DISSERTATION

Twists of quaternionic Kähler manifolds

Vorgelegt von
Arpan SAHA

Betreut von
Prof. Dr. Vicente CORTÉS

zur Erlangung des Doktorgrades

an der

Fakultät für Mathematik, Informatik und Naturwissenschaften
Fachbereich Mathematik

der

UNIVERSITÄT HAMBURG

Hamburg, 2020

1. Gutachter: Prof. Dr. Vicente Cortés
2. Gutachter: Prof. Dr. Diego Conti

Datum der Disputation: 29.04.2020
Vorsitzende der Prüfungskommission: Prof. Dr. Jörg Teschner

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Arpan Saha
Hamburg, 2020

This dissertation is partially based on the following preprints coauthored by me:

Vicente Cortés, Arpan Saha, Danu Thung. *Symmetries of quaternionic Kähler manifolds with S^1-symmetry*. 2020. arXiv: 2001.10026 [math.DG].

Vicente Cortés, Arpan Saha, Danu Thung. *Curvature of quaternionic Kähler manifolds with S^1-symmetry*. 2020. arXiv: 2001.10032 [math.DG].

In addition, this dissertation relies on results in the following publication that grew out of my Masters' thesis:

Vicente Cortés and Arpan Saha. "Quarter-pinched Einstein metrics interpolating between real and complex hyperbolic metrics". In: *Mathematische Zeitschrift* (May 2017). DOI: 10.1007/s00209-017-2013-x.

"Did you ever look at Greenland on a map?"

"I guess I have, once or twice perhaps."

"Did you ever notice that it's never the same size on any two maps? The size of Greenland changes map to map. It also changes year to year. It's very large. It's enormous. But sometimes it's a little less enormous, depending on which map you're looking at."

"I believe it's the largest island in the world."

"The largest island in the world," Marvin said. "But you don't know anyone who's ever been there. And the size keeps changing. What's more, listen to this, the location also changes. Because if you look closely at one map and then another, Greenland seems to move. It's in a slightly different part of the ocean. Which is the whole juxt of my argument."

"What's your argument?"

"You asked so I'll tell you. That the biggest secrets are staring us in the face and we don't see a thing."

Don DeLillo, *Underworld*

Abstract

Twists of quaternionic Kähler manifolds

by Arpan Saha

In [Hay08], Haydys showed that to any hyperkähler manifold, equipped with a Killing field $\tilde{Z}$ that preserves one of its Kähler structures and rotates the other two, one can associate a quaternionic Kähler manifold of the same dimension, which has positive scalar curvature and also carries a Killing field Z. This HK/QK correspondence was extended to indefinite hyperkähler manifolds and quaternionic Kähler manifolds of negative scalar curvature by Alekseevsky, Cortés, and Mohaupt in [ACM13]. It was later described by Macia and Swann in [MS14] in terms of elementary deformations and the twist construction, originally introduced by Swann in [Swa10].

In this dissertation, we use the twist realisation of the HK/QK correspondence to write down an elegant formula relating the Riemann curvature of the quaternionic Kähler manifold to that of the hyperkähler manifold. In particular, the Weyl curvature of the quaternionic Kähler manifold (which is of hyperkähler type) can be interpreted as a sum of two abstract curvature tensors, one coming from the curvature on the hyperkähler side of the correspondence, and one coming from a standard abstract curvature tensor constructed out of the twist form. We furthermore use the twist construction to show that the Lie algebra of Hamiltonian Killing fields of the quaternionic Kähler manifold commuting with Z is at least a central extension of the Lie algbera of Hamiltonian Killing fields on the hyperkähler side that preserve the HK/QK data. As an application of these general results, we prove that that the 1-loop deformation of Ferrara–Sabharwal metrics with quadratic prepotential, obtained using the HK/QK correspondence in [Ale+15], have cohomogeneity 1 in every dimension.

In addition to the above, we also complete the twist-based picture of the HK/QK correspondence by identifying certain canonical twist data on the quaternionic Kähler manifolds and showing that the QK/HK correspondence can be realised as the twist of an elementary deformation of the quaternionic Kähler manifold with respect to this twist data. More generally, we construct 1-loop deformations of quaternionic Kähler manifolds as twists of elementary deformations of the quaternionic Kähler manifold directly. In doing so, we prove an analogue of Macia and Swann's theorem in [MS14] where instead of a hyperkähler manifold, we have a quaternionic Kähler manifold.

In order to be able to efficiently carry out these constructions, we also develop an alternative local formulation of the twist construction which requires weaker hypotheses than that of Swann. The description of 1-loop deformations in terms of a local twist map is finally used to construct geometric flow equations on the space of quaternionic Kähler structures on an open ball.

Zusammenfassung

Twists quaternionisch-Kählerscher Mannigfaltigkeiten

von Arpan Saha

In [Hay08] zeigte Haydys, dass man jeder Hyperkählermannigfaltigkeit (HK), ausgestattet mit einem Killingfeld $\tilde{Z}$, das eine der Kählerstrukturen erhält und die anderen beiden rotiert, eine quaternionisch-Kählersche Mannigfaltigkeit (QK) gleicher Dimension zuordnen kann, die positive Skalarkrümmmung hat und ebenfalls ein Killingfeld Z trägt. Diese sogenannte HK/QK-Korrespondenz wurde von Alekseevsky, Cortés und Mohaupt in [ACM13] auf indefinite Hyperkählermannigfaltigkeiten und quaternionisch-Kählersche Mannigfaltigkeiten negativer Skalarkrümmung erweitert. Sie wurde später von Macia und Swann in [MS14] mithilfe von elementaren Deformationen und Swanns Twistkonstruktion [Swa10] beschrieben.

In dieser Dissertation leiten wir mithilfe Twist-Realisierung der HK/QK-Korrespondenz eine elegante Formel her, die die riemannsche Krümmung der quaternionisch-Kählerschen Mannigfaltigkeit mit der Krümmung der Hyperkählermannigfaltigkeit in Beziehung setzt. Insbesondere lässt sich die Weylkrümmung (von Hyperkähler-Typ) der quaternionisch-Kählerschen Mannigfaltigkeit als die Summe zweier abstrakter Krümmungstensoren interpretieren: einer, der sich aus der Krümmung der hyperkählerschen Mannigfaltigkeit ableitet und einer, der sich aus der Twistform konstruieren lässt. Ferner wird die Twistkonstruktion eingesetzt, um zu zeigen, dass die Lie-Algebra der hamiltonschen Vektorfelder der quaternionisch-Kählerschen Mannigfaltigkeit, die mit dem Vektorfeld Z kommutieren, zumindest eine zentrale Erweiterung der Lie-Algebra der hamiltonschen Vektorfelder auf der hyperkählerschen Seite, die die HK/QK-Daten erhalten, ist. Als Anwendung dieser allgemeinen Ergebnisse wird bewiesen, dass alle durch die HK/QK-Korrespondenz in [Ale+15] erhaltenen 1-Schleifen-Deformationen der von quadratischen Präpotentialen herleitbaren Ferrara–Sabharwal-Metriken in jeder Dimension Kohomogenität 1 haben.

Darüber hinaus vervollständigen wir das Twistbild der HK/QK-Korrespondenz, indem wir gewisse Twistdaten auf den quaternionisch-Kählerschen Mannigfaltigkeiten identifizieren und zeigen, dass sich die QK/HK-Korrespondenz als Twist einer elementaren Deformation der quaternionisch-Kählerschen Mannigfaltigkeit bezüglich dieser Twistdaten realisieren lässt. Allgemein konstruieren wir 1-Schleifen-Deformationen von quaternionisch-Kählerschen Mannigfaltigkeiten direkt als Twists elementarer Deformationen von quaternionisch-Kählerschen Mannigfaltigkeiten. Dabei beweisen wir ein Analogon des Satzes von Macia und Swann [MS14] für quaternionisch-Kählersche Mannigfaltigkeiten statt Hyperkählermannigfaltigkeiten.

Um diese Konstruktionen effizient ausführen zu können, entwickeln wir eine alternative lokale Formulierung der Twistkonstruktion, die unter schwächeren Voraussetzungen ausführbar ist als Swanns Konstruktion. Schließlich wird die Beschreibung von 1-Schleifen-Deformationen durch eine lokale Twistabbildung benutzt, um eine geometrische Flussgleichung auf dem Raum der quaternionisch-Kählerschen Strukturen auf einem offenen Ball zu konstruieren.

Acknowledgements

First and foremost, I am deeply indebted to my supervisor Vicente Cortés for introducing me to the rich world of quaternionic Kähler and hyperkähler geometry, and for his compassionate mentorship and insights over the last four years. These have greatly shaped my own mathematical sensibilities and intuitions, and vastly improved my ability to articulate them.

In my attempts to understand how quaternionic Kähler and hyperkähler geometry fit into a wider mathematical and physical context, I have also immensely benefited from conversations with Bernd Siebert and Michel van Garrel about mirror symmetry; with Murad Alim about variation of Hodge structures; with Florian Beck, Aswin Balasubramanian, Markus Röser, and Sebastian Heller about the moduli spaces of Higgs bundles; and with Thomas Mohaupt about supergravity and string theory. The 2018 summer schools *Quantum fields, geometry and representation theory* in Bangalore, and *Higgs bundles in mathematics and physics* in Hamburg have been particularly helpful in this regard.

I would like to express in addition my gratefulness to Diego Conti for his careful refereeing and detailed suggestions; to Hülya Argüz for more general dissertation-related advice; to Áron Szabó for providing invaluable feedback regarding preliminary versions of parts of this dissertation and help in translating the abstract; to Vincentas Mulevičius for his timely help with the figures; and to Danu Thung, who with his near-encyclopædic knowledge of classification results in Riemannian geometry and differential topology, his meticulous note-taking, and his indulgence of my frequent wild-goose chases, is as close to the platonic ideal of a collaborator as one can hope for. It has truly been a joy to work with him.

This dissertation was of course made possible due to the generosity of many, many people throughout the course of my life. Any attempt at a comprehensive list is doomed from the beginning, so I would like to single out my parents and teachers for encouraging and nurturing my interest in science; Suryateja Gavva and Karthik C. S. for playing an instrumental role in my decision to pursue a PhD in mathematics; Nitul Deshpande for being my online hype man; Amir Ahadzadeh of Arrabbiata Orient Bistro for his constant encouragement and his delicious falafels; and Rickmoy Samanta, Ronak Soni, and Prateek Chandra Jha for being shining beacons of light during times of darkness.

I consider myself especially, especially fortunate to be part of such a closely-knit familial community of friends and colleagues in the University of Hamburg and DESY. I thank in particular our secretary Gerda Mierswa Silva for her uncanny ability to anticipate what I need long before I myself realise; Astrid Benz for her German lessons and her help in finding an apartment a stone's throw from Geomatikum; my officemate Áron Szabó for being my first resort for untangling mathematical and TeXnical knots, in addition to being an endless source of trivia about German etymology and 19th century European history; Martin Vogrin and Troy Figiel for being my first two friends in Hamburg who would regularly invite me to things; Peter-Simon Dieterich for generously donating an espresso machine for use by the 15th floor; Danu Thung and Lóránt Szegedy for reminding us to pay attention to the material conditions underlying work; Ilaria Flandoli and Manuel Araújo for reminding us to pay attention to the material conditions underlying life outside of work; Vincentas Mulevičius for reminding us that an important part of paying attention to said material conditions is remembering to have lunch, and for all the aphorisms and the music; David Krusche for lessons in foresight and stoicism; and finally, Vincent Koppen and Manasa Manjunatha for making Hamburg my home.

Contents

To all my fellow graduate workers across space and time

Chapter 1

Introduction

In this chapter, we summarise the results in this dissertation and situate them in a broader mathematical context.

In Section 1.1, we review the background necessary to make sense of this work. We first give a brief rundown of the history of classification results in quaternionic Kähler geometry and identify the goal of explicitly describing quaternionic Kähler metric deformations as the overarching theme. Following this, we recount work due to Swann, Haydys, Cortés and collaborators, on top of which the results in this dissertation are built.

Section 1.2 meanwhile collects together the main results proved in this dissertation and offers condensed accounts of the arguments behind the proofs.

Finally, the appendix at the end of the chapter reviews some of the physics background relevant to quaternionic Kähler manifolds. This is to better contextualise the recurring examples in this dissertation.

1.1 Background

1.1.1 Overarching theme

The main objects of concern in this dissertation are *quaternionic Kähler manifolds*. The idea behind such manifolds may be traced back to Berger's classification of the possible holonomy groups that may be realised by the Levi-Civita connection on Riemannian manifolds.

Theorem 1.1.1 ([Ber55] Chapitre IV, Théorème 3)**.** *The holonomy of the Levi-Civita connection* ∇^g *of a complete simply connected Riemannian manifold* (M, g) *that is neither a product of two Riemannian manifolds nor a symmetric space belongs to the list in Table 1.1, where* $\mathsf{Sp}(n) \cdot \mathsf{Sp}(1)$ *denotes the* $\mathbb{Z}_2$ *quotient of* $\mathsf{Sp}(n) \times \mathsf{Sp}(1)$ *given by the identification*

$$(\mathrm{id}_{2n}, \mathrm{id}_2) \sim (-\mathrm{id}_{2n}, -\mathrm{id}_2). \tag{1.1}$$

$\dim(M)$	Possible holonomies
n	$\mathsf{SO}(n)$
$2n$	$\mathsf{SU}(n)$, $\mathsf{U}(n)$
$4n$	$\mathsf{Sp}(n)$, $\mathsf{Sp}(n) \cdot \mathsf{Sp}(1)$
7	G_2
8	$\mathsf{Spin}(7)$

TABLE 1.1: Berger's list

G	H	$\dim(G/H)$
$\mathsf{SU}(n+2)$	$\mathsf{S}(\mathsf{U}(n)\times\mathsf{U}(2))$	$4n$
$\mathsf{SO}(n+4)$	$\mathsf{SO}(n)\cdot\mathsf{SO}(4)$	$4n$
$\mathsf{Sp}(n+1)$	$\mathsf{Sp}(n)\cdot\mathsf{Sp}(1)$	$4n$
E_6	$\mathsf{SU}(6)\cdot\mathsf{SU}(2)$	40
E_7	$\mathsf{Spin}(12)\cdot\mathsf{Sp}(1)$	64
E_8	$\mathsf{E}_7\cdot\mathsf{Sp}(1)$	112
F_4	$\mathsf{Sp}(3)\cdot\mathsf{Sp}(1)$	28
G_2	$\mathsf{SO}(4)$	8

TABLE 1.2: List of Wolf spaces of compact type G/H

Note that Berger originally included 16-dimensional manifolds with holonomy $\mathsf{Spin}(9)$ in his list, but this case was later shown to be locally symmetric by Alekseevsky [Ale68] and independently, Brown and Gray [BG72].

The requirement that the manifold be simply connected may be dropped if we instead look at the restricted holonomy groups, i.e. the group of holonomies along contractible loops. Quaternionic Kähler manifolds may then be defined to be (pseudo-)Riemannian manifolds of dimension $4n > 4$ whose restricted holonomy is contained in $\mathsf{Sp}(n)\cdot\mathsf{Sp}(1)$ but not in $\mathsf{Sp}(n)$. Note that the case $n=1$ is excluded because $\mathsf{Sp}(1)\cdot\mathsf{Sp}(1)$ happens to be isomorphic to $\mathsf{SO}(4)$, the generic case. Later in Definition 2.1.3, we will provide an alternative definition that extends to $n=1$. This turns out to coincide with (anti-)self-dual Einstein metrics of nonzero scalar curvature.

Given their occurrence on Berger's list, the question of producing examples of and classifying quaternionic Kähler manifolds is one of great mathematical interest. A preliminary observation in this regard is that excluding manifolds with restricted holonomy contained in $\mathsf{Sp}(n)$ amounts to excluding Ricci-flat manifolds from our definition. It was shown by Berger in [Ber66] that quaternionic Kähler manifolds are necessarily Einstein, so a crude way to categorise them up to isometry and overall scaling would be by sign of the scalar curvature. We shall see that positively curved quaternionic Kähler manifolds are very different from negatively curved ones.

The dichotomy between positively and negatively curved quaternionic Kähler manifolds is reflected in the examples of quaternionic Kähler symmetric spaces that Wolf had earlier constructed, building off the classification of simply connected complex homogeneous contact manifolds due to Boothby.

Theorem 1.1.2 ([Boo62] Theorem 1)**.** *There is a one-to-one correspondence between simply connected complex homogeneous contact manifolds and compact simple Lie groups.*

Theorem 1.1.3 ([Wol65] Theorems 6.1, 6.7)**.** *There is a one-to-one correspondence between compact simply connected quaternionic Kähler symmetric spaces and simply connected complex homogeneous contact manifolds, and a one-to-one correspondence between noncompact quaternionic Kähler symmetric spaces and the noncompact duals of simply connected complex homogeneous contact manifolds.*

The two kinds of quaternionic Kähler symmetric spaces, referred to as *Wolf spaces of compact and noncompact types*, have positive and negative scalar curvatures respectively. The two results taken together imply that for every compact simple Lie group G, there is one Wolf space of compact type, namely G/H, and one of noncompact type, namely G^*/H, where H is an isotropy subgroup and G^* is the noncompact dual of G. The possible pairs (G,H) are listed in Table 1.2.

The correspondence with complex homogeneous contact manifolds is given by realising the complex contact manifolds as $\mathbb{C}\mathbf{P}^1$-bundles over the quaternionic Kähler manifolds. This was generalised by Salamon in [Sal82] to arbitrary quaternionic Kähler manifolds as the *twistor correspondence*, closely related to Hitchin's generalisation [Hit81] of the original twistor construction on $\mathbb{R}^4$ due to Penrose [Pen67].

Salamon's twistor construction has been instrumental in establishing classification results extending those of Wolf. As mentioned above, positively and negatively curved quaternionic Kähler manifolds behave rather differently. No examples of complete positively curved quaternionic Kähler manifolds that are *not* Wolf spaces are known at the time of writing this. In fact, what we do know is the following.

Theorem 1.1.4 ([Hit81], [FK82] Main Theorem)**.** *Any complete connected positively curved quaternionic Kähler manifold of dimension* 4*, in the sense of Definition 2.1.3, is isometric to either the sphere* S^4 *or the complex projective plane* $\mathbb{C}\mathbf{P}^2$ *equipped with the canonical metric. That is, it is necessarily a Wolf space.*

Theorem 1.1.5 ([PS91] Theorem 1.1)**.** *Any complete connected positively curved quaternionic Kähler manifold of dimension* 8 *is necessarily a Wolf space.*

Theorem 1.1.6 ([LS94] Theorem 0.1)**.** *For any positive integer* n*, there are up to isometries and rescalings only finitely many positively curved quaternionic Kähler manifolds of dimension* n*.*

Based on the evidence above, it has been conjectured by LeBrun and Salamon that a complete connected positively curved quaternionic Kähler manifold of *any* dimension is necessarily a Wolf space. This is considered to be one of the major open conjectures in the field of quaternionic Kähler geometry today.

Negatively curved quaternionic Kähler manifolds, by contrast, are a lot less rigid. For instance, using Lie theoretic techniques, Alekseevsky in [Ale75] classified quaternionic Kähler manifolds of negative curvature with transitive, solvable isometries that aren't Wolf spaces. It was pointed out by the physicists Van Proeyen and de Wit in [WVP92] that Alekseevsky's classification was in fact incomplete. This was fixed by Cortés in [Cor96]. We omit the completed list of Alekseevsky spaces here since describing it involves setting up a large number of prerequisite definitions.

More strikingly, applying the deformation theory of complex manifolds to the twistor space, LeBrun has shown that there is an abundance of negatively curved quaternionic Kähler manifolds that are not even homogeneous!

Theorem 1.1.7 ([LeB91] Main Theorem)**.** *The tangent space of the moduli space of unobstructed deformations of the quaternionic hyperbolic space* $\mathbb{H}\mathbf{H}^n$ *is isomorphic to* $H^1(\mathscr{Z}, \mathscr{O}(2))$*, where* $\mathscr{Z}$ *is the twistor space of* $\mathbb{H}\mathbf{H}^n$*. In particular, the moduli space of complete quaternionic Kähler manifolds on* $\mathbb{R}^{4n}$ *is infinite-dimensional.*

Unfortunately, extracting the quaternionic Kähler metric from the twistor space is rather nontrivial. The central theme around which this dissertation is organised is describing at least some of these deformations as explicitly as possible. In the past few decades, insights from physics, in particular, supergravity and superstring theory, have been tremendously helpful in constructing explicit examples of such deformations. This physical context is reviewed in the appendix to this chapter.

1.1.2 Immediate context

In parallel with developments in physics, there has been a lot of progress by mathematicians in translating many of the physical constructions underlying supergravity and superstring theory into independent geometric constructions. In particular, the notion of 1-loop deformations, originally applicable only to the hypermultiplet moduli spaces of the Type II superstring, has been since generalised to arbitrary quaternionic Kähler manifolds equipped with a Killing field. We provide a short account of this progress for establishing the more immediate context for this dissertation.

In his dissertation [Swa90], Swann showed that given a quaternionic Kähler manifold M of dimension $4n$, we may construct a $\mathbb{C}^\times$-bundle $\overline{M}$ over its twistor space $\mathscr{Z}$ which carries a metric with holonomy contained in $\mathsf{Sp}(n+1)$. With the additional assumption that the quaternionic Kähler manifold is Riemannian, this bundle, referred to as the Swann bundle, is positive definite if the curvature of the the quaternionic Kähler manifold is positive and of signature $(4, 4n)$ (minus signs first) if the curvature is negative. (Pseudo-)Riemannian manifolds of dimension $4(n+1) \geqslant 8$ with holonomy contained in $\mathsf{Sp}(n+1)$ are said to be *hyperkähler*. These may be characterised by the existence of three different Kähler structures $\bar{I}_1, \bar{I}_2, \bar{I}_3$ satisfying the quaternionic relation

$$\bar{I}_1 \circ \bar{I}_2 = \bar{I}_3. \tag{1.2}$$

The hyperkähler metric $\bar{g}$ on the Swann bundle of a quaternionic Kähler manifold is special in that it carries an action of $\mathbb{H}^\times$ i.e. it forms a cone. Given a hyperkähler cone, one can always quotient out this $\mathbb{H}^\times$-action to obtain a quaternionic Kähler whose Swann bundle is the given hyperkähler cone. This operation is called the superconformal quotient and involves choosing a level set of the norm of the generator of the $\mathbb{R}^\times$ scaling action and then taking the ordinary (pseudo-)Riemannian quotient by the action of the group of unit quaternions, which is just $\mathsf{Sp}(1)$, on this level set.

Theorem 1.1.8 ([Swa91] Corollary 3.6)**.** *There is a one-to-one correspondence between positively (respectively, negatively) curved Riemannian quaternionic Kähler manifolds of dimension* $4n$ *and Riemannian (respectively, pseudo-Riemannian) hyperkähler cones of dimension* $4(n+1)$*.*

Building on the work of Swann, Haydys in *his* dissertation [Hay06] made use of the fact that Killing fields Z on a positively curved quaternionic Kähler manifold lift to *trihamiltonian* Killing fields on its (Riemannian) Swann bundle, i.e. Killing fields $\overline{Z}$ that are Hamiltonian with respect to all three of the Kähler forms $\bar{\omega}_i := \bar{g} \circ \bar{I}_i$ of the hyperkähler metric $\bar{g}$. Thus, we can take a *hyperkähler quotient* by first restricting to the intersection P of level sets of the three moment maps $\bar{\mu}_i^{\overline{Z}}$ defined by

$$\mathrm{d}\bar{\mu}_i^{\overline{Z}} = -\iota_{\overline{Z}}\bar{\omega}_i, \tag{1.3}$$

and then taking the Riemannian quotient of P by the $\overline{Z}$-action on it. This yields a hyperkähler manifold of same dimension as the original quaternionic Kähler manifold. Although Kähler moment maps are not unique, there is a unique choice which is compatible with conical structure. When the level set chosen is one on which $\bar{\mu}_i^{\overline{Z}}$ don't all vanish, this new hyperkähler manifold inherits from the $\mathbb{H}^\times$-action on the hyperkähler cone a Killing field $\tilde{Z}$ which isn't trihamiltonian but *rotating*, i.e. its action preserves one of the complex structures I_1, but rotates the complex structures I_2, I_3 orthogonal to it into one another.

This result, applicable to positively curved quaternionic Kähler manifolds, was generalised by Alekseevsky, Cortés, and Mohaupt in [ACM13] to include the physically more relevant negatively curved quaternionic Kähler manifolds. In the more general case, the resulting hyperkähler manifold of the same dimension may be pseudo-Riemannian.

In addition to the above, Haydys, and separately Alekseevsky, Cortés, and Mohaupt, showed that the above construction, referred to as the *QK/HK correspondence*, can be inverted. This involves a certain conification procedure that is more generally applicable to Kähler manifolds (of which hyperkähler manifolds are a special case). There is however an ambiguity in the inverse construction, i.e. the *HK/QK correspondence*, which originates from having to make a choice of a certain Hamiltonian function. The resulting construction gives back not only the original quaternionic Kähler manifold but a whole 1-parameter family of such manifolds.

Theorem 1.1.9 ([Hay08] Theorems 2.3, 2.7, [ACM13] Theorem 2)**.** *There is a correspondence between* 1*-parameter families of quaternionic Kähler manifolds with a* $\mathsf{U}(1)$*-action and hyperkähler manifolds of the same dimension with a rotating* $\mathsf{U}(1)$*-action given by hyperkähler reduction of the Swann bundle by the triholomorphic lift of the* $\mathsf{U}(1)$*-action for a choice of nonzero level set of the homogeneous hyperkähler moment map.*

Meanwhile, Swann introduced the *twist construction* in order to unify and generalise several differential geometric constructions arising from T-duality in physics. The construction takes as input the following *twist data* on a manifold M: a vector field Z, an integral closed 2-form ω with respect to which Z is Hamiltonian, and a choice of Hamiltonian function f that is nowhere vanishing.

Theorem 1.1.10 ([Swa10] Propositions 2.1, 2.3)**.** *Given twist data* (Z, ω, f) *such that* Z *generates a* $\mathsf{U}(1)$*-action on* M*, there exists a* $\mathsf{U}(1)$*-principal bundle* $P \to M$ *with connection* $\hat{\eta}$ *having curvature* ω *and fundamental vector field* X_P *such that the lift*

$$\hat{Z} + fX_P \tag{1.4}$$

of Z*, where* $\hat{Z}$ *is the* $\hat{\eta}$*-horizontal lift of* Z *to* P*, generates a* $\mathsf{U}(1)$*-action on* P *and so defines a well-defined quotient manifold*

$$\tilde{M} := P/\langle \hat{Z} + fX_P \rangle. \tag{1.5}$$

Furthermore, the $\hat{\eta}$*-horizontal lift of any* Z*-invariant vector field on* M *and the pullback of any* Z*-invariant function on* M *to* P *descend to a well-defined vector field and function on* $\tilde{M}$ *respectively.*

The well-defined vector field and function on $\tilde{M}$ that the $\hat{\eta}$-horizontal lift of any Z-invariant vector field on M and the pullback of any Z-invariant function on M to P descend to are referred to as the *twists* of the vector field and function on M with respect to the twist data (Z, ω, f). By stipulating compatibility with contractions, this notion of twists can be extended to arbitrary tensor fields, in particular symmetric bilinear forms. This twist construction is moreover an involution; $\tilde{M}$ carries *dual twist data* $(\tilde{Z}, \tilde{\omega}, \tilde{f})$, twists with respect to which are the inverses of twists with respect to (Z, ω, f).

As it turns out, one can always choose the level sets of the moment maps $\overline{\mu}_i^{Z}$ on the hyperkähler cone $\overline{M}$ so that their intersection P, along with the connection $\hat{\eta}$ induced by the Levi-Civita connection ∇^g, forms precisely such a $\mathsf{U}(1)$-principal bundle over the reduced hyperkähler manifold $\tilde{M}$ with respect to some twist data $(\tilde{Z}, \tilde{\omega}_{\mathrm{H}}, \tilde{f}_{\mathrm{H}})$ on it. By identifying appropriate twist data on hyperkähler manifolds with rotating $\mathsf{U}(1)$-action, Swann and Macia were able to give an account of the HK/QK correspondence due to [ACM13] that circumvented the conification procedure. For this, they defined a generalisation of the notion of conformal scaling of hyperkähler metrics $\tilde{g}$ called *elementary deformation by a Killing field* $\tilde{Z}$, whereby the restriction of $\tilde{g}$ to the span of $\tilde{Z}, I_1\tilde{Z}, I_2\tilde{Z}, I_3\tilde{Z}$ and the restriction of $\tilde{g}$ to the subspace orthogonal to this span are rescaled by different conformal factors.

Theorem 1.1.11 ([MS14] Theorem 1). *Given a hyperkähler manifold $(\tilde{M}, \tilde{g})$ equipped with a triple of Kähler forms $(\varpi_1, \varpi_2, \varpi_3)$, a Killing field $\tilde{Z}$, and a nowhere vanishing function $\tilde{f}_1$ satisfying*

$$\iota_{\tilde{Z}}\varpi_1 = -\mathrm{d}\tilde{f}_1, \quad \mathscr{L}_{\tilde{Z}}\varpi_2 = \varpi_3, \quad \mathscr{L}_{\tilde{Z}}\varpi_3 = -\varpi_2, \tag{1.6}$$

the twist of the "standard hyperkähler elementary deformation"

$$\tilde{g}_{\mathrm{H}} := \frac{\tilde{K}}{\tilde{f}_1}\,\tilde{g} + \frac{\tilde{K}}{\tilde{f}_1^2}\left((\iota_{\tilde{Z}}\tilde{g})^2 + \sum_{i=1}^{3}(\iota_{\tilde{Z}}\varpi_i)^2\right) \tag{1.7}$$

with respect to twist data

$$(\tilde{Z}, \tilde{\omega}_{\mathrm{H}}, \tilde{f}_{\mathrm{H}}) := (\tilde{Z}, \tilde{k}(\varpi_1 + \mathrm{d} \circ \iota_{\tilde{Z}}\tilde{g}), \tilde{k}(\tilde{f}_1 + \tilde{g}(\tilde{Z}, \tilde{Z})), \tag{1.8}$$

where $\tilde{k}$ and $\tilde{K}$ are nonzero constants, is quaternionic Kähler. Moreover, these are the only combinations of elementary deformations by Killing fields $\tilde{Z}$ and twists with respect to twist data of the form $(\tilde{Z}, \tilde{\omega}, \tilde{f})$ that yield quaternionic Kähler metrics.

Note that we have the freedom of adding a constant c to the Hamiltonian function $\tilde{f}_1$; this yields the expected 1-parameter family of quaternionic Kähler metrics.

There is a natural hyperkähler metric on the cotangent bundle of an affine special Kähler manifold. This additionally carries a natural rotating $\mathsf{U}(1)$-action when it is a conical affine special Kähler manifold forming a $\mathbb{C}^\times$-bundle over a projective special Kähler manifold. In other words, we have precisely the data that we need for the HK/QK correspondence! Explicit computations carried out by Alekseevsky, Cortés, Dyckmanns, and Mohaupt in [Ale+15] showed that the 1-parameter family of quaternionic Kähler metrics this produces consists precisely of the Ferrara–Sabharwal metrics and their 1-loop deformations, described in equations (1.71) and (1.75) in Section 1.B of the appendix.

Theorem 1.1.12 ([Ale+15] Corollary 1). *The Ferrara–Sabharwal metrics and their* 1*-loop deformations are quaternionic Kähler.*

Of course, this was already expected on physical grounds (and in the case of no deformation, explicitly proved by Ferrara and Sabharwal in [FS90]), but the computation in [Ale+15] established a mathematical proof for this claim. In addition, it motivated a general definition of 1-loop deformations applicable to any quaternionic Kähler manifold, not just the ones arising as hypermultiplet moduli spaces of the Type IIA superstring: the 1-loop deformation of a given quaternionic Kähler metric is roughly defined to be the 1-parameter family of quaternionic Kähler metrics which produces the same hyperkähler metric under the QK/HK correspondence.

1.2 Main results

We now outline the new contributions in this dissertation. Since chapter summaries have been included at the beginning of every chapter, we won't provide a breakdown of results by chapter here. Instead, we provide a holistic summary. In particular, we bring together results that are proved in different chapters but are more naturally stated as a single theorem. The rough statements here have of course been cross-referenced to wherever they have been stated in detail and proved in this dissertation.

A key definition introduced in this work is a local version of Swann's twist construction in [Swa10]. For this, we first need to augment the twist data (Z, ω, f) that Swann makes use of with some auxiliary data: an open set U such that the restriction $\omega|_U$ is exact and the choice of a 1-form η on U such that $f - \eta(Z)$ is nowhere vanishing and $\omega|_U = \mathrm{d}\eta$. Given this extra data, we define a local twist map $\mathrm{tw}_{Z,f,\eta}$ in Definition 3.1.7 to be a $C^\infty(U)$-linear map of tensor fields on U which in the special case of functions h and 1-forms α is given by

$$\mathrm{tw}_{Z,f,\eta}(h) = h, \quad \mathrm{tw}_{Z,f,\eta}(\alpha) = \alpha - \frac{\alpha(Z)}{f}\eta. \tag{1.9}$$

Stipulating compatibility with tensor products and contractions then fixes the map for all tensor fields.

This ostensibly differs from Swann's construction in two significant ways. First of all, the local twist map may be applied to arbitrary tensor fields on U, and not just Z-invariant ones. Secondly, the local twist map depends nontrivially on the auxiliary 1-form η. These two facts are in fact related. An application of Moser's trick gives Proposition 3.3.1 which may be roughly stated as follows.

Proposition 1.2.1. *When restricted to Z-invariant tensor fields in an open set around a given point p, the local twists with respect to two different choices of auxiliary 1-forms η_0 and η_1 such that $f - \eta_0(Z)$ and $f - \eta_1(Z)$ have the same sign, are related by a diffeomorphism of local neighbourhoods of p.*

Thus, in general we can expect a global twist map to be well-defined only for Z-invariant tensor fields. In order to obtain such a map, one would need to glue together the local twist maps on open sets U_Λ equipped with auxiliary 1-forms η_Λ using these diffeomorphisms. Theorems 3.3.9 and 3.3.10 give a necessary and sufficient existence criterion for when this can be consistently done, which in slightly less generality may be stated as follows.

Theorem 1.2.2. *If the vector field Z that is part of the twist data (Z, ω, f) on a manifold M induces a $\mathrm{U}(1)$-action on M, then local twist maps on M may be consistently glued together if and only if there exists a $\mathrm{U}(1)$-principal bundle $P \to M$ with curvature ω to which the action of Z lifts properly.*

This is essentially Swann's construction of the twist. Globally, our local twist construction is in fact equivalent to that of Swann. But even though we don't get anything new, there are a few technical advantages our approach enjoys:

(a) We get to work directly with open sets on M without having to first lift tensor fields to P,

(b) Technical difficulties associated with ensuring properness of group actions on P can be entirely avoided,

(c) We can work with tensor fields which are not Z-invariant to verify local properties that the twists of certain tensor fields need to satisfy.

This allows us to obtain more direct proofs of many known results in addition to some new results such as Proposition 3.2.4 and Corollary 3.2.6. These describe how a Lie algebra of ω-Hamiltonian vector fields, such as the algebra of ω-Hamiltonian Killing fields, interacts with the twist.

Proposition 1.2.3. *Let (Z, ω, f) be twist data with dual twist data*

$$(\tilde{Z}, \tilde{\omega}, \tilde{f}) := \left(-\frac{1}{f}\operatorname{tw}_{Z,f,\eta}(Z), \frac{1}{f}\operatorname{tw}_{Z,f,\eta}(\omega), \frac{1}{f}\right). \tag{1.10}$$

If S is a Z-invariant tensor field annihilated by a Lie algebra generated by ω-Hamiltonian vector fields v_a with Z-invariant Hamiltonian functions f_{v_a} and structure constants C^c_{ab}, then the twist of S is annihilated by the Lie algebra generated by $\tilde{v}_0 := \tilde{Z}$ and the twists $\tilde{v}_a$ of

$$v_a - \frac{f_{v_a} + 1}{f} Z. \tag{1.11}$$

Furthermore, $\tilde{v}_a$ are Hamiltonian with respect to $\tilde{\omega}$ with $\tilde{Z}$-invariant Hamiltonian functions

$$\tilde{f}_{\tilde{v}_a} := \frac{f_{v_a} + 1}{f} - 1, \tag{1.12}$$

and have structure constants

$$\tilde{C}^c_{ab} = \begin{cases} C^c_{ab} & \text{when} \quad c \neq 0, \\ \omega(v_a, v_b) - C^0_{ab} f - \sum_{d \neq 0} C^d_{ab}(f_{v_d} + 1) & \text{when} \quad c = 0. \end{cases} \tag{1.13}$$

The main upshot of the local twist map is that it makes formulating the QK/HK correspondence in terms of a twist much simpler. Recall that on the hyperkähler side, we have a rotating Killing field which preserves only one Kähler structure I_1 but rotates the other two. This complicates a global approach, but in our local formulation, it becomes possible to directly construct I_2 and I_3 locally on the quaternionic Kähler manifold.

In order to perform a (local) twist of a quaternionic Kähler manifold, we need to first identify (local) twist data on it. This is carried out in Lemmata 2.2.7 and 2.2.11. We work with the characterisation of quaternionic Kähler manifolds as special cases of almost quaternionic Hermitian (AQH) manifolds, i.e. (pseudo-)Riemannian manifolds (M, g) with a distinguished rank 3 subbundle Q of Hermitian structures which, together with the identity endomorphism field form a faithful representation of quaternion algebra $\mathbb{H}$. We also make use of the quaternionic moment map μ^Z, defined for any Killing field Z of (M, g) and given by the explicit expression

$$\mu^Z = -\frac{2}{\nu}\operatorname{pr}_Q(\nabla^g Z) =: \|\mu^Z\| J^Z, \tag{1.14}$$

where ν is the constant reduced scalar curvature

$$\nu = \frac{\operatorname{scal}_g}{4n(n+2)}, \tag{1.15}$$

depending on the dimension $4n$ of (M, g), and $\nabla^g Z$ is interpeted as a (skew-self-adjoint) endomorphism field. In other words, μ^Z is the $\mathsf{Sp}(1)$ part of the endomorphism field $\nabla^g Z$.

Proposition 1.2.4. *Any quaternionic Kähler manifold (M, g, Q) of reduced scalar curvature ν with a nowhere vanishing Killing field Z admits on an open everywhere dense submanifold twist data $(Z, \omega_Q, f_Q + b)$ where b is some constant and ω_Q and f_Q are given* *by*

$$\begin{aligned}\omega_Q(u,v) &= -\mathrm{d}\left(\frac{\iota_Z g}{\|\mu^Z\|}\right)(u,v) - \nu g(J^Z u, v) + \langle \nabla^g_u J^Z, J^Z \circ \nabla^g_v J^Z \rangle,\\ f_Q &= -\frac{g(Z,Z)}{\|\mu^Z\|} - \nu\|\mu^Z\|.\end{aligned} \tag{1.16}$$

Furthermore, on any contractible open set $U \subseteq M$, we can find a local oriented orthonormal frame $(J_1 = J^Z, J_2, J_3)$ of Q with auxiliary local twist data (U, η_Q) given by

$$\eta_Q(u) = -\frac{g(Z,u)}{\|\mu^Z\|} - \langle J_2, \nabla^g_u J_3 \rangle. \tag{1.17}$$

With this twist data, we can then prove an analogue of Theorem 1.1.11 due to Macia and Swann for *quaternionic Kähler* manifolds in place of *hyperkähler* manifolds, with an appropriate generalisation of the notion of elementary deformations to AQH manifolds.

Theorem 1.2.5. *Given a quaternionic Kähler manifold (M, g, Q) of reduced scalar curvature ν with a nowhere vanishing Killing field Z, the twist of its elementary deformation*

$$\frac{1}{\nu\|\mu^Z\| - b}\, g|_{\mathbb{H}_Q Z^\perp} - \frac{f_Q + b}{(\nu\|\mu^Z\| - b)^2}\, g|_{\mathbb{H}_Q Z} \tag{1.18}$$

with respect to twist data $(Z, \omega_Q, f_Q + b)$ is locally hyperkähler with a rotating Killing field when $b = 0$ and quaternionic Kähler otherwise. In fact, these are up to an overall scaling the QK/HK dual and 1-loop deformation with deformation parameter $c = \nu/4b$ respectively. Moreover, the $b \neq 0$ case constitutes up to an overall scaling the only combinations of elementary deformations by Killing fields Z and twists with respect to twist data of the form (Z, ω, f) that produce other quaternionic Kähler metrics.

The proof of this statement is split into the proofs of Theorems 4.1.11, 5.2.1, and 5.2.4, and Propositions 4.2.7 and 4.2.10. The key idea is to reduce it to Theorem 1.1.11 using Lemma 5.1.1 describing the composition of local twists that are not dual to each other.

Lemma 1.2.6. *Let (U, Z, ω, f, η) be local twist data with dual local twist data $(U, \tilde{Z}, \tilde{\omega}, \tilde{f}, \tilde{\eta})$. Let $(U, \tilde{Z}, \tilde{\omega}', \tilde{f}', \tilde{\eta}')$ be local twist data as well. Then the composition of local twist maps*

$$\mathrm{tw}_{\tilde{Z}, \tilde{f}', \tilde{\eta}'} \circ \mathrm{tw}_{Z,f,\eta} \tag{1.19}$$

is itself a local twist map with respect to some choice of local twist data.

The local formulation of the twist gives us the following geometric flow on the space of quaternionic Kähler structures on a contractible open set, whose solution is shown to be the 1-loop deformation in Proposition 5.3.2. (Contractibility is not really necessary but we assume it in order to keep the discussion straightforward.)

Proposition 1.2.7. *The "naïve 1-loop flow" defined by the system of differential equations*

$$\begin{aligned}
\frac{\mathrm{d}g^c}{\mathrm{d}c} &= -\frac{8}{\nu}\,\eta^c_{\mathrm{Q}}\,\iota_{Z^c}g^c + 4\|\mu^{Z^c}\|\,g^c - \frac{4}{\nu}\frac{g^c(Z^c,Z^c)}{\|\mu^{Z^c}\|}\,g^c|_{\mathbb{H}_{Q^c}Z^c},\\
\frac{\mathrm{d}Z^c}{\mathrm{d}c} &= -\frac{4}{\nu}\,(f^c_{\mathrm{Q}} - \eta^c_{\mathrm{Q}}(Z^c))Z^c,\\
\frac{\mathrm{d}Q^c}{\mathrm{d}c} &= -\frac{4}{\nu}\,[Q^c, \eta^c_{\mathrm{Q}} \otimes Z^c],\\
\frac{\mathrm{d}\eta^c_{\mathrm{Q}}}{\mathrm{d}c} &= -\frac{4}{\nu}\,f^c_{\mathrm{Q}}\,\eta^c_{\mathrm{Q}}.
\end{aligned} \tag{1.20}$$

defines a flow on the space of analytic quaternionic Kähler metrics (U, g^c, Q^c) *of reduced scalar curvature* ν *equipped with a nowhere vanishing Killing field* Z^c *and a choice of 1-form* η_{Q} *as in* (1.17)*, that is solved by the 1-loop deformation of a quaternionic Kähler metric.*

Analyticity is needed in the statement as the Cauchy–Kovaleskaya theorem only guarantees the uniqueness of the 1-loop flow solution for partial differential equations with analytic coefficients. This geometric flow is called naïve in order to distinguish it from a reparamerised version that preserves the norm of the quaternionic moment map (Proposition 5.3.5) and a rescaled version that interpolates between a quaternionic Kähler manifold and its QK/HK dual (Proposition 5.3.7).

The realisations of the QK/HK and HK/QK correspondences in terms of the twist construction is a powerful tool that enables us to use information on the simpler hyperkähler side to say something about the more complicated quaternionic Kähler side. For instance, Proposition 1.2.3 above allows us to construct Killing fields of a quaternionic Kähler manifold using Killings fields of its QK/HK dual. Carrying out similar computations for the Levi-Civita connections (Propositions 3.2.8 and 4.2.8), we can relate the Riemann curvature of any quaternionic Kähler metric with a Killing field to the Riemann curvature of its QK/HK dual along with an abstract curvature tensor field constructed out of the data on the hyperkähler side that we noted in Theorem 1.1.11 above! This is accomplished in Theorem 4.2.17.

Theorem 1.2.8. *Let* (M, g, Q) *be a quaternionic Kähler manifold that is the HK/QK dual of a locally hyperkähler metric* $\tilde{g}$ *with associated data as in Theorem 1.1.11. Then its Riemann curvature* $g \circ R^g$ *is the twist of*

$$\begin{aligned}
&\frac{\tilde{K}}{\tilde{f}_1}\,\tilde{g}\circ R^{\tilde{g}} + \frac{1}{8\tilde{K}}\left(\tilde{g}_{\mathrm{H}} \owedge \tilde{g}_{\mathrm{H}} + \sum_{i=1}^{3}(\tilde{g}_{\mathrm{H}}\circ I_i) \varobar (\tilde{g}_{\mathrm{H}}\circ I_i)\right)\\
&\quad - \frac{\tilde{K}}{8\tilde{k}}\frac{1}{\tilde{f}_1\tilde{f}_{\mathrm{H}}}\left(\tilde{\omega}_{\mathrm{H}} \varobar \tilde{\omega}_{\mathrm{H}} + \sum_{i=1}^{3}(\tilde{\omega}_{\mathrm{H}}\circ I_i) \owedge (\tilde{\omega}_{\mathrm{H}}\circ I_i)\right),
\end{aligned} \tag{1.21}$$

with respect to the twist data $(\tilde{Z}, \tilde{\omega}_{\mathrm{H}}, \tilde{f}_{\mathrm{H}})$.

Here $\owedge$ and $\varobar$ denote the Kulkarni–Nomizu and Riemann products on the symmetric bilinear forms and 2-forms respectively. These are defined in Definitions 4.2.12 and 4.2.13, and may be thought of as projections of the tensor products of symmetric bilinear forms and 2-forms onto the space of abstract curvature tensor fields. In particular, as pointed out in Remark 4.2.19, this may be regarded as a refinement of Alekseevsky's decomposition of the Riemann curvature of quaternionic Kähler metrics, quoted in Theorem 2.1.12, with the hyperkähler-type quaternionic Weyl curvature being further decomposed into a piece arising from the curvature of the locally hyperkähler QK/HK dual and a piece arising from the twist data.

In the case of 1-loop-deformed Ferrara–Sabharwal metrics $g^{\prime c}_{\mathrm{FS}}$ with prepotential F a quadratic polynomial, the relevant hyperkähler metric $\tilde{g}$ is flat and things become particularly simple. We can always locally choose coordinates X^a so that the prepotential F is given by

$$F(z) := F(z_0, \dots, z_{n-1}) = \sum_{a=0}^{n-1} z_a^2, \tag{1.22}$$

and the metric $g^{\prime c}_{\mathrm{FS}}$ becomes

$$\begin{aligned} g^{\prime c}_{\mathrm{FS}} = 2\tilde{K} \Bigg(& \frac{1}{4\rho^2} \frac{\rho + 2c}{\rho + c} \mathrm{d}\rho^2 + \frac{\rho + c}{\rho} \left(\sum_{a=1}^{n-1} \frac{|\mathrm{d}X_a|^2}{1 - \sum_{b=1}^{n-1} |X_b|^2} + \frac{\left| \sum_{a=1}^{n-1} X_a \mathrm{d}\overline{X}_a \right|^2}{\left(1 - \sum_{b=1}^{n-1} |X_b|^2\right)^2} \right) \\ & + \frac{1}{2\rho} \left(-|\mathrm{d}\zeta_0|^2 + \sum_{a=1}^{n-1} |\mathrm{d}\zeta_a|^2 \right) + \frac{\rho + c}{\rho^2} \frac{\left| \mathrm{d}\zeta_0 + \sum_{a=1}^{n-1} X_a \mathrm{d}\zeta_a \right|^2}{1 - \sum_{b=1}^{n-1} |X_b|^2} \\ & + \frac{1}{4\rho^2} \frac{\rho + c}{\rho + 2c} \left(\frac{\mathrm{d}\tau}{2\tilde{K}} - \sum_{a=1}^{n-1} \frac{2c \,\mathrm{Im}(X_a \mathrm{d}\overline{X}_a)}{1 - \sum_{b=1}^{n-1} |X_b|^2} + \mathrm{Im} \left(\zeta_0 \mathrm{d}\overline{\zeta}_0 - \sum_{a=1}^{n-1} \zeta_a \mathrm{d}\overline{\zeta}_a \right) \right)^2 \Bigg). \end{aligned} \tag{1.23}$$

Then using Proposition 1.2.3, we obtain the following Killing fields for $g^{\prime c}_{\mathrm{FS}}$ in addition to ∂_τ:

$$\begin{aligned} u_a^+ &= \mathrm{Re} \left(- \sum_{b=1}^{n-1} X_a X_b \partial_{X_b} + \partial_{\overline{X}_a} - \zeta_0 \partial_{\zeta_a} - \overline{\zeta}_a \partial_{\overline{\zeta}_0} + 2\mathrm{i}\tilde{K} c X^a \partial_\tau \right), \\ v_0^+ &= \sqrt{2}\, \mathrm{Re}(\partial_{\zeta_0} + \mathrm{i}\tilde{K}\overline{\zeta}_0 \partial_\tau), \quad v_a^+ = \sqrt{2}\, \mathrm{Re}(\partial_{\zeta_a} - \mathrm{i}\tilde{K}\overline{\zeta}_a \partial_\tau), \\ u_a^- &= \mathrm{Im} \left(- \sum_{b=1}^{n-1} X_a X_b \partial_{X_b} + \partial_{\overline{X}_a} - \zeta_0 \partial_{\zeta_a} - \overline{\zeta}_a \partial_{\overline{\zeta}_0} + 2\mathrm{i}\tilde{K} c X^a \partial_\tau \right), \\ v_0^- &= \sqrt{2}\, \mathrm{Im}(\partial_{\zeta_0} + \mathrm{i}\tilde{K}\overline{\zeta}_0 \partial_\tau), \quad v_a^- = \sqrt{2}\, \mathrm{Im}(\partial_{\zeta_a} - \mathrm{i}\tilde{K}\overline{\zeta}_a \partial_\tau). \end{aligned} \tag{1.24}$$

And using Theorem 1.2.8, we may compute the curvature norm of to be $g^{\prime c}_{\mathrm{FS}}$

$$\begin{aligned} & \mathrm{tr}(\mathscr{R}^2) \\ & = \nu^2 \left(n(5n+1) + 3 \left(\frac{\rho^3}{(\rho + 2c)^3} + \frac{(n-1)\rho}{(\rho + 2c)} \right)^2 + 3 \left(\frac{\rho^6}{(\rho + 2c)^6} + \frac{(n-1)\rho^2}{(\rho + 2c)^2} \right) \right). \end{aligned} \tag{1.25}$$

Together, these two results give us Theorem 4.2.21.

