > 0$ smoothing operator given by a kernel

$h_t \in \Gamma(\pi_1 S \otimes \pi_2 S^*)$ $\quad \pi_i : M \times M \to M$

$t \to h_t$ has a precise asymptotic at $t = 0$ along the diagonal in terms of curvature terms

for $t > 0$ the operator is also of trace-class, i.e.

$$\underset{\text{well-def}}{tr\, e^{-tD^2}} = \int_M h_t(p,p)\, dvol_p$$

for D this is boring as D is s.a., i.e. $\ker D = \operatorname{coker} D$

Consider $\dim M$ even: $S = S_+ \oplus S_-$ $\rightsquigarrow$ grading on sections

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \qquad \varsigma = \pm id \text{ on } S^\pm$$

$$tr_s A = tr\, \varsigma \circ A$$

Then $\underset{\text{super}}{ind_s(D)} := ind(D_+) = \dim \ker D_+ - \dim \ker D_-$

let $E_\lambda = Eig(D, \lambda)$

& $E_\lambda^\pm = E_\lambda \cap \Gamma(S^\pm)$

for $\lambda \neq 0$ $\quad D : E_\lambda^\pm \xrightarrow{\sim} E_\lambda^\mp$

thus for tr_s we get

$$tr_s\, e^{-tD^2} = \dim \ker D_+ - \dim \ker D_- + \sum_\lambda e^{-t\lambda^2}(\dim E_\lambda^+ - \dim E_\lambda^-) = ind(D)$$

Now $h_\varepsilon(p,p) \sim \frac{1}{(4\pi\varepsilon)^{n/2}} \left(\underset{\uparrow \atop id}{\theta_0} + \varepsilon \underset{\uparrow \atop \frac{1}{6} scal - K}{\theta_1} + \dots \right.$ θ_4 in terms of curvature & its derivatives

Thus $\operatorname{ind}(D) = \int_M \operatorname{tr}_s \theta_{\frac{n}{2}} = \int_M \hat{A}(TM) \wedge ch(S/\Delta)$

$\hat{A}(TM) = 1 - \frac{1}{24} p_1 + \dots$

$ch(S/\Delta) = \operatorname{tr}_{S/\Delta} \left(\frac{F^S}{2\pi i} \right)$ ← relative trace w.r.t. $S = \Delta \otimes V$

F^S - twisting curv. | $\text{curv} = R^S + F^S$

Ex: $\dim M = 2 \overset{\text{spin}}{\rightsquigarrow} \hat{A} = 1 \quad F^S = F^V$

$\operatorname{ind}(D) = \int_M \operatorname{tr} \frac{F}{2\pi i} = \deg V = \deg(S) - \operatorname{rk}(1-g)$

$\dim M = 4$:

M spin, spin bundle $S = S^+ \oplus S^-$

$\operatorname{ind}(D) = \dim \ker D_+ - \dim \ker D_- = -\frac{1}{24} p_1(TM)(M)$

$= -\frac{1}{8} \sigma(M)$

Cor.: $\sigma(M)$ is divisible by 16 for cpt, or., spin mfd of dim 4

pf: $\dim \ker D_\pm$ are quat. vector-spaces, hence $-\frac{1}{8}\sigma(M)$ is even.

M COS $\quad S \quad$ spin^c: locally $S = \Delta \otimes \tilde{L} \rightsquigarrow \bar{S} = \Delta \otimes \tilde{L}^*$

$\rightsquigarrow F^S = \frac{1}{2} \omega^{F^L}$ $\quad S = \bar{S} \otimes L$, $\tilde{L}^2 = L$

and

$\operatorname{ind}(D) = \int_M \hat{A} \wedge ch(S/\Delta)$

$= \int_M \left(1 - \frac{1}{24} p_1\right) \wedge \left(1 + \frac{1}{4\pi i} \omega^{F^L} - \frac{1}{8\pi^2} \omega^{F^L}\right)$

$= -\sigma(M) + \frac{c_1^2}{8}(L_s)(M)$