Theorem 1.2.9. *The* 1*-loop-deformed quadratic prepotential Ferrara–Sabharwal metrics have cohomogeneity* 1.

For the case $n = 1$, i.e. the case of the 1-loop-deformed universal hypermultiplet metric $g^{\prime c}_{\mathrm{UH}}$, we in fact have the full isometry group explicitly described in Proposition 2.3.6.

Proposition 1.2.10. *The full isometry group of the* 1*-loop-deformed universal hypermultplet metric* $g^{\prime c}_{\mathrm{UH}}$ *is the semidirect product of a Heisenberg group with a rotation group*

$$\mathsf{Heis}_3(\mathbb{R}) \rtimes \mathsf{O}(2) \tag{1.26}$$

consisting of isometries of one of the following two forms:

$$\begin{aligned}(\rho, \zeta, \tau) &\mapsto \left(\rho, e^{\mathrm{i}\theta}(\zeta + \zeta'), \tau + \tau' + \frac{2}{\nu}\operatorname{Im}(\zeta'\overline{\zeta})\right),\\(\rho, \zeta, \tau) &\mapsto \left(\rho, e^{-\mathrm{i}\theta}(\overline{\zeta} + \overline{\zeta}'), -\tau - \tau' - \frac{2}{\nu}\operatorname{Im}(\zeta'\overline{\zeta})\right),\end{aligned} \tag{1.27}$$

where $\tau', \theta \in \mathbb{R}$ *and* $\zeta' \in \mathbb{C}$ *are arbitrary constants.*

The 1-loop-deformed universal hypermultiplet metric was the subject of investigation in our earlier work [CS17], where a computation of the sectional curvature was used to conclude that this metric is different from the family of metrics γ^m on $(0,1) \times \mathbb{S}^3$ constructed by Pedersen in [Ped86] and given in terms of $\varrho \in (0,1)$ and $\mathsf{SU}(2)$-invariant 1-forms $\varsigma_1, \varsigma_2, \varsigma_3$ on $\mathbb{S}^3$ by

$$\gamma^m = \frac{1}{\nu(1-\varrho^2)^2}\left(\frac{1+m^2\varrho^2}{1+m^2\varrho^4}\,\mathrm{d}\varrho^2 + \varrho^2(1+m^2\varrho^2)(\varsigma_1^2+\varsigma_2^2) + \frac{\varrho^2(1+m^2\varrho^4)}{1+m^2\varrho^2}\varsigma_3^2\right). \tag{1.28}$$

This too is a 1-parameter family of quaternionic Kähler metrics of cohomogeneity 1. The relationship between the two families are clarified in Proposition 2.3.7 by their identification as subfamilies of a larger family of quaternionic Kähler metrics constructed in [Ket01], namely

$$\begin{aligned}g^{a,b,c} = -\frac{1}{2\nu\rho^2}\Bigg(&\frac{b\rho+2c}{a\rho^2+b\rho+c}\,\mathrm{d}\rho^2 + \frac{2(b\rho+2c)|\mathrm{d}\zeta|^2}{(1+\frac{a}{2}|\zeta|^2)^2}\\&+\frac{a\rho^2+b\rho+c}{b\rho+2c}\left(-\frac{\nu}{2}\,\mathrm{d}\tau + \frac{b\operatorname{Im}(\zeta\,\mathrm{d}\overline{\zeta})}{1+\frac{a}{2}|\zeta|^2}\right)^2\Bigg).\end{aligned} \tag{1.29}$$

Proposition 1.2.11. *The* 3*-parameter family of quaternionic Kähler metrics* $g^{a,b,c}$ *in* (1.29) *has cohomogeneity generically* 1 *and reduces to the* 1*-loop-deformed universal hypermultiplet metric* $g^{\prime c}_{\mathrm{UH}}$ *when* $a = b = 0$ *and* c *is nonzero, and to a metric isometric to the Pedersen metrics* γ^m *restricted to an open everywhere dense submanifold of* $(0,1) \times \mathbb{S}^3$ *with*

$$m = \sqrt{\frac{4ac}{b^2} - 1} \tag{1.30}$$

when $4ac > b^2$.

Appendix

This appendix provides a digest of various facts about supergravity and string theory drawn from more comprehensive sources such as Polchinski [Pol07a; Pol07b], Cecotti [Cec09], Freedman and Van Proeyen [FVP12], and Alexandrov [Ale13]. In particular, a mathematically precise formulation of Bagger and Witten's result in [BW83] regarding the correspondence between supergravity and quaternionic Kähler geometry is given in Subsection 1.A. For this, we follow Dell and Smolin [DS79] and work in the setting of graded manifolds due to Kostant [Kos77] and Batchelor [Bat79], slightly adapted to account for extended supersymmetry and symplectic Majorana spinors.

1.A Supergravity and quaternionic Kähler geometry

Quaternionic Kähler manifolds naturally arise in physics in the study of supergravity models. Let us recall that gravitational theories on a manifold S have among their dynamical field content a pseudo-Riemmanian metric h_S (typically of Lorentzian signature $(1, d-1)$ for some $d > 1$) and that the dynamical equations of motion are preserved under the action of diffeomorphisms of S. The infinitesimal version of this is that for any vector field u on S, the equations of motions are annihilated by the Lie derivative along u.

A supergravity theory involves enhancing the Lie algebra $\mathscr{X}(S)$ of vector fields to a complex Lie superalgebra parametrised by a positive integer $\mathscr{N}$ and stipulating that the equations of motion be preserved under the action of a real part of the Lie superalgebra. To make this work, we set $d = 4$ and fix the following ingredients:

(a) A complex vector bundle V of complex rank $\mathscr{N}$, equipped with an antilinear map J_V such that $J_V^2 = -\mathrm{id}_V$, a nondegenerate form $\omega_V \in \Gamma(\Lambda^2 V^*)$, and a connection ∇^V preserving these structures,

(b) A spin bundle Σ over S, which comes equipped with a Clifford action $\gamma \in \Gamma(TS \otimes \mathrm{End}(\Sigma))$ subject to the convention

$$\gamma(\alpha, \beta) + \gamma(\beta, \alpha) = -2h_S^{-1}(\alpha, \beta), \tag{1.31}$$

an antilinear map J_Σ such that $J_\Sigma^2 = -\mathrm{id}_\Sigma$, and a nondegenerate form $\epsilon \in \Gamma(\Lambda^2\Sigma^*)$, and a spin connection ∇^Σ compatible with the Levi-Civita connection ∇^{h_S} preserving these structures,

(c) ∇^V-parallel *central charges* $\mathsf{Z} \in \Gamma(\Lambda^2 V)$ and $\overline{\mathsf{Z}} \in \Gamma(\Lambda^2 V^*)$ subject to the reality condition

$$\omega_V \circ J_V(\mathsf{Z}) = \overline{\mathsf{Z}}. \tag{1.32}$$

The Clifford action may be used to define an involution $\gamma(\star_{h_S} 1)$ on Σ and so gives an eigendecomposition of Σ into a *left-handed* part, whose elements are denoted with a subscript L, and a *right-handed* part, whose elements are denoted with a subscript R.

Vector fields on S are derivations on the algebra of smooth functions on S. In analogy with this, we consider the space $\mathrm{Der}_K^\bullet$ of *graded derivations* acting on sections of the $\mathbb{Z}$-graded bundle

$$K^\bullet = \Lambda^\bullet((\Sigma \otimes V) \oplus (\Sigma^* \otimes V^*)). \tag{1.33}$$

The $\mathbb{Z}$-grading on $\mathrm{Der}_K^\bullet$ is given by the canonical action

$$\mathrm{Der}_K^\bullet \times K^\blacklozenge \to K^{\blacklozenge+\bullet}. \tag{1.34}$$

The defining property of graded derivations is that they are $\mathbb{C}$-linear and they satisfy a graded version of the Leibniz rule i.e.

$$D(\Xi \wedge \Theta) = (D\Xi) \wedge \Theta + (-1)^{\deg(D)\deg(\Xi)}\Xi \wedge D\Theta, \tag{1.35}$$

where $D \in \mathrm{Der}_K^\bullet$ and $\Xi, \Theta \in \Gamma(K^\bullet)$ are assumed to be homogeneous. As claimed in [DS79], we may make an identification

$$\mathrm{Der}_K^\bullet \cong \Gamma(K^\bullet \otimes (TS^{\mathbb{C}} \oplus K^1)). \tag{1.36}$$

This identification is explicitly given in terms of the action D of the right-hand side on $\Theta \in \Gamma(K^\bullet)$:

$$\begin{aligned} \mathsf{D}(\Xi \otimes (\xi \otimes a))\Theta &= \Xi \wedge \left(\gamma(\nabla^K\Theta)\xi \otimes a)^\wedge + \frac{1}{2}\,\iota_{\xi\otimes a}\Theta \right), \\ \mathsf{D}(\Xi \otimes (\overline{\xi} \otimes \overline{a}))\Theta &= \Xi \wedge \left((\gamma(\nabla^K\Theta)^*\overline{\xi} \otimes \overline{a})^\wedge + \frac{1}{2}\,\iota_{\overline{\xi}\otimes\overline{a}}\Theta \right), \\ \mathsf{D}(\Xi \otimes u)\Theta &= \Xi \wedge \nabla_u^K\Theta, \end{aligned} \tag{1.37}$$

where

$$\begin{aligned} &\xi \otimes a \in \Gamma(\Sigma \otimes V) \subset \Gamma(K^1), \quad \Xi \in \Gamma(K^\bullet), \\ &\overline{\xi} \otimes \overline{a} \in \Gamma(\Sigma^* \otimes V^*) \subset \Gamma(K^1), \quad u \in \Gamma(TS^{\mathbb{C}}) \end{aligned} \tag{1.38}$$

are arbitrary sections, ∇^K is the connection induced on $K^\bullet$ by ∇^Σ and ∇^V, and $\Xi \wedge (\gamma(\nabla^K\Theta)\xi \otimes a)^\wedge$ denotes the composition

$$\begin{aligned} \Gamma(K^\bullet) &\xrightarrow{\nabla^K} \Gamma(T^*S \otimes K^\bullet) \xrightarrow{\gamma\otimes \mathrm{id}_{K^\bullet}} \Gamma(\mathrm{End}(\Sigma) \otimes K^\bullet) \xrightarrow{\mathrm{ev}(\xi)\otimes \mathrm{id}_{K^\bullet}} \Gamma(\Sigma \otimes K^\bullet) \\ &\xrightarrow{\otimes a} \Gamma((\Sigma \otimes V) \otimes K^\bullet) \xrightarrow{\wedge} \Gamma(K^{\bullet+1}) \xrightarrow{\Xi\,\wedge} \Gamma(K^{\bullet+\deg(\Xi)+1}), \end{aligned} \tag{1.39}$$

with $\mathrm{ev}(\xi)$ denoting the evaluation of an endomorhism field in $\Gamma(\mathrm{End}(\Sigma))$ on $\xi \in \Gamma(\Sigma)$ and the map $\wedge$ denoting full antisymmetrisation. The term $\Xi \wedge (\gamma(\nabla^K\Theta)^*\overline{\xi} \otimes \overline{a})^\wedge$ is to be similarly interpreted.

We finally have all the pieces required to introduce the complex ($\mathbb{Z}_2$-graded) Lie superalgebra $\mathscr{X}^\bullet(S,\Sigma,V) = \mathscr{X}^0(S,\Sigma,V) \oplus \mathscr{X}^1(S,\Sigma,V)$ describing local supersymmetry transformations. As a vector space, it is given by

$$\begin{aligned} \mathscr{X}^0(S,\Sigma,V) &= \Gamma(K_{\mathsf{C}}^{2\bullet}) \oplus \mathrm{Der}_K^{2\bullet}, \\ \mathscr{X}^1(S,\Sigma,V) &= \Gamma(K_{\mathsf{C}}^{2\bullet+1}) \oplus \mathrm{Der}_K^{2\bullet+1}, \end{aligned} \tag{1.40}$$

where the subscript C only serves to distinguish the "central" $K^\bullet$ from any copy of $K^\bullet$

present in $\mathrm{Der}^\bullet_K$ under the identification (1.36). We have an action D' of $\mathscr{X}^\bullet(S,\Sigma,V)$ on $K^\bullet$ given by

$$\begin{aligned} \mathsf{D}'(\Xi\otimes(\xi\otimes a))\Theta &= \mathsf{D}(\Xi\otimes(\xi\otimes a))\Theta + \Xi\wedge(\epsilon\otimes\overline{\mathsf{Z}})(\xi\otimes a)\wedge\Theta,\\ \mathsf{D}'(\Xi\otimes(\overline{\xi}\otimes\overline{a}))\Theta &= \mathsf{D}(\Xi\otimes(\overline{\xi}\otimes\overline{a}))\Theta + \Xi\wedge(\epsilon^{-1}\otimes\mathsf{Z})(\overline{\xi}\otimes\overline{a})\wedge\Theta,\\ \mathsf{D}'(\Xi\otimes u)\Theta &= \mathsf{D}(\Xi\otimes u)\Theta,\quad \mathsf{D}'(\Xi_{\mathbb{C}})\Theta = \Xi_{\mathbb{C}}\wedge\Theta, \end{aligned} \tag{1.41}$$

where $\Xi_{\mathbb{C}}\in K^\bullet_{\mathbb{C}}$. Then the super Lie bracket $[\Psi,\Phi]^{\mathscr{X}}_\bullet$ of $\Psi,\Phi\in\mathscr{X}^\bullet(S,\Sigma,V)$ may be defined in terms of a graded commutator of endomorphisms by the following lemma.

Lemma 1.A.1. *Given elements $\Psi,\Phi\in\mathscr{X}^\bullet(S,\Sigma,V)$, there is a unique element $[\Psi,\Phi]^{\mathscr{X}}_\bullet\in\mathscr{X}^\bullet(S,\Sigma,V)$ satisfying*

$$\mathsf{D}'\left([\Psi,\Phi]^{\mathscr{X}}_\bullet\right) = [\mathsf{D}'(\Psi),\mathsf{D}'(\Phi)]_\bullet. \tag{1.42}$$

Proof. In general, given an element $\Psi\in\mathscr{X}^\bullet(S,\Sigma,V)$, the operator $\mathsf{D}'(\Psi)$ is not a graded derivation but a *first-order graded differential operator* i.e. a $\mathbb{C}$-linear endomorphism on the space of sections $\Gamma(K^\bullet)$ of the form

$$\Theta\mapsto D\Theta+\Xi\wedge\Theta, \tag{1.43}$$

where D is a graded derivation and Ξ is a section of $K^\bullet$. This decomposition of a first-order graded differential operator L into a derivation and a wedge product is canonical and given by

$$L=(L-(L1)\wedge)+(L1)\wedge, \tag{1.44}$$

where $L1$ denotes the evaluation of L on the constant section $1\in\Gamma(K^\bullet)$. As the operator $L-(L1)\wedge$ is a graded derivation, the identification (1.36) gives us a unique section

$$\Psi_L\in\Gamma(K^\bullet\otimes(TS^{\mathbb{C}}\oplus K^1)),\quad \mathsf{D}(\Psi_L):=L-(L1)\wedge. \tag{1.45}$$

If we additionally let $\Xi_L\in\Gamma(K^\bullet)$ be defined as

$$\Xi_L:=(\mathsf{D}(\Psi_L)-\mathsf{D}'(\Psi_L)+L)1, \tag{1.46}$$

then any first-order graded differential operator L can be uniquely written as

$$L=\mathsf{D}'(\Psi_L\oplus\Xi_{L,\mathbb{C}}), \tag{1.47}$$

where $\Xi_{L,\mathbb{C}}$ is just Ξ_L but interpreted as a section of the central copy $K^\bullet_{\mathbb{C}}\subset\mathscr{X}^\bullet(S,\Sigma,V)$.

Now that we have argued that any first-order graded differential operator is canonically the action D' of some element of $\mathscr{X}^\bullet(S,\Sigma,V)$ on $K^\bullet$, all the remains to complete this proof is showing that the graded commutator of two first-order graded differential operators is a first-order graded differential operator. This follows from the fact that the graded commutator of two graded derivations is a graded derivation:

$$[D+\Xi\wedge,D'+\Xi'\wedge]_\bullet=[D,D']_\bullet+((D\Xi')-(-1)^{\deg(D')\deg(\Xi)}(D'\Xi))\wedge. \tag{1.48}$$

□

Remark 1.A.2. The graded Lie bracket $[\Psi,\Phi]^{\mathscr{X}}_\bullet$ given by the above prescription may be described a little more explicitly as follows. In order to do so, we make use of a choice of local frame $\{e_\mu\}$ for TS with dual frame $\{\theta^\mu\}$. In terms of such a frame, we may write

$$\begin{aligned}
\mathsf{D}'(\xi\otimes a)\Theta &= \sum_\mu (\gamma^\mu\xi\otimes a)\wedge\nabla^K_{e_\mu}\Theta + \frac{1}{2}\,\iota_{\xi\otimes a}\Theta + (\epsilon\otimes\overline{\mathsf{Z}})(\xi\otimes a)\wedge\Theta \\
&= \sum_\mu \mathsf{D}'((\gamma^\mu\xi\otimes a)\otimes e_\mu)\Theta + \mathsf{D}'((\epsilon\otimes\overline{\mathsf{Z}})(\xi\otimes a)_{\mathsf{C}})\Theta + \frac{1}{2}\,\iota_{\xi\otimes a}\Theta, \\
\mathsf{D}'(\overline{\xi}\otimes\overline{a})\Theta &= \sum_\mu (\gamma^{\mu*}\overline{\xi}\otimes\overline{a})\wedge\nabla^K_{e_\mu}\Theta + \frac{1}{2}\,\iota_{\overline{\xi}\otimes\overline{a}}\Theta + (\epsilon^{-1}\otimes\mathsf{Z})(\overline{\xi}\otimes\overline{a})\wedge\Theta \\
&= \sum_\mu \mathsf{D}'((\gamma^{\mu*}\overline{\xi}\otimes\overline{a})\otimes e_\mu)\Theta + \mathsf{D}'((\epsilon^{-1}\otimes\mathsf{Z})(\overline{\xi}\otimes\overline{a})_{\mathsf{C}})\Theta + \frac{1}{2}\,\iota_{\overline{\xi}\otimes\overline{a}}\Theta,
\end{aligned} \tag{1.49}$$

where γ^μ denotes $\gamma(\theta^\mu)$ and $(\epsilon\otimes\overline{\mathsf{Z}})(\xi\otimes a)_{\mathsf{C}}$ is just $(\epsilon\otimes\overline{\mathsf{Z}})(\xi\otimes a)$ but interpreted as a section of $K^\bullet_{\mathsf{C}}$. It will be convenient to introduce the notation

$$\begin{aligned}
\mathsf{I}(\Xi\otimes(\xi\otimes a)) &= \Xi\otimes(\xi\otimes a) - \sum_\mu (\Xi\wedge(\gamma^\mu\xi\otimes a))\otimes e_\mu - \Xi\wedge(\epsilon\otimes\overline{\mathsf{Z}})(\xi\otimes a)_{\mathsf{C}}, \\
\mathsf{I}(\Xi\otimes(\overline{\xi}\otimes\overline{a})) &= \Xi\otimes(\overline{\xi}\otimes\overline{a}) - \sum_\mu (\Xi\wedge(\gamma^{\mu*}\overline{\xi}\otimes\overline{a}))\otimes e_\mu - \Xi\wedge(\epsilon^{-1}\otimes\mathsf{Z})(\xi\otimes a)_{\mathsf{C}},
\end{aligned} \tag{1.50}$$

so that we have

$$\Xi\wedge\iota_{\xi\otimes a}\Theta = \mathsf{D}'\circ\mathsf{I}(\Xi\otimes(\xi\otimes a))\Theta, \quad \Xi\wedge\iota_{\overline{\xi}\otimes\overline{a}}\Theta = \mathsf{D}'\circ\mathsf{I}(\Xi\otimes(\overline{\xi}\otimes\overline{a}))\Theta. \tag{1.51}$$

Using (1.49), we now compute the graded commutators

$$\begin{aligned}
&[\mathsf{D}'(u),\mathsf{D}'(v)]_\bullet\Theta = \mathsf{D}'(\mathscr{L}_u v)\Theta + R^K(u,v)\Theta, \\
&[\mathsf{D}'(u),\mathsf{D}'(\xi\otimes a)]_\bullet\Theta = \mathsf{D}'(\nabla^K_u(\xi\otimes a))\Theta + \sum_\mu (\gamma^\mu\xi\otimes a)\wedge R^K(u,e_\mu)\Theta, \\
&[\mathsf{D}'(u),\mathsf{D}'(\overline{\xi}\otimes\overline{a})]_\bullet\Theta = \mathsf{D}'(\nabla^K_u(\overline{\xi}\otimes\overline{a}))\Theta + \sum_\mu (\gamma^{\mu*}\overline{\xi}\otimes\overline{a})\wedge R^K(u,e_\mu)\Theta, \\
&[\mathsf{D}'(\xi\otimes a),\mathsf{D}'(\varsigma\otimes b)]_\bullet\Theta \\
&= \sum_\mu \mathsf{D}'((\gamma^\mu\xi\otimes a)\otimes\nabla^K_{e_\mu}(\varsigma\otimes b) + (\gamma^\mu\varsigma\otimes b)\otimes\nabla^K_{e_\mu}(\xi\otimes a))\Theta \\
&\qquad + \mathsf{D}'(\epsilon(\xi,\varsigma)\overline{\mathsf{Z}}(a,b)_{\mathsf{C}})\Theta + \sum_{\mu,\nu}(\gamma^\mu\xi\otimes a)\wedge(\gamma^\nu\varsigma\otimes b)\wedge R^K(e_\mu,e_\nu)\Theta, \\
&[\mathsf{D}'(\overline{\xi}\otimes\overline{a}),\mathsf{D}'(\overline{\varsigma}\otimes\overline{b})]_\bullet\Theta \\
&= \sum_\mu \mathsf{D}'((\gamma^{\mu*}\overline{\xi}\otimes\overline{a})\otimes\nabla^K_{e_\mu}(\overline{\varsigma}\otimes\overline{b}) + (\gamma^{\mu*}\overline{\varsigma}\otimes\overline{b})\otimes\nabla^K_{e_\mu}(\overline{\xi}\otimes\overline{a}))\Theta \\
&\qquad + \mathsf{D}'(\epsilon^{-1}(\overline{\xi},\overline{\varsigma})\mathsf{Z}(\overline{a},\overline{b})_{\mathsf{C}})\Theta + \sum_{\mu,\nu}(\gamma^{\mu*}\overline{\xi}\otimes\overline{a})\wedge(\gamma^{\nu*}\overline{\varsigma}\otimes\overline{b})\wedge R^K(e_\mu,e_\nu)\Theta, \\
&[\mathsf{D}'(\xi\otimes a),\mathsf{D}'(\overline{\varsigma}\otimes\overline{b})]_\bullet\Theta \\
&= \sum_\mu \mathsf{D}'((\gamma^\mu\xi\otimes a)\otimes\nabla^K_{e_\mu}(\overline{\varsigma}\otimes\overline{b}) + (\gamma^{\mu*}\overline{\varsigma}\otimes\overline{b})\otimes\nabla^K_{e_\mu}(\xi\otimes a))\Theta \\
&\qquad + \sum_\mu \mathsf{D}'(\langle\gamma^\mu\xi,\overline{\varsigma}\rangle\langle a,\overline{b}\rangle e_\mu)\Theta + \sum_{\mu,\nu}(\gamma^\mu\xi\otimes a)\wedge(\gamma^\nu\varsigma\otimes b)\wedge R^K(e_\mu,e_\nu)\Theta,
\end{aligned} \tag{1.52}$$

where R^K is the curvature of the connection ∇^K. By introducing frames $\{\xi_A\}$ and $\{a_p\}$ for Σ and V respectively, with respective dual frames $\{\overline{\xi}_A\}$ and $\{\overline{a}_p\}$, we may write this as a D'-action:

$$\begin{aligned} &R^K(u,v)\Theta \\ &= \sum_{A,p}(R^K(u,v)(\overline{\xi}_A \otimes \overline{a}_p) \wedge \iota_{\xi_A\otimes a_p}\Theta + R^K(u,v)(\xi_A \otimes a_p) \wedge \iota_{\overline{\xi}_A\otimes \overline{a}_p}\Theta) \\ &\overset{(1.51)}{=} \sum_{A,p} \mathsf{D}' \circ \mathsf{I}((R^K(u,v)(\overline{\xi}_A \otimes \overline{a}_p) \otimes (\xi_A \otimes a_p) + (R^K(u,v)(\xi_A \otimes a_p) \otimes (\overline{\xi}_A \otimes \overline{a}_p))\Theta. \end{aligned} \tag{1.53}$$

Any other graded commutator can now be described in terms of the ones above using the following identities:

$$\begin{aligned} &[\Xi \wedge \mathsf{D}'(\Psi), \Upsilon \wedge \mathsf{D}'(\Phi)]_\bullet\Theta \\ &= \Xi \wedge (\mathsf{D}'(\Psi)\Upsilon) \wedge \mathsf{D}'(\Phi)\Theta + (\mathsf{D}'(\Phi)\Xi) \wedge \Upsilon \wedge \mathsf{D}'(\Psi)\Theta + \Xi \wedge \Upsilon \wedge [\mathsf{D}'(\Psi), \mathsf{D}'(\Phi)]_\bullet, \\ &[\mathsf{D}'(\Psi), \mathsf{D}'(1_\mathbb{C})]_\bullet\Theta = 0, \end{aligned} \tag{1.54}$$

where $\Xi, \Upsilon \in \Gamma(K^\bullet)$ and $\Psi, \Phi \in \mathscr{X}^\bullet(S,\Sigma,V)$ are arbitary elements and $1_\mathbb{C}$ is the constant function 1 interpreted as a section of $K^\bullet_\mathbb{C}$.

There is a real structure on $\mathscr{X}^\bullet(S,\Sigma,V)$ induced by the real structure $(\epsilon \circ J_\Sigma) \otimes (\omega_V \circ J_V)$ on $\Sigma \otimes V$. We denote real part of $\mathscr{X}^\bullet(S,\Sigma,V)$ with respect to this real structure as $\mathscr{X}^\bullet_\mathbb{R}(S,\Sigma,V)$. The reality condition (1.32) ensures that this forms a real Lie superalgebra.

The complex rank of the complex vector bundle $\Sigma \otimes V$ is referred to as the *number of supercharges*. The "spacetime" manifold S has dimension $d = 4$, so this is $2^{\lfloor d/2 \rfloor}\mathscr{N} = 4\mathscr{N}$. Henceforth we set $\mathscr{N} = 2$, so the number of supercharges is 8.

A supergravity theory is built out of various representations of the superalgebra $\mathscr{X}^\bullet(S,\Sigma,V)$, which are referred to as *multiplets*. To avoid wading too far into certain subtleties that have no bearing on the statement of the main result of this section, we will not consider here representations of the full superalgebra $\mathscr{X}^\bullet(S,\Sigma,V)$ but only of the superalgebra $\mathscr{X}^\bullet_1(S,\Sigma,V)$ generated by the elements in $K^1 \subset \mathscr{X}^\bullet(S,\Sigma,V)$. This too carries a natural real structure and we denote its real part as $\mathscr{X}^\bullet_{1,\mathbb{R}}(S,\Sigma,V)$. This kind of bait-and-switch is permitted as representations of $\mathscr{X}^\bullet(S,\Sigma,V)$ can be built out of representations of $\mathscr{X}^\bullet_1(S,\Sigma,V)$ (see, for instance, Section 5.4 of [Soh85]). Taking into account this interchangeability, we shall henceforth refer to representations of $\mathscr{X}^\bullet_1(S,\Sigma,V)$ as multiplets as well.

Any supergravity theory is a theory of gravity, so it must contain at least the gravitational multiplet. This consists of

(a) the metric h_S on S,

(b) gravitino sections $\psi \in \Gamma(T^*S^\mathbb{C} \otimes \Sigma \otimes V)$ and $\overline{\psi} \in \Gamma(T^*S^\mathbb{C} \otimes \Sigma^* \otimes V^*)$ subject to the reality condition

$$(\omega_\Sigma \otimes \omega_V) \circ (J_\Sigma \otimes J_V)(\psi(u)) = \overline{\psi}(u), \tag{1.55}$$

(c) a graviphoton field which is a connection on a $\mathsf{U}(1)$-bundle locally represented by a 1-form A^G on S with curvature 2-form $F^\mathrm{G} = \mathrm{d}A^\mathrm{G}$.

We now specify an action Q of $\mathscr{X}_1^\bullet(S,\Sigma,V)$ on these fields, given by

$$\begin{aligned}(\mathsf{Q}(\xi\otimes a)h_S)(u,v) &= \sqrt{-\frac{\nu}{2}}\,(\langle\gamma(\iota_v h_S)\xi,\langle u\otimes a,\overline{\psi}\rangle\rangle+\langle\gamma(\iota_u h_S)\xi,\langle v\otimes a,\overline{\psi}\rangle\rangle),\\ \mathsf{Q}(\xi\otimes a)\psi_{\mathrm{L}} &= \sqrt{-\frac{8}{\nu}}\,\nabla^K(\xi_{\mathrm{L}}\otimes a)+\mathsf{G}^-\left(\frac{1}{\sqrt{2}}F^{\mathrm{G}}-\sqrt{-\frac{\nu}{8}}\langle\overline{\psi}\otimes\psi\rangle\right)(\gamma\xi_{\mathrm{R}}\otimes a),\\ \mathsf{Q}(\overline{\xi}\otimes\overline{a})A^{\mathrm{G}} &= \sqrt{2}\,\langle\overline{\xi}\otimes\overline{a},\psi\rangle,\end{aligned} \tag{1.56}$$

and other similar expressions, where $\nu<0$ is a real parameter related to Newton's constant, and $\mathsf{G}^\pm$ assigns to any bilinear form $F\in\Gamma(T^*S^{\mathbb{C}}\otimes T^*S^{\mathbb{C}})$ an $\mathrm{End}(\Sigma\otimes V)$-valued bilinear form $\mathsf{G}^\pm F$ as follows:

$$\mathsf{G}^\pm F = F\otimes\mathrm{id}_{\Sigma\otimes V}\pm\frac{1}{2}(\star_{h_S}F^\wedge)\otimes\gamma(\star_{h_S}1)\otimes\mathrm{id}_V, \tag{1.57}$$

with the wedge in the superscript denoting antisymmetrisation $\alpha\otimes\beta\mapsto\alpha\wedge\beta$. It may then be verified that the following Lagrangian density is invariant modulo closed terms and equations of motion under the action Q of the real Lie superalgebra $\mathscr{X}_{\mathbb{R}}^\bullet(S,\Sigma,V)$:

$$\begin{aligned}L_{\mathrm{SG}} = \Bigg(&\frac{1}{\nu}\,\mathrm{scal}_{h_S}-\frac{1}{4}\|F^{\mathrm{G}}\|^2-\frac{1}{2}\star_{h_S}\langle\overline{\psi}\wedge(\gamma(\star_{h_S}1)\circ\gamma)\wedge \mathrm{d}^\nabla\psi\rangle\\ &+\left\langle\overline{\psi},h_S^{-1}\circ\mathsf{G}^+\left(\frac{\sqrt{-\nu}}{4}F^{\mathrm{G}}+\frac{\nu}{16}\langle\overline{\psi}\otimes\psi\rangle\right)\circ h_S^{-1}\psi\right\rangle\Bigg)\,\mathrm{dvol}_{h_S},\end{aligned} \tag{1.58}$$

where $\|\cdot\|$ denotes the pointwise norm and d^∇ is covariant exterior derivative with respect to the connection ∇ induced on the gravitino bundle $T^*S\otimes\Sigma\otimes V$ by ∇^{h_S}, ∇^Σ, and ∇^V. This defines the pure supergravity theory, the simplest supergravity theory possible, with the gravitational muliplet being the only field content.

To make contact with quaternionic Kähler geometry, we need to introduce another kind of representation of $\mathscr{X}_1^\bullet(S,\Sigma,V)$, namely a *hypermuliplet*. The bosonic field content of such a multiplet consists just of 4 scalar fields, i.e. smooth functions on the manifold S. To write down the most general Lagrangian density that can be built out of the gravitational multiplet and hypermultiplets, it helps to think of the scalar fields as pullbacks of coordinate functions on some other fixed (pseudo-)Riemannian manifold (M,g) along a map $\varphi: S\to M$. Given that the metric h_S is stationary and has suitable asymptotic behaviour, it can be shown that the values of the scalar fields approach a constant as we go towards infinity. The manifold M may thus be thought of as a *moduli space* of stationary solutions to the supergravity equations of motion, parametrised by the boundary values of the scalar fields.

In order to give the field content of a theory with n hypermultiplets we first need to fix the following data:

(a) A fixed (pseudo-)Riemannian manifold (M,g) of dimension $4n$,

(b) A complex vector bundle $H\to M$ of complex rank 2 equipped with an antilinear map J_H such that $J_H^2=-\mathrm{id}_H$, a nondegenerate form $\omega_H\in\gamma(\Lambda^2H^*)$, and a connection ∇^H preserving all this structure,

(c) A complex vector bundle $E\to M$ of complex rank $2n$ equipped with an antilinear map J_H such that $J_E^2=-\mathrm{id}_E$, a nondegenerate form $\omega_E\in\gamma(\Lambda^2E^*)$, and a connection ∇^E preserving all this structure,

(d) A section $\sigma \in \Gamma(T^*M^{\mathbb{C}} \otimes H \otimes E)$ that defines an isomorphism between $TM^{\mathbb{C}}$ and $H \otimes E$ sending the canonical real structure on $TM^{\mathbb{C}}$ to $J_H \otimes J_E$.

Given this, we can assemble our field content into the additional data below, following notational conventions in [Jos+17]:

(a) A smooth map $\varphi : S \to M$,

(b) An identification of the pullback of H with V^*, i.e.

$$\varphi^{-1}(H, J_H, \omega_H, \nabla^H) \cong (V^*, J_V^*, \omega_V^{-1}, \nabla^V), \tag{1.59}$$

so that the gravitinos ψ and $\overline{\psi}$ may be regarded as sections of $T^*S \otimes \Sigma \otimes \varphi^{-1}H^*$ and $T^*S \otimes \Sigma^* \otimes \varphi^{-1}H$ respectively,

(c) *Hyperino* sections $\chi \in \Gamma(\Sigma \otimes \varphi^{-1}E)$ and $\overline{\chi} \in \Gamma(\Sigma^* \otimes \varphi^{-1}E^*)$ subject to the reality condition

$$(\omega_\Sigma \otimes \omega_E) \circ (J_\Sigma \otimes J_E)(\chi) = \overline{\chi}. \tag{1.60}$$

The situation is now complicated by the fact that the gravitino and hyperino bundles $T^*S^{\mathbb{C}} \otimes \Sigma \otimes \varphi^{-1}H^*$ and $\Sigma \otimes \varphi^{-1}E$ depend on the map φ, which is acted on by the superalgebra $\mathscr{X}^\bullet(S, \Sigma, V)$. (Complexified) infinitesimal variations in the map φ are sections of the pullback bundle $\varphi^{-1}TM^{\mathbb{C}}$, so it makes sense to define the action Q' of $\mathscr{X}_1^\bullet(S, \Sigma, V)$ on φ to be given by

$$\begin{aligned} \mathsf{Q}'(\xi \otimes a)\varphi &= (\varphi^{-1}\sigma^*)(\langle \overline{\chi}_{\mathrm{L}}, \xi_{\mathrm{R}} \otimes a\rangle + \langle \overline{\chi}_{\mathrm{R}}, \xi_{\mathrm{L}} \otimes a\rangle), \\ \mathsf{Q}'(\overline{\xi} \otimes \overline{a})\varphi &= (\varphi^{-1}\sigma^{-1})(\langle \overline{\xi}_{\mathrm{L}} \otimes \overline{a}, \chi_{\mathrm{R}}\rangle + \langle \overline{\xi}_{\mathrm{R}} \otimes \overline{a}, \chi_{\mathrm{L}}\rangle). \end{aligned} \tag{1.61}$$

For notational convenience, we shall henceforth write the pulled back bundle map $\varphi^{-1}\sigma$ as just σ. In order to specify the action Q' on the other fields, let us denote by

$$w(\overline{\xi}, \overline{a}), \overline{w}(\xi, a) \in \Gamma(\varphi^{-1}TM^{\mathbb{C}}) \tag{1.62}$$

the projections of the sections $\mathsf{Q}'(\overline{\xi} \otimes \overline{a})\varphi$ and $\mathsf{Q}'(\xi \otimes a)\varphi$ onto the image of $\mathrm{d}\varphi$. Then, we have a well-defined action Q' of $\mathscr{X}_1^\bullet(S, \Sigma, V)$ on the rest of the fields given by

$$\begin{aligned} (\mathsf{Q}'(\overline{\xi} \otimes \overline{a})h_S)(u, v) &= \sqrt{-\frac{\nu}{2}}\,(\langle \gamma(\iota_v h_S)\xi, \langle u \otimes a, \overline{\psi}\rangle\rangle + \langle \gamma(\iota_u h_S)\xi, \langle v \otimes a, \overline{\psi}\rangle\rangle), \\ \mathsf{Q}'(\xi \otimes a)\psi_{\mathrm{L}} &= \sqrt{-\frac{8}{\nu}}\,\nabla(\xi_{\mathrm{L}} \otimes a) - \nabla^H_{\overline{w}(\xi,a)}\psi_{\mathrm{L}} \\ &\quad + \mathsf{G}^-\left(\frac{1}{\sqrt{2}}F^{\mathrm{G}} - \sqrt{-\frac{\nu}{8}}\langle \overline{\psi} \otimes \psi\rangle\right)(\gamma\xi_{\mathrm{R}} \otimes a), \\ \mathsf{Q}'(\overline{\xi} \otimes \overline{a})A^{\mathrm{G}} &= \sqrt{2}\langle \overline{\xi} \otimes \overline{a}, \psi\rangle, \\ \mathsf{Q}'(\xi \otimes a)\chi_{\mathrm{L}} &= 2\langle \sigma(\mathrm{d}\varphi), \langle \xi_{\mathrm{R}}, \gamma\rangle \otimes a\rangle - \nabla^E_{\overline{w}(\xi,a)}\chi_{\mathrm{L}}, \end{aligned} \tag{1.63}$$

and similar expressions. The pure supergravity Lagrangian density L_{SG} now has to be supplemented with additional terms:

$$L_{\mathrm{HM}} = \Big(-\|\mathrm{d}\varphi\|^2 - \frac{1}{2}\langle \overline{\chi}, \not{D}\chi \rangle + \frac{1}{16}\Big\langle \langle \omega_H, \sigma \otimes \sigma \rangle^{-1}(R^E) + \nu\, \mathrm{id}_{E\otimes E}, \mathrm{tr}_{h_S}\left(\langle \overline{\chi}_{\mathrm{L}}, \gamma\chi_{\mathrm{L}} \rangle_{\Sigma}^{\otimes 2} \right) \Big\rangle + \sqrt{-\frac{\nu}{2}}\left(\langle \overline{\chi}_{\mathrm{L}}, \gamma(\langle \sigma(\mathrm{d}\varphi), \psi_{\mathrm{R}} \rangle) \rangle + \langle \overline{\chi}_{\mathrm{R}}, \gamma(\langle \sigma(\mathrm{d}\varphi), \psi_{\mathrm{L}} \rangle) \rangle \right) \Big) \mathrm{dvol}_{h_S}, \tag{1.64}$$

where $\not{D}$ is the Dirac operator and R^E is the curvature of the connection ∇^E. In contrast to the pure supergravity case, it is not guaranteed that the theory described by the Lagrangian density $L := L_{\mathrm{SG}} + L_{\mathrm{HM}}$ is invariant under the action Q' of the real Lie superalgebra $\mathscr{X}^{\bullet}_{1,\mathbb{R}}(S, \Sigma, V)$. The following result due to Bagger and Witten gives the precise criterion for when this happens.

Folklore 1.A.3 ([BW83])**.** *Given that the (pseudo-)Riemannian manifold of hypermultiplet scalars (M, g) has dimension $4n \geqslant 8$, the Lagrangian density $L := L_{\mathrm{SG}} + L_{\mathrm{HM}}$ is invariant modulo closed forms and equations of motion under the action Q' of the real Lie superalgebra $\mathscr{X}^{\bullet}_{1,\mathbb{R}}(S, \Sigma, V)$ if and only if the restricted holonomy of (M, g) is contained in $\mathsf{Sp}(n) \cdot \mathsf{Sp}(1)$, and its reduced scalar curvature*

$$\frac{\mathrm{scal}_g}{4n(n+2)} \tag{1.65}$$

is the constant $\nu < 0$.

For dimension 4, Bagger and Witten have an additional constraint on the metric g. In fact, this additional constraint can be taken to *define* quaternionic Kähler manifolds in dimension 4. Indeed, Definition 2.1.3 that we later provide in Chapter 2 reflects this additional constraint. Note also that the bundles H and E are precisely the $\mathsf{Sp}(1)$- and $\mathsf{Sp}(n)$-bundles of Salamon's EH formalism in [Sal82].

The takeaway from all this is that there is a correspondence between 4-dimensional $\mathscr{N} = 2$ supergravity coupled to hypermultiplets and quaternionic Kähler geometry. In particular, it should be possible to translate physical recipes for constructing such supergravity Lagrangians into differential-geometric recipes for constructing quaternionic Kähler manifolds.

1.B The Type IIA superstring and its dimensional reduction

Superstring theory is a particularly rich source of such recipes for constructing supergravity Lagrangians.

Recall that string theory describes the propagation of a string through spacetime S. As the string propagates, it traces out a 2-dimensional surface called the *worldsheet*. From the perspective of the worldsheet C, string theory is essentially a conformal field theory (CFT) living on the worldsheet, with the background fields on S realised as various parameters or *couplings* of the CFT. Since a CFT is sensitive to just the conformal structure on C and not the metric, C may be thought of as a Riemann surface or a complex curve. In dimension 2, excitations of the CFT are organised into two sectors, i.e. subspaces of the full Hilbert space of excitations: *holomorphic* and *anti-holomorphic*. Physically, these are the *left-moving* and *right-moving* modes of the string.

Meanwhile, the "super" in superstring theory just means that the worldsheet CFT is promoted to a superconformal field theory (SCFT) by introducing certain fermionic excitation modes on the string. On the closed string, these may be subject to either *periodic* or *anti-periodic* boundary conditions. This again gives us two subsectors each of the left-moving and right-moving sectors: The *Ramond* and the *Neveu–Schwarz* sectors respectively. All in all, we thus have four sectors: NS–NS, R–NS, NS–R, and R–R.

Requiring consistency of the SCFT underlying superstring theory places heavy constraints on the states that are allowed in its Hilbert space. It turns out, up to equivalence, only five different consistent choices may be made. These are the various *Types*: I, IIA, IIB, heterotic $\mathsf{E}_8 \times \mathsf{E}_8$, and heterotic $\mathsf{O}(32)$. We'll be specialising to Type IIA for the rest of this appendix.

Since the backgound S and the fields on it are realised as couplings for string excitations, and are in fact labelled by the excitation modes, a constraint on allowed states translates to a constraint on allowed backgrounds. The executive summary is that the background has to be a solution to the equations of motion of 10-dimensional supergravity. Note that supersymmetry on the background is *not* the same supersymmetry that is present on the worldsheet. Rather, it is a consequence of a symmetry of the SCFT known as *spectral flow* symmetry.

The various fields constituting the 10-dimensional supergravity background are labelled by the string excitations. The bosonic fields are labelled by NS–NS and R–R excitations, while the fermionic fields are labelled by NS–R and R–NS excitations. Let us restrict our attention to only the bosonic fields. The NS–NS fields of the 10d supergravity theory are as follows:

(a) the 10d metric h_S,

(b) the *Kalb–Ramond* field, which is a connection on a gerbe locally represented by a 2-form B with a curvature 3-form $H = \mathrm{d}B$,

(c) a scalar *dilaton* ϕ.

The R–R fields meanwhile comprise of the following:

(a) the 10d graviphoton A^{G} with curvature 2-form F^{G},

(b) a connection on a 2-gerbe locally represented by a 3-form A^{D} with curvature 4-form
$$F^{\mathrm{D}} = \mathrm{d}A^{\mathrm{D}} - A^{\mathrm{G}} \wedge H. \tag{1.66}$$

In order to make contact with the well-established fact that the world we live in is 4-dimensional, we consider background manifolds S of the form $\mathbb{R}^{1,3} \times Y$, where Y is a compact manifold of real dimension 6, whose size is presumed to be smaller than the length scales we can probe with our measuring devices. It turns out that such a background can support supergravity solutions if and only if Y is Calabi–Yau, which is to say a (Ricci-flat) Kähler manifold with vanishing first Chern class.

Keeping only the lowest Fourier modes of a stationary 10d supergravity solution on $\mathbb{R}^{1,3} \times Y$ gives a stationary solution to 4d $\mathcal{N} = 2$ supergravity. We have already observed that the scalars of a supergravity theory form (a partial set of) coordinates on its moduli space of solutions. Thus, we describe the scalars of the dimensionally reduced 4d theory by describing certain coordinates on the moduli space of classical 10d supergravity solutions.

First of all, supersymmetry preserving deformations of 10d metric h_S are essentially Calabi–Yau deformations of the Calabi–Yau metric h_Y. It is known that the moduli space of such deformations has a product structure $N_{\mathrm{C}} \times N'_{\mathrm{K}}$, where N_{C} is a

4d scalar	10d origin	Sector	Multiplet
X_a	h_S	NS–NS	hyper
$\mathrm{Re}(s_i)$	h_S	R–R	vector
$\mathrm{Im}(s_i)$	B	NS–NS	vector
ζ_0, ζ_a	A^{D}	R–R	hyper
τ	B	NS–NS	hyper
ϕ	ϕ	NS–NS	hyper

TABLE 1.B.1: Scalars in dimensionally reduced 4d supergravity

complex manifold of complex dimension $h^{2,1}$ parametrised by $h^{2,1}$ complex numbers X_a, and N'_{K} is a manifold of real dimension $h^{1,1}$ parametrised by $h^{1,1}$ real numbers. Here, $h^{2,1}$ and $h^{1,1}$ are the Hodge numbers of the Calabi–Yau manifold Y:

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} \tag{1.67}$$

It is clear that the full moduli space of scalars forms a bundle over the manifold $N_{\mathrm{C}} \times N'_{\mathrm{K}}$, and that fibre coordinates are obtained by fixing a Calabi–Yau manifold Y and looking for coordinates describing the rest of the fields on $\mathbb{R}^{1,3} \times Y$.

Up to gauge equivalence, the fields $B, A^{\mathrm{G}}, A^{\mathrm{D}}$ are characterised by their curvature forms $H, F^{\mathrm{G}}, F^{\mathrm{D}}$. To extract coordinates from these, we follow the following general prescription. Say α is a harmonic curvature k-form. This means that both α as well as its Hodge dual, the $(10-k)$-form $\star_{h_S}\alpha$, are closed. Since we are considering only stationary solutions, there is by definition an asymptotically timelike Killing field v which preserves all fields. This may be shown to imply that $\iota_v\alpha$ and $\iota_v(\star_{h_S}\alpha)$ are both closed and can therefore be unambiguously integrated along cycles of the appropriate dimension. Integrating $\iota_v\alpha$ along a $(k-1)$-cycle of Y (if available) and $\iota_v(\star_{h_S}\alpha)$ along a $(9-k)$-cycle of Y (if available) thus gives one coordinate each for every $(k-1)$-cycle of Y and one coordinate each for every $(9-k)$-cycle of Y.

The Hodge diamond (1.67) of the Calabi–Yau manifold Y tells us which cycles are present. Following the above prescription, we obtain $2h^{2,1}+2$ real coordinates by integrating $\iota_v F^{\mathrm{D}}$ over the $2h^{2,1}+2$ independent 3-cycles. These can be combined into $h^{2,1}+1$ complex coordinates ζ_0, ζ_a. Meanwhile, integrating $\iota_v H$ over the $h^{1,1}$ independent 2-cycles yields $h^{1,1}$ real coordinates which may be combined with the real Kähler moduli to give $h^{1,1}$ complexified Kähler moduli $s_i \in \mathbb{C}$. We also have the full 6-dimensional Calabi–Yau manifold itself on which we can integrate $\iota_v(\star_{h_S}H)$. This gives the real coordinate τ, called the *axion*.

Finally, the 10d dilaton trivially descends to the 4d dilaton giving us the last scalar moduli ϕ. These are summarised in Table 1.B.1. Not all of these scalars are organised in hypermultiplets; the complexified Kähler moduli belong to another kind of representation referred to as *vector multiplets*. On general physical and representation theoretic grounds, one can conclude that the full 4d scalar moduli space also has a product structure $M \times N_{\mathrm{K}}$, with M being the manifold of hypermultiplet scalars forming a bundle over N_{C}, and N_{K} being the manifold of vector multiplet scalars forming a bundle over N'_{K}.

By Folklore 1.A.3, we know that there has to be a quaternionic Kähler metric on M. Consistent with this, we note that the real dimension of M is $4h^{2,1}+4$, corresponding to

(a) $h^{2,1}$ complex coordinates X_a parametrising the complex structure moduli space N_{C} of the Calabi–Yau manifold Y,

(b) $h^{2,1}+1$ complex coordinates ζ_0, ζ_a parametrising the Weil intermediate Jacobian of Y (see Chapter 7 of [Aal11] for details),

(c) Coordinates ϕ and τ parametrising fibres of a $\mathbb{C}^\times$ bundle over the Weil intermediate Jacobian.

The pairs (X_a, ζ_a) combine into $h^{2,1}$ different hypermultiplets; the number of these depends on the specific Calabi–Yau manifold we compactify on. The triple (ϕ, τ, ζ_0) meanwhile forms the *universal* hypermultiplet, called so because because its presence is universal across all choices of Calabi–Yau manifolds.

The complex structure moduli space N_{C} of a Calabi–Yau manifold is itself known to have an interesting Riemannian structure, namely a *projective special Kähler* metric g_N, first introduced by Cecotti, Ferrara, and Girardello in [CFG89] and later given a precise definition by Freed in [Fre99]. These may be characterised by a local holomorphic homogeneous function F of degree 2 in formal coordinates z_0, z_a called the *prepotential*. The quaternionic Kähler metric g on M in terms of g_N was first described in physical terms (and proved to be indeed quaternionic Kähler) by Ferrara and Sabharwal in [FS90] and later in a mathematically precise way by Hitchin in [Hit09]. To write down the Ferrara–Sabharwal metric, we first introduce some notation, following Section 4.3 of [CHM12].

Given a holomorphic prepotential $F(z_0, \ldots, z_{n-1})$ which is homogeneous of degree 2, we let N_{ab}, with a, b now running from 0 to $n-1$, denote twice the imaginary part of the entries of its Hessian matrix:

$$N_{ab} = 2\,\mathrm{Im}\left(\frac{\partial^2 F}{\partial z_a\, \partial z_b}\right). \tag{1.68}$$

This is a holomorphic homogeneous function of degree 0, and so may be used to define another holomorphic homogeneous function $\mathcal{N}_{ab}$ of degree 0 as follows:

$$\mathcal{N}_{ab}(z_0, \ldots, z_{n-1}) = \frac{F(z_0, \ldots, z_{n-1})}{z_0^2} + \mathrm{i}\,\frac{\sum_{b,d} N_{ab} N_{cd} z_b z_d}{\sum_{e,f} N_{ef} z_e z_f}. \tag{1.69}$$

This may be used to define functions $\Re$ and $\Im$ taking values in real square matrices of size n on the complex structure moduli space N_{C} (and so by pullback on the hypermultiplet moduli space M) in the following way:

$$\Re_{ab}(X_1, \ldots, X_{n-1}) + \mathrm{i}\,\Im_{ab}(X_1, \ldots, X_{n-1}) = \mathcal{N}_{ab}\left(1, X_1, \ldots, X_{n-1}\right). \tag{1.70}$$

It was shown in Corollary 5 of [CHM12] that $\Im$ is invertible everywhere. We may thus write the Ferrara–Sabharwal metric g_{FS} on M as

$$\begin{aligned} g_{\mathrm{FS}} = g_{N_{\mathrm{C}}} + \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{4\rho^2}\left(\mathrm{d}\tau + \sum_{a=0}^{n-1}(y_a \mathrm{d}x_a - x_a \mathrm{d}y_a)\right)^2 + \frac{1}{2\rho}\sum_{a,b=0}^{n-1}\Im_{ab}\mathrm{d}y_a \mathrm{d}y_b \\ + \frac{1}{2\rho}\sum_{a,b=0}^{n-1}(\Im^{-1})_{ab}\left(\mathrm{d}x_a + \sum_{c=0}^{n-1}\Re_{ac}\mathrm{d}y_c\right)\left(\mathrm{d}x_b + \sum_{c=0}^{n-1}\Re_{bd}\mathrm{d}y_d\right), \end{aligned} \tag{1.71}$$

where we have set $\zeta_a = x_a + \mathrm{i}y_a$, with x_a, y_a real.

So far, our discussion of the background supergravity theory and its associated hypermultiplet moduli space metric has been strictly classical. It turns out that when quantum effects are taken into account, this metric undergoes a deformation which is still quaternionic Kähler! In order to make precise what we mean by quantum effects, we first have to interpret the metric g on M in terms of the worldsheet SCFT.

1.C Quantum corrections to Type IIA hypermultiplets

What we have seen so far is that points on the moduli spaces of superstring theory encode the background through which the superstring is propagating. A tangent vector on the quaternionic Kähler manifold that is the hypermultiplet moduli space therefore represents a variation in the hypermultiplet scalars in the background. Such variations of the background may be interpreted as *vertex operators* from the point of view of the worldsheet SCFT (see for instance Chapter 7 of [Ton09]). There is a natural pairing on such operators induced by the 2-point correlator, depicted in Figure 1.C.1a, which is encoded by the quaternionic Kähler metric on the moduli space evaluated on two tangent vectors.

In any quantum theory, classical correlators are expected to receive quantum corrections given by a formal series expansion in some parameter $\hbar$ assumed to be small. Superstring theory has two such parameters: the *string length scale* ℓ_s and the *string coupling* $\rho^{-1} := e^{-2\phi}$. The former is a dimensionful parameter that needs to be put in by hand. The latter is a dimensionless parameter that is by contrast a natural consequence of the presence of a term

$$\frac{1}{4\pi}\,\phi\,\mathrm{scal}_{h_C}\mathrm{dvol}_{h_C} \tag{1.72}$$

in the worldsheet Lagrangian density, coupling the scalar curvature scal_{h_C} of the worldsheet metric h_C with the background dilaton ϕ. From the Gauß–Bonnet theorem we know that the integral

$$\frac{1}{4\pi}\int_C \phi\,\mathrm{scal}_{h_C}\mathrm{dvol}_{h_C} = \phi(2-2g_C) \tag{1.73}$$

is a topological invariant, with g_C being the genus of the worldsheet C. The quantum-corrected correlator is formally given by an appropriate *path integral* over a configuration space parametrising the worldsheets and the fields on them. Such a configuration space consists of several connected components labelled by the genus g_C of the worldsheet. And since the Gauß–Bonnet integral is a topological invariant, it contributes an overall weight

$$e^{-\phi(2-2g_C)} = \rho^{1-g_C} \tag{1.74}$$

to the path integral evaluated on the component with label g_C. In other words, ρ^{-1} acts as the formal parameter for the string genus expansion, analogous to the Feynman loop expansion for point particles in quantum field theory.

Since superstring theory is, from the worldsheet perspective, a superconformal field theory, the string length scale ℓ_s comes into play only through dimensionless ratios with the length scales of the compactification, encoded in the Kähler moduli of the Calabi–Yau manifold Y. For the Type IIA superstring, the Kähler moduli are

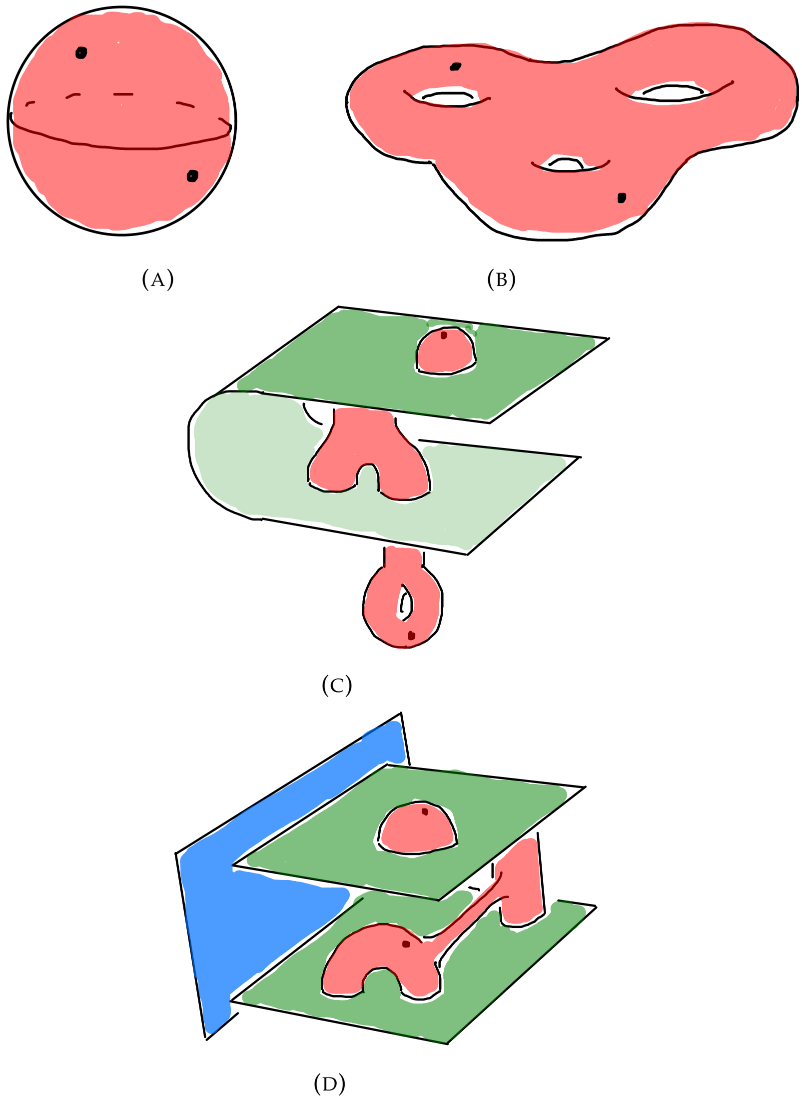

FIGURE 1.C.1: Contributions to the hypermultiplet metric labelled by abstract configurations of worldsheets (red), D-branes (green), and NS5-branes (blue).

not included in the hypermultiplets, hence the hypermultiplet metric admits no ℓ_s corrections. However, there are ρ^{-1} corrections to the hypermultiplet metric.

These corrections are of two kinds: perturbative and nonperturbative. Given a suitable parametrisation of the quaternionic Kähler metric, the perturbative corrections to it consist of terms proportional to ρ^{-k} coming from the genus expansion that we have described above and depicted in Figure 1.C.1b. In a generic quantum theory, the resulting formal series diverges rapidly and has to be interpreted as an asymptotic series. However, in supersymmetric theories, there is improved behaviour. In fact, it was shown by Robles-Llana, Saueressig, and Vandoren in Section 4.3 of [RSV06] that with an appropriate field redefinition, the full set of perturbative corrections of all orders to the Ferrara–Sabharwal metric can be absorbed into an order 1 or 1-*loop* correction. This yields the 1-*loop-deformed* Ferrara–Sabharwal metric

$$
\begin{aligned}
g'^{c}_{\mathrm{FS}} \\
&= \frac{\rho+c}{\rho} g_{N_{\mathrm{C}}} + \frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}\,\mathrm{d}\rho^2 + \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}\left(\mathrm{d}\tau + c\,\mathrm{d}^{\mathrm{c}}\mathscr{K} + \sum_{a=0}^{n-1}(y_a\mathrm{d}x_a - x_a\mathrm{d}y_a)\right)^2 \\
&\quad + \frac{1}{2\rho}\sum_{a,b=0}^{n-1}\Im_{ab}\mathrm{d}y_a\mathrm{d}y_b + \frac{1}{2\rho}\sum_{a,b=0}^{n-1}(\Im^{-1})_{ab}\left(\mathrm{d}x_a + \sum_{c=0}^{n-1}\Re_{ac}\mathrm{d}y_c\right)\left(\mathrm{d}x_b + \sum_{c=0}^{n-1}\Re_{bd}\mathrm{d}y_d\right) \\
&\quad + \frac{2c}{\rho^2}e^{\mathscr{K}}\left|\sum_{a=0}^{n-1}\left(X_a\mathrm{d}x_a + \frac{\partial F}{\partial z_a}(1, X_1, \ldots, X_{n-1})\,\mathrm{d}y_a\right)\right|^2,
\end{aligned}
\tag{1.75}
$$

where $\mathscr{K}$ is the Kähler potential on the projective special Kähler manfold given by evaluating at $(z_0, z_1, \ldots, z_{n-1}) = (1, X_1, \ldots, X_{n-1})$ the expression

$$
-\log\left(\sum_{a,b=0}^{n-1}\frac{N_{ab}z_a\overline{z}_b}{|z_0|^2}\right),
\tag{1.76}
$$

while the parameter c is related to the Euler characteristic χ_Y of the Calabi–Yau manifold Y on which the theory is compactified by

$$
c = -\frac{\chi_Y}{192\pi}.
\tag{1.77}
$$

Nonperturbative corrections on the other hand arise from the presence of higher-dimensional objects in string theory that can be charged under the various background gauge fields. These are again of two kinds: *D-branes* and *NS5-branes*.

D-branes or Dirichlet branes are objects that are charged under the R–R gauge fields that descend to the complex-valued fields $\zeta_a = x_a + \mathrm{i}y_a$ and are called so because the ends of the fundamental string of string theory are restricted to be incident on them, as depicted in Figure 1.C.1c. In other words, they provide Dirichlet boundary conditions on the associated SCFT on the string worldsheet. For the Type IIA superstring, these are defects localised on special Lagrangian 3-cycles of the Calabi–Yau 3-fold Y labelled by tuples of integers (p_a, q_a), one for each $a \in \{0, \ldots, h^{2,1}\}$, referred to as its *dyonic charge*. These contribute corrections proportional to

$$
\exp\left(-2\pi \mathrm{Z}_{p,q}\rho - 2\pi\mathrm{i}\sum_a(q_a x_a - p_a y_a)\right),
\tag{1.78}
$$

where $\mathrm{Z}_{p,q}$ is the pullback of a certain central charge function defined on the complex structure moduli space of the Calabi–Yau manifold Y over which the hypermultiplet moduli space forms a bundle. In particular, the presence of D-branes breaks the isometries generated by the vector fields

$$
\frac{1}{2}(\partial_{x_a} + y_a\mathrm{d}\tau), \quad \frac{1}{2}(\partial_{y_a} - x_a\mathrm{d}\tau).
\tag{1.79}
$$

NS5-branes are objects that are charged under the Kalb–Ramond 2-form B in the NS–NS sector that descends to the axion τ. The Kalb–Ramond 2-form B is a higher analogue of the electromagnetic 1-form A in that it is the field under which the fundamental string is charged, just like how point particles are charged under the electromagnetic field. NS5-branes then are to the fundamental string what magnetic monopoles are to electric charges. An NS5-branes cannot be incident on the fundamental string; however, they can be incident on D-branes, which can in turn be incident on both NS5-branes and the fundamental string. Such a brane configuration is depicted in Figure 1.C.1d. Geometrically, the NS5-branes is the Calabi–Yau manifold itself with an attached integer charge r. It contributes corrections proportional to

$$\exp\left(-2\pi|r|\mathrm{Vol}(Y)\rho^2 - \pi i r\tau\right), \tag{1.80}$$

where $\mathrm{Vol}(Y)$ is the volume of the Calabi–Yau manifold Y. In particular, the presence of NS5-branes breaks the isometries generated by the vector field ∂_τ.

The nonperturbative corrections are exponentially suppressed in comparison to the perturbative corrections. So, it makes sense to retain only the perturbative i.e. the 1-loop correction. Studying the deformation of quaternionic Kähler metrics this induces shall be the focus of this dissertation.

However, it is still worth saying a few words about why the full set of corrections are interesting to mathematicians. First of all, on physical grounds, it is expected that the resulting quaternionic Kähler manifold will be complete. Moreover, physical arguments also suggest that this manifold will have no continuous isometries. Having explicit examples of complete quaternionic Kähler manifolds without any continuous isometries would be valuable for any attempts at a classification of negatively curved quaternionic Kähler manifolds.

Quaternionic Kähler geometry is not the only field of mathematics that would stand to gain from such an investigation. From the point of view of the supergravity theory in dimension 4, in the regime that the associated charges p_a, q_a, r are large, the brane confugurations described above appear as solutions to the supergravity equations of motions that are localised around certain instants in time. (This incidentally is why they are often referred to as *D-instantons* or *NS5-instantons*.) These may be analytically continued in a formal manner to yield multi-centred black hole solutions with Taub-NUT charge r and BPS charges (p_a, q_a), which would be interesting to construct in their own right. But there are additional reasons this is interesting.

As we move about the moduli space, a black hole can split apart into many or many black holes may combine into one. The overall smoothness of the moduli space metric, again expected on physical grounds, severely constrains the precise way in which this happens. In a string theoretic context, these constraints have been shown in [APP11] to encode a lot of nontrivial information about dilogarithm identities and the Donaldson–Thomas invariants of the underlying Calabi–Yau manifold.

Chapter 2

Quaternionic Kähler manifolds

In this chapter, we define quaternionic Kähler manifolds and collect in one place results and facts about them that we shall be relying on later. These are interspersed with several examples illustrating these facts as we proceed.

In particular, we introduce the quaternionic moment map constructed in [GL88] and use it to identify certain *twist data* associated to quaternionic Kähler manifolds, a notion that will be formally introduced in Chapter 3. These are then computed for an important class of examples of dimension 4 related to the continuous Toda integrable system, which includes the 1-loop-deformed universal hypermultiplet metric.

Most of the results and examples in this chapter are well-established in mathematical lore. There are nonetheless a few original results: certain constructions using the quaternionic moment map (Lemmata 2.2.7, 2.2.10, 2.2.11), a complete description of the full isometry group of the 1-loop-deformed universal hypermultiplet metric (Proposition 2.3.6), and the identification of the family of quaternionic Kähler manifolds (2.95) constructed by applying separation of variables to the continuous Toda system in [Ket01], with the well-known Pedersen family of quaternionic Kähler metrics (Proposition 2.3.7).

2.1 Definition and basic properties

We have already seen in Chapter 1 what quaternionic Kähler manifolds are supposed to be: roughly speaking, (pseudo-)Riemannian manifolds with local holonomy contained in $\mathsf{Sp}(1)\cdot\mathsf{Sp}(n)$, at least when the dimension is greater than 4. While this was the characterisation that originally motivated mathematical interest in quaternionic Kähler manifolds, it is not so easy to work with. We shall hence choose a different starting point.

Definition 2.1.1 (Almost quaternionic Hermitian manifolds)**.** An almost quaternionic Hermitian (AQH) manifold (M, g, Q) is a (pseudo-)Riemannian manifold (M, g) with a distinguished subbundle $Q \subset \mathrm{End}(TM)$ locally spanned by three almost Hermitian structures J_1, J_2, J_3 satisfying the following equation:

$$J_1 \circ J_2 = J_3. \tag{2.1}$$

Remark 2.1.2. The "quaternionic" in the name comes from the fact that J_1, J_2, J_3 along with the identity id_{TM} form a representation of the quaternion algebra $\mathbb{H}$. This may be seen as follows:

$$\begin{aligned} J_2 \circ J_1 &= J_1^2 \circ J_2 \circ J_1 \circ J_2^2 = J_1 \circ (J_1 \circ J_2)^2 \circ J_2 = J_1 \circ J_3^2 \circ J_2 = -J_1 \circ J_2, \\ J_2 \circ J_3 &= -J_1^2 \circ J_2 \circ J_3 = -J_1 \circ (J_1 \circ J_2) \circ J_3 = -J_1 \circ J_3^2 = J_1. \end{aligned} \tag{2.2}$$

The bundle Q is therefore referred to as the quaternionic bundle of the almost quaternionic Hermitian manifold.

Definition 2.1.3 (Quaternionic Kähler manifolds)**.** A quaternionic Kähler (QK) manifold (M, g, Q) is a non-Ricci-flat AQH manifold (M, g, Q) such that the quaternionic bundle Q is parallel with respect to the Levi-Civita connection ∇^g associated to g and satisfies

$$[\mathscr{R}, Q^* \wedge \mathrm{id}_{T^*M}] = 0, \tag{2.3}$$

where $\mathscr{R} \in \Gamma(\mathrm{End}(\Lambda^2 T^*M))$ is the Riemann curvature map, and $Q^* \subset \mathrm{End}(T^*M)$ is the transpose of Q.

Remark 2.1.4. The Riemann curvature map $\mathscr{R}$ is related to the more familiar Riemann curvature R^g via

$$\mathscr{R}(\alpha \wedge \beta)(u, v) = \alpha(R^g(u, v)\beta^\sharp), \tag{2.4}$$

where α, β are arbitrary 1-forms and u, v are arbitrary vector fields. The vanishing of $[\mathscr{R}, Q^* \wedge \mathrm{id}_{T^*M}]$ is equivalent to the statement

$$-J \circ R(u, v)w + R(u, v) \circ Jw + R(Ju, v)w + R(u, Jv)w = 0, \tag{2.5}$$

for all sections $J \in \Gamma(Q)$ and all vector fields u, v, w. This is automatic when the dimension of M is greater than 4, and we may drop the condition from the definition in this case.

Remark 2.1.5. In accordance with convention, Ricci-flat manifolds have been explicitly excluded from this definition, so the local holonomy is a subgroup of $\mathsf{Sp}(1) \cdot \mathsf{Sp}(n)$ not contained in an $\mathsf{Sp}(n)$ subgroup. The reason for this is that many constructions related to quaternionic Kähler manifolds, such as the quaternionic moment map (to be introduced in the next section) or the Swann bundle fail to be well-defined in case of hyperkähler manifolds, i.e. manifolds which are Ricci-flat but otherwise satisfy the quaternionic Kähler conditions. These however are pretty important in their own right and in fact very much relevant to the goals of this dissertation. We shall be seeing hyperkähler manifolds again in Chapter 4.

Remark 2.1.6. Since it is only the span of J_1, J_2, J_3 that is required to be ∇^g-parallel and not J_1, J_2, J_3 individually, quaternionic Kähler manifolds are in general not Kähler. In fact, since the J_i are only locally defined, there may not even be a global almost complex structure, as the following example shall demonstrate.

Example 2.1.7 (Quaternionic projective spaces)**.** The quaternionic projective space $\mathbb{HP}^n$ is the space of quaternionic lines in $\mathbb{H}^{n+1}$. By quaternionic lines, we mean subspaces of real dimension 4 that are closed under right multiplication

$$q := (q_0, q_1, \ldots, q_n) \mapsto (q_0 s, q_1 s, \ldots, q_n s) =: qs \tag{2.6}$$

by nonzero quaternions $s \in \mathbb{H}^\times$. This may be realised as the coset space

$$\mathsf{Sp}(n+1)/(\mathsf{Sp}(1) \times \mathsf{Sp}(n)). \tag{2.7}$$

To describe the metric g and quaternionic bundle Q, we note that tangent vectors at

$$[q] := q \cdot \mathbb{H}^\times := (q_0, q_1, \ldots, q_n) \cdot \mathbb{H}^\times \tag{2.8}$$

may be identified with orbits $[q, v]$ of pairs $(q, v) \in [q] \times [q]^\perp$ under the right $\mathbb{H}^\times$-action

$$(q, v) \mapsto (qs, vs). \tag{2.9}$$

Here, $[q]^\perp$ denotes the orthogonal complement of the subspace $[q] \subseteq \mathbb{H}^{n+1}$ with respect to the standard inner product $\hat{g}$ on $\mathbb{H}^{n+1}$. The metric now is simply given by the well-defined expression

$$g([q,v],[q',v']) = \frac{\hat{g}(v,v')}{\hat{g}(q,q')}, \tag{2.10}$$

where q and q' are required to be parallel, which in particular implies $[q]^\perp = [q']^\perp$.

Using Pontryagin classes, Hirzebruch showed in [Hir54] that there are no global almost complex structures on $\mathbb{H}\mathbf{P}^n$. (In fact, there are no almost complex structures on any compact quaternionic Kähler manifolds of positive scalar curvature other than the complex Grassmannians $\mathsf{SU}(n+2)/\mathsf{S}(\mathsf{U}(n)\times\mathsf{U}(2))$ [GMS11].) However, if we make a local choice of representatives $q \in \mathbb{H}^{n+1}$ of points $[q] \in \mathbb{H}\mathbf{P}^n$, then we may define local almost complex structures J_1, J_2, J_3 as follows:

$$J_1([q,v]) = [q,-v\mathrm{i}], \quad J_2([q,v]) = [q,-v\mathrm{j}], \quad J_3([q,v]) = [q,-v\mathrm{k}]. \tag{2.11}$$

These definitions are sensitive to the choice of representative q, but the span Q of them, which just involves v getting right-multiplied by imaginary quaternions, is not. Thus, Q is globally defined even though the basis $\{J_1, J_2, J_3\}$ is not.

There is a natural pointwise inner product $\langle\cdot,\cdot\rangle$ induced on $\mathrm{End}(TM)$ by the metric g:

$$\langle A, B\rangle = \frac{1}{\dim(M)}\operatorname{tr}(A^{\dagger g}\circ B), \tag{2.12}$$

where A and B are sections of $\mathrm{End}(TM)$, and $A^{\dagger g} := g^{-1}\circ A^* \circ g$ is the adjoint of A with respect to the metric g. The adjoint of any Hermitian structure A is just $-A$, so the restriction of the inner product to the bundle Q is just

$$\langle A, B\rangle|_Q = -\frac{1}{\dim(M)}\operatorname{tr}(A\circ B). \tag{2.13}$$

In particular, we see that (J_1, J_2, J_3) is a local oriented orthonomal frame for Q. Two such frames are related by an $\mathsf{SO}(3)$ transformation. As an immediate consequence, we see that the tensor fields $\Omega^\sharp, \Omega, \Omega^\wedge$ given by the local expressions

$$\Omega^\sharp = \sum_i J_i \otimes J_i, \quad \Omega = \sum_i \omega_i \otimes J_i, \quad \Omega^\wedge = \sum_i \omega_i \wedge \omega_i, \tag{2.14}$$

with $\omega_i = g(J_i\cdot,\cdot)$, are in fact global sections of the bundles $Q\otimes Q, Q^\flat \times Q, Q^\flat \wedge Q^\wedge$ respectively. Note that since we have only used the fact that J_i are Hermitian structures satisfying $J_1 \circ J_2 = J_3$, this observation in fact holds true for any AQH manifold and not just quaternionic Kähler ones. The 4-form $\Omega^\wedge$ in particular referred to as the *fundamental 4-form* of the AQH manifold in question.

Example 2.1.8. In the case of $\mathbb{H}\mathbf{P}^n$, the tensor field Ω is given by the following expression:

$$\begin{aligned}\Omega^\sharp([q,v],[q',v']) &= [q,v\mathrm{i}]\otimes_{\mathbb{R}}[q',v'\mathrm{i}] + [q,v\mathrm{j}]\otimes_{\mathbb{R}}[q',v'\mathrm{j}] + [q,v\mathrm{k}]\otimes_{\mathbb{R}}[q',v'\mathrm{k}] \\ &\cong [q\otimes_{\mathbb{R}} q', v\mathrm{i}\otimes_{\mathbb{R}} v'\mathrm{i} + v\mathrm{j}\otimes_{\mathbb{R}} v'\mathrm{j} + v\mathrm{k}\otimes_{\mathbb{R}} v'\mathrm{k}],\end{aligned} \tag{2.15}$$

where q and q' are required to be parallel and the pairs $[q \otimes_{\mathbb{R}} q', v \otimes_{\mathbb{R}} v']$ denote the orbits of the right $(\mathbb{H}^\times)^2$-action

$$(q \otimes_{\mathbb{R}} q', v \otimes_{\mathbb{R}} v') \mapsto (qs \otimes_{\mathbb{R}} q's', vs \otimes_{\mathbb{R}} v's'). \tag{2.16}$$

The fact that $\Omega^\sharp$ is globally well-defined then amounts to the statement that

$$\Omega^\sharp([qs, vs], [q's', v's']) = \Omega^\sharp([q, v], [q', v']), \tag{2.17}$$

for all $s \in \mathbb{H}^\times$. In other words, the definition of $\Omega^\sharp$ is not sensitive to the local choice of representatives $(q, v), (q', v')$ for the vector fields $[q, v], [q', v']$.

Since the inner product $\langle \cdot, \cdot \rangle$ is naturally induced by the metric g, at least up to an overall constant factor, it is compatible with the connection on Q induced by the Levi-Civita connection ∇^g, which we denote by ∇^g as well. This leads us to deduce a rather useful property of $\Omega^\sharp, \Omega, \Omega^\wedge$.

Proposition 2.1.9. *The tensor fields $\Omega^\sharp, \Omega, \Omega^\wedge$ on a quaternionic Kähler manifold (M, g, Q) are ∇^g-parallel. In particular $\Omega^\wedge$ is a closed form.*

Proof. It's enough to show that $\Omega^\sharp \in \Gamma(Q \otimes Q)$ is ∇^g-parallel. Note that if J_i form a local orthonormal basis of sections of Q, then $J_j \otimes J_k$ form a local orthonormal basis of sections of $Q \otimes Q$ (with respect to inner product on $Q \otimes Q$ induced by that on Q). If we can show $\langle J_j \otimes J_k, \nabla^g \Omega^\sharp \rangle$ vanishes for all j, k, we are done. We therefore compute:

$$\begin{aligned} \langle J_j \otimes J_k, \nabla^g \Omega^\sharp \rangle &= \left\langle J_j \otimes J_k, \sum_i (\nabla^g J_i \otimes J_i + J_i \otimes \nabla^g J_i) \right\rangle \\ &= \sum_i (\langle J_j, \nabla^g J_i \rangle \langle J_k, J_i \rangle + \langle J_j, J_i \rangle \langle J_k, \nabla^g J_i \rangle) \\ &= \langle J_j, \nabla^g J_k \rangle + \langle J_k, \nabla^g J_j \rangle = \mathrm{d} \langle J_j, J_k \rangle = 0. \end{aligned} \tag{2.18}$$

□

In fact, something more is true: we can use the closedness of $\Omega^\wedge$ to *characterise* quaternionic Kähler manifolds of dimension greater than 8.

Theorem 2.1.10 ([Swa91] Theorem 2.2)**.** *A non-Ricci-flat AQH manifold (M, g, Q) with $\dim(M) > 8$ is quaternionic Kähler if and only if its fundamental 4-form $\Omega^\wedge$ is closed.*

Remark 2.1.11. One way to think about this is by analogy to Kähler manifolds: a Hermitian manifold (M, g, J) is Kähler if and only if the 2-form $\omega := g(J\cdot, \cdot)$ is closed. Thus, the fundamental 4-form $\Omega^\wedge$ plays a role analogous to the Kähler form.

Meanwhile, for dimensions 4 and 8, we invoke the following result due to Alekseevsky, applicable to quaternionic Kähler manifolds of all dimensions.

Theorem 2.1.12 ([Ale68] Tables 1 and 2)**.** *The Riemann curvature R^g of a quaternionic Kähler manifold (M, g, Q) of dimension $4n$ is of the form*

$$R^g = \nu R_{\mathbb{HP}^n} + W^g_Q, \tag{2.19}$$

where ν is a constant, $R_{\mathbb{HP}^n}$ is formally the Riemann curvature of the quaternionic projective space $\mathbb{HP}^n$, and W^g_Q is the "quaternionic Weyl" curvature. The latter is a traceless abstract curvature tensor field of "hyperkähler type" i.e. it satisfies for all vector fields u, v on M,

$$[W^g_Q(u, v), Q] = 0. \tag{2.20}$$

In particular, (M, g) is Einstein with constant reduced scalar curvature ν and all sections $u, v \in \Gamma(TM)$ and $A \in \Gamma(Q)$ satisfy

$$[R^g(u,v), A] := ((\nabla^g)^2_{u,v} - (\nabla^g)^2_{v,u})A = \frac{\nu}{2}[\iota_u \iota_v \Omega, A]. \tag{2.21}$$

Remark 2.1.13. It is instructive to restate the above result in the Cartan formalism, in which the connection and curvature of Q are respectively realised as a set of 1-forms α_{ij} and 2-forms β_{ij} via

$$\alpha_{ij}(u) = \langle J_i, \nabla^g_u J_j \rangle, \quad \beta_{ij}(u,v) = \langle J_i, [R^g(u,v), J_j] \rangle. \tag{2.22}$$

These are equivalently described in terms of the Cartan structure equations:

$$\alpha_{ij} = -\alpha_{ji}, \quad d\omega_i + \sum_j \alpha_{ij} \wedge \omega_j = 0, \quad d\alpha_{ij} + \sum_k \alpha_{ik} \wedge \alpha_{kj} = \beta_{ij}. \tag{2.23}$$

In particular, the algebraic ideal generated by the locally defined 2-forms ω_i (and therefore the bundle $Q^\flat$ locally spanned by them) is a differential ideal i.e. closed under the action of the exterior derivative d.

As for Theorem 2.1.12, it reduces to the following equation:

$$\beta_{ij} = \nu \sum_k \epsilon_{ijk} \omega_k, \tag{2.24}$$

where ϵ_{ijk} is the Levi-Civita symbol. Equations (2.23) and (2.24) together are equivalent to the self-dual (respectively, anti-self-dual) Einstein condition in dimension 4 when the bundle $Q^\flat$ is identified with the bundle of anti-self-dual (respectively, self-dual) 2-forms $\Lambda^2_-(TM)$ (respectively, $\Lambda^2_+(TM)$).

Theorem 2.1.14 ([Swa91] Theorem 2.2). *An AQH manifold (M, g, Q) of dimension 8 is quaternionic Kähler if and only if its fundamental 4-form $\Omega^\wedge$ is closed and the algebraic ideal generated by $Q^\flat$ is a differential ideal.*

Proposition 2.1.15. *An oriented (pseudo-)Riemannian manifold (M, g) of dimension 4 admits a quaternionic Kähler structure (M, g, Q) if and only if it is self-dual and Einstein or anti-self-dual and Einstein.*

Remark 2.1.16. An anti-self-dual oriented (pseudo-)Riemannian manifold becomes self-dual if its orientation is reversed. So these are not two separate cases.

Example 2.1.17 (Complex hyperbolic plane)**.** Let $\mathbb{C}^{1,2}$ denote $\mathbb{C}^3$ endowed with the standard Hermitian inner product of signature $(1, 2)$ (minus signs first). Then, the complex hyperbolic plane $\mathbb{C}\mathbf{H}^2$ is the space of complex lines of negative norm, and it carries a metric g induced by the Hermitian inner product on $\mathbb{C}^{1,2}$ in a manner analogous to that in the case of quaternionic projective spaces. This space is also a coset space, namely $\mathsf{SU}(1, 2)/\mathsf{U}(2)$. The horospherical coordinatisation for the complex line

$$\mathbb{C}^\times \cdot z := \mathbb{C}^\times \cdot (z_0, z_1, z_2) \in \mathbb{C}\mathbf{H}^2 \tag{2.25}$$

spanned by $z := (z_0, z_1, z_2) \in \mathbb{C}^{1,2}$, subject to

$$\|z\|^2 = -|z_0|^2 + |z_1|^2 + |z_2|^2 < 0, \tag{2.26}$$

is given by

$$\begin{aligned} \mathbb{C}^\times \cdot (z_0, z_1, z_2) &\mapsto (\rho, \tau, \zeta) \\ &= \left(-\frac{\|z\|^2}{|z_0 + z_1|^2}, -\mathrm{Im}\left(\frac{z_0 - z_1}{z_0 + z_1}\right), \frac{\sqrt{2}\, z_2}{z_0 + z_1} \right). \end{aligned} \tag{2.27}$$

This is a diffeomorphism from $\mathbb{C}\mathbf{H}^2$ to $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{C}$ with inverse given by

$$(\rho, \tau, \zeta) \mapsto \mathbb{C}^\times \cdot \left(1, \frac{1 - (\rho + |\zeta|^2 - \mathrm{i}\tau)}{1 - (\rho - |\zeta|^2 - \mathrm{i}\tau)}, \frac{\sqrt{2}\,\zeta}{1 - (\rho - |\zeta|^2 - \mathrm{i}\tau)} \right). \tag{2.28}$$

Moreover, the metric g in these coordinates is given by

$$g = \frac{1}{4\rho^2} \left(\mathrm{d}\rho^2 + 2\rho\, |\mathrm{d}\zeta|^2 + (\mathrm{d}\tau + \mathrm{Im}(\zeta\, \mathrm{d}\bar{\zeta}))^2 \right). \tag{2.29}$$

In [CS17], we computed its Ricci curvature map $g^{-1} \circ \mathrm{Ric} \in \Gamma(\mathrm{End}(T^*\mathbb{C}\mathbf{H}^2))$ and Weyl curvature map $\mathscr{W} \in \Gamma(\mathrm{End}(\Lambda^2 T^*\mathbb{C}\mathbf{H}^2))$ to be

$$g^{-1} \circ \mathrm{Ric} = -6\, \mathrm{id}_{T^*M}, \quad \mathscr{W} = \frac{1}{2}(1 + \star)(\mathscr{R} + 2\, \mathrm{id}_{\Lambda^2 T^* M}). \tag{2.30}$$

Since $\mathbb{C}\mathbf{H}^2$ is both Einstein and self-dual, it follows that it admits a quaternionic Kähler structure. In order to describe the quaternionic Kähler sructure explicitly, let us set $\zeta = x + \mathrm{i}y$ and choose the following orthonormal frame for the tangent bundle:

$$e_1 = 2\rho\, \partial_\rho, \quad e_2 = \sqrt{2\rho}\, (\partial_x - y\, \partial_\tau), \quad e_3 = \sqrt{2\rho}\, (\partial_y + x\, \partial_\tau), \quad e_4 = 2\rho\, \partial_\tau. \tag{2.31}$$

This induces a dual frame ϑ^a for the cotangent bundle:

$$\vartheta^1 = \frac{\mathrm{d}\rho}{2\rho}, \quad \vartheta^2 = \frac{\mathrm{d}x}{\sqrt{2\rho}}, \quad \vartheta^3 = \frac{\mathrm{d}y}{\sqrt{2\rho}}, \quad \vartheta^4 = \frac{1}{2\rho}\, (\mathrm{d}\tau + y\, \mathrm{d}x - x\, \mathrm{d}y). \tag{2.32}$$

Then we can use e_a and ϑ^a to construct the following orthonormal frame for Q:

$$\begin{aligned} J_1 &= \vartheta^2 \wedge_g e_3 - \vartheta^1 \wedge_g e_4, \\ J_2 &= \vartheta^3 \wedge_g e_1 - \vartheta^2 \wedge_g e_4, \\ J_3 &= \vartheta^1 \wedge_g e_2 - \vartheta^3 \wedge_g e_4, \end{aligned} \tag{2.33}$$

where we have introduced the notation

$$\alpha \wedge_g u := \alpha \otimes u - u^\flat \otimes \alpha^\sharp. \tag{2.34}$$

One may check that the connection 1-forms α_{ij} associated with this frame are given by

$$\alpha_{23} = -\vartheta^4, \quad \alpha_{31} = -2\vartheta^3, \quad \alpha_{12} = 2\vartheta^2, \tag{2.35}$$

while the curvature 2-forms β_{ij} are given by

$$\begin{aligned} \beta_{23} &= -2(\vartheta^2 \wedge \vartheta^3 - \vartheta^1 \wedge \vartheta^4) = -2\omega_1, \\ \beta_{31} &= -2(\vartheta^3 \wedge \vartheta^1 - \vartheta^2 \wedge \vartheta^4) = -2\omega_2, \\ \beta_{12} &= -2(\vartheta^1 \wedge \vartheta^2 - \vartheta^3 \wedge \vartheta^4) = -2\omega_3, \end{aligned} \tag{2.36}$$

where $\omega_i = g(J_i\cdot, \cdot)$.

2.2 The quaternionic moment map

2.2.1 Definition and examples

The quaternionic moment map was introduced by Galicki and Lawson [GL88] in order to define quaternionic quotients by "quaternionic" Killing fields, analogous to the Marsden–Weinstein quotient construction for symplectic manifolds. A quaternionic Killing field was defined by them to be a Killing field Z of a quaternionic Kähler manifold such that the quaternionic bundle Q and the fundamental 4-form $\Omega^\wedge$ are Z-invariant. However, this definition was later shown to be superfluous.

Proposition 2.2.1 ([Ale+03] p. 529)**.** *Any Killing vector field Z of a quaternionic Kähler manifold (M, g, Q) with fundamental 4-form $\Omega^\wedge$ satisfies*

$$\mathscr{L}_Z Q \subseteq Q, \quad \mathscr{L}_Z \Omega^\wedge = 0. \tag{2.37}$$

Thus, following [Dyc15], we may drop the hypothesis (2.37) from Galicki and Lawson's definition of the quaternionic moment map.

Definition 2.2.2 (Quaternionic moment map, [GL88] Theorem 2.4)**.** The quaternionic moment map μ^Z of a quaternionic Kähler manifold (M, g, Q) with respect to a Killing vector field Z thereof is the unique section μ^Z of Q satisfying

$$\nabla^g \mu^Z = -\iota_Z \Omega. \tag{2.38}$$

Remark 2.2.3. This definition may be motivated by the analogy between the fundamental 4-form $\Omega^\wedge$ on a quaternionic Kähler manifold and the Kähler form on a Kähler manifold. Using the metric to lower an index on μ^Z, we get a 2-form $\mu^{Z\flat} \in \Gamma(Q^\flat)$. Fully antisymmetrising $\nabla^g \mu^{Z\flat}$ then gives the following:

$$\mathrm{d}\mu^{Z\flat} = -\iota_Z \Omega^\wedge. \tag{2.39}$$

Remark 2.2.4. The solution to (2.38) can be written down rather explicitly as

$$\mu^Z = -\frac{2}{\nu}\,\mathrm{pr}_Q(\nabla^g Z), \tag{2.40}$$

where ν is the (constant) reduced scalar curvature, $\nabla^g Z$ is interpreted as a section of $\mathrm{End}(TM)$, and $\mathrm{pr}_Q(\nabla^g Z)$ denotes its orthogonal projection onto Q, with respect to the inner product $\langle \cdot, \cdot \rangle$. In terms of a local oriented orthonormal frame (J_1, J_2, J_3), the explicit solution (2.40) may also be written as

$$\mu^Z = \frac{1}{2\nu} \sum_{i,j,k} \epsilon_{ijk} \langle J_i, (\nabla^g_Z - \mathscr{L}_Z) J_j \rangle J_k. \tag{2.41}$$

Remark 2.2.5. The definition of quaternionic moment maps $\mu^Z \in \Gamma(Q)$ with respect to Killing fields Z may be straightforwardly generalised to quaternionic moment maps $\mu^{\mathfrak{g}} \in \mathfrak{g}^* \otimes_{\mathbb{R}} \Gamma(Q)$ with respect to a Killing algebra $\mathfrak{g}$ via

$$\iota_x \mu^{\mathfrak{g}} = \mu^{Z_x}, \tag{2.42}$$

where Z_x is the fundamental Killing vector field associated to the Lie algebra element $x \in \mathfrak{g}$.

Example 2.2.6. Continuing with Example 2.1.17, let $\mathfrak{g}$ be the algebra of the complex hyperbolic plane $\mathbb{C}\mathbf{H}^2$ spanned by the Killing fields

$$\begin{aligned} Z_1 &= \frac{1}{2}\left(e_1 + \frac{x}{\sqrt{2\rho}}\, e_2 + \frac{y}{\sqrt{2\rho}}\, e_3 + \frac{\tau}{\rho}\, e_4\right) = \rho\,\partial_\rho + \frac{1}{2}(x\,\partial_x + y\,\partial_y) + \tau\,\partial_\tau, \\ Z_2 &= \frac{1}{2}\left(\frac{1}{\sqrt{2\rho}}\, e_2 + \frac{y}{\rho}\, e_4\right) = \frac{1}{2}(\partial_x + y\,\partial_\tau), \\ Z_3 &= \frac{1}{2}\left(\frac{1}{\sqrt{2\rho}}\, e_3 - \frac{x}{\rho}\, e_4\right) = \frac{1}{2}(\partial_y - x\,\partial_\tau), \quad Z_4 = \frac{1}{2\rho}\, e_4 = \partial_\tau. \end{aligned} \tag{2.43}$$

Note that the frame (J_1, J_2, J_3) as chosen in (2.33) is rather conveniently invariant with respect to all of the Killing fields Z^a above. So, (2.41) just reduces to

$$\mu^Z = \frac{1}{\nu}(\alpha_{23}(Z)J_1 + \alpha_{31}(Z)J_2 + \alpha_{12}(Z)J_3). \tag{2.44}$$

From the above, we may read off $\mu^a := \mu^{Z_a}$ as

$$\begin{aligned} \mu^1 &= \frac{\tau}{4\rho}\, J_1 + \frac{y}{2\sqrt{2\rho}}\, J_2 - \frac{x}{2\sqrt{2\rho}}\, J_3, \\ \mu^2 &= \frac{y}{4\rho}\, J_1 - \frac{1}{2\sqrt{2\rho}}\, J_3, \quad \mu^3 = -\frac{x}{4\rho}\, J_1 + \frac{1}{2\sqrt{2\rho}}\, J_2, \quad \mu^4 = \frac{1}{4\rho}\, J_1. \end{aligned} \tag{2.45}$$

Note that the quaternionic moment map behaves quite differently from the Kähler moment map. Recall that Kähler moment maps are only locally defined and unique only up to addition by a constant function, so if we want to take Kähler quotients, we are free to choose any level set of Kähler moment map. However, the quaternionic moment map is globally defined and unique—Theorem 2.1.12 tells us there cannot be any local parallel nonzero sections of the quaternionic bundle Q. So, if we want to take quaternionic Kähler quotients, the uniqueness of μ^Z forces us to consider only the zero-set.

From Example 2.2.6, we know that the zero-set of μ^Z may be empty for a certain Killing field Z, so the existence of quaternionic Kähler quotients is far more restricted than that of Kähler quotients. However, for the purposes of this dissertation, we are interested precisely in the complement of the zero-set of μ^Z.

2.2.2 Anticipatory lemmata

On the complement of the zero set of μ^Z, there is a global Hermitian structure $J^Z \in \Gamma(Q)$ given by its normalisation

$$J^Z = \frac{\mu^Z}{\|\mu^Z\|} := \frac{\mu^Z}{\sqrt{\langle \mu^Z, \mu^Z \rangle}}. \tag{2.46}$$

This may be used to introduce *twist data* on the quaternionic Kähler manifold, a notion we shall later define in Chapter 3, which we shall then use to construct hyperkähler manifolds in Chapter 4. In anticipation of this, we collect here a few lemmata that will be later useful.

Lemma 2.2.7. *Given a quaternionic Kähler manifold (M, g, Q) of reduced scalar curvature ν equipped with Killing field Z with nowhere vanishing quaternionic moment map μ^Z with normalisation J^Z, the "quaternionic twist form" ω_{Q} given by*

$$\omega_{\mathrm{Q}}(u,v) = -\mathrm{d}\left(\frac{\iota_Z g}{\|\mu^Z\|}\right)(u,v) - \nu g(J^Z u, v) + \langle \nabla^g_u J^Z, J^Z \circ \nabla^g_v J^Z \rangle, \tag{2.47}$$

is a closed 2-form and Z is Hamiltonian with respect to it.

Proof. We work in a local oriented orthonormal frame $(J'_1 = J^Z, J'_2, J'_3)$ for Q, where we have inoduced the prime to avoid confusion between this choice of frame and that in (2.33) when we discuss examples. In particular, this means that the quaternionic moment map is given by

$$\mu^Z = \|\mu^Z\| J'_1. \tag{2.48}$$

Substituting the above into the defining property (2.38) of μ^Z then gives us

$$\mathrm{d}\|\mu^Z\| = -\iota_Z \omega'_1, \quad \|\mu^Z\|\alpha'_{21} = -\iota_Z \omega'_2, \quad \|\mu^Z\|\alpha'_{31} = -\iota_Z \omega'_3, \tag{2.49}$$

where as usual, $\omega'_i = g(J'_i \cdot, \cdot)$, and α'_{ij} are the connection 1-forms associated to the frame. As an immediate consequence, we have

$$\alpha'_{21}(Z) = \alpha'_{31}(Z) = 0. \tag{2.50}$$

Meanwhile, we also have the following chain of equalities:

$$\begin{aligned}
\langle \nabla^g_u J'_1, J'_1 \circ \nabla^g_v J'_1 \rangle &= \langle \nabla^g_u J'_1, J'_2 \rangle \langle J'_2, J'_1 \circ \nabla^g_v J'_1 \rangle + \langle \nabla^g_u J'_1, J'_3 \rangle \langle J'_3, J'_1 \circ \nabla^g_v J'_1 \rangle \\
&\overset{(2.13)}{=} \langle \nabla^g_u J'_1, J_2 \rangle \langle J'_2 \circ J'_1, \nabla^g_v J'_1 \rangle + \langle \nabla^g_u J'_1, J'_3 \rangle \langle J'_3 \circ J'_1, \nabla^g_v J'_1 \rangle \\
&= -\langle \nabla^g_u J'_1, J'_2 \rangle \langle J'_3, \nabla^g_v J'_1 \rangle + \langle \nabla^g_u J'_1, J'_3 \rangle \langle J'_2, \nabla^g_v J'_1 \rangle \\
&\overset{(2.22)}{=} -(\alpha'_{21} \wedge \alpha'_{31})(u,v) \overset{(2.23)}{=} (\alpha'_{21} \wedge \alpha'_{23})(u,v).
\end{aligned} \tag{2.51}$$

This gives the following local expression for ω_{Q}:

$$\omega_{\mathrm{Q}} = -\mathrm{d}\left(\frac{\iota_Z g}{\|\mu^Z\|}\right) - \nu\omega'_1 + \alpha'_{21} \wedge \alpha'_{13}. \tag{2.52}$$

By (2.23) and (2.24), we know that

$$\mathrm{d}\alpha'_{23} + \alpha'_{21} \wedge \alpha'_{13} = \nu\omega'_1. \tag{2.53}$$

So we have the following manifestly exact and thus closed local expression for ω_{Q}:

$$\omega_{\mathrm{Q}} = -\mathrm{d}\left(\frac{\iota_Z g}{\|\mu^Z\|} + \alpha'_{23}\right). \tag{2.54}$$

Contracting (2.52) with Z and using the Z-invariance of $\iota_Z g$ and μ^Z meanwhile yields

$$\begin{aligned}\iota_Z\omega_Q &= -\iota_Z \mathrm{d}\left(\frac{\iota_Z g}{\|\mu^Z\|}\right) - \nu\iota_Z\omega_1' + \iota_Z(\alpha_{21}'\wedge\alpha_{13}') \\ &= -\mathscr{L}_Z \mathrm{d}\left(\frac{\iota_Z g}{\|\mu^Z\|}\right) + \mathrm{d}\left(\frac{g(Z,Z)}{\|\mu^Z\|}\right) - \nu\iota_Z\omega_1' + \iota_Z(\alpha_{21}'\wedge\alpha_{13}') \\ &= \mathrm{d}\left(\frac{g(Z,Z)}{\|\mu^Z\|}\right) - \nu\iota_Z\omega_1' + \alpha_{21}'(Z)\alpha_{13}' - \alpha_{13}'(Z)\alpha_{21}' \\ &\overset{(2.49)}{=} -\mathrm{d}\left(-\frac{g(Z,Z)}{\|\mu^Z\|} - \nu\|\mu^Z\|\right).\end{aligned} \tag{2.55}$$

□

Remark 2.2.8. The 2-form ω_Q was constructed in a different way by Haydys in the proof of Theorem 14 in [Hay08], where he additionally proved that J^Z is in fact integrable.

Definition 2.2.9 (Elementary deformation)**.** The elementary deformation of an AQH manifold (M, g, Q) by a nowhere vanishing Killing field Z is an AQH manifold (M, g', Q) of the form

$$g' = h_1 g|_{\mathbb{H}_Q Z^\perp} + h_2 g|_{\mathbb{H}_Q Z}, \tag{2.56}$$

where h_1 and h_2 are nowhere vanishing functions, and $\mathbb{H}_Q Z$ denotes the quaternionic span of Z, i.e. the subbundle of TM spanned by the vector fields Z, J_1Z, J_2Z, J_3Z, and $\mathbb{H}_Q Z^\perp$ denotes the subbundle g-orthogonal to it.

Lemma 2.2.10. *Let (M, g, Q) be a quaternionic Kähler manifold of reduced scalar curvature ν equipped with Killing field Z with nowhere vanishing quaternionic moment map μ^Z with normalisation J^Z such that*

$$f_Q := -\frac{g(Z,Z)}{\|\mu^Z\|} - \nu\|\mu^Z\| \tag{2.57}$$

is nowhere vanishing, and let ω_Q be the quaternionic twist form. Then, the "standard quaternionic elementary deformation" g_Q, defined for a fixed nonzero constant parameter K by

$$g_Q(u,v) = \frac{K}{\|\mu^Z\|}\,\omega_Q(J^Z u, v) + K\,\mathrm{d}\left(\frac{\iota_Z g}{\|\mu^Z\|^2}\right)(J^Z u, v), \tag{2.58}$$

is an elementary deformation of (M, g, Q) by Z.

Proof. We continue to work in the frame $(J_1' = J^Z, J_2', J_3')$ introduced in the proof of Lemma 2.2.7. Substituting the local expression (2.52) for ω_Q into (2.58) and simplifying, we get

$$g_Q(u,v) = -\frac{K}{\|\mu^Z\|^3}\left(\|\mu^Z\|^2(\nu\omega_1' - \alpha_{21}'\wedge\alpha_{13}') + \mathrm{d}\|\mu\|\wedge\iota_Z g\right)(J_1'u, v). \tag{2.59}$$

Next we use (2.49) to see that

$$\begin{aligned}
g_Q(u,v) &= -\frac{K}{\|\mu^Z\|^3}(\nu\|\mu^Z\|^2\omega_1' + \iota_Z\omega_2' \wedge \iota_Z\omega_3') - \iota_Z\omega_1 \wedge \iota_Z g)(J_1'u,v) \\
&= -\frac{K}{\|\mu^Z\|^3}(-\nu\|\mu^Z\|^2 g(u,v) + g(J_2'Z, J_1'u)g(J_3'Z,v) - g(J_3'Z,J_1'u)g(J_2'Z,v) \\
&\quad - g(J_1'Z,J_1'u)g(Z,v) + g(Z,J_1'u)g(J_1'Z,v)) \\
&= \frac{K}{\|\mu^Z\|^3}(\nu\|\mu^Z\|^2 g(u,v) + g(Z,u)g(Z,v) + g(J_1'Z,u)g(J_1'Z,v) \\
&\quad + g(J_2'Z,u)g(J_2'Z,v) + g(J_3'Z,u)g(J_3'Z,v)).
\end{aligned} \tag{2.60}$$

It is now easy to see that this metric is the elementary deformation

$$g_Q = \frac{K\nu}{\|\mu^Z\|} g|_{\mathbb{H}_Q Z^\perp} - \frac{K f_Q}{\|\mu^Z\|^2} g|_{\mathbb{H}_Q Z}. \tag{2.61}$$

□

Lemma 2.2.11. *Given a constant κ and a quaternionic Kähler manifold (M, g, Q) equipped with Killing field Z with nowhere vanishing quaternionic moment map μ^Z with normalisation J^Z, we can always find a local oriented orthonormal frame $(J_1' = J^Z, J_2', J_3')$ for the quaternionic bundle Q such that*

$$\mathscr{L}_Z J_2' = \kappa J_3', \quad \mathscr{L}_Z J_3' = -\kappa J_1'. \tag{2.62}$$

Proof. Since the quaternionic bundle Q, the quaternionic moment map μ^Z, and hence its normalisation J^Z, are all Z-invariant, the Lie derivative of any section of Q orthogonal to J^Z must necessarily be a section of Q orthogonal to J^Z. So, any local oriented orthonormal frame of the form $(J_1'' = J^Z, J_2'', J_3'')$ must necessarily satisfy

$$\begin{aligned}
\mathscr{L}_Z J_2'' &= \langle J_2'', \mathscr{L}_Z J_2''\rangle J_2'' + \langle J_3'', \mathscr{L}_Z J_2''\rangle J_3'' = -\langle J_2'', \mathscr{L}_Z J_3''\rangle J_3'', \\
\mathscr{L}_Z J_3'' &= \langle J_2'', \mathscr{L}_Z J_3''\rangle J_2'' + \langle J_3'', \mathscr{L}_Z J_3''\rangle J_3'' = \langle J_2'', \mathscr{L}_Z J_3''\rangle J_2'',
\end{aligned} \tag{2.63}$$

where we have used the fact that

$$\langle J_i'', \mathscr{L}_Z J_j''\rangle + \langle J_j'', \mathscr{L}_Z J_i''\rangle = \mathscr{L}_Z\langle J_i'', J_j''\rangle = 0. \tag{2.64}$$

Now we define a new local oriented orthonormal frame (J_1', J_2', J_3') given by

$$J_1' = J_1'' = J^Z, \quad J_2' = \cos(\chi)J_2'' - \sin(\chi)J_3'', \quad J_3' = \sin(\chi)J_2'' + \cos(\chi)J_3'', \tag{2.65}$$

where χ is a smooth function satisfying the following inhomogeneous linear PDE:

$$Z(\chi) + \langle J_2'', \mathscr{L}_Z J_3''\rangle + \kappa = 0. \tag{2.66}$$

Then we may check that

$$\begin{aligned}
\mathscr{L}_Z J_2' &= -(Z(\chi) + \langle J_2'', \mathscr{L}_Z J_3''\rangle)J_3' = \kappa J_3', \\
\mathscr{L}_Z J_3' &= (Z(\chi) + \langle J_2'', \mathscr{L}_Z J_3''\rangle)J_3' = -\kappa J_2'.
\end{aligned} \tag{2.67}$$

□

Example 2.2.12. Continuing with Example 2.2.6, we note that the quaternionic moment map for Z_1 vanishes on the line

$$x = y = \tau = 0, \tag{2.68}$$

but those of Z_2, Z_3, Z_4 are nowhere vanishing, with norms given by

$$\|\mu^2\| = \frac{1}{4\rho}\sqrt{y^2 + 2\rho}, \quad \|\mu^3\| = \frac{1}{4\rho}\sqrt{x^2 + 2\rho}, \quad \|\mu^4\| = \frac{1}{4\rho}. \tag{2.69}$$

Therefore, it ought to be possible to find for each of these Z_a, local oriented orthonormal frames (J_1', J_2', J_3') for Q such that

$$\mathscr{L}_{Z_a} J_1' = 0, \quad \mathscr{L}_{Z_a} J_2' = \kappa J_3', \quad \mathscr{L}_{Z_a} J_3' = -\kappa J_2'. \tag{2.70}$$

In case of Z_2, such a frame is given by

$$\begin{aligned} J_1' &= \frac{\mu^2}{\|\mu^2\|} = \frac{yJ_1 - \sqrt{2\rho}\, J_3}{\sqrt{y^2 + 2\rho}}, \\ J_2' &= \cos(-2\kappa x)J_2 - \sin(-2\kappa x)J_1' \circ J_2, \\ J_3' &= \sin(-2\kappa x)J_2 + \cos(-2\kappa x)J_1' \circ J_2. \end{aligned} \tag{2.71}$$

In case of Z_3, such a frame is given by

$$\begin{aligned} J_1' &= \frac{\mu^2}{\|\mu^2\|} = \frac{-xJ_1 + \sqrt{2\rho}\, J_2}{\sqrt{x^2 + 2\rho}}, \\ J_2' &= \cos(2\kappa y)J_3 \circ J_1' - \sin(2\kappa y)J_3, \\ J_3' &= \sin(2\kappa y)J_3 \circ J_1' + \cos(2\kappa y)J_3. \end{aligned} \tag{2.72}$$

And finally, in case of Z_4, such a frame is given by

$$\begin{aligned} J_1' &= \frac{\mu^2}{\|\mu^2\|} = J_1, \\ J_2' &= \cos(-\kappa\tau)J_2 - \sin(-\kappa\tau)J_3, \\ J_3' &= \sin(-\kappa\tau)J_2 + \cos(-\kappa\tau)J_3. \end{aligned} \tag{2.73}$$

In particular, for the final case we have

$$\alpha_{23}' = \alpha_{23} - \kappa\, \mathrm{d}\tau = -\vartheta_4 - \kappa\, \mathrm{d}\tau. \tag{2.74}$$

2.3 Przanowski–Tod Ansatz and continuous Toda

2.3.1 Quaternionic moment map for the Ansatz

The next thing to do is to generalise the computation of the quaternionic moment map for $\mathbb{C}\mathbf{H}^2$ to a general quaternionic Kähler manifold of dimension 4 admitting a Killing vector field. It turns out that such quaternionic Kähler manifolds are completely characterised by solutions to an integrable system known as the continuous Toda system in dimension 3.

Theorem 2.3.1 ([Prz91], [Tod95]). *Any quaternionic Kähler manifold of dimension* 4 *with nonzero scalar curvature and at least one* $\mathrm{U}(1)$ *isometry is locally isometric to an open set* *in $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{C}$, coordinatised by (ρ, τ, ζ), and equipped with a metric g admitting the following Ansatz:*

$$g = \frac{1}{4\rho^2} \left(P\, \mathrm{d}\rho^2 + 2Pe^u |\mathrm{d}\zeta|^2 + \frac{1}{P} (\mathrm{d}\tau + \Theta)^2 \right), \tag{2.75}$$

where P and u are ∂_τ-invariant smooth functions, and Θ is a ∂_τ-invariant 1-form satisfying

$$\begin{aligned} &\partial_\zeta \partial_{\bar\zeta} u = -\frac{1}{2} \partial_\rho^2 (e^u), \quad P = \frac{2}{\nu} (\rho\, \partial_\rho u - 2) > 0 \\ &\mathrm{d}\Theta = \mathrm{i} \left(\left(\partial_\zeta P\, \mathrm{d}\zeta - \partial_{\bar\zeta} P\, \mathrm{d}\bar\zeta \right) \wedge \mathrm{d}\rho - \partial_\rho (Pe^u)\, \mathrm{d}\zeta \wedge \mathrm{d}\bar\zeta \right), \end{aligned} \tag{2.76}$$

with ν a nonzero constant. Conversely, any metric that locally admits the above Ansatz admits a quaternionic Kähler structure with nonzero scalar curvature.

Proof. The proof that every quaternionic Kähler manifold M of dimension 4 with nonzero scalar curvature and one $\mathsf{U}(1)$ isometry locally admits this Ansatz is omitted. As for the converse, we prove it by computing the curvature of the bundle of anti-self-dual 2-forms $\Lambda^2_- T^*M$.

As earlier, let us set $\zeta = x + \mathrm{i}y$ and choose the following orthonormal frame ϑ^a for the cotangent bundle:

$$\vartheta^1 = \frac{\sqrt{P}}{2\rho} \mathrm{d}\rho, \quad \vartheta^2 = \frac{e^{u/2}\sqrt{P}}{\sqrt{2}\,\rho} \mathrm{d}x, \quad \vartheta^3 = \frac{e^{u/2}\sqrt{P}}{\sqrt{2}\,\rho} \mathrm{d}y, \quad \vartheta^4 = \frac{1}{2\rho\sqrt{P}} (\mathrm{d}\tau + \Theta). \tag{2.77}$$

Then we can choose the orthonormal frame for $\Lambda^2_- T^*M$ to be

$$\omega_1 = \vartheta^2 \wedge \vartheta^3 - \vartheta^1 \wedge \vartheta^4, \quad \omega_2 = \vartheta^3 \wedge \vartheta^1 - \vartheta^2 \wedge \vartheta^4, \quad \omega_3 = \vartheta^1 \wedge \vartheta^2 - \vartheta^3 \wedge \vartheta^4. \tag{2.78}$$

The associated connection 1-forms α_{ij} are then given by

$$\begin{aligned} \alpha_{23} &= \frac{1}{2} (\partial_y u\, \mathrm{d}x - \partial_x u\, \mathrm{d}y) + \frac{\nu}{4\rho} (\mathrm{d}\tau + \Theta), \\ \alpha_{31} &= \frac{e^{u/2}}{\sqrt{2}\,\rho} \left(\rho\, \partial_\rho u - \frac{\nu}{2} P - 4 \right) \mathrm{d}y = -\frac{2\, e^{u/2}}{\sqrt{2}\,\rho} \mathrm{d}y, \\ \alpha_{12} &= -\frac{e^{u/2}}{\sqrt{2}\,\rho} \left(\rho\, \partial_\rho u - \frac{\nu}{2} P - 4 \right) \mathrm{d}x = \frac{2\, e^{u/2}}{\sqrt{2}\,\rho} \mathrm{d}x. \end{aligned} \tag{2.79}$$

One may now readily check that the curvature β_{ij} satisfies the quaternionic Kähler condition:

$$\beta_{ij} = \mathrm{d}\alpha_{ij} + \sum_k \alpha_{ik} \wedge \alpha_{kj} = \nu \sum_k \epsilon_{ijk} \omega_k, \tag{2.80}$$

where ϵ_{ijk} is the Levi-Civita symbol. □

Remark 2.3.2. This representation of quaternionic Kähler metrics in terms of a Toda potential u (which fixes P and Θ up to a locally exact term that may be absorbed into $\mathrm{d}\tau$) is far from unique. In fact, we can easily see that the equations (2.76) admit the following gauge symmetry:

$$u(\rho, \zeta, \bar\zeta) \mapsto u(\rho, h(\zeta), \overline{h(\zeta)}) - \ln \left| \frac{\mathrm{d}h}{\mathrm{d}\zeta} \right|^2, \tag{2.81}$$

where h is a holomorphic function whose derivative is nowhere vanishing on the domain of definition.

We now compute the quaternionic moment map μ^Z and the tensor fields f_Q, ω_Q, g_Q built out of it for the Przanowski–Tod Ansatz.

Example 2.3.3. The Hermitian structures J_i associated to the ω_i given in (2.78) are again invariant under $Z = \partial_\tau$, so we can again use (2.44) to obtain

$$\mu^Z = \frac{1}{4\rho} J_1, \quad \|\mu^Z\| = \frac{1}{4\rho}. \tag{2.82}$$

Using this, we can directly read off f_Q and g_Q as follows:

$$\begin{aligned} f_Q &= -\frac{1}{P\rho} - \frac{\nu}{4\rho} \overset{(2.76)}{=} -\frac{\partial_\rho u}{2P}, \\ g_Q &= -\frac{K f_Q}{\|\mu^Z\|^2} g|_{\mathbb{H}_Q Z} = \frac{8K}{P} \rho^2 (\partial_\rho u)\, g \\ &= 2K(\partial_\rho u) \left(\mathrm{d}\rho^2 + 2e^u |\mathrm{d}\zeta|^2 + \frac{1}{P^2} (\mathrm{d}\tau + \Theta)^2 \right). \end{aligned} \tag{2.83}$$

As for the 2-form ω_Q, in anticipation of future use, it will be convenient to just leave it as

$$\begin{aligned} \omega_Q &\overset{(2.54)}{=} -\mathrm{d} \left(\frac{1}{P\rho} (\mathrm{d}\tau + \Theta) + \frac{1}{2} (\partial_y u \, \mathrm{d}x - \partial_x u \, \mathrm{d}y) + \frac{\nu}{4\rho} (\mathrm{d}\tau + \Theta) \right) \\ &\overset{(2.76)}{=} -\frac{1}{2} \mathrm{d} \left(\frac{1}{P} (\partial_\rho u)(\mathrm{d}\tau + \Theta) + \partial_y u \, \mathrm{d}x - \partial_x u \, \mathrm{d}y \right). \end{aligned} \tag{2.84}$$

2.3.2 Explicit solutions of cohomogeneity one

Following Ketov in [Ket01], we now use the Przanowski–Tod Ansatz to obtain explicit examples of quaternionic Kähler manifolds of dimension 4.

Example 2.3.4. A natural class of examples of Toda potentials u to consider are potentials that depend only on ρ. The continuous Toda equation in dimension 3, i.e.

$$\partial_\zeta \partial_{\bar\zeta} u = -\frac{1}{2} \partial_\rho^2 (e^u), \tag{2.85}$$

then implies that e^u is at most a linear polynomial in ρ. The most general solution is therefore given by

$$e^u = b\rho + c. \tag{2.86}$$

However, since we can always absorb an overall nonzero factor into a gauge shift, there are essentially two cases:

$$e^u = 1, \quad \text{or} \quad e^u = \rho + c. \tag{2.87}$$

In the first case, we have the Poincaré half-space model of the real hyperbolic 4-space, which is a symmetric space as well:

$$g = -\frac{1}{\nu\rho^2} \left(\mathrm{d}\rho^2 + 4|\mathrm{d}\zeta|^2 + \left(\frac{\nu}{4} \mathrm{d}\tau \right)^2 \right). \tag{2.88}$$

Meanwhile, in the second case, we have the following family of metrics:

$$g'^c = -\frac{1}{2\nu\rho^2}\left(\frac{\rho+2c}{\rho+c}\,\mathrm{d}\rho^2 + 2(\rho+2c)|\mathrm{d}\zeta|^2 + \frac{\rho+c}{\rho+2c}\left(-\frac{\nu}{2}\,\mathrm{d}\tau + \mathrm{Im}(\zeta\,\mathrm{d}\overline{\zeta})\right)^2\right). \tag{2.89}$$

The prime is included for notational consistency with Chapter 5. Note that we retrieve the $\mathbb{C}\mathbf{H}^2$ metric (2.29) when $\nu = -2$ and $c = 0$. The parameter ν may be absorbed into a rescaling of τ and thus contributes only to an overall factor in front on the metric. The parameter c meanwhile gives what is referred to in physics literature as the 1-*loop deformation* g'^c_{UH} of the *universal hypermultiplet*.

Example 2.3.5. Another natural class of metrics arise from separable exact solutions to the Toda equation i.e. solutions of the form

$$u(\rho,\zeta,\overline{\zeta}) = F(\rho) + G(\zeta,\overline{\zeta}), \tag{2.90}$$

where F and G satisfy

$$\partial_\zeta\partial_{\overline{\zeta}} G = -ae^G, \quad \partial_\rho^2 e^F = 2a \neq 0. \tag{2.91}$$

The equation for F implies that e^F is a quadratic polynomial in ρ with leading coefficient a. That is, its general solutions are given by

$$e^F = a\rho^2 + b\rho + c. \tag{2.92}$$

Note that the left-hand side is always positive, so a has to be positive as well. Meanwhile, the equation for G is just the 2d Liouville equation whose general solutions are known to be of the form

$$e^G = \frac{4}{(1+2a|h(\zeta)|^2)^2}\left|\frac{\mathrm{d}h}{\mathrm{d}\zeta}\right|^2, \tag{2.93}$$

where h is some holomorphic function which is nonvanishing in the domain of definition. The freedom to choose h may be absorbed into a gauge transformation (2.81). In particular, we may set $h(\zeta) = \frac{1}{2}\zeta$ without any loss of generality. This gives us the following exact solution for u:

$$e^u = e^F e^G = \frac{a\rho^2 + b\rho + c}{(1+\frac{a}{2}|\zeta|^2)^2}. \tag{2.94}$$

The quaternionic Kähler metric this then yields is

$$\begin{aligned} g^{a,b,c} = &-\frac{1}{2\nu\rho^2}\left(\frac{b\rho+2c}{a\rho^2+b\rho+c}\,\mathrm{d}\rho^2 + \frac{2(b\rho+2c)|\mathrm{d}\zeta|^2}{(1+\frac{a}{2}|\zeta|^2)^2}\right. \\ &\left.+\frac{a\rho^2+b\rho+c}{b\rho+2c}\left(-\frac{\nu}{2}\,\mathrm{d}\tau + \frac{b\,\mathrm{Im}(\zeta\,\mathrm{d}\overline{\zeta})}{1+\frac{a}{2}|\zeta|^2}\right)^2\right). \end{aligned} \tag{2.95}$$

It has been known in the physics community that the cohomogeneity of the metric g'^c in (2.89) is at most 1 and that of the metric $g^{a,b,c}$ in (2.95) is at most 2. Here, we show that both their cohomogeneities happen to be exactly 1 when $a,b,c > 0$.

In addition to this, we also retrieve the Pedersen family of quaternionic Kähler metrics γ^m on the product manifold $(0,1) \times \mathbb{S}^3$, given in equation (1.6) of [Ped86], from $g^{a,b,c}$. In Theorem 12 of [CS17], we had shown that the Pedersen family is indeed different from the 1-loop deformation g'^c_{UH} of the universal hypermultiplet. The proof of Proposition 2.3.7 thus clarifies how the two families are related.

Proposition 2.3.6. *Given a constant $c > 0$, the full isometry group of the 1-loop deformation g'^c_{UH} of the universal hypermultplet metric given in (2.89) is the semidirect product of a Heisenberg group with a rotation group*

$$\mathsf{Heis}_3(\mathbb{R}) \rtimes \mathsf{O}(2) \tag{2.96}$$

consisting of isometries of one of the following two forms:

$$\begin{aligned} (\rho,\zeta,\tau) &\mapsto \left(\rho, e^{\mathrm{i}\theta}(\zeta+\zeta'), \tau+\tau'+\frac{2}{\nu}\operatorname{Im}(\zeta'\overline{\zeta})\right), \\ (\rho,\zeta,\tau) &\mapsto \left(\rho, e^{-\mathrm{i}\theta}(\overline{\zeta}+\overline{\zeta}'), -\tau-\tau'-\frac{2}{\nu}\operatorname{Im}(\zeta'\overline{\zeta})\right), \end{aligned} \tag{2.97}$$

where $\tau', \theta \in \mathbb{R}$ and $\zeta' \in \mathbb{C}$ are arbitrary constants. In particular, g'^c is of cohomogeneity 1.

Proof. The curvature norm of the metric g, which may be computed using the results of [CS17] to be

$$\operatorname{tr}(\mathscr{R}^2) = 6\nu^2\left(1+\left(\frac{\rho}{\rho+2c}\right)^6\right), \tag{2.98}$$

is an injective function of ρ, and so any isometry of g must necessarily preserve constant ρ hypersurfaces. Moreover, it must preserve the unit normal field of these hypersurfaces, which is

$$\sqrt{-2\nu}\,\rho\sqrt{\frac{\rho+c}{\rho+2c}}\,\partial_\rho. \tag{2.99}$$

This implies that an isometry of g'^c must necessarily be of the form

$$(\rho,\zeta,\tau) \mapsto (\rho,(\phi(\zeta,\tau)), \tag{2.100}$$

for some automorphism ϕ of $\mathbb{C}\times\mathbb{R}$.

This already significantly constrains the form of any possible isometry of g'^c. In order to constrain this even further, we adopt the following strategy. First we describe the full Killing algebra of g'^c and then we'll use the fact that any isometry should induce an automorphism of the Killing algebra.

Note that (2.100) in particular implies that any Killing field Z of g must be of the form

$$Z = A_2(\zeta,\tau)Z_2 + A_3(\zeta,\tau)Z_3 + A_4(\zeta,\tau)Z_4, \tag{2.101}$$

where the A_a are some functions of $\zeta = x + \mathrm{i}y$ and τ, and the Z_a are minor modifications of the Z_a defined in (2.43), namely

$$Z_2 = \frac{1}{2}\left(\partial_x - \frac{2y}{\nu}\partial_\tau\right), \quad Z_3 = \frac{1}{2}\left(\partial_y + \frac{2x}{\nu}\partial_\tau\right), \quad Z_4 = \partial_\tau. \tag{2.102}$$

We can deduce the following from the above form of the Killing field:

$$-2\nu\rho^2\mathscr{L}_Z g'^c = 2(\rho+2c)\mathscr{L}_Z(|\mathrm{d}\zeta|^2) + \frac{\rho+c}{\rho+2c}\mathscr{L}_Z\left(\left(-\frac{\nu}{2}\mathrm{d}\tau + \operatorname{Im}(\zeta\,\mathrm{d}\overline{\zeta})\right)^2\right). \tag{2.103}$$

Since this must hold for all $\rho > 0$, it can be concluded that the Lie derivatives vanish separately:

$$\mathscr{L}_Z(|d\zeta|^2) = \mathscr{L}_Z\left(\left(-\frac{\nu}{2}\,d\tau + \mathrm{Im}(\zeta\,d\bar{\zeta})\right)^2\right) = 0. \tag{2.104}$$

Substituting (2.101) into the above and using the fact that Z_2, Z_3, Z_4 are themselves Killing fields satisfying the above, we get the following system of differential equations for the functions A_a:

$$dA_2\,dx + dA_3\,dy = 0, \tag{2.105a}$$

$$y\,dA_2 - x\,dA_3 - \frac{\nu}{2}\,dA_4 = 0. \tag{2.105b}$$

The only way (2.105a) can hold is if A_2 is a function solely of y, A_3 is a function solely of x and they satisfy

$$\partial_y A_2 = -\partial_x A_3. \tag{2.106}$$

Since this is an equality of a function of y and a function of x, it follows that they are both equal to some constant k, which in particular means that A_2 and A_3 are both affine linear. As we already know Z_2, Z_3, Z_4 to be Killing fields, we are only interested in the functions A_2, A_3, A_4 up to a constant term. So, wihout loss of generality, we can take A_2 and A_3 to be linear and given by

$$A_2 = ky, \quad A_3 = -kx. \tag{2.107}$$

Substituting this into (2.105b), we get

$$d\left(\frac{k}{2}\,(x^2+y^2) - \frac{\nu}{2}\,A_4\right) = 0. \tag{2.108}$$

This means that the function under the exterior derivative must be constant. Again, since we are interested in A_4 only up to a constant term, we may take this to be zero. This gives us

$$A_4 = -\frac{k}{\nu}\,(x^2+y^2) = -\frac{k}{\nu}\,|\zeta|^2. \tag{2.109}$$

Thus any Killing field Z of g'^c is necessarily an $\mathbb{R}$-linear combination of Z_2, Z_3, Z_4, and

$$Z_5 := y\,Z_2 - x\,Z_3 - \frac{1}{\nu}\,|\zeta|^2 Z_4 = \frac{1}{2}\,(y\,\partial_x - x\,\partial_y). \tag{2.110}$$

This determines the full Killing algebra of g'^c.

As argued earlier, any isometry must induce an automorphism of the Killing algebra, which in particular has to send the centre to itself. Here, the centre is just spanned by $Z_4 = \partial_\tau$, so ϕ in (2.100) must necessarily be of the form

$$\phi(\zeta,\tau) = (\phi_1(\zeta), \phi_2(\zeta,\tau)). \tag{2.111}$$

As in the case of Killing fields, ϕ must separately satisfy

$$\phi^*|d\zeta|^2 \overset{(2.111)}{=} \phi_1^*|d\zeta|^2 = |d\zeta|^2, \tag{2.112a}$$

$$\phi^*\left(-\frac{\nu}{2}\,d\tau + \mathrm{Im}(\zeta\,d\bar{\zeta})\right)^2 = \left(-\frac{\nu}{2}\,d\tau + \mathrm{Im}(\zeta\,d\bar{\zeta})\right)^2. \tag{2.112b}$$

where ϕ^* denotes the linear map on tensor fields induced by the diffeomorphism

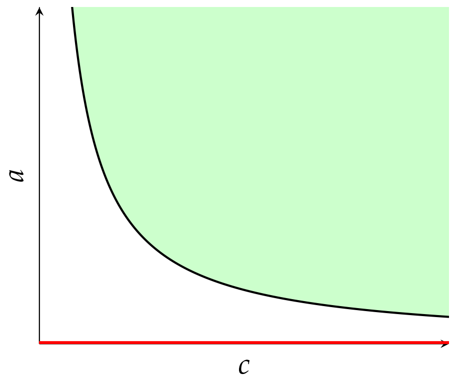

FIGURE 1: Pedersen metrics (light green) versus 1-loop deformed universal hypermultiplet (red).

ϕ via Lie dragging. Note that (2.112a) just means that ϕ_1 is a Euclidean motion (inclusive of reflections) and so entails one of the following two possibilities:

$$\phi_1(\zeta) = e^{\mathrm{i}\theta}(\zeta + \zeta'), \tag{2.113a}$$

$$\phi_1(\zeta) = e^{-\mathrm{i}\theta}(\bar{\zeta} + \bar{\zeta}'), \tag{2.113b}$$

where $\theta \in \mathbb{R}$ and $\zeta' \in \mathbb{C}$ are arbitrary constants. Equation (2.112b) also entails two possibilities:

$$\phi^* \left(-\frac{\nu}{2}\,\mathrm{d}\tau + \mathrm{Im}(\zeta\,\mathrm{d}\bar{\zeta})\right) = \pm\left(-\frac{\nu}{2}\,\mathrm{d}\tau + \mathrm{Im}(\zeta\,\mathrm{d}\bar{\zeta})\right). \tag{2.114}$$

These can be written out more explictly as

$$\frac{\nu}{2}\,\mathrm{d}(\phi_2(\zeta,\tau) - \tau) = \mathrm{Im}\left(\phi_1(\zeta)\,\mathrm{d}(\overline{\phi_1(\zeta)}) - \zeta\,\mathrm{d}\bar{\zeta}\right), \tag{2.115a}$$

$$\frac{\nu}{2}\,\mathrm{d}(\phi_2(\zeta,\tau) + \tau) = \mathrm{Im}\left(\phi_1(\zeta)\,\mathrm{d}(\overline{\phi_1(\zeta)}) + \zeta\,\mathrm{d}\bar{\zeta}\right). \tag{2.115b}$$

Since the left-hand sides are exact, so must be the right-hand sides. Therefore, (2.115a) is compatible only with (2.113a), and likewise, (2.115b) is compatible only with (2.113b). These two choices gives us two forms of isometries in (2.97). □

Proposition 2.3.7. *Given constants $a, b, c > 0$, the metric $g^{a,b,c}$ given in (2.95) is of cohomogeneity 1, and when $4ac > b^2$, isometric to a metric in the Pedersen family*

$$\gamma^m = \frac{1}{\nu(1-\varrho^2)^2}\left(\frac{1+m^2\varrho^2}{1+m^2\varrho^4}\,\mathrm{d}\varrho^2 + \varrho^2(1+m^2\varrho^2)(\varsigma_1^2+\varsigma_2^2) + \frac{\varrho^2(1+m^2\varrho^4)}{1+m^2\varrho^2}\varsigma_3^2\right), \tag{2.116}$$

restricted to an open everywhere dense subset of $(0,1)\times\mathbb{S}^3$, where $\varsigma_1, \varsigma_2, \varsigma_3$ are a choice of $\mathsf{SU}(2)$-invariant 1-forms on the 3-sphere $\mathbb{S}^3 \cong \mathsf{SU}(2)$ satisfying

$$\mathrm{d}\varsigma_1 = \varsigma_2 \wedge \varsigma_3. \tag{2.117}$$

Proof. That the cohomogeneity of the metric is a least 1 when $b > 0$ can be seen from the fact that its curvature norm, computed using `SageManifolds` to be

$$\mathrm{tr}(\mathscr{R}^2) = 6\nu^2 \left(1 + b^2((b^2 - 2ac)^2 + 4a^2c^2)\left(\frac{\rho}{\rho + 2c}\right)^6\right), \tag{2.118}$$

is an injective function of $\rho > 0$ whenever

$$b^2((b^2 - 2ac)^2 + 4a^2c^2) \neq 0. \tag{2.119}$$

But for this to vanish, we either need b to vanish or for $b^2 - 2ac$ and ac to simultaneously vanish, which again necessarily implies the vanishing of b.

Meanwhile, that the cohomogoneity is exactly 1 when $a, b > 0$ may be seen from the fact that the constant ρ submanifolds are (at least locally) transitively acted on by isometries

$$(\rho, \tau, \zeta) \mapsto \left(\rho, \tau + \tau' + \frac{4b}{\nu a}\,\mathrm{Im}\left(\ln\left(\sqrt{\frac{a}{2}}\,v\zeta + \overline{w}\right)\right), \frac{w\zeta - \sqrt{\frac{a}{2}}\,\overline{v}}{\sqrt{\frac{a}{2}}\,v\zeta + \overline{w}}\right), \tag{2.120}$$

where $v, w \in \mathbb{C}$ and satisfy $|v|^2 + |w|^2 = 1$. Geometrically, we can interpret ζ as the stereographic coordinate on a Riemann sphere with (v, w) parametrising rotations of the sphere.

Next we turn to the demonstrating the isometry with the Pedersen family γ^m. This is in fact given by the following change of coordinates:

$$\rho = \frac{b}{2a}\left(\frac{1}{\varrho^2} - 1\right), \quad \zeta = \sqrt{\frac{2}{a}}\,\xi, \quad \tau = \frac{2b}{a}\,\theta. \tag{2.121}$$

Note that this is invertible when $0 < \varrho < 1$, with the inverse coordinate transformation given by

$$\varrho = \sqrt{\frac{b}{2a\rho + b}}, \quad \xi = \sqrt{\frac{a}{2}}\,\zeta, \quad \theta = \frac{a}{2b}\,\tau. \tag{2.122}$$

Under this coordinate transformation, the metric in (2.95) becomes

$$g = \frac{1}{\nu(1 - \varrho^2)^2}\left(\frac{1 + k\varrho^2}{1 + k\varrho^4}\,\mathrm{d}\varrho^2 + \varrho^2(1 + k\varrho^2)(\varsigma_1^2 + \varsigma_2^2) + \frac{\varrho^2(1 + k\varrho^4)}{1 + k\varrho^2}\varsigma_3^2\right), \tag{2.123}$$

where k is given by

$$k = \frac{4ac}{b^2} - 1, \tag{2.124}$$

and $\varsigma_1, \varsigma_2, \varsigma_3$ are 1-forms given by

$$\varsigma_1 = \frac{\mathrm{Re}\left(e^{\mathrm{i}\theta}\mathrm{d}\xi\right)}{1 + |\xi|^2}, \quad \varsigma_2 = \frac{\mathrm{Im}\left(e^{\mathrm{i}\theta}\mathrm{d}\xi\right)}{1 + |\xi|^2}, \quad \varsigma_3 = \frac{1}{2}\,\mathrm{d}\theta + \frac{\mathrm{Im}(\xi\,\mathrm{d}\overline{\xi})}{1 + |\xi|^2}. \tag{2.125}$$

These constitute a choice of $\mathrm{SU}(2)$-invariant 1-forms on the unit 3-sphere $\mathbb{S}^3$ parametrised by the coordinates $(\xi, \theta) \in \mathbb{C} \times \mathbb{R}/2\pi\mathbb{Z}$ as follows:

$$\left(\frac{e^{\mathrm{i}\theta}\xi}{1 + |\xi|^2}, \frac{e^{\mathrm{i}\theta}}{1 + |\xi|^2}\right) \in \mathbb{S}^3 \subset \mathbb{C}^2. \tag{2.126}$$

When $k \geqslant 0$, this family is precisely the Pedersen family of metrics with $k = m^2$. □

Remark 2.3.8. The distinction between the 1-loop-deformed universal hypermultiplet metric and the Pedersen family may be more easily visualised by setting $b = 1$ so that (2.124) becomes

$$k = 4ac - 1. \tag{2.127}$$

In order for the resulting metric (2.95) to be Pedersen, k must be nonnegative and so $4ac$ must be at least 1. By contrast, we obtain the 1-loop-deformed universal hypermultiplet if we set $a = 0$. This is depicted in Figure 1.

Chapter 3

The twist construction

In this chapter, we review the twist construction due to Swann, and derive some of its basic properties, especially how it interacts with various derivative operators such as the exterior derivative, the Lie derivative, and the Levi-Civita connection.

The approach to the twist construction undertaken here is rather different from that of Swann. While Swann considered a $\mathsf{U}(1)$-principal bundle P fibred over both the given manifold M and its twist $\tilde{M}$, we define the twist directly on local patches of M. In the process, we introduce some additional local auxiliary data, but later prove in Proposition 3.3.1 that for tensor fields invariant with respect to the vector field Z that is part of the twist data, the choice of the auxiliary data doesn't matter. The main advantage this approach offers is that it allows us to

(a) avoid having to lift tensor fields on M to P,

(b) bypass several technical difficulties associated with ensuring properness of group actions on P,

(c) use tensor fields which are not Z-invariant to verify local properties that the twists of certain tensor fields need to satisfy.

As a result of this, we give much more direct proofs of certain results originally obtained by Swann in [Swa10], such as Propositions 3.1.13, 3.2.1, and 3.2.2. We are also able to derive some new results, such as Proposition 3.2.4, which, along with Corollary 3.2.6 (proved in a different way in collaboration with Danu Thung in [CST20b]), allows us to transport Hamiltonian Killing fields through the twist, and Proposition 3.2.8, which relates the Levi-Civita connection of the twist of a metric to the Levi-Civita connection of the given metric.

Finally, in Definition 3.3.4, we introduce a global version of the twist as a map sending Z-invariant tensor fields on some manifold M to tensor fields on another manifold $\tilde{M}$. As a necessary existence criterion, we retrieve (a slight generalisation of) the $\mathsf{U}(1)$-principal bundle P in Swann's construction as a submanifold of $M \times \tilde{M}$ in Theorem 3.3.9 and show that the existence of such a double surjection is sufficient in Theorem 3.3.10. We further illustrate the obstruction that the irrationality of ω poses to the existence of a global twist map through Example 3.3.11.

3.1 The local twist map

The twist construction was introduced by Swann in [Swa10] in order to unify and generalise several constructions involving manifolds endowed with hypercomplex structures inspired by the celebrated T-duality in physics. The construction is essentially a correspondence of manifolds equipped with certain twist data.

Definition 3.1.1 (Twist data)**.** A manifold M is said to be equipped with twist data (Z, ω, f) where Z is a nowhere vanishing vector field, ω is a closed 2-form, and $f : M \to \mathbb{R}$ is a nowhere vanishing smooth Z-invariant function satisfying

$$\iota_Z \omega = -\mathrm{d}f. \tag{3.1}$$

Definition 3.1.2 (Twist automorphism)**.** A twist automorphism ϕ of a manifold equipped with twist data (Z, ω, f) is a diffeomorphism $\phi : M \to M$ preserving all the twist data:

$$\phi^* Z = Z, \quad \phi^* \omega = \omega, \quad \phi^* f = f. \tag{3.2}$$

The group of twist automorphisms shall be denoted $\mathsf{Aut}(Z, \omega, f)$.

Example 3.1.3. From Lemma 2.2.7 we know that a quaternionic Kähler manifold with a nowhere vanishing Killing field Z with nowhere vanishing quaternionic moment map μ^Z such that f_Q as defined in (2.57) is nowhere vanishing, admits twist data (Z, ω_Q, f_Q), where ω_Q is as defined in (2.47).

In Swann's formulation, the above data, along with the hypothesis that ω belongs to an integral cohomology class, was sufficient to construct the twist. In our approach we need some extra data that in general is only locally defined. We shall later show that this extra data, although necessary to explicitly construct the twist, is ultimately superfluous up to local diffeomorphisms.

Definition 3.1.4 (Auxiliary local twist data)**.** Auxiliary local twist data (U, η) associated to twist data (Z, ω, f) on a manifold M consists of an open set $U \subseteq M$ on which ω is exact, and a choice of local 1-form $\eta \in \Gamma(T^*U)$ such that the function $f - \eta(Z)$ is nowhere vanishing on U and

$$\omega|_U = \mathrm{d}\eta. \tag{3.3}$$

Moreover, the full tuple (U, Z, ω, f, η) consisting of the twist data and auxiliary local twist data shall be referred to as just *local twist data*.

Example 3.1.5. Let (Z, ω, f) be twist data on some manifold M such that Z acts properly on M. Then let $\pi_B : M \to B$ denote the quotient by the Z-action and let $\tau \in \mathbb{R}/\mathbb{Z}$ be the fibre coordinate so that $Z = \partial_\tau$ and

$$\omega =: \mathrm{d}f \wedge \mathrm{d}\tau + \pi_B^*(\omega^B). \tag{3.4}$$

Note that ω^B is closed. We choose an open set $U = \pi_B^{-1}(U^B)$ such that U^B is contractible. In particular, the restriction $\omega^B|_{U^B}$ is exact. Let $\omega^B|_{U^B} = \mathrm{d}\eta^B$, and let

$$\eta = (f+1)\,\mathrm{d}\tau + \pi_B^*(\eta^B). \tag{3.5}$$

Then $\mathrm{d}\eta = \omega|_U$ and $f - \eta(Z) = -1$. This therefore gives us auxiliary local twist data (U, η).

Example 3.1.6. In general, local oriented orthonormal frames for the quaternionic bundle Q on a quaternionic Kähler manifold M is guaranteed to exist only over contractible open sets of M. Let U be such a set. If we choose a frame $(J_1' = J^Z, J_2', J_3')$ over U as in the proof of Lemma 2.2.7, then (2.54) provides a potential candidate for the auxiliary 1-form η_Q:

$$\eta_Q = -\frac{\iota_Z g}{\|\mu^Z\|} - \alpha_{23}'. \tag{3.6}$$

For this to actually qualify, we need to ensure that

$$f_Q - \eta_Q(Z) = -\nu\|\mu^Z\| + \alpha'_{23}(Z) \overset{(2.41)}{=} -\langle J'_2, (\nabla_Z - \mathscr{L}_Z)J'_3\rangle + \langle J'_2, \nabla_Z J'_3\rangle \\ = \langle J'_2, \mathscr{L}_Z J'_3\rangle \tag{3.7}$$

is nowhere vanishing. But from Lemma 2.2.11, we know that this can be chosen to be any (nonzero) constant κ.

Note that J'_1 is globally well-defined while $\alpha'_{23} = \langle J'_2, \nabla^g J'_3\rangle$ (and hence η_Q) remains preserved under a constant rotation

$$(J'_2, J'_3) \mapsto (\cos(a)J'_2 + \sin(a)J'_3, -\sin(a)J'_2 + \cos(a)J'_3). \tag{3.8}$$

When the Z-action is proper, this observation can be used to define local twist data on Z-invariant open sets U even when they are not contractible (which is indeed the case when the Z-action is a $\mathrm{U}(1)$-action) with the help of a contractible open cover of U.

Using this extra data, we can now define a local linear map acting on tensor fields

$$S \in \Gamma(T^*U^{\otimes\bullet} \otimes TU^{\otimes\bullet}) =: \Gamma(\mathbb{T}^{\bullet,\bullet}U). \tag{3.9}$$

Definition 3.1.7 (Local twist map)**.** The local twist map $\mathrm{tw}_{Z,f,\eta}$ with respect to local twist data (U, Z, ω, f, η) is a graded $C^\infty(U)$-linear map

$$\mathrm{tw}_{Z,f,\eta} : \Gamma(\mathbb{T}^{\bullet,\bullet}U) \to \Gamma(\mathbb{T}^{\bullet,\bullet}U) \tag{3.10}$$

of tensor fields, compatible with tensor products and contractions, whose action on an arbitrary function h and 1-form α is given by

$$\mathrm{tw}_{Z,f,\eta}(h) = h, \quad \mathrm{tw}_{Z,f,\eta}(\alpha) = \alpha - \frac{\alpha(Z)}{f}\eta. \tag{3.11}$$

Remark 3.1.8. As it stands, the above definition obscures why we need $f - \eta(Z)$ to be nowhere vanishing, but this requirement is crucial for the local twist map to be well-defined. In order to see this, we work out the local twist of an arbitrary vector field u. Let us choose an arbitrary 1-form α. Then, compatiblity with contraction means that

$$\begin{aligned} \alpha(u) &= \mathrm{tw}_{Z,f,\eta}(\alpha)(\mathrm{tw}_{Z,f,\eta}(u)) \\ &= \alpha(\mathrm{tw}_{Z,f,\eta}(u)) - \frac{1}{f}\alpha(Z)\eta(\mathrm{tw}_{Z,f,\eta}(u)) \\ &= \alpha\left(\mathrm{tw}_{Z,f,\eta}(u) - \frac{1}{f}\eta(\mathrm{tw}_{Z,f,\eta}(u))Z\right). \end{aligned} \tag{3.12}$$

Since α was arbitrarily chosen, it follows that

$$u = \mathrm{tw}_{Z,f,\eta}(u) - \frac{1}{f}\eta(\mathrm{tw}_{Z,f,\eta}(u))Z. \tag{3.13}$$

Plugging both sides of the above equation into the 1-form η, we find that

$$\eta(u) = \frac{f - \eta(Z)}{f}\eta(\mathrm{tw}_{Z,f,\eta}(u)). \tag{3.14}$$

Thus, if $f - \eta(Z)$ vanished at some point, then u would be forced to lie in the kernel of η at that point. This contradicts the fact that u is arbitrary and so it follows that $f - \eta(Z)$ must be nowhere vanishing. Given this, we may substitute

$$\eta(\mathrm{tw}_{Z,f,\eta}(u)) = \frac{f}{f - \eta(Z)}\,\eta(u) \tag{3.15}$$

into (3.13) and rearrange to obtain the following expression for the action of the local twist map on u:

$$\mathrm{tw}_{Z,f,\eta}(u) = u + \frac{\eta(u)}{f - \eta(Z)}\,Z. \tag{3.16}$$

Remark 3.1.9. Since the local twist map is linear and compatible with tensor products, it preserves symmetries of tensor fields. In particular, the local twists of a k-form α and a symmetric bilinear form field g, given by

$$\begin{aligned}
\mathrm{tw}_{Z,f,\eta}(\alpha) &= \alpha - \frac{1}{f}\,\eta \wedge \iota_Z\alpha,\\
\mathrm{tw}_{Z,f,\eta}(g) &= g - \frac{2}{f}\,\eta\,\iota_Z g + \frac{g(Z,Z)}{f^2}\,\eta^2\\
&= g - \frac{(\iota_Z g)^2}{g(Z,Z)} + \frac{g(Z,Z)}{f^2}\left(\eta - \frac{f}{g(Z,Z)}\,\iota_Z g\right)^2\\
&= g - \frac{1}{g(Z,Z)}((\iota_Z g)^2 - (\mathrm{tw}_{Z,f,\eta}(\iota_Z g))^2).
\end{aligned} \tag{3.17}$$

are also a k-form and a symmetric bilinear form field, respectively.

Moreover, since the local twist is compatible with contractions, the local twist of a nondegenerate symmetric bilinear form is again nondegenerate with the same signature and the local twist of endomorphism fields A_i, given by

$$\begin{aligned}
\mathrm{tw}_{Z,f,\eta}(A_i) = A_i &+ \frac{1}{f - \eta(Z)}\,[\eta \otimes Z, A_i]\\
&+ \frac{1}{f(f - \eta(Z))}(\eta(Z)A_i - \mathrm{tr}((\eta \otimes Z) \circ A_i)\mathrm{id}) \circ (\eta \otimes Z),
\end{aligned} \tag{3.18}$$

satisfy any algebraic relations that A_i does. In particular, the local twist of the identity endomorphism field is the identity endomorphism field itself and that of local AQH structures J_i with respect to some metric g are again local AQH structures $\mathrm{tw}_{Z,f,\eta}(J_i)$ with respect to $\mathrm{tw}_{Z,f,\eta}(g)$.

Note however that if J_i are quaternionic *Kähler* structures with respect to g, the local twists $\mathrm{tw}_{Z,f,\eta}(J_i)$ will *not* be quaternionic Kähler structures with respect to $\mathrm{tw}_{Z,f,\eta}(g)$. This is because the quaternionic Kähler property is a differential criterion and not an algebraic one.

Example 3.1.10 (Trivial twist)**.** The tuple (U, ω, Z, f, η) with $\omega = \eta = 0$ and f a nonzero constant constitute valid local twist data. The local twist with respect to it is just the identity map.

Example 3.1.11. The local twist data associated with the quaternionic Kähler metric (2.29) on $\mathbb{C}\mathbf{H}^2$, equipped with the Killing field $Z = Z_4 = \partial_\tau$, may be taken to be

$$\begin{aligned} f_Q &= -\frac{g(Z,Z)}{\|\mu^Z\|} - \nu\|\mu^Z\| = -\frac{(4\rho^2)^{-1}}{(4\rho)^{-1}} + \frac{2}{4\rho} = -\frac{1}{2\rho}, \\ \eta_Q &= -\frac{\iota_Z g}{\|\mu^Z\|} - \alpha'_{23} = -\frac{(2\rho)^{-1}}{(4\rho)^{-1}}\vartheta^4 + \vartheta^4 + \kappa\,\mathrm{d}\tau = -\vartheta^4 + \kappa\,\mathrm{d}\tau, \\ \omega_Q &= \mathrm{d}\eta_Q = -\mathrm{d}\vartheta^4 = \frac{1}{\rho}\,\mathrm{d}x\wedge\mathrm{d}y + \frac{1}{2\rho^2}\,\mathrm{d}\rho\wedge(\mathrm{d}\tau + y\,\mathrm{d}x - x\,\mathrm{d}y) \\ &= 2(\vartheta^2\wedge\vartheta^3 + \vartheta^1\wedge\vartheta^4). \end{aligned} \tag{3.19}$$

In particular, we have

$$\begin{aligned} \mathrm{tw}_{Z,f_Q,\eta_Q}(\iota_Z g) &= \iota_Z g - \frac{g(Z,Z)}{f_Q}\eta_Q \\ &= \frac{1}{2\rho}\vartheta^4 + \frac{(4\rho^2)^{-1}}{(2\rho)^{-1}}(-\vartheta^4 + \kappa\,\mathrm{d}\tau) = \frac{\kappa}{2\rho}\,\mathrm{d}\tau, \\ \mathrm{tw}_{Z,f_Q,\eta_Q}(g) &= g - \frac{1}{g(Z,Z)}((\iota_Z g)^2 - (\mathrm{tw}_{Z,f,\eta}(\iota_Z g))^2) \\ &= \frac{1}{4\rho^2}(\mathrm{d}\rho^2 + 2\rho|\mathrm{d}\zeta|^2) + \kappa^2\mathrm{d}\tau^2. \end{aligned} \tag{3.20}$$

We can now check that

$$\begin{aligned} \mathrm{tw}_{Z,f_Q,\eta_Q}(J_1) &= \mathrm{d}x\otimes\partial_y - \mathrm{d}y\otimes\partial_x - \frac{1}{2\kappa\rho}\,\mathrm{d}\rho\otimes\partial_\tau + 2\kappa\rho\,\mathrm{d}\tau\otimes\partial_\rho, \\ \mathrm{tw}_{Z,f_Q,\eta_Q}(J_2) &= \sqrt{2\rho}\,\mathrm{d}y\otimes\partial_\rho - \frac{1}{\sqrt{2\rho}}\,\mathrm{d}\rho\otimes\partial_y - \frac{1}{\kappa\sqrt{2\rho}}\,\mathrm{d}x\otimes\partial_\tau + \kappa\sqrt{2\rho}\,\mathrm{d}\tau\otimes\partial_x, \\ \mathrm{tw}_{Z,f_Q,\eta_Q}(J_2) &= \frac{1}{\sqrt{2\rho}}\,\mathrm{d}\rho\otimes\partial_x - \sqrt{2\rho}\,\mathrm{d}x\otimes\partial_\rho - \frac{1}{\kappa\sqrt{2\rho}}\,\mathrm{d}y\otimes\partial_\tau + \kappa\sqrt{2\rho}\,\mathrm{d}\tau\otimes\partial_y, \end{aligned} \tag{3.21}$$

form an AQH structure with respect to $\mathrm{tw}_{Z,f_Q,\eta_Q}(g)$. This structure however is not quaternionic Kähler.

Example 3.1.12. The local twist of the standard elementary quaternionic deformation g_Q with respect to the quaternionic twist data (Z, ω_Q, f_Q) is given by

$$\begin{aligned} \mathrm{tw}_{Z,f_Q,\eta_Q}(g_Q) &= g_Q - \frac{1}{g_Q(Z,Z)}((\iota_Z g_Q)^2 - (\mathrm{tw}_{Z,f,\eta}(\iota_Z g_Q))^2) \\ &\overset{(2.61)}{=} g_Q + \frac{Kf_Q}{\|\mu^Z\|^2}\frac{1}{g(Z,Z)}((\iota_Z g)^2 - (\mathrm{tw}_{Z,f,\eta}(\iota_Z g))^2). \end{aligned} \tag{3.22}$$

It makes sense to apply the local twist map to the twist data on an open set U itself. These can in fact be used to obtain a second tuple of local twist data on the same domain U such that the local twist with respect to this second tuple of twist data is inverse to that with respect to the first! Thus, the local twist map may be regarded as an involution of (local) twist data.

Proposition 3.1.13. *If (U, Z, ω, f, η) is local twist data on some manifold M, then so is its "dual"*

$$(U, \tilde{Z}, \tilde{\omega}, \tilde{f}, \tilde{\eta}) := \left(U, -\frac{1}{f}\,\mathrm{tw}_{Z,f,\eta}(Z), \frac{1}{f}\,\mathrm{tw}_{Z,f,\eta}(\omega), \frac{1}{f}, \frac{\eta}{f}\right). \tag{3.23}$$

Moreover, the local twist map with respect to local twist data $(U, \tilde{Z}, \tilde{\omega}, \tilde{f}, \tilde{\eta})$ satisfies

$$\begin{aligned}\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}} \circ \mathrm{tw}_{Z,f,\eta} &= \mathrm{tw}_{Z,f,\eta} \circ \mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}} = \mathrm{id}_{\mathbb{T}^{\bullet,\bullet}U},\\ (U, Z, \omega, f, \eta) &= \left(U, -\frac{1}{\tilde{f}}\,\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}(\tilde{Z}), \frac{1}{\tilde{f}}\,\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}(\tilde{\omega}), \frac{1}{\tilde{f}}, \frac{\tilde{\eta}}{\tilde{f}}\right).\end{aligned} \tag{3.24}$$

Proof. That $\tilde{f}$ and $\tilde{Z}$ are nowhere vanishing is clear. To check if $\tilde{f} - \tilde{\eta}(\tilde{Z})$ is also nowhere vanishing, we first work out the explicit expression for $\tilde{Z}$ as follows:

$$\tilde{Z} = -\frac{1}{f}\,\mathrm{tw}_{Z,f,\eta}(Z) \overset{(3.16)}{=} -\frac{1}{f}\left(Z + \frac{\eta(Z)}{f - \eta(Z)}\,Z\right) = -\frac{1}{f - \eta(Z)}\,Z, \tag{3.25}$$

Now we may check that

$$\tilde{f} - \tilde{\eta}(\tilde{Z}) \overset{(3.25)}{=} \frac{1}{f} + \frac{1}{f(f - \eta(Z))}\,\eta(Z) = \frac{1}{f - \eta(Z)}. \tag{3.26}$$

Next we work out an explicit expression for $\tilde{\omega}$:

$$\begin{aligned}\tilde{\omega} &= \frac{1}{f}\,\mathrm{tw}_{Z,f,\eta}(\omega) \overset{(3.17)}{=} \frac{1}{f}\left(\omega - \frac{1}{f}\,\eta\,\iota_Z\omega\right)\\ &= \frac{1}{f}\,\mathrm{d}\eta - \frac{1}{f^2}\,\mathrm{d}f \wedge \eta = \mathrm{d}\left(\frac{\eta}{f}\right) = \mathrm{d}\tilde{\eta}.\end{aligned} \tag{3.27}$$

From the above we see that $\tilde{\omega}$ is closed and that $\tilde{\eta}$ is an auxiliary 1-form for $\tilde{\omega}$. The last thing to check is that the vector field $\tilde{Z}$ is Hamiltonian with respect to $\tilde{\omega}$ with Hamiltonian function $\tilde{f}$. For this we make use of the fact that the local twist map preserves contractions to see that

$$\begin{aligned}\iota_{\tilde{Z}}\tilde{\omega} &= -\frac{1}{f^2}\,\mathrm{tw}_{Z,f,\eta}(\omega)(\mathrm{tw}_{Z,f,\eta}(Z), \cdot) = -\frac{1}{f^2}\,\mathrm{tw}_{Z,f,\eta}(\iota_Z\omega)\\ &= -\frac{1}{f^2}\left(\iota_Z\omega - \frac{\omega(Z,Z)}{f^2}\,\eta\right) = \frac{\mathrm{d}f}{f^2} = -\mathrm{d}\left(\frac{1}{f}\right) = -\mathrm{d}\tilde{f}.\end{aligned} \tag{3.28}$$

Now that we have verified that $(U, \tilde{Z}, \tilde{\omega}, \tilde{f}, \tilde{\eta})$ satisfies all the defining poperties of local twist data, we describe the local twist map with respect to it in terms of the local twist data (U, Z, ω, f, η). It suffices to do this just for a 1-form α:

$$\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}(\alpha) = \alpha - \frac{\alpha(\tilde{Z})}{\tilde{f}}\,\tilde{\eta} = \alpha + \frac{\alpha(Z)}{f - \eta(Z)}\,\eta. \tag{3.29}$$

Straightforward substitution then gives us

$$\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}} \circ \mathrm{tw}_{Z,f,\eta}(\alpha) = \mathrm{tw}_{Z,f,\eta} \circ \mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}(\alpha) = \alpha. \tag{3.30}$$

If $\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}$ and $\mathrm{tw}_{Z,f,\eta}$ restricted to 1-forms are inverses of each other, then they must be inverses for all tensor fields. In particular,

$$\begin{aligned}
&\left(U, -\frac{1}{\tilde{f}}\,\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}(\tilde{Z}), \frac{1}{\tilde{f}}\,\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}(\tilde{\omega}), \frac{1}{\tilde{f}}, \frac{\tilde{\eta}}{\tilde{f}}\right) \\
&= \left(U, \frac{1}{\tilde{f}f}\,\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}} \circ \mathrm{tw}_{Z,f,\eta}(Z), \frac{1}{\tilde{f}f}\,\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}} \circ \mathrm{tw}_{Z,f,\eta}(\omega), \frac{1}{\tilde{f}}, \frac{\eta}{\tilde{f}f}\right) \\
&= (U, Z, \omega, f, \eta).
\end{aligned} \tag{3.31}$$

□

3.2 Derivatives under the local twist

3.2.1 Exterior derivative

In Remark 3.1.9, it was mentioned that the local twist preserves algebraic relations but not differential ones. Since we are eventually interested in describing what happens to the quaternionic Kähler condition under the twist, we need to first investigate what happens to derivatives, such as the exterior derivative, the Lie derivative, and the Levi-Civita connection with respect to some metric. We take these up one by one, beginning with the exterior derivative.

Proposition 3.2.1. *Any k-form α on a manifold M equipped with local twist data (U, Z, ω, f, η) satisfies over U the following equation:*

$$\mathrm{d} \circ \mathrm{tw}_{Z,f,\eta}(\alpha) = \mathrm{tw}_{Z,f,\eta}\left(\mathrm{d}\alpha - \frac{1}{f}\,\omega \wedge \iota_Z\alpha\right) + \frac{1}{f}\,\eta \wedge \mathscr{L}_Z\alpha. \tag{3.32}$$

Proof. This follows from a direct computation:

$$\begin{aligned}
\mathrm{d} \circ \mathrm{tw}_{Z,f,\eta}(\alpha) &= \mathrm{d}\left(\alpha - \frac{1}{f}\,\eta \wedge \iota_Z\alpha\right) \\
&= \mathrm{d}\alpha + \frac{1}{f^2}\,\mathrm{d}f \wedge \eta \wedge \iota_Z\alpha - \frac{1}{f}\,\mathrm{d}\eta \wedge \iota_Z\alpha + \frac{1}{f}\,\eta \wedge \mathrm{d} \circ \iota_Z\alpha \\
&= \mathrm{d}\alpha - \frac{1}{f^2}\,\iota_Z\omega \wedge \eta \wedge \iota_Z\alpha - \frac{1}{f}\,\omega \wedge \iota_Z\alpha + \frac{1}{f}\,\eta \wedge \mathrm{d} \circ \iota_Z\alpha \\
&= \mathrm{d}\alpha - \frac{1}{f}\,\omega \wedge \iota_Z\alpha + \frac{1}{f^2}\,\eta \wedge \iota_Z\omega \wedge \iota_Z\alpha \\
&\quad - \frac{1}{f}\,\eta \wedge \iota_Z\mathrm{d}\alpha + \frac{1}{f}\,\eta \wedge \mathscr{L}_Z\alpha \\
&= \mathrm{tw}_{Z,f,\eta}\left(\mathrm{d}\alpha - \frac{1}{f}\,\omega \wedge \iota_Z\alpha\right) + \frac{1}{f}\,\eta \wedge \mathscr{L}_Z\alpha.
\end{aligned} \tag{3.33}$$

□

3.2.2 Lie derivatives

As the Cartan formula relates exterior derivatives to Lie derivatives, we can bootstrap our way to a relation between local twists and the Lie derivative using the above result.

Proposition 3.2.2. *Any vector fields u, v and k-form α on a manifold M equipped with local twist data (U, Z, ω, f, η) satisfy over U the following equations:*

$$\mathscr{L}_{\mathrm{tw}_{Z,f,\eta}(u)} \circ \mathrm{tw}_{Z,f,\eta}(\alpha) = \mathrm{tw}_{Z,f,\eta}\left(\mathscr{L}_u\alpha - \frac{1}{f}\,\iota_u\omega \wedge \iota_Z\alpha + \frac{\eta(u)}{f-\eta(Z)}\,\mathscr{L}_Z\alpha\right) + \frac{1}{f}\,\eta \wedge \iota_{\mathscr{L}_Z u}\alpha, \tag{3.34a}$$

$$\mathscr{L}_{\mathrm{tw}_{Z,f,\eta}(u)} \circ \mathrm{tw}_{Z,f,\eta}(v) = \mathrm{tw}_{Z,f,\eta}\left(\mathscr{L}_u v + \frac{1}{f}\,\omega(u,v)Z + \frac{\eta(u)}{f-\eta(Z)}\,\mathscr{L}_Z v - \frac{\eta(v)}{f-\eta(Z)}\,\mathscr{L}_Z u\right). \tag{3.34b}$$

Proof. In order to show (3.34a), we use the Cartan formula to rewrite the left-hand side as

$$\begin{aligned}\mathscr{L}_{\mathrm{tw}_{Z,f,\eta}(u)} \circ \mathrm{tw}_{Z,f,\eta}(\alpha) &= \mathrm{d} \circ \iota_{\mathrm{tw}_{Z,f,\eta}(u)} \circ \mathrm{tw}_{Z,f,\eta}(\alpha) + \iota_{\mathrm{tw}_{Z,f,\eta}(u)} \circ \mathrm{d} \circ \mathrm{tw}_{Z,f,\eta}(\alpha) \\ &= \mathrm{d} \circ \mathrm{tw}_{Z,f,\eta}(\iota_u\alpha) + \iota_{\mathrm{tw}_{Z,f,\eta}(u)} \circ \mathrm{d} \circ \mathrm{tw}_{Z,f,\eta}(\alpha).\end{aligned} \tag{3.35}$$

Using (3.32), collecting terms under the local twist, and simplifying, we get

$$\begin{aligned}\mathscr{L}_{\mathrm{tw}_{Z,f,\eta}(u)} \circ \mathrm{tw}_{Z,f,\eta}(\alpha) &= \mathrm{tw}_{Z,f,\eta}\left(\mathscr{L}_u\alpha - \frac{1}{f}\,\iota_u\omega \wedge \iota_Z\alpha\right) + \frac{1}{f}\,\eta \wedge \iota_{\mathscr{L}_Z u}\alpha \\ &\quad + \frac{1}{f}(\eta \wedge \iota_u(\mathscr{L}_Z\alpha) + \iota_{\mathrm{tw}_{Z,f,\eta}(u)}(\eta \wedge \mathscr{L}_Z\alpha)).\end{aligned} \tag{3.36}$$

Equation (3.34a) then follows from the observation that

$$\begin{aligned}&\eta \wedge \iota_u(\mathscr{L}_Z\alpha) + \iota_{\mathrm{tw}_{Z,f,\eta}(u)}(\eta \wedge \mathscr{L}_Z\alpha) \\ &= \eta(u)\mathscr{L}_Z\alpha - \iota_u(\eta \wedge \mathscr{L}_Z\alpha) + \iota_{\mathrm{tw}_{Z,f,\eta}(u)}(\eta \wedge \mathscr{L}_Z\alpha) \\ &\overset{(3.16)}{=} \eta(u)\mathscr{L}_Z\alpha + \frac{\eta(u)}{f-\eta(Z)}\,\iota_Z(\eta \wedge \mathscr{L}_Z\alpha) \\ &= \frac{f\eta(u)}{f-\eta(Z)}\left(\mathscr{L}_Z\alpha - \frac{1}{f}\,\eta \wedge \iota_Z(\mathscr{L}_Z\alpha)\right) \overset{(3.17)}{=} \frac{f\eta(u)}{f-\eta(Z)}\,\mathrm{tw}_{Z,f,\eta}(\mathscr{L}_Z\alpha).\end{aligned} \tag{3.37}$$

As for showing (3.34b), we plug its left-hand side into the local twist of an arbitrary 1-form α and rewrite it as

$$\begin{aligned}&\mathrm{tw}_{Z,f,\eta}(\alpha)(\mathscr{L}_{\mathrm{tw}_{Z,f,\eta}(u)} \circ \mathrm{tw}_{Z,f,\eta}(v)) \\ &= \mathrm{d}(\alpha(v))(\mathrm{tw}_{Z,f,\eta}(u)) - (\mathscr{L}_{\mathrm{tw}_{Z,f,\eta}(u)} \circ \mathrm{tw}_{Z,f,\eta}(\alpha))(\mathrm{tw}_{Z,f,\eta}(v)).\end{aligned} \tag{3.38}$$

Then we may use (3.16) and (3.34a) followed by collecting terms together under α to see that

$$\begin{aligned}&\mathrm{tw}_{Z,f,\eta}(\alpha)(\mathscr{L}_{\mathrm{tw}_{Z,f,\eta}(u)} \circ \mathrm{tw}_{Z,f,\eta}(v)) \\ &= \alpha\left(\mathscr{L}_u v + \frac{1}{f}\,\omega(u,v)Z + \frac{\eta(u)}{f-\eta(Z)}\,\mathscr{L}_Z v - \frac{\eta(v)}{f-\eta(Z)}\,\mathscr{L}_Z u\right),\end{aligned} \tag{3.39}$$

from which (3.34b) follows by compatibility of the local twist with contraction. □

The above proposition has an important consequence.

Corollary 3.2.3. *Any tensor field S on a manifold M equipped with local twist data (U, Z, ω, f, η) and dual twist data $(U, \tilde{Z}, \tilde{\omega}, \tilde{f}, \tilde{\eta})$ satisfies over U the following equation:*

$$\mathscr{L}_{\tilde{Z}} \circ \mathrm{tw}_{Z,f,\eta}(S) = -\frac{1}{f - \eta(Z)} \mathrm{tw}_{Z,f,\eta}(\mathscr{L}_Z S). \tag{3.40}$$

In particular, the local twist map sends Z-invariant tensor fields and Z-invariant tensor distributions to $\tilde{Z}$-invariant tensor fields and $\tilde{Z}$-invariant tensor distributions respectively.

Proof. Substituting $u = -\frac{1}{f} Z$ into (3.34a) and simplifying gives us for any k-form α

$$\mathscr{L}_{\tilde{Z}} \circ \mathrm{tw}_{Z,f,\eta}(\alpha) = \mathscr{L}_{-\frac{1}{f}\mathrm{tw}_{Z,f,\eta}(Z)} \circ \mathrm{tw}_{Z,f,\eta}(\alpha) = -\frac{1}{f - \eta(Z)} \mathrm{tw}_{Z,f,\eta}(\mathscr{L}_Z \alpha). \tag{3.41}$$

In particular, the above holds for 1-forms and 0-forms i.e. functions. Since any tensor field may be defined in terms of these via contractions, tensor products, and $\mathbb{R}$-linear combinations, and since the Lie derivative distributes over these via linearity and the Leibniz rule, it follows that the above holds for an arbitrary tensor field S. □

In fact, this corollary admits a useful generalisation.

Proposition 3.2.4. *Given a manifold equipped with local twist data (U, Z, ω, f, η), if a vector field v is Hamiltonian with respect to ω with Z-invariant Hamiltonian function f_v, then*

$$\tilde{v} := \mathrm{tw}_{Z,f,\eta}(v) + (f_v + 1)\tilde{Z} = \mathrm{tw}_{Z,f,\eta}\left(v - \frac{f_v + 1}{f} Z\right) \tag{3.42}$$

is Hamiltonian with respect to $\tilde{\omega}$ with Hamiltonian function

$$\tilde{f}_{\tilde{v}} := \frac{f_v + 1}{f} - 1 \tag{3.43}$$

and satisfies for any tensor field S over U, the following equation:

$$\mathscr{L}_{\tilde{v}} \circ \mathrm{tw}_{Z,f,\eta}(S) = \mathrm{tw}_{Z,f,\eta}\left(\mathscr{L}_v S - \frac{f_v - \eta(v) + 1}{f - \eta(Z)} \mathscr{L}_Z S\right). \tag{3.44}$$

In particular, the local twist map sends (Z, v)-invariant tensor fields to $(\tilde{Z}, \tilde{v})$-invariant tensor fields.

Proof. To check that $\tilde{v}$ is indeed Hamiltonian with respect to $\tilde{\omega}$ with Hamiltonian function $\tilde{f}_{\tilde{v}}$, we compute

$$\begin{aligned}
\iota_{\tilde{v}}\tilde{\omega} &= \frac{1}{f} \mathrm{tw}_{Z,f,\eta}(\omega)\left(\mathrm{tw}_{Z,f,\eta}\left(v - \frac{f_v + 1}{f} Z\right), \cdot\right) \\
&= \frac{1}{f} \mathrm{tw}_{Z,f,\eta}\left(\iota_v \omega - \frac{f_v + 1}{f} \iota_Z \omega\right) \\
&= -\mathrm{tw}_{Z,f,\eta}\left(\frac{f\,\mathrm{d}f_v - (f_v + 1)\,\mathrm{d}f}{f^2}\right) \\
&= -\mathrm{tw}_{Z,f,\eta}\left(\mathrm{d}\left(\frac{f_v + 1}{f} - 1\right)\right) \\
&\overset{(3.32)}{=} -\mathrm{d}\left(\frac{f_v + 1}{f} - 1\right) + \eta \wedge \mathscr{L}_Z\left(\frac{f_v + 1}{f} - 1\right) \\
&= -\mathrm{d}\left(\frac{f_v + 1}{f} - 1\right) = -\mathrm{d}\tilde{f}_{\tilde{v}}.
\end{aligned} \tag{3.45}$$

Meanwhile to show (3.44), we substitute into (3.34a)

$$u = v - \frac{f_v + 1}{f} Z. \tag{3.46}$$

Using the fact that if v has a Z-invariant Hamiltonian function with respect to a Z-invariant form ω, then v itself is Z-invariant, we obtain for any k-form α the following equation upon simplification:

$$\mathscr{L}_{\tilde{v}} \circ \mathrm{tw}_{Z,f,\eta}(\alpha) = \mathrm{tw}_{Z,f,\eta} \left(\mathscr{L}_v \alpha - \frac{f_v - \eta(v) + 1}{f - \eta(Z)} \mathscr{L}_Z \alpha \right). \tag{3.47}$$

Then the same argument as in the proof of Corollary 3.2.3 implies the above holds for all tensor fields S. □

Remark 3.2.5. Note that $(v, \omega, f_v) \leftrightarrow (\tilde{v}, \tilde{\omega}, \tilde{f}_{\tilde{v}})$ is an involution and generalises the involution $(Z, \omega, f) \leftrightarrow (\tilde{Z}, \tilde{\omega}, \tilde{f})$ of twist data dual to each other.

The reason this result is useful is that if we already know that a given Z-invariant metric g admits some Killing fields v_a with specified Z-invariant Hamiltonians f_{v_a}, then we automatically obtain Killing fields $\tilde{v}_a$ for the local twist $\mathrm{tw}_{Z,f,\eta}(g)$. In fact, we can directly read off the structure constants of the Lie algebra of such Killing fields using the following corollary.

Corollary 3.2.6. *Given a manifold equipped with local twist data (U, Z, ω, f, η), and vector fields v_1, v_2, v_3 Hamiltonian with respect to ω with Z-invariant Hamiltonian functions $f_{v_1}, f_{v_2}, f_{v_3}$ respectively such that*

$$[v_1, v_2] := \mathscr{L}_{v_1} v_2 = v_3, \tag{3.48}$$

then the vector fields

$$\tilde{v}_a = \mathrm{tw}_{Z,f,\eta}(v_a) + (f_{v_a} + 1)\tilde{Z}, \tag{3.49}$$

satisfy the following equation:

$$[\tilde{v}_1, \tilde{v}_2] := \mathscr{L}_{\tilde{v}_1} \tilde{v}_2 = \tilde{v}_3 + (\omega(v_1, v_2) - f_{v_3} - 1)\tilde{Z}. \tag{3.50}$$

Proof. We make the following substitutions into (3.44):

$$v = v_1, \quad S = v_2 - \frac{f_{v_2} + 1}{f} Z. \tag{3.51}$$

Then using the assumed Z-invariance of f_{v_2} and hence v_2, we obtain

$$\begin{aligned} \mathscr{L}_{\tilde{v}_1} \circ \mathrm{tw}_{Z,f,\eta}(v_2) &= \mathrm{tw}_{Z,f,\eta} \circ \mathscr{L}_{v_1} \left(v_2 - \frac{f_{v_2} + 1}{f} Z \right) \\ &= \mathrm{tw}_{Z,f,\eta} \left(\mathscr{L}_{v_1} v_2 - v_1 \left(\frac{f_{v_2} + 1}{f} \right) Z \right). \end{aligned} \tag{3.52}$$

Equation (3.50) then follows from the observation that

$$\begin{aligned} v_1(f) = \mathrm{d}f(v_1) = -\omega(Z, v_1) = \mathrm{d}f_1(Z) = 0, \\ v_1(f_{v_1}) = \mathrm{d}f_{v_2}(v_1) = \omega(v_1, v_2). \end{aligned} \tag{3.53}$$

□

Example 3.2.7. From Example 2.2.6 and the proof of Proposition 2.3.6, we know that the following vector fields are Killing with respect to the metric (2.29) on $\mathbb{C}\mathbf{H}^2$:

$$\begin{aligned} &Z_1 = \rho\,\partial_\rho + \frac{1}{2}(x\,\partial_x + y\,\partial_y) + \tau\,\partial_\tau, \quad Z_2 = \frac{1}{2}(\partial_x + y\,\partial_\tau), \quad Z_3 = \frac{1}{2}(\partial_y - x\,\partial_\tau), \\ &Z_4 = \partial_\tau, \quad Z_5 = \frac{1}{2}\,(y\,\partial_x - x\,\partial_y). \end{aligned} \tag{3.54}$$

Of these, all the Killing fields other than Z_1 Lie-commute with $Z = Z_4$ and are Hamiltonian with respect to ω_Q:

$$\begin{aligned} \iota_{Z_2}\omega_Q &= -\mathrm{d}\left(-\frac{y}{2\rho}\right), \quad \iota_{Z_3}\omega_Q = -\mathrm{d}\left(\frac{x}{2\rho}\right), \\ \iota_{Z_4}\omega_Q &= -\mathrm{d}\left(-\frac{1}{2\rho}\right), \quad \iota_{Z_5}\omega_Q = -\mathrm{d}\left(-\frac{x^2+y^2}{4\rho}\right). \end{aligned} \tag{3.55}$$

It will be convenient to make the following choice of Hamiltonian functions:

$$f_{Z_1} = -\frac{y}{2\rho} - 1, \quad f_{Z_3} = \frac{x}{2\rho} - 1, \quad f_{Z_4} = -\frac{1}{2\rho}, \quad f_{Z_5} = -\frac{x^2+y^2}{4\rho} - 1. \tag{3.56}$$

Note that any $\mathbb{R}$-linear combination

$$u = c_2 Z_2 + c_3 Z_3 + c_4 Z_4 + c_5 Z_5 \tag{3.57}$$

of these vector fields is also going to be an ω_Q-Hamiltonian Killing field commuting with Z. We choose the Hamiltonian function f_u of u so that $f_u + 1$, rather than f_u itself, is linear in the coefficients c_a. This is to ensure that $\tilde{u}$ vanishes when u vanishes. Hence, we choose

$$f_u = -1 + c_2(f_{Z_2} + 1) + c_3(f_{Z_3} + 1) + c_4(f_{Z_4} + 1) + c_5(f_{Z_5} + 1). \tag{3.58}$$

If we now choose η_Q as in (3.19), i.e.

$$\eta_Q = -\frac{1}{2\rho}\,(\mathrm{d}\tau + y\,\mathrm{d}x - x\,\mathrm{d}y) + \kappa\,\mathrm{d}\tau, \tag{3.59}$$

so that the vector fields $\tilde{Z}_a$ are given by

$$\begin{aligned} \tilde{Z}_2 &= Z_2 + \frac{1}{\kappa}\,(f_{Z_2} - \eta_Q(Z_2) + 1)Z = Z_2 - \frac{y}{2}\,Z = \frac{1}{2}\,\partial_x, \\ \tilde{Z}_3 &= Z_3 + \frac{1}{\kappa}\,(f_{Z_3} - \eta_Q(Z_3) + 1)Z = Z_3 + \frac{x}{2}\,Z = \frac{1}{2}\,\partial_y, \\ \tilde{Z}_4 &= Z_4 + \frac{1}{\kappa}\,(f_{Z_4} - \eta_Q(Z_3) + 1)Z = \frac{1}{\kappa}\,Z = \frac{1}{\kappa}\,\partial_\tau, \\ \tilde{Z}_5 &= Z_5 + \frac{1}{\kappa}\,(f_{Z_5} - \eta_Q(Z_5) + 1)Z = Z_5 = \frac{1}{2}\,(y\,\partial_x - x\,\partial_y), \end{aligned} \tag{3.60}$$

then $\tilde{u}$ is given by

$$\begin{aligned} \tilde{u} &= u + \frac{1}{\kappa}\,(f_u - \eta_Q(u) + 1)Z = \sum_{a=2}^{5} c_a \tilde{Z}_a \\ &= \frac{1}{2}\,(c_2 + c_5 y)\,\partial_x + \frac{1}{2}\,(c_3 - c_5 x)\,\partial_y + \frac{c_4}{\kappa}\,\partial_\tau. \end{aligned} \tag{3.61}$$

It's easy to see that this is a Killing field of the metric

$$\mathrm{tw}_{Z,f_Q,\eta_Q}(g) = \frac{1}{4\rho^2}\left(\mathrm{d}\rho^2 + 2\rho(\mathrm{d}x^2 + \mathrm{d}y^2)\right) + \kappa^2\mathrm{d}\tau^2. \tag{3.62}$$

Moreover, we note that

$$\begin{aligned} \mathscr{L}_{\tilde{Z}_2}\tilde{Z}_3 &= 0 = \tilde{Z}_4 + (\omega_Q(Z_2, Z_3) - f_{Z_4} - 1)\tilde{Z}, \\ \mathscr{L}_{\tilde{Z}_5}\tilde{Z}_2 &= \tilde{Z}_3 = \tilde{Z}_3 + (\omega_Q(Z_5, Z_2) - f_{Z_3} - 1)\tilde{Z}, \\ \mathscr{L}_{\tilde{Z}_5}\tilde{Z}_3 &= -\tilde{Z}_2 = -\tilde{Z}_2 + (\omega_Q(Z_5, Z_3) + f_{Z_2} - 1)\tilde{Z}. \end{aligned} \tag{3.63}$$

3.2.3 Levi-Civita connection

Finally, we end this section by deriving an expression for how the Levi-Civita connection transforms under the local twist. This is accomplished using the Koszul formula, which relates the Levi-Civita connection to Lie derivatives.

Proposition 3.2.8. *Any vector fields u, v and metric g on a manifold M equipped with local twist data (U, Z, ω, f, η) satisfy over U the following equation:*

$$\nabla^{\mathrm{tw}_{Z,f,\eta}(g)}_{\mathrm{tw}_{Z,f,\eta}(u)} \circ \mathrm{tw}_{Z,f,\eta}(v) = \mathrm{tw}_{Z,f,\eta}\left(\nabla^g_u v + S^\omega_u v + S^Z_u v + \frac{\eta(u)}{f - \eta(Z)}\mathscr{L}_Z v\right), \tag{3.64}$$

where $\nabla^{\mathrm{tw}_{Z,f,\eta}(g)}$ and ∇^g are Levi-Civita connections associated to the metrics $\mathrm{tw}_{Z,f,\eta}(g)$ and g, and

$$S^\omega \in \Gamma(\mathbb{T}^{1,2}M) := \Gamma(T^*M^{\otimes 2} \otimes TM), \quad S^Z \in \Gamma(\mathbb{T}^{1,2}U) := \Gamma(T^*U^{\otimes 2} \otimes TU) \tag{3.65}$$

are tensor fields given by

$$\begin{aligned} 2f\, g(S^\omega_u v, w) &= \omega(u, v)g(Z, w) - \omega(v, w)g(Z, u) - \omega(u, w)g(Z, v), \\ 2(f - \eta(Z))g(S^Z_u v, w) &= \eta(u)(\mathscr{L}_Z g)(v, w) + \eta(v)(\mathscr{L}_Z g)(u, w) - \eta(w)(\mathscr{L}_Z g)(u, v). \end{aligned} \tag{3.66}$$

Proof. We introduce the shorthands g', u', v', w' for the local twists of g, u, v, w, with w an arbitrary vector field. Then plugging in the left-hand side of (3.64) and w' into g', we may use the Koszul formula:

$$\begin{aligned} 2g'(\nabla^{g'}_{u'}v', w') = u'(g'(v', w')) &+ v'(g'(u', w')) - w(g'(u', v')) \\ &+ g'(\mathscr{L}_{u'}v', w') - g'(\mathscr{L}_{u'}w', v') - g'(\mathscr{L}_{v'}w', u'). \end{aligned} \tag{3.67}$$

By (3.16), we know that

$$\begin{aligned} u'(g'(v', w')) = u'(g(v, w)) &= u(g(v, w)) + \frac{\eta(u)}{f - \eta(Z)} Z(g(v, w)) \\ &= u(g(v, w)) + \frac{\eta(u)}{f - \eta(Z)}(g(\mathscr{L}_Z v, w) + g(\mathscr{L}_Z w, v) + (\mathscr{L}_Z g)(v, w)). \end{aligned} \tag{3.68}$$

Likewise, (3.34b) tells us that

$$\begin{aligned} g'(\mathscr{L}_{u'}v', w') = g(\mathscr{L}_u v, w) + \frac{1}{f}\,\omega(u,v)g(Z,w) \\ + \frac{1}{f-\eta(Z)}(\eta(u)g(\mathscr{L}_Z v, w) - \eta(v)g(\mathscr{L}_Z u, w)). \end{aligned} \tag{3.69}$$

Permuting u, v, w and u', v', w' in the above expressions and summing them up with signs in accordance with the Koszul formula, we get

$$g'(\nabla^{g'}_{u'}v', w') = g\left(\nabla^g_u v + S^\omega_u v + S^Z_u v + \frac{\eta(u)}{f-\eta(Z)}\,\mathscr{L}_Z v, w\right), \tag{3.70}$$

from which (3.64) follows by compatibility of the local twist with contractions. □

3.3 Global aspects of the twist construction

3.3.1 Auxiliary data are indeed auxiliary

So far we have been dealing with local twist maps over open sets on which closed twist 2-form ω is exact. In order to develop a global version of the local twist map applicable to domains on which auxiliary 1-forms η may not exist, we must first investigate the dependence of the local twist map on η.

As it turns out, if we restrict to Z-invariant tensor fields or to Z-invariant tensor subbundles, then local twists with respect to two different choices of auxiliary 1-forms η_0 and η_1 may be identified up to local diffeomorphisms.

Proposition 3.3.1. *Given two tuples of auxiliary local twist data (U, η_0) and (U, η_1) associated to twist data (Z, ω, f) such that $f - \eta_0(Z)$ and $f - \eta_1(Z)$ have the same sign and an arbitrary point $p_0 \in U$, there exist neighbourhoods $V_0, V_1 \subseteq U$ of p_0 and a diffeomorphism $\phi : V_0 \to V_1$ such that $\phi(p_0) = p_0$ and*

$$\phi^*\mathrm{tw}_{Z,f,\eta_0}(S) = \mathrm{tw}_{Z,f,\eta_1}(S), \tag{3.71}$$

for all Z-invariant tensor fields S.

Proof. Note that it is enough to show this for an arbitrary Z-invariant k-form α since any other tensor field may be defined using 0-forms and 1-forms through contraction, tensor products, and linearity, all operations compatible with diffeomorphisms. We shall accomplish this using Moser's trick.

Consider a smooth 1-parameter family of 1-forms

$$\eta_t = (1-t)\eta_0 + t\eta_1 \tag{3.72}$$

interpolating between η_0 and η_1 as t varies over the closed interval $[0, 1]$. Then the function

$$f - \eta_t(Z) = (1-t)(f - \eta_0(Z)) + t(f - \eta_1(Z)) \tag{3.73}$$

has the same sign as $f - \eta_0(Z)$ and $f - \eta_1(Z)$ and so is nowhere vanishing, while

$$\mathrm{d}\eta_t = (1-t)\mathrm{d}\eta_0 + t\,\mathrm{d}\eta_1 = \omega. \tag{3.74}$$

So, (U, η_t) constitutes local data for every $t \in [0,1]$ and we may take the local twist with respect to it. Given an arbitrary Z-invariant k-form α, we introduce the short-hands

$$\begin{aligned} \tilde{Z}_t &= -\frac{1}{f}\operatorname{tw}_{Z,f,\eta_t}(Z) = -\frac{1}{f-\eta_t(Z)}Z, \\ \alpha'_t &= \operatorname{tw}_{Z,f,\eta_t}(\alpha) = \alpha - \frac{1}{f}\eta_t \wedge \iota_Z\alpha. \end{aligned} \tag{3.75}$$

Note that $\eta_1 - \eta_0$ is a closed 1-form on U and so by Poincaré's lemma, we can find a function h on a contractible neighbourhood V of p_0 which vanishes at the point p_0 and satisfies

$$\eta_1|_V - \eta_0|_V = \mathrm{d}h. \tag{3.76}$$

Thus we have the following chains of equalities by virtue of Corollary 3.2.3:

$$\begin{aligned} \mathscr{L}_{h\tilde{Z}_t}\alpha'_t &= h\mathscr{L}_{\tilde{Z}_t}\alpha'_t + \mathrm{d}h \wedge \iota_{\tilde{Z}_t}\alpha'_t \\ &= 0 - \frac{1}{f}\mathrm{d}h \wedge \operatorname{tw}_{Z,f,\eta_t}(\iota_Z\alpha) \\ &= -\frac{1}{f}\mathrm{d}h \wedge \iota_Z\alpha = \frac{\mathrm{d}\alpha'_t}{\mathrm{d}t}. \end{aligned} \tag{3.77}$$

If we now have a 1-parameter family of diffeomorphisms $\phi_t : V_0 \to V_t$, with $V_0, V_1 \subseteq V$, which satisfies

$$\phi_0 = \mathrm{id}_{V_0}, \quad \left.\frac{\mathrm{d}\phi_s}{\mathrm{d}s}\right|_{s=t}(\phi_t^{-1}(p)) = h(p)\tilde{Z}_{t,p}, \tag{3.78}$$

for all $p \in V_t$ and all $t \in [-\varepsilon, 1+\varepsilon]$ for some small $\varepsilon > 0$, then the chain rule tells us that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi_t^{-1})^*\alpha'_t = (\phi_t^{-1})^*\left(-\mathscr{L}_{h\tilde{Z}_t}\alpha'_t + \frac{\mathrm{d}\alpha'_t}{\mathrm{d}t}\right) \overset{(3.77)}{=} 0. \tag{3.79}$$

In particular, this would mean that

$$(\phi_1^{-1})^*\alpha'_1 = (\phi_0^{-1})^*\alpha'_0 = \alpha_0, \tag{3.80}$$

which may be rearranged into

$$\phi_1^*\alpha'_0 = \alpha'_1. \tag{3.81}$$

Since h vanishes at the point p_0, it follows that $\phi_t(p) = p$ for all $t \in [-\varepsilon, 1+\varepsilon]$ and that we may indeed find sufficiently small neighbourhoods $V_t \subseteq U$ of p_0 for which a solution to (3.78) exists for all $t \in [-\varepsilon, 1+\varepsilon]$. So, $\phi = \phi_1$ solves (3.71) for k-forms and hence for all tensor fields. □

Remark 3.3.2. Since Z-invariant tensor subbundles are locally spanned by Z-invariant tensor fields, it follows that

$$\phi^*\operatorname{tw}_{Z,f,\eta_0}(\mathscr{S}) = \operatorname{tw}_{Z,f,\eta_1}(\mathscr{S}), \tag{3.82}$$

for all Z-invariant tensor subbundles $\mathscr{S}$.

Example 3.3.3. Continuing with Example 3.1.5, let $\mathrm{d}\eta_0^B = \mathrm{d}\eta_1^B = \omega^B$ on U^B. Then there exists a function h on U^B (and hence by pullback on U) such that

$$\eta_0^B - \eta_1^B = \mathrm{d}h. \tag{3.83}$$

Then for α a Z-invariant form on M, we have on U

$$\mathrm{tw}_{Z,f,\eta_0}(\alpha) = \phi^*\mathrm{tw}_{Z,f,\eta_1}(\alpha), \tag{3.84}$$

where the diffeomorphism ϕ sends (τ, p^B) to $(\tau + h(p^B), p^B)$ for all $p^B \in U^B$. Note that since $f - \eta(Z) = -1$, (3.25) implies $\tilde{Z} = Z = \partial_\tau$.

What this suggests is that in general, one should expect that only twists of Z-invariant tensor fields and Z-invariant tensor subbundles can be patched together from local data to have some globally well-defined meaning. The transition functions on the overlap of various local patches U_Λ on which auxiliary local twist data $(U_\Lambda, \eta_\Lambda)$ are defined are modified by diffeomorphisms such as ϕ as constructed in the above proof. So, in general, a global twist map shouldn't be expected to be between tensor fields or tensor subbundles on the same manifold M.

3.3.2 Global twists

The idea of gluing together local twists in order to get a consistent map of invariant tensor fields on two different manifolds is made precise by the following definition.

Definition 3.3.4 (Global twist map)**.** Given manifolds M and $\tilde{M}$ equipped with twist data (Z, ω, f) and $(\tilde{Z}, \tilde{\omega}, \tilde{f})$, a global twist map T is an $\mathbb{R}$-linear map

$$T : \Gamma(\mathbb{T}^{\bullet,\bullet} M)^Z \to \Gamma(\mathbb{T}^{\bullet,\bullet} \tilde{M})^{\tilde{Z}} \tag{3.85}$$

sending Z-invariant tensor fields on M to $\tilde{Z}$-invariant tensor fields on $\tilde{M}$, in particular

$$\left(-\frac{1}{f}Z, \frac{1}{f}\omega, \frac{1}{f}\right) \mapsto (\tilde{Z}, \tilde{\omega}, \tilde{f}), \tag{3.86}$$

such that there exist

- open covers $\{U_\Lambda\}$ of M and $\{\tilde{U}_\Lambda\}$ of $\tilde{M}$ and surjective local diffeomorphisms $\{\psi_\Lambda : U_\Lambda \to \tilde{U}_\Lambda\}$, all indexed by the same set $\{\Lambda\}$,
- local twist data $(U_\Lambda, Z, \omega, f, \eta_\Lambda)$ satisfying for any Z-invariant tensor field S

$$\psi_\Lambda^* T(S)|_{\tilde{U}_\Lambda} = \mathrm{tw}_{Z,f,\eta_\Lambda}(S|_{U_\Lambda}). \tag{3.87}$$

Furthermore, if such a map T exists, $(\tilde{M}, \tilde{Z}, \tilde{\omega}, \tilde{f})$ is said to be a *twist* of (M, Z, ω, f).

Remark 3.3.5. Any Z-invariant tensor subbundle is generated by (compactly supported) Z-invariant tensor fields. So, if we have a global twist map T sending Z-invariant tensor fields on M to $\tilde{Z}$-invariant tensor fields on $\tilde{M}$, we automatically have a map, identified with T itself, sending Z-invariant tensor subbundles on M to $\tilde{Z}$-invariant tensor subbundles on $\tilde{M}$.

Example 3.3.6. We specialise Example 3.3.3 to the case where the Z-action is a $\mathrm{U}(1)$-action and ω is integral. Let U_Λ^B be a contractible open cover of B and let $h_{\Lambda\Sigma}$ be functions on the (contractible) overlaps $U_\Lambda^B \cap U_\Sigma^B$ such that

$$\eta_\Lambda^B - \eta_\Sigma^B = \mathrm{d}h_{\Lambda\Sigma}. \tag{3.88}$$

Then for α a Z-invariant form on M, we have on $U_\Lambda \cap U_\Sigma$

$$\mathrm{tw}_{Z,f,\eta_\Lambda}(\alpha) = \phi^*_{\Lambda\Sigma}\mathrm{tw}_{Z,f,\eta_\Sigma}(\alpha), \tag{3.89}$$

where $\phi_{\Lambda\Sigma}$ sends (τ, p^B) to $(\tau + h_{\Lambda\Sigma}(p^B), p^B)$. Since $Z = \partial_\tau$ is Hamiltonian, the integrality of ω is equivalent to the integrality of ω^B. This in turn is equivalent to saying that we can choose the sets U^B_Λ and the functions $h_{\Sigma\Lambda}$ so that on the triple overlaps $U^B_\Lambda \cap U^B_\Sigma \cap U^B_\Pi$, we have the cocycle condition

$$h_{\Sigma\Pi} - h_{\Lambda\Pi} + h_{\Lambda\Sigma} \equiv 0 \pmod 1. \tag{3.90}$$

Phrased differently, this means that if we regard M and $\tilde{M}$ as $\mathsf{U}(1)$-bundles on B, then their first Chern classes are related by $c_1(\tilde{M}) = c_1(M) + [\omega^B]$, where $[\omega^B]$ denotes the (integral) cohomology class of ω^B.

The definition of twists $\tilde{M}$ of a manifold M and of global twist maps T between tensor fields on M and $\tilde{M}$ does not make any guarantees about existence or uniqueness. It's not too hard to see that global twist maps, if they exist at all, are generally not unique. This is because the composition of any global twist map between tensor fields on M and $\tilde{M}$ with the linear action induced by either a twist automorphism of M or of $\tilde{M}$ is again going to be a global twist map. As for existence, the construction in Examples 3.1.5, 3.3.3, 3.3.6 can always be carried out whenever the twist vector field Z induces a $\mathsf{U}(1)$-action and the twist form ω is integral. So in these cases, a twist certainly exists.

A necessary criterion of existence is given by Theorem 3.3.9 below, which also establishes the relationship of our formulation of the twist with that of Swann. But first, we need the following lemma and an easy corollary thereof.

Lemma 3.3.7. *Let $(\tilde{M}, \tilde{Z}, \tilde{\omega}, \tilde{f})$ be a twist of (M, Z, ω, f) realised by a global twist map T and let $p \in M$ and $\tilde{p} \in \tilde{M}$ be points such that for all Z-invariant functions h on M, we have $h(p) = T(h)(\tilde{p})$. Then for any Z-invariant function h_0 on M that is identically zero in some open neighbourhood V of p, the function $T(h_0)$ is identically zero in some open neighbourhood $\tilde{V}$ of $\tilde{p}$ that is independent of h_0.*

Proof. Assume without loss of generality that V is Z-invariant. Then there exists a Z-invariant function h_1 such that $h_1(p) \neq 0$ and whose support is contained in V. Thus h_0h_1 is identically zero on M, and so $T(h_0)T(h_1) = T(h_0h_1)$ is identically zero on $\tilde{M}$.

However, $T(h_1)(\tilde{p}) = h_1(p) \neq 0$ by hypothesis. Since $T(h_1)$ is continuous, there exists a neighbourhood $\tilde{V}$ of $\tilde{p}$ on which $T(h_1)$ is nowhere vanishing. But since $T(h_0)T(h_1)$ is identically zero on $\tilde{V}$, so must be $T(h_0)$. Note that $\tilde{V}$ is independent of h_0. □

Corollary 3.3.8. *Let $(\tilde{M}, \tilde{Z}, \tilde{\omega}, \tilde{f})$ be a twist of (M, Z, ω, f) realised by a global twist map T and let $p \in M$ and $\tilde{p} \in \tilde{M}$ be points such that for all Z-invariant functions h on M, we have $h(p) = T(h)(\tilde{p})$. Then if two Z-invariant functions h, h' on M coincide in some open neighbourhood V of p, the functions $T(h), T(h')$ coincide in some open neighbourhood $\tilde{V}$ of $\tilde{p}$ that is independent of h, h'.*

Theorem 3.3.9. *Let M be a manifold of dimension n equipped with twist data (Z, ω, f) such that every point $p \in M$ is contained in a hypersurface transversal to Z which intersects any Z-flowline in at most one point. Let $(\tilde{M}, \tilde{Z}, \tilde{\omega}, \tilde{f})$ be a twist of (M, Z, ω, f) realised by* *a global twist map T. Then there exists a double surjection*

$$\begin{array}{ccc} & P & \\ {}^{\pi}\swarrow & & \searrow^{\tilde{\pi}} \\ M & & \tilde{M} \end{array} \tag{3.91}$$

where P is a manifold of dimension $n+1$ *equipped with a* 1*-form* $\hat{\eta}$ *such that*

$$\mathrm{d}\hat{\eta} = \pi^*(\omega), \tag{3.92}$$

and for any Z-invariant vector field u on M with $\hat{\eta}$*-horiziontal lift* $\hat{u}$*, we have*

$$\tilde{\pi}(\hat{u}) = T(u), \tag{3.93}$$

where $T(u)$ *is understood as a section of the pullback bundle* $\tilde{\pi}^{-1}T\tilde{M}$ *constant along the map* $\tilde{\pi}$*.*

Proof. Define P to be the set of pairs of points $(p, \tilde{p}) \in M \times \tilde{M}$ such that for all Z-invariant functions h on M, we have

$$h(p) = T(h)(\tilde{p}). \tag{3.94}$$

We shall show that P has all the required properties.

Given a point $(p, \tilde{p}) \in P$, we can by hypothesis find a hypersurface N' containing p and transversal to Z such that any Z-flowline intersects it at most once. Let N be an open set of N' containing p and let $h_a|_N$ be $n-1$ coordinate functions on N. These can be extended to $n-1$ functions $h_a|_{N'}$ with compact support on N'. Then we define the value of the function h_a at any other point $p' \in M$ to be equal to the value of the function $h_a|_{N'}$ at p'' if the Z-flowline through p' intersects N' in some point p'' and equal to zero if the Z-flowline through p' does not intersect N'. This extends the coordinate functions $h_a|_N$ on N to $n-1$ Z-invariant functions h_a on M.

Without loss of generality, we may assume h_a yields a diffeomorphism $\phi : N \to \mathbb{R}^{n-1}$. Given any Z-invariant function h on M, this yields a map $F_h : \mathbb{R}^{n-1} \to \mathbb{R}$ given by

$$F_h = h|_N \circ \phi^{-1}. \tag{3.95}$$

Construct a new Z-invariant function $h' = F_h(h_a) := F_h(h_1, h_2, \dots)$ on M that restricts to h on N. If we let V_N be the (Z-invariant) open neighbourhood of p in M given by the union of all the Z-translates of N, h and h' agree on V_N. By Corollary 3.3.8, the functions $T(h), T(h')$ agree on some open neighbourhood $\tilde{V}_N$ of $\tilde{p}$ that is independent of h, h'.

Restrictions of the global twist map T are local twists. This may be used to locally verify that

$$T(F_h(h_a)) = F_h(T(h_a)). \tag{3.96}$$

So if a point $(p', \tilde{p}') \in V_N \times \tilde{V}_N$ satisfies $h_a(p') = T(h_a)(\tilde{p}')$ for all the h_a, then it satisfies $h'(p') = T(h')(\tilde{p}')$. Thus for any $(p', \tilde{p}') \in V_N \times \tilde{V}_N$, we have $h_a(p') = T(h_a)(\tilde{p}')$ if and only if $h(p') = T(h)(\tilde{p}')$ for all h (i.e. if and only if $(p', \tilde{p}') \in P$).

This allows us to realise the intersection of P with the open neighbourhood $V_N \times \tilde{V}_N$ of an arbitrary point $(p, \tilde{p}) \in P$ as a level set of a map from $V_N \times \tilde{V}_N \subseteq M \times \tilde{M}$ to $\mathbb{R}^{n-1}$ given by

$$(p', \tilde{p}') \mapsto h_a(p') - T(h_a)(\tilde{p}'). \tag{3.97}$$

Since the functions h_a restrict to a coordinate chart containing p on N, it follows that the differential of the above map is surjective. Hence, we conclude using the implicit function theorem that P is a submanifold of $M \times \tilde{M}$ of codimension $n-1$. From the local realisation of the twist, we know that $\tilde{M}$ has the same dimension as M i.e. n. So, P is a manifold of dimension

$$2n - (n-1) = n+1 \tag{3.98}$$

which inherits from the canonical projections pr_M and $\mathrm{pr}_{\tilde{M}}$ on $M \times \tilde{M}$, the maps π and $\tilde{\pi}$ to M and $\tilde{M}$ respectively.

In particular, if two points p, p' are related by the flow along Z, then $(p, \tilde{p})$ lies in P if and only if $(p', \tilde{p})$ does so as well. Likewise, since functions on $\tilde{M}$ of the form $T(f)$ are $\tilde{Z}$-invariant, if two points $\tilde{p}, \tilde{p}'$ are related by the flow along $\tilde{Z}$, then $(p, \tilde{p})$ lies in P if and only if $(p, \tilde{p}')$ does so as well.

All this tells us so far is that the maps π and $\tilde{\pi}$ are submersions. To see that they are indeed surjections, we make use of the fact that the local diffeomorphisms ψ_Λ are surjective. Note that the graphs of the maps ψ_Λ are contained in P. So, given a point $p \in U_\Lambda$, we have $(p, \psi_\Lambda(p))$ in the preimage $\pi^{-1}(p)$ of p. Likewise, given $\tilde{p} \in \tilde{U}_\Lambda$, there exists some p such that $\psi_\Lambda(p) = \tilde{p}$ and we have $(p, \tilde{p})$ in the preimage $\tilde{\pi}^{-1}(\tilde{p})$ of $\tilde{p}$.

The space of tangent vector fields on P regarded as a $C^\infty(P)$-module is spanned by vector fields of the form

$$u \oplus u' = u \oplus (T(u) + a\tilde{Z}), \tag{3.99}$$

where u is a Z-invariant vector field on M, a is a constant, and we think of TP as being a subbundle of the pullback of the bundle

$$T(M \times \tilde{M}) \cong \mathrm{pr}_M^{-1} TM \oplus \mathrm{pr}_{\tilde{M}}^{-1} TM \tag{3.100}$$

along the inclusion map from P into $M \times \tilde{M}$. Note that we have slightly abused notation to let u and u' denote sections of the pullback bundles $\mathrm{pr}_M^{-1} TM$ and $\mathrm{pr}_{\tilde{M}}^{-1} T^M$ constant along pr_M and $\mathrm{pr}_{\tilde{M}}$ respectively. Since such vector fields span the space of all tangent vector fields on P, in order to define the 1-form $\hat{\eta}$, it's enough to specify how it acts on sections of the form (3.99):

$$\hat{\eta}(u \oplus (T(u) + a\tilde{Z})) = a. \tag{3.101}$$

In particular the $\hat{\eta}$-horizontal lift $\hat{u}$ of any Z-invariant vector field u is given by

$$\hat{u} = u \oplus T(u), \tag{3.102}$$

from which (3.93) immediately follows.

Meanwhile, to deduce (3.92), we make use of the following consequence of the naturality of Lie-commutators. If u, v are vector fields on M and u', v' vector fields on $\tilde{M}$, then we have

$$\mathscr{L}_{u \oplus u'}(v \oplus v') = \mathscr{L}_u v \oplus \mathscr{L}_{u'} v'. \tag{3.103}$$

In conjunction with (3.50), this implies that if u, v, w are Z-invariant vector fields on M satisfying

$$\mathscr{L}_u v = w, \tag{3.104}$$

then their $\hat{\eta}$-horizontal lifts $\hat{u}, \hat{v}, \hat{w}$ satisfy

$$\begin{aligned} \mathscr{L}_{\hat{u}}\hat{v} &= \mathscr{L}_u v \oplus \mathscr{L}_{T(u)} \circ T(v) = w \oplus (T(w) - \omega(u,v)\tilde{Z}) \\ &= \hat{w} - 0 \oplus \omega(u,v)\tilde{Z}, \\ \mathscr{L}_{0\oplus\tilde{Z}}\hat{v} &= \mathscr{L}_{\tilde{Z}} \circ T(v) = 0. \end{aligned} \tag{3.105}$$

This in turn implies the following:

$$\begin{aligned} &\mathrm{d}\hat{\eta}(\hat{u},\hat{v}) = \hat{u}(\hat{\eta}(\hat{v})) - \hat{v}(\hat{\eta}(\hat{u})) - \hat{\eta}(\mathscr{L}_{\hat{u}}\hat{v}) = \omega(u,v), \\ &\mathrm{d}\hat{\eta}(0\oplus\tilde{Z},\hat{v}) = (0\oplus\tilde{Z})(\hat{\eta}(\hat{v})) - \hat{v}(\hat{\eta}(0\oplus\tilde{Z})) - \hat{\eta}(\mathscr{L}_{0\oplus\tilde{Z}}\hat{v}) = 0. \end{aligned} \tag{3.106}$$

It follows that $\mathrm{d}\hat{\eta} = \pi^*(\omega)$ since vector fields of the form $\hat{u}$ and $0 \oplus \tilde{Z}$ span $\Gamma(TP)$. □

We have essentially retrieved Swann's construction of the twist map in [Swa10]. From the above proof, we see that if Z and $\tilde{Z}$ induced $\mathrm{U}(1)$-actions on M and $\tilde{M}$, then P would be a $\mathrm{U}(1)$-principal bundle over both M and $\tilde{M}$, with $\hat{\eta}$ being a connection 1-form for $\pi : P \to M$ with curvature ω.

We can go the other way as well and show that the existence of a double surjection is also a sufficient criterion for the existence of global twists.

Theorem 3.3.10. *Let M be a manifold of dimension n equipped with twist data (Z, ω, f) and let*

$$\begin{array}{ccc} & P & \\ {}^{\pi}\swarrow & & \searrow^{\tilde{\pi}} \\ M & & \tilde{M} \end{array} \tag{3.107}$$

be a double surjection, with P being a manifold of dimension $n+1$ equipped with a 1-form $\hat{\eta}$ such that

$$\mathrm{d}\hat{\eta} = \pi^*(\omega). \tag{3.108}$$

Let $X_P, \hat{Z}, Z_P$ be vector fields on P satisfying

$$\begin{aligned} &\pi_*(X_P) = 0, \quad \tilde{\pi}_*(Z_P) = 0, \quad \pi_*(\hat{Z}) = Z, \\ &\hat{\eta}(X_P) = 1, \quad \hat{\eta}(\hat{Z}) = 0, \quad Z_P = \hat{Z} + fX_P. \end{aligned} \tag{3.109}$$

Let T be the map sending Z-invariant tensor fields on M to tensor fields on $\tilde{M}$ induced by pulling back functions on M along π and taking the $\hat{\eta}$-horizontal lift of vector fields on M and pushing them down onto $\tilde{M}$ along $\tilde{\pi}$. Then

$$\left(T\left(-\frac{1}{f}Z\right), T\left(\frac{1}{f}\omega\right), T\left(\frac{1}{f}\right)\right) \tag{3.110}$$

is twist data on $\tilde{M}$ and T is a global twist map.

Proof. Let h be a Z-invariant function and α be a Z-invariant 1-form on M. Then the pullbacks $\pi^*(h)$ and $\tilde{\pi}^*(T(h))$ on P are equal while the pullbacks $\pi^*(\alpha)$ and $\tilde{\pi}^*(T(\alpha))$ on P agree on $\hat{\eta}$-horizontal vector fields. So if $\hat{u}$ is the $\hat{\eta}$-horizontal lift of a vector field u on M, we have

$$\begin{aligned} \tilde{\pi}^*(T(\alpha))(\hat{u}) &= \pi^*(\alpha)(\hat{u}) = \alpha(u), \\ \tilde{\pi}^*(T(\alpha))(X_P) &= \frac{1}{f}\tilde{\pi}^*(T(\alpha))(Z_P) - \frac{1}{f}\tilde{\pi}^*(T(\alpha))(\hat{Z}) = -\frac{1}{f}\alpha(Z). \end{aligned} \tag{3.111}$$

Note that the choice of auxiliary local twist data (U, η) on M gives a 1-form $\theta := \hat{\eta} - \pi^*(\eta)$ on $P|_U$ satisfying

$$d\theta = 0, \quad \theta(X_P) = \hat{\eta}(X_P) = 1, \quad \theta(Z_P) = \hat{\eta}(Z_P) - \eta(Z) = f - \eta(Z). \tag{3.112}$$

The closed 1-form θ defines integral hypersurfaces in P and the nonvanishing of $\theta(X_P) = 1$ and $\theta(Z_P) = f - \eta(Z)$ corresponds to the fact that the integral hypersurfaces are transversal to both π and $\tilde{\pi}$. We may identify one such hypersurface U_P with $U = \pi(U_P)$. This gives us a surjective local diffeomorphism $\psi : U \to \tilde{\pi}(U_P) =: \tilde{U}$.

The tangent vectors to U_P are elements of the kernel of θ. We may check

$$\theta(\hat{u} + \eta(u)X_P) = \eta(u)\hat{\eta}(X_P) - \pi^*(\eta)(\hat{u}) = \eta(u) - \eta(u) = 0. \tag{3.113}$$

The vector fields $\hat{u} + \eta(u)X_P$ are thus tangent to U_P and may be identified with u on U. To obtain $\psi^*T(h)$ and $\psi^*T(\alpha)$, we simply pull back $\tilde{\pi}^*(T(h))$ and $\tilde{\pi}^*(T(\alpha))$ along the map $\iota : U \cong U_P \hookrightarrow P$ given by the inclusion under the above identification:

$$\begin{aligned} \psi^*T(h) &= \iota^* \circ \tilde{\pi}^*(T(h)) = \iota^* \circ \pi^*(h) = h = \mathrm{tw}_{Z,f,\eta}(h), \\ \psi^*T(\alpha)(u) &= \iota^* \circ \tilde{\pi}^*(T(\alpha))(u) = \tilde{\pi}^*(T(\alpha))(\hat{u} + \eta(u)X_P) \\ &\overset{(3.111)}{=} \alpha(u) - \frac{1}{f}\alpha(Z)\eta(u) = (\mathrm{tw}_{Z,f,\eta}(\alpha))(u). \end{aligned} \tag{3.114}$$

Compatibility with tensor products and contractions then tells us that T restricts to local twist maps for all tensor fields. Proposition 3.1.13 may then be used to locally verify that

$$(\tilde{Z}, \tilde{\omega}, \tilde{f}) := \left(T\left(-\frac{1}{f}Z\right), T\left(\frac{1}{f}\omega\right), T\left(\frac{1}{f}\right)\right) \tag{3.115}$$

constitutes twist data.

The only thing remaining to check in order to conclude that T is indeed a global twist map is to show that there exist tuples of local twist data $(U_\Lambda, \eta_\Lambda)$ such that $\{U_\Lambda\}$ and $\{\tilde{U}_\Lambda := \tilde{\pi}(U_{\Lambda,P})\}$ constitute open covers of M and $\tilde{M}$ respectively. For this, we note that for any arbitrary point $(p, \tilde{p}) \in P$, we can find a sufficiently small hypersurface U_P containing $(p, \tilde{p})$ which is transversal to both π and $\tilde{\pi}$ such that $\pi|_{U_P}$ is a diffeomorphism onto its image U in M. There is then a unique (closed) X_P-invariant 1-form which vanishes on U_P such that $\theta(X_P) = 1$. Then $\hat{\eta} - \theta$ is an X_P-invariant horizontal 1-form and so it is the pullback $\pi^*(\eta)$ of some 1-form η on U such that $d\eta = \omega|_U$ and $f - \eta(Z) = \theta(Z_P)$ is nowhere vanishing. □

Thus, our construction of the twist and that of Swann are essentially equivalent. That said, there is a slight increase of generality in that the requirement of integrality of the twist form ω in Swann's construction may be relaxed to rationality. This is illustrated in the following example.

Example 3.3.11. Let $M = \mathbb{R}_{>0} \times \mathbb{R} \times T^4$ be coordinatised by $(r, \tau, x_1, y_1, x_2, y_2)$ subject to the identifications

$$\begin{aligned} (x_1, y_1, x_2, y_2) &\sim (x_1 + 1, y_1, x_2, y_2) \sim (x_1, y_1 + 1, x_2, y_2) \\ &\sim (x_1, y_1, x_2 + 1, y_2) \sim (x_1, y_1, x_2, y_2 + 1). \end{aligned} \tag{3.116}$$

Then we have twist data (Z, ω, f) on it given by

$$Z = \partial_\tau, \quad \omega = \mathrm{d}r \wedge \mathrm{d}\tau + \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \lambda\, \mathrm{d}x_2 \wedge \mathrm{d}y_2, \quad f = r. \tag{3.117}$$

where λ is some real parameter that shall be used to control whether ω is rational or not. Working with an atlas is pretty cumbersome, so we will instead go to the universal cover $\underline{M}$ of M (which happens to be contractible in this case), while keeping track of the identifications (3.116). On $\underline{M}$, we introduce for every pair (a, b) of integers, the auxiliary 1-form

$$\eta_{a,b} = (r+1)\,\mathrm{d}\tau + (x_1 + a)\,\mathrm{d}y_1 + \lambda(x_2 + b)\,\mathrm{d}y_2. \tag{3.118}$$

Note that these all satisfy

$$f - \eta_{a,b}(Z) = -1, \tag{3.119}$$

which is nowhere vanishing. As a result

$$\tilde{Z} = -\frac{Z}{f - \eta_{a,b}(Z)} = Z = \partial_\tau. \tag{3.120}$$

The various $\eta_{a,b}$ differ from $\eta_{0,0}$ by the exterior derivative of the function

$$h_{a,b} = a y_1 + \lambda b y_2. \tag{3.121}$$

From the proof of Proposition 3.3.1, we know that the local twists with respect to $\eta_{a,b}$ differ from that with respect to $\eta_{0,0}$ by a diffeomorphism ϕ_1 given by the differential equation

$$\phi_0 = \mathrm{id}_{\underline{M}}, \quad \left.\frac{\mathrm{d}\phi_s}{\mathrm{d}s}\right|_{s=t} (\phi_t^{-1}(\cdot)) = h_{a,b}(\cdot)\partial_\tau. \tag{3.122}$$

The solution to this is

$$\phi_1(r, \tau, x_1, y_1, x_2, y_2) = (r, \tau + a y_1 + \lambda b y_2, x_1, y_1, x_2, y_2). \tag{3.123}$$

Note that making the identifications (3.116) forces us to make the following identifications in addition:

$$\tau \sim \tau + a + \lambda b, \tag{3.124}$$

where (a, b) is an arbitrary pair of integers. If λ is rational with standard form p/q, then the set of integer linear combinations of 1 and λ is $q^{-1}\mathbb{Z}$. Otherwise, the set is dense in $\mathbb{R}$. Thus, it is only when λ (and hence ω) is rational that the local twist map on $\underline{M}$ descends to a well-defined global twist map sending ∂_τ-invariant tensor fields on M to ∂_τ-invariant tensor fields on a manifold $\tilde{M}$ obtained from $\underline{M}$ by making the identifications

$$\begin{aligned}(\tau, x_1, y_1, x_2, y_2) &\sim (\tau - y_1, x_1 + 1, y_1, x_2, y_2) \sim (\tau, x_1, y_1 + 1, x_2, y_2)\\ &\sim (\tau - \lambda y_2, x_1, y_1, x_2 + 1, y_2) \sim (\tau, x_1, y_1, x_2, y_2 + 1).\end{aligned} \tag{3.125}$$

Chapter 4

To locally hyperkähler manifolds and back again

In this chapter, we describe Haydys' QK/HK correspondence in terms of the twist (Theorem 4.1.11). Note that though the correspondence itself is not new, the original formulation in [Hay08] was different and made use of Swann bundles and the hyperkähler quotient. We then identify in Propositions 4.2.7 and 4.2.10 the precise conditions for which this is inverse to the twist description of the opposite HK/QK correspondence due to Macia and Swann [MS14]. Finally, as a generalisation of the results in [Cor+17], we use the various twist formulae developed in Chapter 3 to derive identities relating the Levi-Civita connection and Riemann curvature of a quaternionic Kähler manifold to that of the locally hyperkähler manifold associated to it via the QK/HK correspondence (Proposition 4.2.8 and Theorem 4.2.17). We then use this to show that the 1-loop-deformed Ferrara–Sabharwal metrics with quadratic prepotential have cohomogeneity exactly 1 (Theorem 4.2.21).

All the results in this chapter with the exception of Proposition 4.2.5 due to Macia and Swann [MS14] are original. Proposition 4.2.8 and and Theorems 4.2.17 and 4.2.21 have been proved in collaboration with Danu Thung [CST20b; CST20a].

4.1 Locally hyperkähler structures on quaternionic twists

4.1.1 Locally hyperkähler manifolds and rotating Killing fields

In Definition 2.1.3 of quaternionic Kähler manifolds, we explicitly excluded Ricci-flat manifolds. Since Theorem 2.1.12 requires only the existence of a parallel quaternionic bundle, it continues to hold for Ricci-flat manifolds which satisfy all the other defining properties of quaternionic Kähler manifolds. In particular, this implies that the quaternionic bundle is flat and admits local parallel sections.

Definition 4.1.1 (Locally hyperkähler manifolds)**.** A locally hyperkähler (HK) manifold $(\tilde{M}, \tilde{g}, H)$ is an AQH manifold $(\tilde{M}, \tilde{g}, H)$ such that the quaternionic bundle H admits a local oriented orthonormal frame (I_1, I_2, I_3) of Kähler structures i.e. Hermitian structures parallel with respect to the Levi-Civita connection $\nabla^{\tilde{g}}$ associated to $\tilde{g}$.

Remark 4.1.2. To show that a (local) almost Hermitian structure I is Kähler, it is not enough to show that associated 2-form $\tilde{g}(I\cdot, \cdot)$ is closed. One would also have to show that I is integrable. However, when we have three such local almost Hermitian structures I_1, I_2, I_3, Hitchin shows in Lemma 6.8 of [Hit87], that the integrability of I_i automatically follows from the closedness of all three 2-forms $\omega_i = \tilde{g}(I_j\cdot, \cdot)$.

Example 4.1.3 (Complex cotangent spaces)**.** Let N be an open subset of $\mathbb{C}^n$, coordinatised by $(z_0, z_1, \ldots, z_{n-1})$ and equipped with the pseudo-Riemannian metric

$$g_N = -|\mathrm{d}z_0|^2 + \sum_{a=1}^{n-1} |\mathrm{d}z_i|^2. \tag{4.1}$$

Then the metric induced on the cotangent bundle

$$\tilde{M} := T^*N \cong N \times \mathbb{C}^n, \tag{4.2}$$

coordinatised by $(z_0, \ldots, z_{n-1}, w_0, \ldots, w_{n-1})$, i.e.

$$\tilde{g} = -(|\mathrm{d}z_0|^2 + |\mathrm{d}w_0|^2) + \sum_{a=1}^{n-1} (|\mathrm{d}z_a|^2 + |\mathrm{d}w_a|^2), \tag{4.3}$$

admits a natural locally hyperkähler structure given by

$$\begin{aligned}
I_1 &= \mathrm{i}\left(\mathrm{d}z_0 \wedge_{\tilde{g}} \partial_{z_0} + \mathrm{d}w_0 \wedge_{\tilde{g}} \partial_{w_0} + \sum_{a=1}^{n-1} (\mathrm{d}z_a \wedge_{\tilde{g}} \partial_{z_a} + \mathrm{d}w_a \wedge_{\tilde{g}} \partial_{w_a})\right), \\
I_2 &= \mathrm{i}\left(-\mathrm{d}z_0 \wedge_{\tilde{g}} \partial_{\overline{w}_0} + \mathrm{d}\overline{z}_0 \wedge_{\tilde{g}} \partial_{w_0} + \sum_{a=1}^{n-1} (\mathrm{d}z_a \wedge_{\tilde{g}} \partial_{\overline{w}_a} - \mathrm{d}\overline{z}_a \wedge_{\tilde{g}} \partial_{w_a})\right), \\
I_3 &= I_1 \circ I_2 = -\mathrm{d}z_0 \wedge_{\tilde{g}} \partial_{\overline{w}_0} - \mathrm{d}\overline{z}_0 \wedge_{\tilde{g}} \partial_{w_0} + \sum_{a=1}^{n-1} (\mathrm{d}z_a \wedge_{\tilde{g}} \partial_{\overline{w}_a} + \mathrm{d}\overline{z}_a \wedge_{\tilde{g}} \partial_{w_a}).
\end{aligned} \tag{4.4}$$

The corresponding Kähler forms are then given by

$$\begin{aligned}
\omega_1 &= \frac{\mathrm{i}}{2}\left(-\mathrm{d}z_0 \wedge \mathrm{d}\overline{z}_0 - \mathrm{d}w_0 \wedge \mathrm{d}\overline{w}_0 + \sum_{a=1}^{n-1} (\mathrm{d}z_a \wedge \mathrm{d}\overline{z}_a + \mathrm{d}w_a \wedge \mathrm{d}\overline{w}_a)\right), \\
\omega_2 &= \frac{\mathrm{i}}{2}\left(\mathrm{d}z_0 \wedge \mathrm{d}w_0 - \mathrm{d}\overline{z}_0 \wedge \mathrm{d}\overline{w}_0 + \sum_{a=1}^{n-1} (\mathrm{d}z_a \wedge \mathrm{d}w_a - \mathrm{d}\overline{z}_a \wedge \mathrm{d}\overline{w}_a)\right), \\
\omega_3 &= \frac{1}{2}\left(\mathrm{d}z_0 \wedge \mathrm{d}w_0 + \mathrm{d}\overline{z}_0 \wedge \mathrm{d}\overline{w}_0 + \sum_{a=1}^{n-1} (\mathrm{d}z_a \wedge \mathrm{d}w_a + \mathrm{d}\overline{z}_a \wedge \mathrm{d}\overline{w}_a)\right).
\end{aligned} \tag{4.5}$$

Unsurprisingly, the data of locally hyperkähler manifolds behave much more rigidly than that of quaternionic Kähler manifolds. While in the case of quaternionic Kähler manifolds (M, g, Q) equipped with a Killing field Z, one can, by virtue of the quaternionic moment map construction and Lemma 2.2.11, generically find a local oriented orthonormal frame (J_1, J_2, J_3) for Q such that

$$\mathscr{L}_Z J_1 = 0, \quad \mathscr{L}_Z J_2 = J_3, \quad \mathscr{L}_Z J_3 = -J_2, \tag{4.6}$$

it is not clear if a similar statement can be made about locally hyperkähler manifolds. This is certainly because the quaternionic moment map construction breaks down in case of locally hyperkähler manifolds, but also because in case of locally hyperkähler manifolds, we are not interested in some arbitrary local oriented orthonormal frame for H but a local oriented orthonormal *Kähler* frame. This motivates the following definition.

Definition 4.1.4 (Rotating Killing field)**.** A rotating Killing field $\tilde{Z}$ of a locally hyperkähler manifold $(\tilde{M}, \tilde{g}, H)$ is a Killing field such that there exists a local oriented orthonormal Kähler frame (I_1, I_2, I_3) of H such that

$$\mathscr{L}_Z I_1 = 0, \quad \mathscr{L}_Z I_2 = I_3, \quad \mathscr{L}_Z I_3 = -I_2. \tag{4.7}$$

Remark 4.1.5. Note that the definition implies that I_1 is a global section of H.

Example 4.1.6. The locally hyperkähler manifold $\tilde{M}$ in Example 4.1.3 admits the following rotating Killing field:

$$\tilde{Z} = -\mathrm{i}\left(z_0 \partial_{z_0} - \bar{z}_0 \partial_{\bar{z}_0} + \sum_{a=1}^{n-1} (z_a \partial_{z_a} - \bar{z}_a \partial_{\bar{z}_a}) \right). \tag{4.8}$$

In fact, it's possible to give a more intrinsic characterisation of rotating Killing fields, one that doesn't refer to any particular choice of frames.

Proposition 4.1.7. *A Killing field $\tilde{Z}$ of a locally hyperkähler manifold $(\tilde{M}, \tilde{g}, H)$ is rotating if and only if there exists a global endomorphism field I_{H} commuting with H such that*

$$I_1 := I_{\mathrm{H}} - 2\nabla^{\tilde{g}} \tilde{Z} \tag{4.9}$$

is a Kähler section of H.

Proof. Recall the general identity for an arbitrary 1-form α, vector fields u, v, and torsion-free connection ∇ that

$$(\nabla\alpha)(u, v) - (\nabla\alpha)(v, u) = \mathrm{d}\alpha(u, v). \tag{4.10}$$

Specialising to the case of the 1-forms $\iota_{\tilde{Z}}\omega_i$ and Levi-Civita connection $\nabla^{\tilde{g}}$ on a locally hyperkähler manifold, where $(\omega_1, \omega_2, \omega_3)$ is a local oriented orthonormal frame of Kähler forms, we obtain

$$\omega_i(\nabla^{\tilde{g}}_u \tilde{Z}, v) - \omega_i(\nabla^{\tilde{g}}_v \tilde{Z}, u) = \mathrm{d} \circ \iota_{\tilde{Z}}\omega_i = \mathscr{L}_{\tilde{Z}}\omega_i. \tag{4.11}$$

Next we use the Killing equation for $\tilde{Z}$, namely

$$\tilde{g}(\nabla^{\tilde{g}}_u \tilde{Z}, v) = -\tilde{g}(\nabla^{\tilde{g}}_v \tilde{Z}, u), \tag{4.12}$$

to rewrite the left-hand side of (4.11) as

$$\begin{aligned} \omega_i(\nabla^{\tilde{g}}_u \tilde{Z}, v) - \omega_i(\nabla^{\tilde{g}}_v \tilde{Z}, u) &= \tilde{g}(I_i \nabla^{\tilde{g}}_u \tilde{Z}, v) - \tilde{g}(I_i \nabla^{\tilde{g}}_v \tilde{Z}, u) \\ &= \tilde{g}(I_i \nabla^{\tilde{g}}_u \tilde{Z}, v) + \tilde{g}(\nabla^{\tilde{g}}_v \tilde{Z}, I_i u) \\ &\overset{(4.12)}{=} \tilde{g}(I_i \nabla^{\tilde{g}}_u \tilde{Z}, v) - \tilde{g}(\nabla^{\tilde{g}}_{I_i u} \tilde{Z}, v) \\ &= -\tilde{g}([\nabla^{\tilde{g}} \tilde{Z}, I_i] u, v). \end{aligned} \tag{4.13}$$

Now if we assume that $\tilde{Z}$ is a rotating Killing field so that the right-hand side of (4.11) becomes

$$\begin{aligned} \mathscr{L}_{\tilde{Z}}\varpi_1 = 0 &= \frac{1}{2}\,\tilde{g}([I_1, I_1]u, v), \\ \mathscr{L}_{\tilde{Z}}\varpi_2 = \varpi_3 &= \frac{1}{2}\,\tilde{g}([I_1, I_2]u, v), \\ \mathscr{L}_{\tilde{Z}}\varpi_3 = -\varpi_2 &= \frac{1}{2}\,\tilde{g}([I_1, I_3]u, v), \end{aligned} \tag{4.14}$$

then we may rearrange terms to get the required endomorphism field I_{H}:

$$\tilde{g}([I_{\mathrm{H}}, I_i]u, v) := \tilde{g}([I_1 + 2\nabla^{\tilde{g}}\tilde{Z}, I_i]u, v) = 0. \tag{4.15}$$

Conversely, if we were given I_{H}, then we could extend the I_1 specified by (4.9) to a local oriented orthonormal frame (I_1, I_2, I_3) of H. Then the same argument as above tells us that

$$\begin{aligned} \mathscr{L}_{\tilde{Z}}\varpi_1 &= -\tilde{g}([\nabla^{\tilde{g}}\tilde{Z}, I_1]u, v) = \frac{1}{2}\,\tilde{g}([I_1, I_1]u, v) = 0, \\ \mathscr{L}_{\tilde{Z}}\varpi_2 &= -\tilde{g}([\nabla^{\tilde{g}}\tilde{Z}, I_2]u, v) = \frac{1}{2}\,\tilde{g}([I_1, I_2]u, v) = \varpi_3, \\ \mathscr{L}_{\tilde{Z}}\varpi_3 &= -\tilde{g}([\nabla^{\tilde{g}}\tilde{Z}, I_3]u, v) = \frac{1}{2}\,\tilde{g}([I_1, I_3]u, v) = -\varpi_2. \end{aligned} \tag{4.16}$$

□

Corollary 4.1.8. *Given a locally hyperkähler manifold $(\tilde{M}, \tilde{g}, H)$ with (I_1, I_2, I_3) a local oriented orthonormal Kähler frame for H, and a rotating Killing field $\tilde{Z}$ preserving I_1, with associated endomorphism field $I_{\mathrm{H}} = I_1 + 2\nabla^{\tilde{g}}Z$, the tensor fields $\tilde{g}(I_{\mathrm{H}} \circ I_i\cdot, \cdot)$ are symmetric bilinear.*

Proof. Note that the Killing equation implies that I_{H} is skew-self-adjoint. Therefore we have for all vector fields u, v on M, the following chain of equalities:

$$\tilde{g}(I_{\mathrm{H}} \circ I_i u, v) = -\tilde{g}(I_i u, I_{\mathrm{H}} v) = \tilde{g}(u, I_i \circ I_{\mathrm{H}} v) = \tilde{g}(u, I_{\mathrm{H}} \circ I_i v) = \tilde{g}(I_{\mathrm{H}} \circ I_i v, u). \tag{4.17}$$

□

Remark 4.1.9. Making the replacement $v \mapsto I_i v$ in the above tells us that $\tilde{g} \circ I_{\mathrm{H}}$ is a 2-form which is of type $(1, 1)$ with respect to all three local complex structures I_i. This was proved in a different way by Hitchin in Proposition 1 of [Hit13].

Example 4.1.10. The endomorphism field I_{H} associated with the rotating Killing field $\tilde{Z}$ in Example 4.1.6 is given by

$$\begin{aligned} I_{\mathrm{H}} = I_1 + 2\nabla^{\tilde{g}}\tilde{Z} &= I_1 - 2\mathrm{i}\left(\mathrm{d}z_0 \wedge_{\tilde{g}} \partial_{z_0} + \sum_{a=1}^{n-1} \mathrm{d}z_a \wedge_{\tilde{g}} \partial_{z_a}\right) \\ &= \mathrm{i}\left(-\mathrm{d}z_0 \wedge_{\tilde{g}} \partial_{z_0} + \mathrm{d}w_0 \wedge_{\tilde{g}} \partial_{w_0} + \sum_{a=1}^{n-1}(-\mathrm{d}z_a \wedge_{\tilde{g}} \partial_{z_a} + \mathrm{d}w_a \wedge_{\tilde{g}} \partial_{w_a})\right). \end{aligned} \tag{4.18}$$

Thus we may compute

$$\begin{aligned}
\tilde{g}(I_1 \circ I_{\mathrm{H}}\cdot,\cdot) &= \tilde{g}(I_{\mathrm{H}} \circ I_1\cdot,\cdot) = -|\mathrm{d}z_0|^2 + |\mathrm{d}w_0|^2 + \sum_{a=1}^{n-1}(|\mathrm{d}z_a|^2 - |\mathrm{d}w_a|^2), \\
\tilde{g}(I_2 \circ I_{\mathrm{H}}\cdot,\cdot) &= \tilde{g}(I_{\mathrm{H}} \circ I_2\cdot,\cdot) = 2\,\mathrm{Re}\left(\mathrm{d}z_0\,\mathrm{d}w_0 + \sum_{a=1}^{n-1}\mathrm{d}z_a\,\mathrm{d}w_a\right), \\
\tilde{g}(I_3 \circ I_{\mathrm{H}}\cdot,\cdot) &= \tilde{g}(I_{\mathrm{H}} \circ I_3\cdot,\cdot) = 2\,\mathrm{Im}\left(\mathrm{d}z_0\,\mathrm{d}w_0 + \sum_{a=1}^{n-1}\mathrm{d}z_a\,\mathrm{d}w_a\right).
\end{aligned} \tag{4.19}$$

4.1.2 The QK/HK correspondence

It has been known for a while owing to the work of both mathematicians (e.g. [Hay08; Hit13; ACM13; MS14; Ale+15]) and physicists (e.g. [RVV06; APP11]) that one can associate to any quaternionic Kähler manifold (M, g, Q) with a nowhere vanishing Killing field Z a locally hyperkähler manifold $\tilde{M}, \tilde{g}, H)$ with a rotating Killing field $\tilde{Z}$. It turns out that this correspondence can be realised in terms of the twist construction in the case where the quaternionic moment map μ^Z is nowhere vanishing.

Theorem 4.1.11. *Let (M, g, Q) be a quaternionic Kähler manifold equipped with a Killing field Z with nowhere vanishing quaternionic moment map μ^Z with normalisation J^Z and well-defined quaternionic twist data $(Z, \omega_{\mathrm{Q}}, f_{\mathrm{Q}})$ and let $(\tilde{M}, \tilde{Z}, \tilde{\omega}_{\mathrm{H}}, \tilde{f}_{\mathrm{H}})$ be its twist with a global twist map*

$$T : \Gamma(\mathbb{T}^{\bullet,\bullet}M)^Z \to \Gamma(\mathbb{T}^{\bullet,\bullet}\tilde{M})^{\tilde{Z}}. \tag{4.20}$$

Then the tuple

$$(\tilde{M}, \tilde{g}, H) := (\tilde{M}, T(g_{\mathrm{Q}}), T(Q)), \tag{4.21}$$

where g_{Q} is the standard quaternionic elementary deformation, is locally hyperkähler with $\tilde{Z}$ a rotating Killing field which is Hamiltonian with respect to $T(g_{\mathrm{Q}}(J^Z\cdot,\cdot))$.

Proof. Since being quaternionic Kähler or locally hyperkähler with rotating Killing field are local properties, it's enough to work with local twist data $(U, Z, \omega_{\mathrm{Q}}, f_{\mathrm{Q}}, \eta_{\mathrm{Q}})$ on the quaternionic Kähler manifold (M, g, Q) given by the choice of local oriented orthonormal frame $(J_1' = J^Z, J_2', J_3')$ of Q. In fact, in a slight abuse of notation, we will identify U with the relevant open set $\tilde{U}$ in $\tilde{M}$ and, as a consequence, all the relevant tensor fields and distributions on U and $\tilde{U}$.

By Lemma 2.2.11, we know that we can always choose J_2' and J_3' so that

$$\mathscr{L}_Z J_2' = J_3', \quad \mathscr{L}_Z J_3' = -J_2'. \tag{4.22}$$

Let us then make such a choice. Note that this implies that

$$f_{\mathrm{Q}} - \eta_{\mathrm{Q}}(Z) = \langle J_2', \mathscr{L}_Z J_3' \rangle = -1. \tag{4.23}$$

By Lemma 2.2.10, we know that (M, g_{Q}, Q) is almost quaternionic Hermitian and therefore, so must be

$$(U, \tilde{g}, H) = (U, \mathrm{tw}_{Z, f_{\mathrm{Q}}, \eta_{\mathrm{Q}}}(g_{\mathrm{Q}}), \mathrm{tw}_{Z, f_{\mathrm{Q}}, \eta_{\mathrm{Q}}}(Q)). \tag{4.24}$$

Let (I_1, I_2, I_3) be the local twist of (J_1', J_2', J_3'). By Corollary 3.2.3, we have that

$$\mathscr{L}_{\tilde{Z}} I_i = -\frac{1}{f_Q - \eta_Q(Z)} \mathrm{tw}_{Z,f_Q,\eta_Q}(\mathscr{L}_Z J_i') = \mathrm{tw}_{Z,f_Q,\eta_Q}(\mathscr{L}_Z J_i'). \tag{4.25}$$

So in order to prove $(\tilde{M}, \tilde{g}, H)$ is locally hyperkähler with rotating Killing field $\tilde{Z}$, it suffices to show that these I_i are Kähler structures, or equivalently, the 2-forms $\varpi_i = g(I_i\cdot, \cdot)$ are closed. To this end, we introduce the following 2-forms:

$$\begin{aligned}
\sigma_1 := g_Q(J_1'\cdot, \cdot) &\overset{(2.58)}{=} -K\left(\frac{1}{\|\mu^Z\|}\,\omega_Q + \mathrm{d}\left(\frac{\iota_Z g}{\|\mu^Z\|^2}\right)\right) \\
&\overset{(2.60)}{=} \frac{K}{\|\mu^Z\|^3}\,(\nu\|\mu^Z\|^2\omega_1' - \iota_Z\omega_1' \wedge \iota_Z g + \iota_Z\omega_2' \wedge \iota_Z\omega_3'), \\
\sigma_2 := g_Q(J_2'\cdot, \cdot) &\overset{(2.60)}{=} \frac{K}{\|\mu^Z\|^3}\,(\nu\|\mu^Z\|^2\omega_2' - \iota_Z\omega_2' \wedge \iota_Z g + \iota_Z\omega_3' \wedge \iota_Z\omega_1'), \\
\sigma_3 := g_Q(J_3'\cdot, \cdot) &\overset{(2.60)}{=} \frac{K}{\|\mu^Z\|^3}\,(\nu\|\mu^Z\|^2\omega_3' - \iota_Z\omega_3' \wedge \iota_Z g + \iota_Z\omega_1' \wedge \iota_Z\omega_2').
\end{aligned} \tag{4.26}$$

We have argued earlier that it's enough to show that all three 2-forms

$$\varpi_1 = \mathrm{tw}_{Z,\omega_Q,f_Q}(\sigma_1), \quad \varpi_2 = \mathrm{tw}_{Z,\omega_Q,f_Q}(\sigma_2), \quad \varpi_3 = \mathrm{tw}_{Z,\omega_Q,f_Q}(\sigma_3) \tag{4.27}$$

are closed. First of all, we note that

$$\begin{aligned}
\iota_Z\sigma_1 &= \frac{K}{\|\mu^Z\|^3}\,(\nu\|\mu^Z\|^2\iota_Z\omega_1' + g(Z,Z)\iota_Z\omega_1') = -\frac{Kf_Q}{\|\mu^Z\|^2}\,\iota_Z\omega_1', \\
\iota_Z\sigma_2 &= \frac{K}{\|\mu^Z\|^3}\,(\nu\|\mu^Z\|^2\iota_Z\omega_2' + g(Z,Z)\iota_Z\omega_2') = -\frac{Kf_Q}{\|\mu^Z\|^2}\,\iota_Z\omega_2'.
\end{aligned} \tag{4.28}$$

To show that ϖ_1 is closed, we invoke (3.32) to see that

$$\begin{aligned}
\mathrm{d}\varpi_1 &= \mathrm{d} \circ \mathrm{tw}_{Z,\omega_Q,f_Q}(\sigma_1) = \mathrm{tw}_{Z,\omega_Q,f_Q}\left(\mathrm{d}\sigma_1 - \frac{1}{f_Q}\,\omega_Q \wedge \iota_Z\sigma_1\right) \\
&= K\,\mathrm{tw}_{Z,\omega_Q,f_Q}\left(\frac{1}{\|\mu^Z\|^2}\,\mathrm{d}\|\mu^Z\| \wedge \omega_Q + \frac{1}{\|\mu^Z\|^2}\,(\iota_Z\omega_1) \wedge \omega_Q\right) \overset{(2.49)}{=} 0.
\end{aligned} \tag{4.29}$$

In order to prove that ϖ_2 is closed, we first rewrite σ_2 as follows:

$$\begin{aligned}
\sigma_2 &= \frac{K}{\|\mu^Z\|^3}\,(\nu\|\mu^Z\|^2\omega_2' - \iota_Z\omega_2' \wedge \iota_Z g + \iota_Z\omega_3' \wedge \iota_Z\omega_1') \\
&\overset{(2.53)}{=} \frac{K}{\|\mu^Z\|^3}\,(\|\mu^Z\|^2(\mathrm{d}\alpha_{31}' + \alpha_{32}' \wedge \alpha_{21}') - \iota_Z\omega_2' \wedge \iota_Z g + \iota_Z\omega_3' \wedge \iota_Z\omega_1') \\
&\overset{(2.49)}{=} \frac{K}{\|\mu^Z\|^3}\,(\|\mu^Z\|^2\mathrm{d}\alpha_{31}' - \|\mu^Z\|\alpha_{32}' \wedge \iota_Z\omega_2' - \iota_Z\omega_2' \wedge \iota_Z g + \|\mu^Z\|\alpha_{31}' \wedge \mathrm{d}\|\mu^Z\|) \\
&= K\,\mathrm{d}\left(\frac{\alpha_{31}'}{\|\mu^Z\|}\right) - \frac{K}{\|\mu^Z\|^2}\,\eta_Q \wedge \iota_Z\omega_2' = K\,\mathrm{d}\left(\frac{\alpha_{31}'}{\|\mu^Z\|}\right) + \frac{1}{f_Q}\,\eta_Q \wedge \iota_Z\sigma_2.
\end{aligned} \tag{4.30}$$

Thus we find that its local twist

$$\varpi_2 = \mathrm{tw}_{Z,\omega_Q,f_Q}(\sigma_2) = \sigma_2 - \frac{1}{f_Q}\,\eta_Q \wedge \iota_Z\sigma_2 = K\,\mathrm{d}\left(\frac{\alpha_{31}'}{\|\mu^Z\|}\right) \tag{4.31}$$

is in fact closed. An analogous computation tells us that

$$\varpi_3 = \mathrm{tw}_{Z,\omega_Q,f_Q}(\sigma_3) = \sigma_3 - \frac{1}{f_Q}\,\eta_Q \wedge \iota_Z \sigma_3 = K\,\mathrm{d}\left(\frac{\alpha'_{12}}{\|\mu^Z\|}\right) \tag{4.32}$$

is also closed.

The only thing remaining to be proved is that $\tilde{Z}$ is Hamiltonian with respect to ϖ_1. This follows from a short computation:

$$\begin{aligned} \iota_{\tilde{Z}}\varpi_1 &= -\frac{1}{f_Q}\,\mathrm{tw}_{Z,\omega_Q,f_Q}(\sigma_1)(\mathrm{tw}_{Z,\omega_Q,f_Q}(Z),\cdot) \\ &= -\frac{1}{f_Q}\,\mathrm{tw}_{Z,\omega_Q,f_Q}(\iota_Z\sigma_1) = \frac{K}{\|\mu^Z\|^2}\,\mathrm{tw}_{Z,\omega_Q,f_Q}(\iota_Z\omega'_1) \\ &\overset{(2.49)}{=} -\frac{K}{\|\mu^Z\|^2}\,\mathrm{tw}_{Z,\omega_Q,f_Q}\circ \mathrm{d}\|\mu^Z\| \overset{(3.32)}{=} -\mathrm{d}\left(-\frac{K}{\|\mu^Z\|}\right). \end{aligned} \tag{4.33}$$

□

Example 4.1.12. We specialise Example 3.1.12 to the Przanowski–Tod Ansatz, whose quaternionic twist data and standard elementary deformation we had previously computed in Example 2.3.3 to be

$$\begin{aligned} f_Q &= -\frac{1}{P\rho} - \frac{\nu}{4\rho} = -\frac{\partial_\rho u}{2P}, \\ \omega_Q &= -\frac{1}{2}\,\mathrm{d}\left(\frac{1}{P}\,(\partial_\rho u)(\mathrm{d}\tau + \Theta) + \partial_y u\,\mathrm{d}x - \partial_x u\,\mathrm{d}y\right) \\ g_Q &= 2K(\partial_\rho u)\left(\mathrm{d}\rho^2 + 2e^u|\mathrm{d}\zeta|^2 + \frac{1}{P^2}\,(\mathrm{d}\tau+\Theta)^2\right). \end{aligned} \tag{4.34}$$

To do this, we make the following choice of an oriented orthonormal frame for the quaternionic bundle Q:

$$J'_1 = J_1, \quad J'_2 = \cos(\tau)J_2 + \sin(\tau)J_3, \quad J'_3 = -\sin(\tau)J_2 + \cos(\tau)J_3, \tag{4.35}$$

where J_i are the Hermitian structures associated to the ω_i given in (2.78). The connection 1-form α_{23} transforms under this change of basis as

$$\alpha'_{23} = \alpha_{23} - \mathrm{d}\tau. \tag{4.36}$$

Thus, the auxiliary 1-form η_Q is given by

$$\eta_Q = -\frac{1}{2}\left(\frac{1}{P}\,(\partial_\rho u)(\mathrm{d}\tau+\Theta) + \partial_y u\,\mathrm{d}x - \partial_x u\,\mathrm{d}y\right) + \mathrm{d}\tau, \tag{4.37}$$

and we may read off

$$\mathrm{tw}_{Z,f_Q,\eta_Q}(\iota_Z g) = \iota_Z g - \frac{g(Z,Z)}{f_Q}\,\eta_Q = \frac{1}{2\rho^2\partial_\rho u}\left(\mathrm{d}\tau - \frac{1}{2}\,(\partial_y u\,\mathrm{d}x - \partial_x u\,\mathrm{d}y)\right). \tag{4.38}$$

The metric we hence obtain upon applying the QK/HK correspondence is

$$\begin{aligned}\tilde{g} &= g_Q + \frac{K f_Q}{\|\mu^Z\|^2} \frac{1}{g(Z,Z)} \left((\iota_Z g)^2 - (\mathrm{tw}_{Z,f,\eta}(\iota_Z g))^2\right) \\ &= 2K(\partial_\rho u)(\mathrm{d}\rho^2 + 2e^u |\mathrm{d}\zeta|^2) + \frac{8K}{\partial_\rho u}\left(\mathrm{d}\tau - \frac{1}{2}\left(\partial_y u\,\mathrm{d}x - \partial_x u\,\mathrm{d}y\right)\right)^2.\end{aligned} \tag{4.39}$$

This is the well-known Boyer–Finley Ansatz for self-dual Ricci-flat (and so, locally hyperkähler) metrics in dimension 4 with a rotating Killing field

$$\tilde{Z} = -\frac{1}{f_Q - \eta_Q(Z)} Z = Z = \partial_\tau, \tag{4.40}$$

introduced in [BF82]. The correspondence between the Przanowski–Tod Ansatz and the Boyer–Finley Ansatz has also been pointed out in Section 2.3.2 of [APP11].

Example 4.1.13. Substituting, as in Example 2.3.4, the solution

$$u = \ln(\rho + c) \tag{4.41}$$

of the $\mathrm{SU}(\infty)$ Toda equation into the Boyer–Finley Ansatz, we get

$$\tilde{g} = 2K\left(\frac{\mathrm{d}\rho^2}{\rho + c} + 2|\mathrm{d}\zeta|^2 + 4(\rho + c)\,\mathrm{d}\tau^2\right) \tag{4.42}$$

and carrying out a change of coordinates

$$\rho = r^2 - c, \quad \zeta = \sqrt{2}\,w \tag{4.43}$$

where $r > \sqrt{c}$, we get

$$\tilde{g} = 8K(\mathrm{d}r^2 + r^2\mathrm{d}\tau^2 + |\mathrm{d}w|^2). \tag{4.44}$$

This describes a flat metric on the universal cover of

$$(\mathbb{R}^2 \backslash B^2_{\sqrt{c}}[0]) \times \mathbb{C}, \tag{4.45}$$

where $B^2_{\sqrt{c}}[0]$ denotes the closed ball of radius $\sqrt{c}$ about the origin in $\mathbb{R}^2$.

4.2 Inverting the QK/HK correspondence

4.2.1 The HK/QK correspondence

Although the the twist realisation of the QK/HK correspondence is an original result, the opposite correspondence associating a quaternionic Kähler (M, g, Q) with Killing fields Z to locally hyperkähler manifolds $(\tilde{M}, \tilde{g}, H)$ with rotating Killing field $\tilde{Z}$—originally discovered by Haydys [Hay08] and generalised by Alekseevsky, Cortés and Mohaupt [ACM13]—was described in terms of a twist by Macia and Swann. As in the case of quaternionic Kähler manifolds, this involves first taking an elementary deformation.

Definition 4.2.1 (Standard hyperkähler elementary deformation)**.** The standard hyperkähler elementary deformation $\tilde{g}_H$ of a locally hyperkähler manifold $(\tilde{M}, \tilde{g}, H)$ with a rotating Killing field $\tilde{Z}$ given to be Hamiltonian with respect to the Kähler form

$$\omega_1 = \tilde{g}(I_1, \cdot, \cdot) \in \Gamma(H^\flat) \tag{4.46}$$

that it preserves, is a metric defined for fixed constant nonzero parameters $\tilde{K}$ and $\tilde{k}$ by

$$\tilde{g}_{\mathrm{H}} = \frac{\tilde{K}}{\tilde{f}_1}\,\tilde{g}|_{\mathbb{H}_H\tilde{Z}^{\perp}} + \frac{\tilde{K}}{\tilde{k}}\frac{\tilde{f}_{\mathrm{H}}}{\tilde{f}_1^2}\,\tilde{g}|_{\mathbb{H}_H\tilde{Z}}, \tag{4.47}$$

where $\mathbb{H}_H Z$ is the subspace spanned by $\tilde{Z}, I_1\tilde{Z}, I_2\tilde{Z}, I_3\tilde{Z}$, while $\mathbb{H}_H Z^{\perp}$ is the subspace $\tilde{g}$-orthogonal to it, and $\tilde{f}_1$ and $\tilde{f}_{\mathrm{H}}$ are nowhere vanishing functions satisfying

$$\iota_{\tilde{Z}}\varpi_1 = -\mathrm{d}\tilde{f}_1, \quad \tilde{f}_{\mathrm{H}} = \tilde{k}(\tilde{f}_1 + \tilde{g}(\tilde{Z},\tilde{Z})). \tag{4.48}$$

Example 4.2.2. Continuing with Example 4.1.6, we have

$$\begin{aligned}
\iota_{\tilde{Z}}\tilde{g} &= \mathrm{Im}\left(-z_0\mathrm{d}\bar{z}_0 + \sum_{a=1}^{n-1} z_a\mathrm{d}\bar{z}_a\right), \quad \iota_{\tilde{Z}}\varpi_1 = \mathrm{Re}\left(-z_0\mathrm{d}\bar{z}_0 + \sum_{a=1}^{n-1} z_a\mathrm{d}\bar{z}_a\right), \\
\iota_{\tilde{Z}}\varpi_2 &= \mathrm{Re}\left(z_0\mathrm{d}w_0 + \sum_{a=1}^{n-1} z_a\mathrm{d}w_a\right), \quad \iota_{\tilde{Z}}\varpi_3 = \mathrm{Im}\left(z_0\mathrm{d}w_0 + \sum_{a=1}^{n-1} z_a\mathrm{d}w_a\right).
\end{aligned} \tag{4.49}$$

From this we may read off $\tilde{f}_1$ and $\tilde{f}_{\mathrm{H}}$ to be of the form

$$\begin{aligned}
\tilde{f}_1 &= \frac{1}{2}\left(|z_0|^2 - \sum_{a=1}^{n-1}|z_a|^2 - c\right) = -\frac{1}{2}(\tilde{g}(\tilde{Z},\tilde{Z}) + c), \\
\tilde{f}_{\mathrm{H}} &= \frac{\tilde{k}}{2}\left(-|z_0|^2 + \sum_{a=1}^{n-1}|z_a|^2 - c\right) = \frac{\tilde{k}}{2}(\tilde{g}(\tilde{Z},\tilde{Z}) - c),
\end{aligned} \tag{4.50}$$

where c is some choice of a constant. In order to ensure that $\tilde{f}_1$ and $\tilde{f}_{\mathrm{H}}$ are nowhere vanishing, we choose N to be

$$N = \left\{ (z_0, \ldots, z_{n-1}) \in \mathbb{C}^n \;\middle|\; |z_0|^2 - \sum_{a=1}^{n-1}|z_a|^2 > c \right\}, \tag{4.51}$$

and c to be positive. Once this is taken care of, we have the following well-defined expression for the standard hyperkähler elementary deformation:

$$\begin{aligned}
\tilde{g}_{\mathrm{H}} &= \frac{\tilde{K}}{\tilde{f}_1}\,\tilde{g}|_{\mathbb{H}_H\tilde{Z}^{\perp}} + \frac{\tilde{K}}{\tilde{k}}\frac{\tilde{f}_{\mathrm{H}}}{\tilde{f}_1^2}\,\tilde{g}|_{\mathbb{H}_H\tilde{Z}} = \frac{\tilde{K}}{\tilde{f}_1}\,\tilde{g} + \frac{\tilde{K}}{\tilde{f}_1^2}\left((\iota_{\tilde{Z}}\tilde{g})^2 + \sum_{i=1}^{3}(\iota_{\tilde{Z}}\varpi_i)^2\right) \\
&= 2\tilde{K}\left(\frac{-|\mathrm{d}z_0|^2 + \sum_{a=1}^{n-1}|\mathrm{d}z_a|^2}{|z_0|^2 - \sum_{b=1}^{n-1}|z_b|^2 - c}\right. \\
&\quad \left. + 2\,\frac{\left|-z_0\mathrm{d}\bar{z}_0 + \sum_{a=1}^{n-1} z_a\mathrm{d}\bar{z}_a\right|^2 + \left|z_0\mathrm{d}w_0 + \sum_{a=1}^{n-1} z_a\mathrm{d}w_a\right|^2}{\left(|z_0|^2 - \sum_{b=1}^{n-1}|z_b|^2 - c\right)^2}\right).
\end{aligned} \tag{4.52}$$

Proposition 4.2.3. *Let $(\tilde{M}, \tilde{g}, H)$ be a locally hyperkähler manifold equipped with a rotating Killing field $\tilde{Z}$ given to be Hamiltonian with respect to the Kähler form*

$$\varpi_1 = \tilde{g}(I_1\cdot,\cdot) \in \Gamma(H^{\tilde{\flat}}) \tag{4.53}$$

that it preserves, with nowhere vanishing Z-invariant Hamiltonian function $\tilde{f}_1$, and let

$$I_{\mathrm{H}} = I_1 + 2\nabla^{\tilde{g}}\tilde{Z} \tag{4.54}$$

be the endomorphism field associated with the rotating Killing field Z. Then

$$(\tilde{Z}, \tilde{\omega}_{\mathrm{H}}, \tilde{f}_{\mathrm{H}}) := (\tilde{Z}, \tilde{k}\,\tilde{g} \circ I_{\mathrm{H}}, \tilde{k}(\tilde{f}_1 + \tilde{g}(\tilde{Z}, \tilde{Z}))) \tag{4.55}$$

constitute twist data on $\tilde{M}$.

Proof. Firstly, we check that

$$\begin{aligned} \iota_{\tilde{Z}}\tilde{\omega}_{\mathrm{H}} &= \tilde{k}\,\tilde{g}(I_{\mathrm{H}}\tilde{Z}, \cdot) = \tilde{k}\,\tilde{g}(I_1\tilde{Z}, \cdot) + \tilde{k}\,\tilde{g}(2\nabla^{\tilde{g}}_{\tilde{Z}}\tilde{Z}, \cdot) \\ &\overset{(4.12)}{=} \tilde{k}\,\iota_{\tilde{Z}}\varpi - 2\tilde{k}\,\tilde{g}(\nabla^{\tilde{g}}\tilde{Z}, \tilde{Z}) = -\tilde{k}\,\mathrm{d}(f_1 + g(Z, Z)) = -\mathrm{d}\tilde{f}_{\mathrm{H}}. \end{aligned} \tag{4.56}$$

Next, we use (4.11) in conjunction with the Killing equation (4.12) to see

$$2\tilde{g}(\nabla^{\tilde{g}}\tilde{Z}, \cdot) = \mathrm{d} \circ \iota_{\tilde{Z}}\tilde{g}. \tag{4.57}$$

It follows that

$$\tilde{\omega}_{\mathrm{H}} = \tilde{k}(\varpi_1 + \mathrm{d} \circ \iota_{\tilde{Z}}\tilde{g}) \tag{4.58}$$

is closed. □

We will henceforth refer to this twist data as *hyperkähler* twist data.

Example 4.2.4. In the case of Example 4.1.3, the hyperkähler twist data is given by

$$\begin{aligned} \tilde{Z} &= -\mathrm{i}\left(z_0\partial_{z_0} - \bar{z}_0\partial_{\bar{z}_0} + \sum_{a=1}^{n-1}(z_a\partial_{z_a} - \bar{z}_a\partial_{\bar{z}_a})\right), \\ \tilde{\omega}_{\mathrm{H}} &= \frac{\mathrm{i}\tilde{k}}{2}\left(\mathrm{d}z_0 \wedge \mathrm{d}\bar{z}_0 - \mathrm{d}w_0 \wedge \mathrm{d}\overline{w}_0 + \sum_{a=1}^{n-1}(-\mathrm{d}z_a \wedge \mathrm{d}\bar{z}_a + \mathrm{d}w_a \wedge \mathrm{d}\overline{w}_a)\right), \\ \tilde{f}_{\mathrm{H}} &= \frac{\tilde{k}}{2}\left(-|z_0|^2 + \sum_{a=1}^{n-1}|z_a|^2 - c\right) = \frac{\tilde{k}}{2}(\tilde{g}(\tilde{Z}, \tilde{Z}) - c). \end{aligned} \tag{4.59}$$

Let $\tilde{U}$ be a contractible open set over whch the function $\ln(z_0/\bar{z}_0)$ is well-defined, for instance

$$\tilde{U} = \tilde{M}\backslash\{z_0 \in \mathbb{R}_{\leqslant 0}\}. \tag{4.60}$$

Then we can take the auxiliary local twist data to be $(\tilde{U}, \tilde{\eta}_{\mathrm{H}})$, where

$$\begin{aligned} \tilde{\eta}_{\mathrm{H}} &= -\frac{\tilde{k}}{2}\,\mathrm{Im}\left(z_0\mathrm{d}\bar{z}_0 - w_0\mathrm{d}\overline{w}_0 + \sum_{a=1}^{n-1}(-z_a\mathrm{d}\bar{z}_a + w_a\mathrm{d}\overline{w}_a) - \left(c - \frac{1}{4\tilde{K}}\right)\frac{\mathrm{d}z_0}{z_0}\right) \\ &= \frac{\tilde{k}}{2}\left(\iota_{\tilde{Z}}\tilde{g} + \mathrm{Im}\left(w_0\mathrm{d}\overline{w}_0 - \sum_{a=1}^{n-1}w_a\mathrm{d}\overline{w}_a + \left(c - \frac{1}{4\tilde{K}}\right)\frac{\mathrm{d}z_0}{z_0}\right)\right). \end{aligned} \tag{4.61}$$

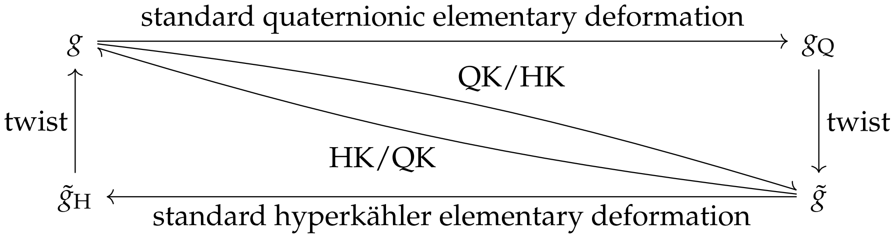

FIGURE 1: The QK/HK and HK/QK correspondences.

We have in particular

$$\begin{aligned}\mathrm{tw}_{\tilde{Z},\tilde{f}_{\mathrm{H}},\tilde{\eta}_{\mathrm{H}}}(\iota_{\tilde{Z}}\tilde{g}) &= \iota_{\tilde{Z}}\tilde{g} - \frac{\tilde{g}(\tilde{Z},\tilde{Z})}{\tilde{f}_{\mathrm{H}}}\,\tilde{\eta}_{\mathrm{H}} \\ &= \iota_{\tilde{Z}}\tilde{g} - \frac{\tilde{g}(\tilde{Z},\tilde{Z})}{\tilde{g}(\tilde{Z},\tilde{Z})-c}\left(\iota_{\tilde{Z}}\tilde{g} + \mathrm{Im}\left(w_0 \mathrm{d}\overline{w}_0 - \sum_{a=1}^{n-1} w_a \mathrm{d}\overline{w}_a + \left(c - \frac{1}{4\tilde{K}}\right)\mathrm{d}\ln(z_0)\right)\right), \\ &= -\frac{\tilde{k}}{2\tilde{f}_{\mathrm{H}}}\left(c\,\iota_{\tilde{Z}}\tilde{g} + \tilde{g}(\tilde{Z},\tilde{Z})\,\mathrm{Im}\left(w_0 \mathrm{d}\overline{w}_0 - \sum_{a=1}^{n-1} w_a \mathrm{d}\overline{w}_a + \left(c - \frac{1}{4\tilde{K}}\right)\mathrm{d}\ln(z_0)\right)\right).\end{aligned} \tag{4.62}$$

We state without proof Macia and Swann's theorem regarding the HK/QK correspondence cast into our notation and conventions. The two correspondences are summarised in Figure 1.

Theorem 4.2.5 ([MS14] Theorem 1)**.** *Let $(\tilde{M},\tilde{g},H)$ be a locally hyperkähler manifold equipped with a rotating Killing field $\tilde{Z}$ given to be Hamiltonian with respect to the Kähler form ϖ_1 that it preserves, with a choice of nowhere vanishing Hamiltonian function f_1, giving rise to hyperkähler twist data $(\tilde{Z},\tilde{\omega}_{\mathrm{H}},\tilde{f}_{\mathrm{H}})$. Let $(M,Z,\omega_{\mathrm{Q}},f_{\mathrm{Q}})$ be its twist with a global twist map*

$$\tilde{T} : \Gamma(\mathbb{T}^{\bullet,\star}\tilde{M})^{\tilde{Z}} \to \Gamma(\mathbb{T}^{\bullet,\star}M)^{Z}. \tag{4.63}$$

Then the tuple

$$(M,g,Q) := (M,\tilde{T}(\tilde{g}_{\mathrm{H}}),\tilde{T}(H)), \tag{4.64}$$

where g_{H} is the standard hyperkähler elementary deformation, is quaternionic Kähler with Killing field $\tilde{Z}$. Moreover, these are the only combinations of elementary deformations by Killing fields $\tilde{Z}$ and twists of of locally hyperkähler metrics with respect to twist data of the form $(\tilde{Z},\tilde{\omega},\tilde{f})$ that produce quaternionic Kähler metrics.

Example 4.2.6 (Ferrara–Sabharwal metrics)**.** Now we apply the HK/QK correspondence to the complex cotangent spaces of Example 4.1.3. It will be convenient to carry out the following change of coordinates:

$$z_0 = \sqrt{\frac{\rho+c}{1-\sum_{b=1}^{n-1}|X_b|^2}}\,e^{-\mathrm{i}\tau}, \quad z_a = X_a\sqrt{\frac{\rho+c}{1-\sum_{b=1}^{n-1}|X_b|^2}}\,e^{-\mathrm{i}\tau}, \quad w_0 = \frac{\zeta_0}{\sqrt{2}}, \quad w_a = \frac{\zeta_a}{\sqrt{2}}, \tag{4.65}$$

where a runs from 1 to $n-1$ and

$$\rho > 0, \quad \sum_{a=1}^{n-1}|X_a|^2 < 1, \quad -\pi < \tau < \pi. \tag{4.66}$$

This has as inverse the following change of coordinates:

$$X_a = \frac{z_a}{z_0}, \quad \rho = |z_0|^2 - \sum_{b=1}^{n-1} |z_b|^2 - c, \quad \tau = \frac{\mathrm{i}}{2} \ln\left(\frac{z_0}{\overline{z}_0}\right), \quad \zeta_0 = \sqrt{2}\, w_0, \quad \zeta_a = \sqrt{2}\, w_a. \tag{4.67}$$

In these new coordinates, we have

$$\begin{aligned}
&\tilde{Z} = \partial_\tau, \quad \tilde{f}_1 = \frac{\rho}{2}, \quad \tilde{f}_{\mathrm{H}} = -\frac{\tilde{k}}{2}(\rho + 2c), \quad \tilde{g}(\tilde{Z}, \tilde{Z}) = -(\rho + c), \\
&\iota_{\tilde{Z}}\tilde{g} = -(\rho + c)\left(\mathrm{d}\tau - \sum_{a=1}^{n-1} \frac{\mathrm{Im}(X_a \mathrm{d}\overline{X}_a)}{1 - \sum_{b=1}^{n-1} |X_b|^2}\right), \\
&\mathrm{tw}_{\tilde{Z}, \tilde{f}_{\mathrm{H}}, \tilde{\eta}_{\mathrm{H}}}(\iota_{\tilde{Z}}\tilde{g}) = -\frac{1}{2}\frac{\rho + c}{\rho + 2c}\left(\frac{\mathrm{d}\tau}{2\tilde{K}} - \sum_{a=1}^{n-1} \frac{2c\,\mathrm{Im}(X_a \mathrm{d}\overline{X}_a)}{1 - \sum_{b=1}^{n-1} |X_b|^2} + \mathrm{Im}\left(\zeta_0 \mathrm{d}\overline{\zeta}_0 - \sum_{a=1}^{n-1} \zeta_a \mathrm{d}\overline{\zeta}_a\right)\right).
\end{aligned} \tag{4.68}$$

Meanwhile, the standard hyperkähler elementary deformation $\tilde{g}_{\mathrm{H}}$ becomes

$$\begin{aligned}
g_{\mathrm{H}} = 2\tilde{K}\Bigg(&\frac{1}{4\rho^2}\frac{\rho + 2c}{\rho + c}\mathrm{d}\rho^2 + \frac{\rho + c}{\rho}\left(\sum_{a=1}^{n-1} \frac{|\mathrm{d}X_a|^2}{1 - \sum_{b=1}^{n-1} |X_b|^2} + \frac{\left|\sum_{a=1}^{n-1} X_a \mathrm{d}\overline{X}_a\right|^2}{\left(1 - \sum_{b=1}^{n-1} |X_b|^2\right)^2}\right) \\
&+ \frac{1}{2\rho}\left(-|\mathrm{d}\zeta_0|^2 + \sum_{a=1}^{n-1} |\mathrm{d}\zeta_a|^2\right) + \frac{\rho + c}{\rho^2}\frac{\left|\mathrm{d}\zeta_0 + \sum_{a=1}^{n-1} X_a \mathrm{d}\zeta_a\right|^2}{1 - \sum_{b=1}^{n-1} |X_b|^2} \\
&+ \frac{1}{\rho^2}(\rho + c)(\rho + 2c)\left(\mathrm{d}\tau - \sum_{a=1}^{n-1} \frac{\mathrm{Im}(X_a \mathrm{d}\overline{X}_a)}{1 - \sum_{b=1}^{n-1} |X_b|^2}\right)^2\Bigg).
\end{aligned} \tag{4.69}$$

Its local twist g is given by

$$\begin{aligned}
g &= \tilde{g}_{\mathrm{H}} - \frac{1}{\tilde{g}_{\mathrm{H}}(\tilde{Z}, \tilde{Z})}\left((\iota_{\tilde{Z}}\tilde{g}_{\mathrm{H}})^2 - (\mathrm{tw}_{\tilde{Z}, \tilde{f}_{\mathrm{H}}, \tilde{\eta}_{\mathrm{H}}}(\iota_{\tilde{Z}}\tilde{g}_{\mathrm{H}}))^2\right) \\
&= \tilde{g}_{\mathrm{H}} - \frac{\tilde{K}}{\tilde{k}}\frac{\tilde{f}_{\mathrm{H}}}{\tilde{f}_1^2}\frac{1}{\tilde{g}(\tilde{Z}, \tilde{Z})}\left((\iota_{\tilde{Z}}\tilde{g})^2 - (\mathrm{tw}_{\tilde{Z}, \tilde{f}_{\mathrm{H}}, \tilde{\eta}_{\mathrm{H}}}(\iota_{\tilde{Z}}\tilde{g}))^2\right) \\
&= \tilde{g}_{\mathrm{H}} - 2\tilde{K}\frac{\rho + 2c}{\rho^2(\rho + c)}\left((\iota_{\tilde{Z}}\tilde{g})^2 - (\mathrm{tw}_{\tilde{Z}, \tilde{f}_{\mathrm{H}}, \tilde{\eta}_{\mathrm{H}}}(\iota_{\tilde{Z}}\tilde{g}))^2\right).
\end{aligned} \tag{4.70}$$

Now, substituting (4.68) and (4.69) into the above yields

$$g = 2\tilde{K}\left(\frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}\mathrm{d}\rho^2 + \frac{\rho+c}{\rho}\left(\sum_{a=1}^{n-1}\frac{|\mathrm{d}X_a|^2}{1-\sum_{b=1}^{n-1}|X_b|^2} + \frac{\left|\sum_{a=1}^{n-1}X_a\mathrm{d}\overline{X}_a\right|^2}{\left(1-\sum_{b=1}^{n-1}|X_b|^2\right)^2}\right)\right.$$
$$+\frac{1}{2\rho}\left(-|\mathrm{d}\zeta_0|^2+\sum_{a=1}^{n-1}|\mathrm{d}\zeta_a|^2\right)+\frac{\rho+c}{\rho^2}\frac{\left|\mathrm{d}\zeta_0+\sum_{a=1}^{n-1}X_a\mathrm{d}\zeta_a\right|^2}{1-\sum_{b=1}^{n-1}|X_b|^2}$$
$$\left.+\frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}\left(\frac{\mathrm{d}\tau}{2\tilde{K}}-\sum_{a=1}^{n-1}\frac{2c\,\mathrm{Im}(X_a\mathrm{d}\overline{X}_a)}{1-\sum_{b=1}^{n-1}|X_b|^2}+\mathrm{Im}\left(\zeta_0\mathrm{d}\overline{\zeta}_0-\sum_{a=1}^{n-1}\zeta_a\mathrm{d}\overline{\zeta}_a\right)\right)^2\right). \tag{4.71}$$

These are the 1-*loop-deformed quadratic prepotential Ferrara–Sabharwal* quaternionic Kähler metrics $g^{\prime c}_{\mathrm{FS}}$ first introduced in [FS90] in the context of the moduli space of vacua of Type II superstring theory and described more explicitly in the context of the HK/QK correspondence in Corollary 15 of [CDS17]. Note that when $n = 1$, we retrieve the 1-loop deformation $g^{\prime c}_{\mathrm{UH}}$ of the universal hypermultiplet, as discussed in Example 2.3.4, with $\nu = -\tilde{K}^{-1}$.

Both the QK/HK as well as the HK/QK correspondence entail certain degrees of freedom. For the QK/HK correspondence, this is the constant parameter K; for the the HK/QK correspondence, these are the Hamiltonian function $\tilde{f}_1$ (to which we can add a constant), and the constant parameters $\tilde{K}$ and $\tilde{k}$. For a certain choice of the Hamiltonian $\tilde{f}_1$ and the parameters $\tilde{K}$ and $\tilde{k}$, the two correspondences are the inverses of each other. We'll show that the HK/QK correspondence is locally left-inverse to the QK/HK correspondence now and postpone a proof that it's also locally right-inverse to the next subsection.

Proposition 4.2.7. *Let (M, g, Q) be a quaternionic Kähler manifold of reduced scalar curvature ν equipped with a Killing field Z with nowhere vanishing quaternionic moment map $\|\mu^Z\|$, let $(\tilde{M}, \tilde{g}, H)$ be a locally hyperkähler manifold with a rotating Killing field $\tilde{Z}$ that is the image of (M, g, Q) under the QK/HK correspondence. Then the image of $(\tilde{M}, \tilde{g}, H)$ under the HK/QK correspondence is locally isometric to (M, g, Q), for the choices*

$$\tilde{f}_1 = T\left(-\frac{K}{\|\mu^Z\|}\right), \quad \tilde{K} = -\frac{1}{\nu}, \quad \tilde{k} = \frac{1}{K\nu}, \tag{4.72}$$

where T is a global twist map realising the QK/HK correspondence and $\tilde{f}_1$ is a choice of Hamiltonian function for $\tilde{Z}$ with respect to the Kähler form ϖ_1 preserved by it.

Proof. Again, we work locally within a contractible open set $U \subseteq M$ which we identify with the corresponding open set $\tilde{U} \subseteq \tilde{M}$.

By (4.33), we know that

$$\tilde{f}_1 = -\frac{K}{\|\mu^Z\|} \tag{4.73}$$

is a valid choice of Hamiltonian function for $\tilde{Z}$ with respect to ϖ_1. Using this, we check that $\tilde{f}_{\mathrm{H}}$ as defined in (4.55) can be written in terms of the quaternionic Kähler data as

$$\begin{aligned}\tilde{f}_{\rm H} = \tilde{k}(\tilde{f}_1 + \tilde{g}(\tilde{Z},\tilde{Z})) &= \tilde{k}\left(-\frac{K}{\|\mu^Z\|} + \frac{1}{f_{\rm Q}^2}\, g_{\rm Q}(Z,Z)\right)\\ &\overset{(2.61)}{=} \tilde{k}\left(-\frac{K}{\|\mu^Z\|} - \frac{K}{\|\mu^Z\|^2}\frac{g(Z,Z)}{f_{\rm Q}}\right)\\ &\overset{(2.57)}{=} -\frac{\tilde{k}K}{\|\mu^Z\|}\left(1 - \frac{g(Z,Z)}{g(Z,Z)+\nu\|\mu^Z\|^2}\right) \overset{(2.57)}{=} \frac{\tilde{k}K\nu}{f_{\rm Q}} = \frac{1}{f_{\rm Q}}.\end{aligned} \tag{4.74}$$

Likewise we may check that $\tilde{\omega}_{\rm H}$ as defined in (4.55) may be written in terms of the quaternionic Kähler data as

$$\begin{aligned}\tilde{\omega}_{\rm H} &= \tilde{k}(\varpi_1 + {\rm d}\circ\iota_{\tilde{Z}}\tilde{g})\\ &= \tilde{k}\left({\rm tw}_{Z,f_{\rm Q},\eta_{\rm Q}}(\sigma_1) + {\rm d}\left(-\frac{1}{f_{\rm Q}}\,{\rm tw}_{Z,f_{\rm Q},\eta_{\rm Q}}(\iota_Z g_{\rm Q})\right)\right)\\ &\overset{(2.61)}{=} \tilde{k}\left({\rm tw}_{Z,f_{\rm Q},\eta_{\rm Q}}(\sigma_1) + {\rm d}\left(\frac{K}{\|\mu^Z\|^2}\,{\rm tw}_{Z,f_{\rm Q},\eta_{\rm Q}}(\iota_Z g)\right)\right)\\ &\overset{(3.32)}{=} \tilde{k}\,{\rm tw}_{Z,f_{\rm Q},\eta_{\rm Q}}\left(\sigma_1 + K\,{\rm d}\left(\frac{\iota_Z g}{\|\mu^Z\|^2}\right) - \frac{K}{\|\mu^Z\|^2}\frac{g(Z,Z)}{f_{\rm Q}}\,\omega_{\rm Q}\right)\\ &\overset{(4.26)}{=} \tilde{k}\,{\rm tw}_{Z,f_{\rm Q},\eta_{\rm Q}}\left(\left(-\frac{K}{\|\mu^Z\|} - \frac{K}{\|\mu^Z\|^2}\frac{g(Z,Z)}{f_{\rm Q}}\right)\omega_{\rm Q}\right)\\ &= \frac{\tilde{k}K\nu}{f_{\rm Q}}\,{\rm tw}_{Z,f_{\rm Q},\eta_{\rm Q}}(\omega_{\rm Q}) = \frac{1}{f_{\rm Q}}\,{\rm tw}_{Z,f_{\rm Q},\eta_{\rm Q}}(\omega_{\rm Q}).\end{aligned} \tag{4.75}$$

Thus, the twist data $(\tilde{Z},\tilde{\omega}_{\rm H},\tilde{f}_{\rm H})$ is dual to $(Z,\omega_{\rm Q},f_{\rm Q})$. Hence, if we choose an auxiliary 1-form

$$\tilde{\eta}_{\rm H} = \frac{1}{f_{\rm Q}}\,{\rm tw}_{Z,f_{\rm Q},\eta_{\rm Q}}(\eta_{\rm Q}) \tag{4.76}$$

for the twist data $(\tilde{Z},\tilde{\omega}_{\rm H},\tilde{f}_{\rm H})$, then the local twists of H and $\tilde{g}_{\rm H}$ with respect to it are

$$\begin{aligned}{\rm tw}_{\tilde{Z},\tilde{f}_{\rm H},\tilde{\eta}_{\rm H}}(H) &= {\rm tw}_{\tilde{Z},\tilde{f}_{\rm H},\tilde{\eta}_{\rm H}}\circ{\rm tw}_{Z,f_{\rm Q},\eta_{\rm Q}}(Q) = Q,\\ {\rm tw}_{\tilde{Z},\tilde{f}_{\rm H},\tilde{\eta}_{\rm H}}(\tilde{g}_{\rm H}) &= {\rm tw}_{\tilde{Z},\tilde{f}_{\rm H},\tilde{\eta}_{\rm H}}\left(\frac{\tilde{K}}{\tilde{f}_1}\,\tilde{g}|_{\mathbb{H}_H\tilde{Z}^\perp} + \frac{\tilde{K}}{\tilde{k}}\frac{f_{\rm H}}{\tilde{f}_1^2}\,\tilde{g}|_{\mathbb{H}_H\tilde{Z}}\right)\\ &= \frac{\tilde{K}}{\tilde{f}_1}\,g_{\rm Q}|_{\mathbb{H}_Q Z^\perp} + \frac{\tilde{K}}{\tilde{k}}\frac{f_{\rm H}}{\tilde{f}_1^2}\,g_{\rm Q}|_{\mathbb{H}_Q Z}\\ &= \frac{\|\mu^Z\|}{K\nu}\,g_{\rm Q}|_{\mathbb{H}_Q Z^\perp} - \frac{\|\mu^Z\|^2}{Kf_{\rm Q}}\,g_{\rm Q}|_{\mathbb{H}_Q Z} = g.\end{aligned} \tag{4.77}$$

□

4.2.2 Levi-Civita connection under HK/QK

We will now use earlier results regarding how the Levi-Civita connection behaves under the twist to say something about how it behaves under the HK/QK correspondence.

Proposition 4.2.8. *Let $(\tilde{M},\tilde{g},H)$ be a locally hyperkähler manifold equipped with a rotating Killing field $\tilde{Z}$ which preserves a Kähler structure I_1 in a local oriented orthonormal frame*

(I_1, I_2, I_3) of H and is Hamiltonian with respect to $\varpi_1 = \tilde{g}(I_1\cdot,\cdot)$ with nowhere vanishing Hamiltonian function $\tilde{f}_1$. Let $(\tilde{Z}, \tilde{\omega}_{\mathrm{H}}, \tilde{f}_{\mathrm{H}})$ be the associated hyperkähler twist data and let (M, g, Q) be a quaternionic Kähler manifold that is its image under the HK/QK correspondence. Then the Levi-Civita connections ∇^g and $\nabla^{\tilde{g}}$ are related via

$$\nabla^g_{\tilde{T}(u)} \circ \tilde{T}(v) = \tilde{T}\left(\nabla^{\tilde{g}}_u v + S^{\mathrm{HQ}}_u v\right), \tag{4.78}$$

where u and v are arbitrary $\tilde{Z}$-invariant vector fields on $\tilde{M}$, $\tilde{T}$ is the global twist map realising the HK/QK correspondence and $S^{\mathrm{HQ}} \in \Gamma(\mathbb{T}^{1,2}\tilde{M})$ is a tensor field given by

$$S^{\mathrm{HQ}}_u v = \frac{1}{2}\sum_{\alpha=0}^{3}\left(\frac{1}{\tilde{f}_{\mathrm{H}}}\tilde{\omega}_{\mathrm{H}}(I_\alpha u, v) I_\alpha \tilde{Z} - \frac{1}{\tilde{f}_1}\left(\lambda_\alpha(u) I_\alpha \circ I_1 v + \lambda_\alpha(v) I_\alpha \circ I_1 u\right)\right), \tag{4.79}$$

with $I_0 = \mathrm{id}_{TM}$ and $\lambda_\alpha = \tilde{g}(I_\alpha \tilde{Z}, \cdot)$.

Proof. We begin by noting that the difference

$$S^{\tilde{g}_{\mathrm{H}}, \tilde{g}} = \nabla^{\tilde{g}_{\mathrm{H}}} - \nabla^{\tilde{g}} \tag{4.80}$$

between the Levi-Civita connections associated to two different metrics is a tensor field given by

$$\tilde{g}_{\mathrm{H}}(S^{\tilde{g}_{\mathrm{H}}, \tilde{g}}_u v, w) = (\nabla^{\tilde{g}}_u \tilde{g}_{\mathrm{H}})(v, w) + (\nabla^{\tilde{g}}_v \tilde{g}_{\mathrm{H}})(u, w) - (\nabla^{\tilde{g}}_w \tilde{g}_{\mathrm{H}})(u, v). \tag{4.81}$$

We have due to (3.64) the following equation for any Z-invariant vector fields u, v, w on $\tilde{M}$ and their global twists u', v', w' on M:

$$\nabla^g_{u'} v' = \tilde{T}(\nabla^{\tilde{g}_{\mathrm{H}}}_u v + S^{\tilde{\omega}_{\mathrm{H}}}_u v) = \tilde{T}(\nabla^{\tilde{g}}_u v + S^{\mathrm{HQ}}_u v), \tag{4.82}$$

where S^{HQ} is the combination

$$S^{\mathrm{HQ}} = S^{\tilde{g}_{\mathrm{H}}, \tilde{g}} + S^{\tilde{\omega}_{\mathrm{H}}}. \tag{4.83}$$

An expression for S^{HQ} in terms of the hyperkähler data may be obtained by combining (3.66) and (4.81):

$$\begin{aligned}\frac{2\tilde{f}_1^2}{\tilde{K}}\tilde{g}_{\mathrm{H}}(S^{\mathrm{HQ}}_u v, t) &= \left(\frac{\tilde{f}_1^2}{\tilde{K}\tilde{f}_{\mathrm{H}}}\iota_{\tilde{Z}}\tilde{g}_{\mathrm{H}} \otimes \tilde{\omega}_{\mathrm{H}} - \frac{\tilde{f}_1^2}{\tilde{K}}\nabla^{\tilde{g}}\tilde{g}_{\mathrm{H}}\right)((t, u, v) - (u, v, t) - (v, u, t)) \\ &= \left(\frac{1}{\tilde{k}}\iota_{\tilde{Z}}\tilde{g} \otimes \tilde{\omega}_{\mathrm{H}} - \frac{\tilde{f}_1^2}{\tilde{K}}\nabla^{\tilde{g}}\tilde{g}_{\mathrm{H}}\right)((t, u, v) - (u, v, t) - (v, u, t)).\end{aligned} \tag{4.84}$$

Introducing the shorthand

$$\tilde{g}_\lambda := \sum_{\alpha=0}^{3}\lambda_\alpha^2 = (\iota_{\tilde{Z}}\tilde{g})^2 + \sum_{i=1}^{3}\iota_{\tilde{Z}}\varpi_i, \tag{4.85}$$

we may write

$$\begin{aligned}
&\left(\frac{1}{\tilde{k}}\,\iota_{\tilde{Z}}\tilde{g}\otimes\tilde{\omega}_{\mathrm{H}}-\frac{\tilde{f}_1^2}{\tilde{K}}\,\nabla^{\tilde{g}}\tilde{g}_{\mathrm{H}}\right)(u,v,t)\\
&=\left(\lambda_0\otimes\omega_1-\lambda_1\otimes\omega_0-\frac{2}{\tilde{f}_1}\,\lambda_1\otimes\tilde{g}_\lambda\right)(u,v,t)\\
&\quad+2\,\lambda_0(u)\omega_0(\nabla^{\tilde{g}}_v\tilde{Z},t)-\sum_{\alpha=0}^{3}(\lambda_\alpha(t)\omega_\alpha(\nabla^{\tilde{g}}_u\tilde{Z},v)+\lambda_\alpha(v)\omega_\alpha(\nabla^{\tilde{g}}_u\tilde{Z},t)).
\end{aligned}\tag{4.86}$$

Then substituting the above into (4.84) and using the identity (4.11) to simplify, we obtain

$$\begin{aligned}
&\frac{2\tilde{f}_1^2}{\tilde{K}}\,\tilde{g}_{\mathrm{H}}(S^{\mathrm{HQ}}_u v,t)\\
&=\left(\sum_{\alpha=0}^{3}\lambda_\alpha\otimes\tilde{g}(I_\alpha\circ I_{\mathrm{H}}\cdot,\cdot)-\frac{2}{\tilde{f}_1}\,\lambda_1\otimes\tilde{g}_\lambda\right)(t,u,v)\\
&\quad-\left(\sum_{\alpha=0}^{3}\lambda_\alpha\otimes\tilde{g}(I_\alpha\circ I_1\cdot,\cdot)-\frac{2}{\tilde{f}_1}\,\lambda_1\otimes\tilde{g}_\lambda\right)((u,v,t)+(v,u,t))\\
&=\tilde{g}\left(\sum_{\alpha=0}^{3}\left(\tilde{g}(I_\alpha\circ I_{\mathrm{H}}u,v)I_\alpha\tilde{Z}-\frac{2}{\tilde{f}_1}\,\lambda_\alpha(u)\lambda_\alpha(v)I_1\tilde{Z}-\lambda_\alpha(u)I_\alpha\circ I_1 v-\lambda_\alpha(v)I_\alpha\circ I_1 u\right.\right.\\
&\quad\left.\left.+\,\frac{2}{\tilde{f}_1}\,(\lambda_1(u)\lambda_\alpha(v)+\lambda_1(v)\lambda_\alpha(u))I_\alpha\tilde{Z}\right),t\right).
\end{aligned}\tag{4.87}$$

Thus, introducing an endomorphism field

$$A=\frac{\tilde{K}}{\tilde{f}_1}\,\tilde{g}_{\mathrm{H}}^{-1}\circ\tilde{g}=\mathrm{id}_{TM}-\frac{\tilde{k}}{\tilde{f}_{\mathrm{H}}}\sum_{\beta=0}^{3}\lambda_\beta\otimes I_\beta\tilde{Z},\tag{4.88}$$

allows us to isolate S^{HQ} as follows:

$$\begin{aligned}
S^{\mathrm{HQ}}_u v&=\frac{1}{2\tilde{f}_1}\sum_{\alpha=0}^{3}A\left(\tilde{g}(I_\alpha\circ I_{\mathrm{H}}u,v)I_\alpha\tilde{Z}-\frac{2}{\tilde{f}_1}\,\lambda_\alpha(u)\lambda_\alpha(v)I_1\tilde{Z}-\lambda_\alpha(u)I_\alpha\circ I_1 v\right.\\
&\qquad\left.-\,\lambda_\alpha(v)I_\alpha\circ I_1 u+\frac{2}{\tilde{f}_1}\,(\lambda_1(u)\lambda_\alpha(v)+\lambda_1(v)\lambda_\alpha(u))I_\alpha\tilde{Z}\right)\\
&=\frac{\tilde{k}}{2\tilde{f}_{\mathrm{H}}}\sum_{\alpha=0}^{3}\tilde{g}(I_\alpha\circ I_{\mathrm{H}}u,v)I_\alpha\tilde{Z}-\frac{1}{2\tilde{f}_1}\sum_{\alpha=0}^{3}(\lambda_\alpha(u)I_\alpha\circ I_1 v+\lambda_\alpha(v)I_\alpha\circ I_1 u)\\
&\quad+\frac{\tilde{k}}{\tilde{f}_1\tilde{f}_{\mathrm{H}}}\sum_{\alpha=0}^{3}(\lambda_1(u)\lambda_\alpha(v)I_\alpha\tilde{Z}+\lambda_\alpha(u)\lambda_1(v)I_\alpha\tilde{Z}-\lambda_\alpha(u)\lambda_\alpha(v)I_1\tilde{Z})\\
&\quad+\frac{\tilde{k}}{2\tilde{f}_1\tilde{f}_{\mathrm{H}}}\sum_{\alpha,\beta=0}^{3}(\lambda_\beta(u)\lambda_\alpha(I_\beta\circ I_1 v)+\lambda_\beta(v)\lambda_\alpha(I_\beta\circ I_1 u))I_\alpha\tilde{Z}.
\end{aligned}\tag{4.89}$$

Note that in order to obtain the above expression, we've had to switch the indices α,β in the double summation. Furthermore, making the replacement $I_\beta\mapsto I_\beta\circ I_1$ in the double sum (which leaves it unchanged) and using the fact that $I_i^{-1}=-I_i$ for

$i \in \{1,2,3\}$ but $I_0^{-1} = I_0$ allows us to simplify the above to

$$\begin{aligned} S_u^{\mathrm{HQ}} v = & \frac{\tilde{k}}{2\tilde{f}_{\mathrm{H}}} \sum_{\alpha=0}^{3} \tilde{g}(I_\alpha \circ I_{\mathrm{H}} u, v) I_\alpha \tilde{Z} - \frac{1}{2\tilde{f}_1} \sum_{\alpha=0}^{3} (\lambda_\alpha(u) I_\alpha \circ I_1 v + \lambda_\alpha(v) I_\alpha \circ I_1 u) \\ & - \frac{\tilde{k}}{\tilde{f}_1 \tilde{f}_{\mathrm{H}}} \sum_{\alpha=0}^{3} \lambda_\alpha(u) \lambda_\alpha(v) I_1 \tilde{Z} + \frac{\tilde{k}}{2\tilde{f}_1 \tilde{f}_{\mathrm{H}}} \sum_{\alpha,\beta=0}^{3} (\tilde{g}(I_\beta \circ I_1 \tilde{Z}, u) \tilde{g}(I_\beta \circ I_\alpha \circ I_1 \tilde{Z}, v) \\ & + \tilde{g}(I_\beta \circ I_1 \tilde{Z}, v) \tilde{g}(I_\beta \circ I_\alpha \circ I_1 \tilde{Z}, u)) I_\alpha \circ I_1 \tilde{Z}. \end{aligned} \tag{4.90}$$

A final simplification follows from the observation that the double summation is manifestly symmetric in u and v for $\alpha = 0$ and antisymmetric for $\alpha \neq 0$, as the replacement $I_\beta \mapsto I_\beta \circ I_\alpha$ shows. This gives us the desired expression for S^{HQ}. □

Remark 4.2.9. With the choice of auxiliary local twist data $(\tilde{U}, \tilde{\eta}_{\mathrm{H}})$, this statement may be easily generalised to vector fields u, v which aren't Z-invariant:

$$\nabla^g_{u'} v' = \mathrm{tw}_{\tilde{Z},\tilde{f}_{\mathrm{H}},\tilde{\eta}_{\mathrm{H}}} \left(\nabla^{\tilde{g}}_u v + S_u^{\mathrm{HQ}} v + \frac{\tilde{\eta}_{\mathrm{H}}(u)}{\tilde{f}_{\mathrm{H}} - \tilde{\eta}_{\mathrm{H}}(\tilde{Z})} \mathscr{L}_{\tilde{Z}} v \right), \tag{4.91}$$

where u' and v' are the local twists of u and v. In particular, if J_i are the local twists of the local Kähler structures I_i (of which only I_1 is $\tilde{Z}$-invariant), we have

$$\begin{aligned} \nabla^g_{u'} J_i &= \mathrm{tw}_{\tilde{Z},\tilde{f}_{\mathrm{H}},\tilde{\eta}_{\mathrm{H}}} \left(\nabla^{\tilde{g}}_u I_i + \left[S_u^{\mathrm{HQ}}, I_i \right] + \frac{\tilde{\eta}_{\mathrm{H}}(u)}{\tilde{f}_{\mathrm{H}} - \tilde{\eta}_{\mathrm{H}}(\tilde{Z})} \mathscr{L}_{\tilde{Z}} I_i \right) \\ &= \mathrm{tw}_{\tilde{Z},\tilde{f}_{\mathrm{H}},\tilde{\eta}_{\mathrm{H}}} \left(\left[S_u^{\mathrm{HQ}}, I_i \right] + \frac{\tilde{\eta}_{\mathrm{H}}(u)}{\tilde{f}_{\mathrm{H}} - \tilde{\eta}_{\mathrm{H}}(\tilde{Z})} \mathscr{L}_{\tilde{Z}} I_i \right). \end{aligned} \tag{4.92}$$

By making the replacement $I_\alpha \mapsto I_i \circ I_\alpha$, we see that

$$\begin{aligned} & \frac{1}{2} \sum_{\alpha=0}^{3} \left(\frac{1}{\tilde{f}_{\mathrm{H}}} \tilde{\omega}_{\mathrm{H}}(I_\alpha u, I_i v) I_\alpha \tilde{Z} - \frac{1}{\tilde{f}_1} \tilde{g}(I_\alpha \tilde{Z}, I_i v) I_\alpha \circ I_1 u \right) \\ & = \frac{1}{2} \sum_{\alpha=0}^{3} I_i \left(\frac{1}{\tilde{f}_{\mathrm{H}}} \tilde{\omega}_{\mathrm{H}}(I_\alpha u, v) I_\alpha \tilde{Z} - \frac{1}{\tilde{f}_1} \tilde{g}(I_\alpha \tilde{Z}, v) I_\alpha \circ I_1 u \right). \end{aligned} \tag{4.93}$$

This implies that the commutators of S^{HQ} with the Kähler structures I_i are given by

$$\left[S_u^{\mathrm{HQ}}, I_i \right] = -\frac{1}{2\tilde{f}_1} \sum_{\alpha=0}^{3} \lambda_\alpha(u) [I_\alpha \circ I_1, I_i]. \tag{4.94}$$

Written out more explicitly, this becomes

$$\begin{aligned} \left[S_u^{\mathrm{HQ}}, I_1 \right] &= \frac{1}{\tilde{f}_1} (\lambda_2(u) I_2 + \lambda_3(u) I_3), \\ \left[S_u^{\mathrm{HQ}}, I_2 \right] &= -\frac{1}{\tilde{f}_1} (\lambda_0(u) I_3 + \lambda_2(u) I_1), \\ \left[S_u^{\mathrm{HQ}}, I_3 \right] &= \frac{1}{\tilde{f}_1} (\lambda_0(u) I_2 - \lambda_3(u) I_1), \end{aligned} \tag{4.95}$$

which gives us

$$\begin{aligned}
\nabla^g J_1 &= \mathrm{tw}_{\tilde{Z},\tilde{f}_\mathrm{H},\tilde{\eta}_\mathrm{H}}\left(\frac{1}{\tilde{f}_1}\left(\lambda_2\otimes I_2+\lambda_3\otimes I_3\right)\right),\\
\nabla^g J_2 &= \mathrm{tw}_{\tilde{Z},\tilde{f}_\mathrm{H},\tilde{\eta}_\mathrm{H}}\left(-\frac{1}{\tilde{f}_1}\left(\lambda_0\otimes I_3+\lambda_2\otimes I_1\right)+\frac{1}{\tilde{f}_\mathrm{H}-\tilde{\eta}_\mathrm{H}(\tilde{Z})}\,\tilde{\eta}_\mathrm{H}\otimes I_3\right),\\
\nabla^g J_3 &= \mathrm{tw}_{\tilde{Z},\tilde{f}_\mathrm{H},\tilde{\eta}_\mathrm{H}}\left(\frac{1}{\tilde{f}_1}\left(\lambda_0\otimes I_2-\lambda_3\otimes I_1\right)-\frac{1}{\tilde{f}_\mathrm{H}-\tilde{\eta}_\mathrm{H}(\tilde{Z})}\,\tilde{\eta}_\mathrm{H}\otimes I_2\right).
\end{aligned} \tag{4.96}$$

Now we're in a position to show that the HK/QK correspondence is also locally right-inverse to the QK/HK correspondence.

Proposition 4.2.10. *Let $(\tilde{M},\tilde{g},H)$ be a locally hyperkähler manifold equipped with a rotating Killing field $\tilde{Z}$ which preserves the Kähler structure I_1 in a local oriented orthonormal Kähler frame (I_1,I_2,I_3) of H and is Hamiltonian with respect to $\varpi_1=\tilde{g}(I_1\cdot,\cdot)$ with nowhere vanishing Hamiltonian function $\tilde{f}_1$. Let $(\tilde{Z},\tilde{\omega}_\mathrm{H},\tilde{f}_\mathrm{H})$ be the associated hyperkähler twist data and let (M,g,Q) be a quaternionic Kähler manifold equipped with a Killing field Z that is its image under the HK/QK correspondence. Then the image of (M,g,Q) under the QK/HK correspondence is locally isometric to $(\tilde{M},\tilde{g},H)$ for the choice*

$$K=-\frac{\tilde{K}}{\tilde{k}}. \tag{4.97}$$

Proof. As in the case of Proposition 4.2.7, we work locally within a contractible open set $\tilde{U}\subseteq\tilde{M}$ which we identify with the corresponding open set $U\subseteq M$.

First of all, we show using (4.96) that the local twist

$$-\frac{K}{\tilde{f}_1}\,J_1=\mathrm{tw}_{\tilde{Z},\tilde{f}_\mathrm{H},\tilde{\eta}_\mathrm{H}}\left(\frac{\tilde{K}}{\tilde{k}\tilde{f}_1}\,I_1\right) \tag{4.98}$$

with respect to any auxiliary 1-form $\tilde{\eta}_\mathrm{H}$ is the quaternionic moment map of (M,g,Q). Introduce the shorthand u' for the local twist of a vector field u. Then, we have

$$\begin{aligned}
\nabla^g_{u'}\left(-\frac{K}{\tilde{f}_1}\,J_1\right) &= \frac{\tilde{K}}{\tilde{k}}\,\mathrm{tw}_{\tilde{Z},\tilde{f}_\mathrm{H},\tilde{\eta}_\mathrm{H}}\left(-\frac{\mathrm{d}\tilde{f}_1(u)}{\tilde{f}_1^2}\,I_1+\frac{1}{\tilde{f}_1^2}\left(\lambda_2\otimes I_2+\lambda_3\otimes I_3\right)\right)\\
&= \frac{\tilde{K}}{\tilde{k}}\,\mathrm{tw}_{\tilde{Z},\tilde{f}_\mathrm{H},\tilde{\eta}_\mathrm{H}}\left(\frac{1}{\tilde{f}_1^2}\sum_{i=1}^3\lambda_i(u)I_i\right)=\frac{\tilde{K}}{\tilde{k}}\,\mathrm{tw}_{\tilde{Z},\tilde{f}_\mathrm{H},\tilde{\eta}_\mathrm{H}}\left(\frac{1}{\tilde{f}_1^2}\sum_{i=1}^3\tilde{g}(I_i\tilde{Z},u)I_i\right)\\
&= -\mathrm{tw}_{\tilde{Z},\tilde{f}_\mathrm{H},\tilde{\eta}_\mathrm{H}}\left(\sum_{i=1}^3\tilde{g}_\mathrm{H}\left(-\frac{1}{\tilde{f}_\mathrm{H}}\,I_i\tilde{Z},u\right)I_i\right)=-\sum_{i=1}^3 g(J_iZ,u')J_i.
\end{aligned} \tag{4.99}$$

This is the defining property of the quaternionic moment map written in terms of local frame (J_1, J_2, J_2). Therefore, we can now obtain

$$\begin{aligned}
f_{\mathrm{Q}} &= -\frac{g(Z,Z)}{\|\mu^Z\|} - \nu\|\mu^Z\| = \frac{\tilde{f}_1}{K\tilde{f}_{\mathrm{H}}^2}\,\tilde{g}_{\mathrm{H}}(\tilde{Z},\tilde{Z}) + \frac{K\nu}{\tilde{f}_1} \\
&= -\frac{1}{\tilde{f}_1\tilde{f}_{\mathrm{H}}}\,\tilde{g}(\tilde{Z},\tilde{Z}) + \frac{1}{\tilde{k}\tilde{f}_1} = \frac{1}{\tilde{f}_{\mathrm{H}}}, \\
\frac{\iota_Z g}{\|\mu^Z\|} &= -\frac{1}{\tilde{f}_{\mathrm{H}}\|\mu^Z\|}\,\mathrm{tw}_{\tilde{Z},\tilde{f}_{\mathrm{H}},\tilde{\eta}_{\mathrm{H}}}(\iota_{\tilde{Z}}\tilde{g}_{\mathrm{H}}) = -\frac{1}{\tilde{f}_1}\,\mathrm{tw}_{\tilde{Z},\tilde{f}_{\mathrm{H}},\tilde{\eta}_{\mathrm{H}}}(\lambda_0), \\
\frac{\iota_Z\omega_i}{\|\mu^Z\|} &= -\frac{1}{\tilde{f}_1}\,\mathrm{tw}_{\tilde{Z},\tilde{f}_{\mathrm{H}},\tilde{\eta}_{\mathrm{H}}}(\lambda_i), \\
\eta_{\mathrm{Q}} &= -\left(\frac{\iota_Z g}{\|\mu^Z\|} + \langle J_2, \nabla^g J_3\rangle\right) \overset{(4.96)}{=} -\frac{\iota_Z g}{\|\mu^Z\|} - \mathrm{tw}_{\tilde{Z},\tilde{f}_{\mathrm{H}},\tilde{\eta}_{\mathrm{H}}}\left(\frac{\lambda_0}{\tilde{f}_1} - \frac{\tilde{\eta}_{\mathrm{H}}}{\tilde{f}_{\mathrm{H}} - \tilde{\eta}_{\mathrm{H}}(\tilde{Z})}\right) \\
&= \mathrm{tw}_{\tilde{Z},\tilde{f}_{\mathrm{H}},\tilde{\eta}_{\mathrm{H}}}\left(\frac{\tilde{\eta}_{\mathrm{H}}}{\tilde{f}_{\mathrm{H}} - \tilde{\eta}_{\mathrm{H}}(\tilde{Z})}\right) = \frac{\tilde{\eta}_{\mathrm{H}}}{\tilde{f}_{\mathrm{H}}}.
\end{aligned} \tag{4.100}$$

Thus, the local twist data $(U, Z, \omega_{\mathrm{Q}}, f_{\mathrm{Q}}, \eta_{\mathrm{Q}})$ is indeed dual to $(\tilde{U}, \tilde{Z}, \tilde{\omega}_{\mathrm{H}}, \tilde{f}_{\mathrm{H}}, \tilde{\eta}_{\mathrm{Q}})$, leading to the following conclusion:

$$\begin{aligned}
\mathrm{tw}_{Z,\omega_{\mathrm{Q}},f_{\mathrm{Q}}}(Q) &= \mathrm{tw}_{Z,\omega_{\mathrm{Q}},f_{\mathrm{Q}}} \circ \mathrm{tw}_{\tilde{Z},\tilde{\omega}_{\mathrm{H}},\tilde{f}_{\mathrm{H}}}(H) = H, \\
\mathrm{tw}_{Z,\omega_{\mathrm{Q}},f_{\mathrm{Q}}}(g_{\mathrm{Q}}) &= \mathrm{tw}_{Z,\omega_{\mathrm{Q}},f_{\mathrm{Q}}}\left(\frac{K\nu}{\|\mu^Z\|}\,g|_{\mathbb{H}_{\mathrm{Q}}Z^\perp} - \frac{Kf_{\mathrm{Q}}}{\|\mu^Z\|^2}\,g|_{\mathbb{H}_{\mathrm{Q}}Z}\right) \\
&= \frac{K\nu}{\|\mu^Z\|}\,\tilde{g}_{\mathrm{H}}|_{\mathbb{H}_H\tilde{Z}^\perp} - \frac{Kf_{\mathrm{Q}}}{\|\mu^Z\|^2}\,\tilde{g}_{\mathrm{H}}|_{\mathbb{H}_H\tilde{Z}} \\
&= \frac{\tilde{f}_1}{\tilde{K}}\,\tilde{g}_{\mathrm{H}}|_{\mathbb{H}_H\tilde{Z}^\perp} + \frac{\tilde{k}}{\tilde{K}}\frac{\tilde{f}_1^2}{f_{\mathrm{H}}}\,\tilde{g}_{\mathrm{H}}|_{\mathbb{H}_H\tilde{Z}} = \tilde{g}.
\end{aligned} \tag{4.101}$$

□

Remark 4.2.11. Now that we know that the HK/QK correspondence is a two-sided inverse of the QK/HK correspondence (for appropriate choices of the parameters involved), we shall refer to the tuple of HK/QK data $(\tilde{M}, \tilde{g}, H, \tilde{Z}, I_1, \tilde{f}_1, \tilde{\omega}_{\mathrm{H}}, \tilde{f}_{\mathrm{H}}, \tilde{g}_{\mathrm{H}})$ as being the *hyperkähler dual* of the tuple of QK/HK data $(M, g, Q, Z, \omega_{\mathrm{Q}}, f_{\mathrm{Q}}, g_{\mathrm{Q}})$, and the QK/HK data $(M, g, Q, Z, \omega_{\mathrm{Q}}, f_{\mathrm{Q}}, g_{\mathrm{Q}})$ as being the *quaternionic dual* of the HK/QK data $(\tilde{M}, \tilde{g}, H, \tilde{Z}, I_1, \tilde{f}_1, \tilde{\omega}_{\mathrm{H}}, \tilde{f}_{\mathrm{H}}, \tilde{g}_{\mathrm{H}})$.

4.2.3 Riemann curvature under HK/QK

In order to state an analogous result describing how the Riemann curvature behaves under the HK/QK corespondence, we need to set up some notation.

Definition 4.2.12 (Kulkarni–Nomizu product)**.** The Kulkarni–Nomizu product $\owedge$ of two $(0,2)$-tensor fields $g, h \in \Gamma(\mathbb{T}^{0,2}M)$ on a manifold M is given by

$$\begin{aligned}
(g \owedge h)(s,t,u,v) = g(s,u)h(t,v) - g(s,v)h(t,u) \\
- g(t,u)h(s,v) + g(t,v)h(s,u),
\end{aligned} \tag{4.102}$$

where s, t, u, v are arbitrary vector fields on M.

Definition 4.2.13 (Riemann product)**.** The Riemann product $\varobar$ of two 2-forms $\alpha, \beta \in \Omega^2 M$ on a manifold M is given by

$$\alpha \varobar \beta = \alpha \owedge \beta + 2\,\alpha \otimes \beta + 2\,\beta \otimes \alpha. \tag{4.103}$$

The point of introducing these products is that they can be used to construct tensor fields which have the same symmetries as the (lowered) Riemann curvature.

Definition 4.2.14 (Abstract curvature tensor field)**.** A $(0,4)$-tensor field $C \in \Gamma(\mathbb{T}^{0,4}M$ on a manifold M is said to be an abstract curvature tensor field if it satisfies for all vector fields s, t, u, v on M the following equations:

$$\begin{aligned} C(s,t,u,v) &= -C(t,s,u,v) = -C(s,t,v,u), \\ C(s,t,u,v) &+ C(t,u,s,v) + C(u,s,t,v) = 0. \end{aligned} \tag{4.104}$$

Lemma 4.2.15. *For any two symmetric bilinear forms g, h and any two 2-forms α, β, the tensor fields $g \owedge h$ and $\alpha \varobar \beta$ are abstract curvature tensor fields.*

Example 4.2.16. The Riemann curvature $R_{\mathbb{H}\mathbf{P}^n}$ of the quaternionic Kähler metric g on the quaternionic projective space $\mathbb{H}\mathbf{P}^n$ is given in terms of a local oriented orthonormal frame (J_1, J_2, J_3) of its quaternionic bundle Q by

$$g \circ R_{\mathbb{H}\mathbf{P}^n} = -\frac{1}{8}\left(g \owedge g + \sum_{i=1}^{3} (g \circ J_i) \varobar (g \circ J_i)\right). \tag{4.105}$$

The minus sign may seem strange when compared to other references which make use of the abstract index notation, but this is just a consequence of taking the (lowered) Riemann curvature in the abstract index notation to be

$$R_{abcd} = g(R^g(e_c, e_d)e_b, e_a) = -g(R^g(e_a, e_b)e_c, e_d). \tag{4.106}$$

In terms of these products we can thus express the lowered Riemann curvature $g \circ R^g$ of a quaternionic manifold as the twist of the sum of the lowered Riemann curvature $\tilde{g}_{\mathrm{H}} \circ R^{\tilde{g}_{\mathrm{H}}}$ of its hyperkähler dual and certain tensor fields algebraically constructed out of $\tilde{g}_{\mathrm{H}}$ and $\tilde{\omega}_{\mathrm{H}}$ that are manifestly abstract curvature tensor fields.

Theorem 4.2.17. *Let $(\tilde{M}, \tilde{g}, H, \tilde{Z}, I_1, \tilde{f}_1, \tilde{\omega}_{\mathrm{H}}, \tilde{f}_{\mathrm{H}}, \tilde{g}_{\mathrm{H}})$ be HK/QK data with quaternionic dual $(M, g, Q, Z, \omega_{\mathrm{Q}}, f_{\mathrm{Q}}, g_{\mathrm{Q}})$, and let $\tilde{T}$ be the global twist map realising the HK/QK correspondence. Then the Riemann curvatures R^g and $R^{\tilde{g}}$ of the metrics g and $\tilde{g}$ satisfy*

$$\begin{aligned} g \circ R^g = \tilde{T}\Bigg(&\frac{\tilde{K}}{\tilde{f}_1}\tilde{g} \circ R^{\tilde{g}} + \frac{1}{8\tilde{K}}\left(\tilde{g}_{\mathrm{H}} \owedge \tilde{g}_{\mathrm{H}} + \sum_{i=1}^{3} (\tilde{g}_{\mathrm{H}} \circ I_i) \varobar (\tilde{g}_{\mathrm{H}} \circ I_i)\right) \\ &- \frac{\tilde{K}}{8\tilde{k}}\frac{1}{\tilde{f}_1 \tilde{f}_{\mathrm{H}}}\left(\tilde{\omega}_{\mathrm{H}} \varobar \tilde{\omega}_{\mathrm{H}} + \sum_{i=1}^{3} (\tilde{\omega}_{\mathrm{H}} \circ I_i) \owedge (\tilde{\omega}_{\mathrm{H}} \circ I_i)\right)\Bigg), \end{aligned} \tag{4.107}$$

where (I_1, I_2, I_3) is a local oriented orthonormal Kähler frame of H.

Proof. Since the Riemann curvature R^h with respect to any metric h is given by

$$R^h(u,v)t = [\nabla^h_u, \nabla^h_v]t - \nabla^h_{\mathscr{L}_u v} t, \tag{4.108}$$

we have as a consequence of (4.78) and (3.34b)

$$R^{g}(u',v')t' = \tilde{T}\left(R^{\nabla^{\tilde{g}}+S^{\mathrm{HQ}}}(u,v)t - \frac{1}{\tilde{f}_{\mathrm{H}}}\tilde{\omega}_{\mathrm{H}}(u,v)\left(\nabla^{\tilde{g}}_{\tilde{Z}}t + S^{\mathrm{HQ}}_{\tilde{Z}}t\right)\right). \tag{4.109}$$

Here u, v, t are $\tilde{Z}$-invariant vector fields on $\tilde{M}$, with global twists u', v', t' on M, and $R^{\nabla^{\tilde{g}}+S^{\mathrm{HQ}}}$ is the curvature of the connection $\nabla^{\tilde{g}} + S^{\mathrm{HQ}}$, given by

$$\begin{aligned}
R^{\nabla^{\tilde{g}}+S^{\mathrm{HQ}}}(u,v)t &= \left[\nabla^{\tilde{g}}_u + S^{\mathrm{HQ}}_u, \nabla^{\tilde{g}}_v + S^{\mathrm{HQ}}_v\right]t - \nabla^{\tilde{g}}_{\mathscr{L}_u v}t - S^{\mathrm{HQ}}_{\mathscr{L}_u v}t \\
&= R^{\tilde{g}}(u,v)t + \left[\nabla^{\tilde{g}}_u, S^{\mathrm{HQ}}_v\right]t - \left[\nabla^{\tilde{g}}_v, S^{\mathrm{HQ}}_u\right]t \\
&\quad + \left[S^{\mathrm{HQ}}_u, S^{\mathrm{HQ}}_v\right]t - S^{\mathrm{HQ}}_{\nabla^{\tilde{g}}_u v}t + S^{\mathrm{HQ}}_{\nabla^{\tilde{g}}_v u}t \\
&= R^{\tilde{g}}(u,v)t + \left(\nabla^{\tilde{g}}_u S^{\mathrm{HQ}}\right)(v,t) - \left(\nabla^{\tilde{g}}_v S^{\mathrm{HQ}}\right)(u,t) + \left[S^{\mathrm{HQ}}_u, S^{\mathrm{HQ}}_v\right]t,
\end{aligned} \tag{4.110}$$

where we have used the fact that $\nabla^{\tilde{g}}$ is torsion-free. Moreover, since t is $\tilde{Z}$-invariant, we have

$$\nabla^{\tilde{g}}_{\tilde{Z}}t = \mathscr{L}_{\tilde{Z}}t + \nabla^{\tilde{g}}_t\tilde{Z} = \nabla^{\tilde{g}}_t\tilde{Z}. \tag{4.111}$$

Putting (4.110) and (4.111) together, we get

$$\begin{aligned}
R^{g}(u',v')t' = \tilde{T}\Big(R^{\tilde{g}}(u,v)t &+ \left(\nabla^{\tilde{g}}_u S^{\mathrm{HQ}}\right)(v,t) - \left(\nabla^{\tilde{g}}_v S^{\mathrm{HQ}}\right)(u,t) \\
&+ \left[S^{\mathrm{HQ}}_u, S^{\mathrm{HQ}}_v\right]t - \frac{1}{\tilde{f}_{\mathrm{H}}}\tilde{\omega}_{\mathrm{H}}(u,v)\left(\nabla^{\tilde{g}}_t\tilde{Z} + S^{\mathrm{HQ}}_{\tilde{Z}}t\right)\Big).
\end{aligned} \tag{4.112}$$

Substituting (4.79) into the above, making use of the identity

$$(\nabla^{\tilde{g}}_u I_{\mathrm{H}})v - (\nabla^{\tilde{g}}_v I_{\mathrm{H}})u = R^{\tilde{g}}(u,v)\tilde{Z}, \tag{4.113}$$

and carrying out simplifications (deferred to Section 4.A in the appendix) then yields

$$\begin{aligned}
&R^{g}(u',v')t' \\
&= \tilde{T}\Big(R^{\tilde{g}}(u,v)t \\
&\quad - \frac{1}{2\tilde{f}_{\mathrm{H}}^2}\Big(\frac{1}{2}\sum_{\alpha,\beta=0}^{3}(\tilde{\omega}_{\mathrm{H}}(I_\alpha\tilde{Z},u)\omega_{\mathrm{H}}(I_\beta v,t) - \omega_{\mathrm{H}}(I_\alpha\tilde{Z},v)\tilde{\omega}_{\mathrm{H}}(I_\beta u,t))I_\alpha\circ I_\beta\tilde{Z} \\
&\qquad + \sum_{\alpha=0}^{3}\tilde{\omega}_{\mathrm{H}}(u,v)\tilde{\omega}_{\mathrm{H}}(I_\alpha\tilde{Z},t)I_\alpha\tilde{Z}\Big) \\
&\quad + \frac{1}{2\tilde{f}_{\mathrm{H}}}\Big(\frac{1}{2}\sum_{\alpha=0}^{3}\left(\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)I_\alpha\circ I_{\mathrm{H}}u - \tilde{\omega}_{\mathrm{H}}(I_\alpha u,t)I_\alpha\circ I_{\mathrm{H}}v + 4\tilde{g}(I_\alpha\circ R^{\tilde{g}}(u,v)\tilde{Z},t)I_\alpha\tilde{Z}\right) \\
&\qquad - \tilde{\omega}_{\mathrm{H}}(u,v)I_{\mathrm{H}}t\Big) \\
&\quad + \frac{1}{4}\sum_{\alpha=0}^{3}\left(\tilde{g}_{\mathrm{H}}(I_\alpha u,t)I_\alpha v - \tilde{g}_{\mathrm{H}}(I_\alpha v,t)I_\alpha u + (\tilde{g}_{\mathrm{H}}(I_\alpha u,v) - g_{\mathrm{H}}(I_\alpha v,u))I_\alpha t\right)\Big).
\end{aligned} \tag{4.114}$$

Recall that any complex structure Kähler with respect to some metric h necessarily commutes with the Riemann curvature $R^h(u,v)$ for any any vector fields u and v and that $R^h(u,v)$ is skew-self-adjoint with respect to h. In this case, I_1, I_2, I_3 are all Kähler with respect to $\tilde{g}$, and so commute with $R^{\tilde{g}}$, as does I_0 on account of just being the identity endomorphism field. Thus, we have

$$\tilde{g}(I_\alpha \circ R^{\tilde{g}}(u,v)\tilde{Z},t) = \tilde{g}(R^{\tilde{g}}(u,v)\circ I_\alpha \tilde{Z},t) = -\tilde{g}(I_\alpha \tilde{Z}, R^{\tilde{g}}(u,v)t). \tag{4.115}$$

As a result, we may now succinctly write (4.114) using the endomorphism field A introduced in (4.88) as

$$\begin{aligned} R^g(u',v')t' &= \tilde{T}\Bigg(A\circ R^{\tilde{g}}(u,v)t - \frac{1}{2\tilde{f}_{\mathrm{H}}}\tilde{\omega}_{\mathrm{H}}(u,v)A\circ I_{\mathrm{H}}t \\ &\quad + \frac{1}{4\tilde{f}_{\mathrm{H}}}\sum_{\alpha=0}^{3}\Big(\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)A\circ I_{\mathrm{H}}\circ I_\alpha u - \tilde{\omega}_{\mathrm{H}}(I_\alpha u,t)A\circ I_{\mathrm{H}}\circ I_\alpha v\Big) \\ &\quad + \frac{1}{4}\sum_{\alpha=0}^{3}\Big(\tilde{g}_{\mathrm{H}}(I_\alpha u,t)I_\alpha v - \tilde{g}_{\mathrm{H}}(I_\alpha v,t)I_\alpha u + \big(\tilde{g}_{\mathrm{H}}(I_\alpha u,v) - \tilde{g}_{\mathrm{H}}(I_\alpha v,u)\big)I_\alpha t\Big)\Bigg). \end{aligned} \tag{4.116}$$

Note that we have made use of the fact that A commutes with all the I_α, a consequence of the fact that I_1, I_2, I_3 are Hermitian with respect to both $\tilde{g}$ and $\tilde{g}_{\mathrm{H}}$. Additionally, because we have

$$\tilde{g}_{\mathrm{H}}(A\cdot,\cdot) = \frac{\tilde{K}}{\tilde{f}_1}\tilde{g}(\cdot,\cdot), \tag{4.117}$$

we may contract (4.116) with $\iota_s\tilde{g}_{\mathrm{H}}$ under the twist, where s is some $\tilde{Z}$-invariant vector field on $\tilde{M}$ with global twist s' on M, to obtain

$$\begin{aligned} g(R^g(u',v')t',s') &= \frac{\tilde{K}}{\tilde{f}_1}\tilde{g}(R^{\tilde{g}}(u,v)t,s) - \frac{1}{2\tilde{f}_{\mathrm{H}}}\frac{\tilde{K}}{\tilde{f}_1}\tilde{\omega}_{\mathrm{H}}(u,v)\tilde{g}(I_{\mathrm{H}}t,s) \\ &\quad + \frac{1}{4\tilde{f}_{\mathrm{H}}}\frac{\tilde{K}}{\tilde{f}_1}\sum_{\alpha=0}^{3}\Big(\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)\tilde{g}(I_{\mathrm{H}}\circ I_\alpha u,s) - \tilde{\omega}_{\mathrm{H}}(I_\alpha u,t)\tilde{g}(I_{\mathrm{H}}\circ I_\alpha v,s)\Big) \\ &\quad + \frac{1}{4\tilde{K}}\sum_{\alpha=0}^{3}\Big(\tilde{g}_{\mathrm{H}}(I_\alpha u,t)\tilde{g}_{\mathrm{H}}(I_\alpha v,s) - \tilde{g}_{\mathrm{H}}(I_\alpha v,t)\tilde{g}_{\mathrm{H}}(I_\alpha u,s) \\ &\qquad + \big(\tilde{g}_{\mathrm{H}}(I_\alpha u,v) - \tilde{g}_{\mathrm{H}}(I_\alpha v,u)\big)\tilde{g}_{\mathrm{H}}(I_\alpha t,s)\Big) \\ &= \frac{\tilde{K}}{\tilde{f}_1}\tilde{g}(R^{\tilde{g}}(u,v)t,s) - \frac{\tilde{K}}{2\tilde{k}}\frac{1}{\tilde{f}_1\tilde{f}_{\mathrm{H}}}\tilde{\omega}_{\mathrm{H}}(u,v)\tilde{\omega}_{\mathrm{H}}(t,s) \\ &\quad + \frac{\tilde{K}}{4\tilde{k}}\frac{1}{\tilde{f}_1\tilde{f}_{\mathrm{H}}}\sum_{\alpha=0}^{3}\Big(\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)\tilde{\omega}_{\mathrm{H}}(I_\alpha u,s) - \tilde{\omega}_{\mathrm{H}}(I_\alpha u,t)\tilde{\omega}_{\mathrm{H}}(I_\alpha v,s)\Big) \\ &\quad + \frac{1}{4\tilde{K}}\sum_{\alpha=0}^{3}\Big(\tilde{g}_{\mathrm{H}}(I_\alpha u,t)\tilde{g}_{\mathrm{H}}(I_\alpha v,s) - \tilde{g}_{\mathrm{H}}(I_\alpha v,t)\tilde{g}_{\mathrm{H}}(I_\alpha u,s) \\ &\qquad + \big(\tilde{g}_{\mathrm{H}}(I_\alpha u,v) - \tilde{g}_{\mathrm{H}}(I_\alpha v,u)\big)\tilde{g}_{\mathrm{H}}(I_\alpha t,s)\Big). \end{aligned} \tag{4.118}$$

This may be rewritten in terms of the products $\owedge$ and $\obar$ to yield (4.107). □

Remark 4.2.18. The HK/QK Levi-Civita connection formula (4.78) and HK/QK curvature formula (4.107) are generalisations of the Levi-Civita and curvature formulae for a special class of quaternionic Kähler manifolds known as *q-map spaces* derived by Cortés, Dyckmanns, Jüngling, and Lindemann in Sections 2.5 and 2.6 of [Cor+17].

Remark 4.2.19. The HK/QK curvature formula (4.107) is also a refinement of Alekseevsky's decomposition of the curvature of quaternionic Kähler metrics, quoted in Theorem 2.1.12, in the following sense. The $\mathbb{H}\mathbf{P}^n$ part arises from

$$\begin{aligned} &\tilde{T}\left(\frac{1}{8\tilde{K}}\left(\tilde{g}_{\mathrm{H}} \owedge \tilde{g}_{\mathrm{H}} + \sum_{i=1}^{3}(\tilde{g}_{\mathrm{H}} \circ I_i) \varobar (\tilde{g}_{\mathrm{H}} \circ I_i)\right)\right) \\ &= \frac{1}{8\tilde{K}}\left(g \owedge g + \sum_{i=1}^{3}(g \circ J_i) \varobar (g \circ J_i)\right) = \nu\, g \circ R_{\mathbb{H}\mathbf{P}^n}. \end{aligned} \tag{4.119}$$

Meanwhile, the quaternionic Weyl curvature W_{Q}^{g} arises from

$$\tilde{T}\left(\frac{\tilde{K}}{\tilde{f}_1}\,\tilde{g} \circ R^{\tilde{g}} - \frac{\tilde{K}}{8\tilde{k}}\frac{1}{\tilde{f}_1\tilde{f}_{\mathrm{H}}}\left(\tilde{\omega}_{\mathrm{H}} \varobar \tilde{\omega}_{\mathrm{H}} + \sum_{i=1}^{3}(\tilde{\omega}_{\mathrm{H}} \circ I_i) \owedge (\tilde{\omega}_{\mathrm{H}} \circ I_i)\right)\right). \tag{4.120}$$

The quaternionic Weyl curvature is by definition a tensor field with all the symmetries of the Riemann or Weyl curvature which additionally commutes with the quaternionic bundle Q. In the lowered form, this amounts to requiring $g \circ W_{\mathrm{Q}}^{g}$ to be an abstract curvature tensor field satisfying

$$g \circ W_{\mathrm{Q}}^{g}(s,t,J_j u, J_j v) = g \circ W_{\mathrm{Q}}^{g}(s,t,u,v), \tag{4.121}$$

where (J_1, J_2, J_3) is any local oriented orthonormal frame for Q. That it is an abstract curvature tensor field is manifest. Meanwhile, (4.121) follows from the proposition below.

Proposition 4.2.20. *The tensor field*

$$W := \tilde{T}\left(\frac{\tilde{K}}{\tilde{f}_1}\,\tilde{g} \circ R^{\tilde{g}} - \frac{\tilde{K}}{8\tilde{k}}\frac{1}{\tilde{f}_1\tilde{f}_{\mathrm{H}}}\left(\tilde{\omega}_{\mathrm{H}} \varobar \tilde{\omega}_{\mathrm{H}} + \sum_{i=1}^{3}(\tilde{\omega}_{\mathrm{H}} \circ I_i) \owedge (\tilde{\omega}_{\mathrm{H}} \circ I_i)\right)\right) \tag{4.122}$$

satisfies for any local oriented orthonormal frame (J_1, J_2, J_3) *of* Q

$$W(s,t,J_j u, J_j v) = W(s,t,u,v). \tag{4.123}$$

Proof. Let $W = \tilde{T}(W')$. Then it suffices to show that W' satisfies

$$W'(s,t,I_j u, I_j v) = W'(s,t,u,v) \tag{4.124}$$

for local Kähler structures I_1, I_2, I_3. But this is indeed the case as can be seen from the fact that $R^{\tilde{g}}$ is the curvature of a locally hyperkähler metric and so commutes with the local Kähler structures and the following short computation:

$$\begin{aligned}
&\left(\tilde{\omega}_{\mathrm{H}} \owedge \tilde{\omega}_{\mathrm{H}} + \sum_{i=1}^{3} (\tilde{\omega}_{\mathrm{H}} \circ I_i) \owedge (\tilde{\omega}_{\mathrm{H}} \circ I_i)\right)(s, t, I_j u, I_j v) \\
&= 4\,\tilde{\omega}_{\mathrm{H}}(s,t)\tilde{\omega}_{\mathrm{H}}(I_j u, I_j v) + 2\sum_{\alpha=0}^{3} (\tilde{\omega}_{\mathrm{H}}(I_\alpha s, I_j u)\tilde{\omega}_{\mathrm{H}}(I_\alpha t, I_j v) - \tilde{\omega}_{\mathrm{H}}(I_\alpha s, I_j v)\tilde{\omega}_{\mathrm{H}}(I_\alpha t, I_j u)) \\
&= 4\,\tilde{\omega}_{\mathrm{H}}(s,t)\tilde{\omega}_{\mathrm{H}}(u, v) + 2\sum_{\alpha=0}^{3} (\tilde{\omega}_{\mathrm{H}}(I_\alpha s, u)\tilde{\omega}_{\mathrm{H}}(I_\alpha t, v) - \tilde{\omega}_{\mathrm{H}}(I_\alpha s, v)\tilde{\omega}_{\mathrm{H}}(I_\alpha t, u)) \\
&= \left(\tilde{\omega}_{\mathrm{H}} \owedge \tilde{\omega}_{\mathrm{H}} + \sum_{i=1}^{3} (\tilde{\omega}_{\mathrm{H}} \circ I_i) \owedge (\tilde{\omega}_{\mathrm{H}} \circ I_i)\right)(s, t, u, v),
\end{aligned} \tag{4.125}$$

where the penultimate step follows from making a replacement $I_\alpha \mapsto I_j \circ I_\alpha$ in the sum and then using the fact that I_{H} commutes with I_j, so that $\tilde{\omega}_{\mathrm{H}}(I_j u, I_j v) = \tilde{\omega}_{\mathrm{H}}(u, v)$. □

The Riemann curvature and associated curvature invariants are in general much harder to compute in case of a quaternionic Kähler manifold than a locally hyperkähler manifold. The above relation therefore simplifies such computation and makes computing, say, the curvature norm of the quadratic prepotential Ferrara–Sabharwal metrics tractable. As a result, we can establish the following fact.

Theorem 4.2.21. *The* 1*-loop-deformed quadratic prepotential Ferrara–Sabharwal metrics have cohomogeneity* 1*.*

Proof. We shall first show that the cohomogeneity of the 1-loop-deformed quadratic prepotential Ferrara–Sabharwal metrics $g_{\mathrm{FS}}^{\prime c} =: g$ is at least 1 by computing the curvature norm, given by

$$\operatorname{tr}(\mathscr{R}^2) = \frac{1}{4} \sum_{a,b,c,d} g(R^g(E'_a, E'_b)E'_c, E'_d)^2, \tag{4.126}$$

where the vector fields E'_a constitute an orthonormal frame with respect to the Riemannian metric g. We have already seen in Example 4.2.6 that these metrics arise as images of the flat metric

$$\tilde{g} = -(|\mathrm{d}z_0|^2 + |\mathrm{d}w_0|^2) + \sum_{a=1}^{n-1} (|\mathrm{d}z_a|^2 + |\mathrm{d}w_a|^2), \tag{4.127}$$

on complex cotagent spaces of dimension real $4n$, under the HK/QK correspondence. Therefore, by (4.107), the curvature norm is given by

$$\operatorname{tr}(\mathscr{R}^2) = \frac{1}{4} \sum_{a,b,c,d} C(E_a, E_b, E_c, E_d)^2, \tag{4.128}$$

where C is a $(0,4)$-tensor given by

$$\begin{aligned} C = \frac{1}{8\tilde{K}} &\left(\tilde{g}_{\mathrm{H}} \owedge \tilde{g}_{\mathrm{H}} + \sum_{i=1}^{3} (\tilde{g}_{\mathrm{H}} \circ I_i) \mathbin{\bigcirc\kern-0.8em{|}} (\tilde{g}_{\mathrm{H}} \circ I_i) \right) \\ &- \frac{\tilde{K}}{8\tilde{k}} \frac{1}{\tilde{f}_1 \tilde{f}_{\mathrm{H}}} \left(\tilde{\omega}_{\mathrm{H}} \mathbin{\bigcirc\kern-0.8em{|}} \tilde{\omega}_{\mathrm{H}} + \sum_{i=1}^{3} (\tilde{\omega}_{\mathrm{H}} \circ I_i) \owedge (\tilde{\omega}_{\mathrm{H}} \circ I_i) \right), \end{aligned} \tag{4.129}$$

and the vector fields E_a constitute an orthonormal frame with respect to $\tilde{g}_{\mathrm{H}}$. Note that E_a do not need to be the twists of E'_a, or even Z-invariant for that matter. A straightforward computation (deferred to Section 4.B in the appendix) now gives us

$$\begin{aligned} &\mathrm{tr}(\mathscr{R}^2) \\ &= \frac{1}{\tilde{K}^2} \left(n(5n+1) + 3\left(\tilde{k}^3 \frac{\tilde{f}_1^3}{\tilde{f}_{\mathrm{H}}^3} + (n-1)\tilde{k}\frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}} \right)^2 + 3\left(\tilde{k}^6 \frac{\tilde{f}_1^6}{\tilde{f}_{\mathrm{H}}^6} + (n-1)\tilde{k}^2 \frac{\tilde{f}_1^2}{\tilde{f}_{\mathrm{H}}^2} \right) \right) \\ &= \nu^2 \left(n(5n+1) + 3\left(\frac{\rho^3}{(\rho+2c)^3} + \frac{(n-1)\rho}{(\rho+2c)} \right)^2 + 3\left(\frac{\rho^6}{(\rho+2c)^6} + \frac{(n-1)\rho^2}{(\rho+2c)^2} \right) \right). \end{aligned} \tag{4.130}$$

We can write the map $\rho \mapsto \mathrm{tr}(\mathscr{R}^2)$ as a composition of two maps, both $\mathbb{R}_{>0} \to \mathbb{R}_{>0}$, given by

$$\rho \mapsto \frac{\rho}{\rho + 2c} =: Y, \quad Y \mapsto \nu^2(n(5n+1) + 3(Y^3 + (n-1)Y)^2 + 3(Y^6 + (n-1)Y^2). \tag{4.131}$$

The first map is clearly injective when $c > 0$. As for the second map, suppose there existed Y_1 and Y_2 such that

$$\begin{aligned} &\nu^2(n(5n+1) + 3(Y_1^3 + (n-1)Y_1)^2 + 3(Y_1^6 + (n-1)Y_1^2) \\ &\quad = \nu^2(n(5n+1) + 3(Y_2^3 + (n-1)Y_2)^2 + 3(Y_2^6 + (n-1)Y_2^2). \end{aligned} \tag{4.132}$$

This can be rearranged into

$$(Y_1 - Y_2)(Y_1 + Y_2)((Y_1^2 + Y_2^2 + n - 1)^2 + Y_1^4 + Y_2^4 + n - 1) = 0. \tag{4.133}$$

The only way the left-hand side can vanish for $n \geqslant 1$ and Y_1 and Y_2 both positive is if we had $Y_1 = Y_2$. So, the second map and hence the composition $\rho \mapsto \mathrm{tr}(\mathscr{R}^2)$ is injective. Since the curvature norm must be preserved by any isometry, it follows that any isometry of g necessarily sends any given constant ρ hypersurface to itself. In other words, g has cohomogeneity at least 1.

To prove that it has cohomogeneity at most and hence equal to 1, we make use of the involution $v \leftrightarrow \tilde{v}$ in (3.42) to construct Killing fields of g from $\tilde{\omega}_{\mathrm{H}}$-Hamiltonian Killing fields of $\tilde{g}_{\mathrm{H}}$ which Lie-commute with $\tilde{Z}$. Killing fields of $\tilde{g}_{\mathrm{H}}$ which Lie-commute with $\tilde{Z}$ may be obtained by considering Killing fields of $\tilde{g}$ which preserve $\tilde{f}_1, \varpi_1, \varpi_2, \varpi_3$ separately. (This is a sufficient condition, but not a necessary one, as the example of $\tilde{Z}$ shows.)

More concretely, we consider the following vector fields:

$$\begin{aligned} \tilde{u}_a^+ &= \mathrm{Re}(z_a \partial_{z_0} + \bar{z}_0 \partial_{\bar{z}_a} - w_0 \partial_{w_a} - \overline{w}_a \partial_{\overline{w}_o}), \quad \tilde{v}_0^+ = \mathrm{Re}(\partial_{w_0}), \quad \tilde{v}_a^+ = \mathrm{Re}(\partial_{w_a}), \\ \tilde{u}_a^- &= \mathrm{Im}(z_a \partial_{z_0} + \bar{z}_0 \partial_{\bar{z}_a} - w_0 \partial_{w_a} - \overline{w}_a \partial_{\overline{w}_o}), \quad \tilde{v}_0^- = \mathrm{Im}(\partial_{w_0}), \quad \tilde{v}_a^- = \mathrm{Im}(\partial_{w_a}), \end{aligned} \tag{4.134}$$

with a ranging from 1 to $n-1$. These Lie-commute with $\tilde{Z}$, preserve $\varpi_1, \varpi_2, \varpi_3$, and are $\tilde{\omega}_{\mathrm{H}}$-Hamiltonian. We make the following choice of Hamiltonian functions:

$$\begin{aligned} \tilde{f}_{\tilde{u}_a^+} &= \frac{\tilde{k}}{2}\operatorname{Re}(-\mathrm{i}(z_a\bar{z}_0 - \overline{w}_a w_0)) - 1, \quad \tilde{f}_{\tilde{v}_0^+} = \frac{\tilde{k}}{2}\operatorname{Re}(\mathrm{i}\overline{w}_0) - 1, \quad \tilde{f}_{\tilde{v}_a^+} = \frac{\tilde{k}}{2}\operatorname{Re}(-\mathrm{i}\overline{w}_a) - 1, \\ \tilde{f}_{\tilde{u}_a^-} &= \frac{\tilde{k}}{2}\operatorname{Im}(-\mathrm{i}(z_a\bar{z}_0 - \overline{w}_a w_0)) - 1, \quad \tilde{f}_{\tilde{v}_0^-} = \frac{\tilde{k}}{2}\operatorname{Im}(\mathrm{i}\overline{w}_0) - 1, \quad \tilde{f}_{\tilde{v}_a^-} = \frac{\tilde{k}}{2}\operatorname{Im}(-\mathrm{i}\overline{w}_a) - 1. \end{aligned} \tag{4.135}$$

With the choice of $\tilde{\eta}_{\mathrm{H}}$ as in (4.61), the involution $v \leftrightarrow \tilde{v}$ is given by

$$\begin{aligned} v = \mathrm{tw}_{\tilde{Z},\tilde{f}_{\mathrm{H}},\tilde{\eta}_{\mathrm{H}}}(\tilde{v}) + (\tilde{f}_{\tilde{v}} + 1)Z &= \tilde{v} - \frac{\tilde{f}_{\tilde{v}} - \tilde{\eta}_{\mathrm{H}}(\tilde{v}) + 1}{\tilde{f}_{\mathrm{H}} - \tilde{\eta}_{\mathrm{H}}(\tilde{Z})}\tilde{Z} \\ &= \tilde{v} + \frac{8\tilde{K}}{\tilde{k}}(\tilde{f}_{\tilde{v}} - \tilde{\eta}_{\mathrm{H}}(\tilde{v}) + 1)\tilde{Z}. \end{aligned} \tag{4.136}$$

Therefore, we obtain

$$\begin{aligned} u_a^+ &= \tilde{u}_a^+ + \left(2\tilde{K}c - \frac{1}{2}\right)\operatorname{Re}\left(\frac{\mathrm{i}z^a}{z_0}\right)\tilde{Z}, \\ v_0^+ &= \tilde{v}_0^+ + 2\tilde{K}\operatorname{Re}(\mathrm{i}\overline{w}_0)\tilde{Z}, \quad v_a^+ = \tilde{v}_a^+ - 2\tilde{K}\operatorname{Re}(\mathrm{i}\overline{w}_a)\tilde{Z}, \\ u_a^- &= \tilde{u}_a^- + \left(2\tilde{K}c - \frac{1}{2}\right)\operatorname{Re}\left(\frac{\mathrm{i}z^a}{z_0}\right)\tilde{Z}, \\ v_0^- &= \tilde{v}_0^- + 2\tilde{K}\operatorname{Re}(\mathrm{i}\overline{w}_0)\tilde{Z}, \quad v_a^- = \tilde{v}_a^- - 2\tilde{K}\operatorname{Re}(\mathrm{i}\overline{w}_a)\tilde{Z}. \end{aligned} \tag{4.137}$$

Carrying out the change of coordinates given in (4.65), we finally get

$$\begin{aligned} u_a^+ &= \operatorname{Re}\left(-\sum_{b=1}^{n-1} X_a X_b \partial_{X_b} + \partial_{\overline{X}_a} - \zeta_0\partial_{\zeta_a} - \overline{\zeta}_a\partial_{\overline{\zeta}_0} + 2\mathrm{i}\tilde{K}cX^a\partial_\tau\right), \\ v_0^+ &= \sqrt{2}\operatorname{Re}(\partial_{\zeta_0} + \mathrm{i}\tilde{K}\overline{\zeta}_0\partial_\tau), \quad v_a^+ = \sqrt{2}\operatorname{Re}(\partial_{\zeta_a} - \mathrm{i}\tilde{K}\overline{\zeta}_a\partial_\tau), \\ u_a^- &= \operatorname{Im}\left(-\sum_{b=1}^{n-1} X_a X_b \partial_{X_b} + \partial_{\overline{X}_a} - \zeta_0\partial_{\zeta_a} - \overline{\zeta}_a\partial_{\overline{\zeta}_0} + 2\mathrm{i}\tilde{K}cX^a\partial_\tau\right), \\ v_0^- &= \sqrt{2}\operatorname{Im}(\partial_{\zeta_0} + \mathrm{i}\tilde{K}\overline{\zeta}_0\partial_\tau), \quad v_a^- = \sqrt{2}\operatorname{Im}(\partial_{\zeta_a} - \mathrm{i}\tilde{K}\overline{\zeta}_a\partial_\tau). \end{aligned} \tag{4.138}$$

These are Killing with respect to g. The diffeomorphisms generated by them and ∂_τ act transitively on the constant ρ hypersurfaces. Hence, the 1-loop-deformed quadratic prepotential Ferrara–Sabharwal metrics have cohomogeneity exactly 1. □

Remark 4.2.22. Note that (4.130) reduces to the curvature norm for the universal hypermultiplet in (2.98) for $n = 1$.

Appendix

This appendix includes details of the computation of the curvature formula in (4.107) and the curvature norm in (4.130) that were skipped over earlier in this chapter. These computations have been carried out in collaboration with Danu Thung [CST20b; CST20a].

4.A HK/QK curvature formula

In this section, we'll show how to obtain (4.114) from (4.112). We'll divide the computation into several lemmata describing the various pieces on the right-hand side of (4.112).

Lemma 4.A.1. *The antisymmetrised covariant derivative of S^{HQ} is given by*

$$\begin{aligned}
&\left(\nabla^{\tilde{g}}_u S^{\mathrm{HQ}}\right)(v,t) - \left(\nabla^{\tilde{g}}_v S^{\mathrm{HQ}}\right)(u,t) \\
&= \frac{1}{2}\sum_{\alpha=0}^{3}\Bigg(\frac{1}{\tilde{f}_{\mathrm{H}}^2}\,\tilde{\omega}_{\mathrm{H}}(\tilde{Z},u)\omega_{\mu}(I_{\mathrm{H}}v,t)I_\alpha\tilde{Z} + \frac{1}{\tilde{f}_1^2}\,\lambda_1(u)\big(\tilde{g}(I_\alpha\circ I_1 Z,v)I_\alpha t + \tilde{g}(I_\alpha\circ I_1\tilde{Z},t)I_\alpha v\big) \\
&\qquad - \frac{1}{\tilde{f}_{\mathrm{H}}^2}\,\tilde{\omega}_{\mathrm{H}}(\tilde{Z},v)\tilde{\omega}_{\mathrm{H}}(I_\alpha u,t)I_\alpha\tilde{Z} - \frac{1}{\tilde{f}_1^2}\,\lambda_1(v)\big(\tilde{g}(I_\alpha\circ I_1\tilde{Z},u)I_\alpha t + \tilde{g}(I_\alpha\circ I_1\tilde{Z},t)I_\alpha u\big) \\
&\qquad + \frac{1}{2\tilde{f}_{\mathrm{H}}}\big(\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)I_\alpha\circ(I_{\mathrm{H}}-I_1)u - \tilde{\omega}_{\mathrm{H}}(I_\alpha u,t)I_\alpha\circ(I_{\mathrm{H}}-I_1)v\big) \\
&\qquad + \frac{1}{2\tilde{f}_1}\Big(\big(\tilde{\omega}_{\mathrm{H}}(I_\alpha\circ I_1 u,t) + \tilde{g}(I_\alpha u,t)\big)I_\alpha v - \big(\tilde{\omega}_{\mathrm{H}}(I_\alpha\circ I_1 v,t) + \tilde{g}(I_\alpha v,t)\big)I_\alpha u\Big) \\
&\qquad + \frac{1}{2\tilde{f}_1}\big(\tilde{g}(I_\alpha u,v) - \tilde{g}(I_\alpha v,u)\big)I_\alpha t\Bigg) - \frac{1}{2\tilde{f}_1}\,\tilde{\omega}_{\mathrm{H}}(u,v)I_1 t + \frac{2\tilde{k}}{\tilde{f}_{\mathrm{H}}}\,\tilde{g}(I_\alpha\circ R^{\tilde{g}}(u,v)\tilde{Z},t)I_\alpha\tilde{Z}.
\end{aligned} \tag{4.139}$$

Proof. We begin by computing the covariant derivative

$$\begin{aligned}
&\left(\nabla^{\tilde{g}}_u S^{\mathrm{HQ}}\right)(v,t) \\
&= \frac{1}{2}\sum_{\alpha=0}^{3}\Bigg(-\frac{\mathrm{d}\tilde{f}_{\mathrm{H}}(u)}{\tilde{f}_{\mathrm{H}}^2}\,\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)I_\alpha\tilde{Z} + \frac{2\tilde{k}}{\tilde{f}_{\mathrm{H}}}\,\tilde{g}(I_\alpha\circ(\nabla^{\tilde{g}})^2_{u,v}\tilde{Z}),t)I_\alpha\tilde{Z} \\
&\qquad + \frac{1}{\tilde{f}_{\mathrm{H}}}\,\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)I_\alpha\circ\nabla^{\tilde{g}}_u\tilde{Z} + \frac{\mathrm{d}\tilde{f}_1(u)}{\tilde{f}_1^2}\,(\lambda_\alpha(v)I_\alpha\circ I_1 t + \lambda_\alpha(t)I_\alpha\circ I_1 v) \\
&\qquad - \frac{1}{\tilde{f}_1}\,(\tilde{g}(I_\alpha\circ\nabla^{\tilde{g}}_u\tilde{Z},v)I_\alpha\circ I_1 t + \tilde{g}(I_\alpha\circ\nabla^{\tilde{g}}_u\tilde{Z},t)I_\alpha\circ I_1 v)\Bigg).
\end{aligned} \tag{4.140}$$

Using the definitions of $\tilde{f}_1, \tilde{f}_{\mathrm{H}}, I_{\mathrm{H}}$ and replacing I_α by $I_\alpha \circ I_1$ in some of the terms, we can rewrite this as

$$\begin{aligned}
&\left(\nabla^{\tilde{g}}_u S^{\mathrm{HQ}}\right)(v,t) \\
&= \frac{1}{2}\sum_{\alpha=0}^{3}\Bigg(\frac{1}{\tilde{f}_{\mathrm{H}}^2}\,\tilde{\omega}_{\mathrm{H}}(\tilde{Z},u)\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)I_\alpha\tilde{Z} + \frac{2\tilde{k}}{\tilde{f}_{\mathrm{H}}}\,\tilde{g}(I_\alpha\circ(\nabla^{\tilde{g}})^2_{u,v}\tilde{Z}),t)I_\alpha\tilde{Z} \\
&\qquad + \frac{1}{2\tilde{f}_{\mathrm{H}}}\,\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)I_\alpha\circ(I_{\mathrm{H}}-I_1)u + \frac{1}{\tilde{f}_1^2}\,\lambda_1(v)(\tilde{g}(I_\alpha\circ I_1\tilde{Z},v)I_\alpha t + \tilde{g}(I_\alpha\circ I_1\tilde{Z},t)I_\alpha v) \\
&\qquad + \frac{1}{2\tilde{f}_1}\,(\tilde{g}(I_\alpha\circ I_1\circ(I_{\mathrm{H}}-I_1)u,v)I_\alpha t + \tilde{g}(I_\alpha\circ I_1\circ(I_{\mathrm{H}}-I_1)u,t)I_\alpha v)\Bigg) \\
&= \frac{1}{2}\sum_{\alpha=0}^{3}\Bigg(\frac{1}{\tilde{f}_{\mathrm{H}}^2}\,\tilde{\omega}_{\mathrm{H}}(\tilde{Z},u)\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)I_\alpha\tilde{Z} + \frac{2\tilde{k}}{\tilde{f}_{\mathrm{H}}}\,\tilde{g}(I_\alpha\circ(\nabla^{\tilde{g}})^2_{u,v}\tilde{Z}),t)I_\alpha\tilde{Z} \\
&\qquad + \frac{1}{2\tilde{f}_{\mathrm{H}}}\,\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)I_\alpha\circ(I_{\mathrm{H}}-I_1)u + \frac{1}{\tilde{f}_1^2}\,\lambda_1(v)(\tilde{g}(I_\alpha\circ I_1\tilde{Z},v)I_\alpha t + \tilde{g}(I_\alpha\circ I_1\tilde{Z},t)I_\alpha v) \\
&\qquad + \frac{1}{2\tilde{f}_1}\,((\tilde{\omega}_{\mathrm{H}}(I_\alpha\circ I_1 u,v) + \tilde{g}(I_\alpha u,v))I_\alpha t + (\tilde{\omega}_{\mathrm{H}}(I_\alpha\circ I_1 u,t) + \tilde{g}(I_\alpha u,t))I_\alpha v)\Bigg).
\end{aligned} \tag{4.141}$$

Now, we can use the fact that $\tilde{\omega}_{\mathrm{H}}\circ I_\alpha\circ I_1$ is antisymmetric when $\alpha = 1$ and symmetric otherwise to conclude that antisymmetrising u and v in (4.141) gives (4.139). □

Lemma 4.A.2. *The commutator of S^{HQ} with itself is given by*

$$\begin{aligned}
&\left[S^{\mathrm{HQ}}_u, S^{\mathrm{HQ}}_v\right]t \\
&= \frac{1}{4}\sum_{\alpha,\beta=0}^{3}\Bigg(\frac{1}{\tilde{f}_{\mathrm{H}}^2}\big(\tilde{\omega}_{\mathrm{H}}(I_\mu u,\tilde{Z})\tilde{\omega}_{\mathrm{H}}(I_\beta v,t) - \tilde{\omega}_{\mathrm{H}}(I_\alpha v,\tilde{Z})\tilde{\omega}_{\mathrm{H}}(I_\beta u,t)\big)I_\beta\circ I_\alpha\tilde{Z} \\
&\quad + \frac{1}{\tilde{f}_1^2}\Big(\big(\tilde{g}(I_\alpha\circ I_1\tilde{Z},u)\tilde{g}(I_\beta\circ I_1 Z,v) - g(I_\alpha\circ I_1\tilde{Z},v)\tilde{g}(I_\beta\circ I_1\tilde{Z},u)\big)I_\alpha\circ I_\beta t \\
&\quad + \tilde{g}(I_\alpha\circ I_1\tilde{Z},v)g(I_\beta\circ I_1\tilde{Z},t)I_\beta\circ I_\alpha u - \tilde{g}(I_\alpha\circ I_1\tilde{Z},u)\tilde{g}(I_\beta\circ I_1\tilde{Z},t)I_\beta\circ I_\alpha v\Big)\Bigg) \\
&\quad + \frac{1}{4\tilde{f}_1\tilde{f}_{\mathrm{H}}}\sum_{\alpha=0}^{3}\Big(2\tilde{\omega}_{\mathrm{H}}(u,v)\tilde{g}(I_\alpha\circ I_1\tilde{Z},t)I_\alpha\tilde{Z} \\
&\quad - \left(\frac{1}{\tilde{k}}\,\tilde{f}_{\mathrm{H}} - \tilde{f}_1\right)\big(\omega_{\mathrm{H}}(I_\alpha v,t)I_\alpha\circ I_1 u - \omega_{\mathrm{H}}(I_\alpha u,t)I_\alpha\circ I_1 v\big)\Big).
\end{aligned} \tag{4.142}$$

Proof. First we compute the composition of S^{HQ} with itself to be

$$
\begin{aligned}
&S^{\mathrm{HQ}}_u \circ S^{\mathrm{HQ}}_v t \\
&= \frac{1}{2}\sum_{\alpha=0}^{3}\left(\frac{1}{\tilde{f}_{\mathrm{H}}}\tilde{\omega}_{\mathrm{H}}(I_\alpha u, S^{\mathrm{HQ}}_v t)I_\alpha\tilde{Z} + \frac{1}{\tilde{f}_1}\big(\tilde{g}(I_\alpha\circ I_1\tilde{Z},u)I_\alpha\circ S^{\mathrm{HQ}}_v t + \tilde{g}(I_\alpha\circ I_1\tilde{Z}, S^{\mathrm{HQ}}_v t)I_\alpha u\big)\right) \\
&= \frac{1}{4}\sum_{\alpha,\beta=0}\left(\frac{1}{\tilde{f}_{\mathrm{H}}^2}\tilde{\omega}_{\mathrm{H}}(I_\alpha u, I_\beta\tilde{Z})\tilde{\omega}(I_\beta v,t)I_\alpha\tilde{Z}\right. \\
&\qquad + \frac{1}{\tilde{f}_1\tilde{f}_{\mathrm{H}}}\Big(\big(\tilde{\omega}_{\mathrm{H}}(I_\alpha u, I_\beta t)\tilde{g}(I_\beta\circ I_1\tilde{Z},v) + \tilde{\omega}_{\mathrm{H}}(I_\alpha u, I_\beta v)g(I_\beta\circ I_1\tilde{Z},t)\big)I_\alpha\tilde{Z} \\
&\qquad\quad + \tilde{g}(I_\alpha\circ I_1\tilde{Z},u)\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)I_\alpha\circ I_\beta\tilde{Z} + \tilde{g}(I_\alpha\circ I_1\tilde{Z}, I_\beta\tilde{Z})\tilde{\omega}_{\mathrm{H}}(I_\beta v,t)I_\alpha u\Big) \\
&\qquad + \frac{1}{\tilde{f}_1^2}\Big(\tilde{g}(I_\alpha\circ I_1\tilde{Z},u)\big(\tilde{g}(I_\beta\circ I_1\tilde{Z},v)I_\alpha\circ I_\beta t + \tilde{g}(I_\beta\circ I_1\tilde{Z},t)I_\alpha\circ I_\beta v\big) \\
&\qquad\quad \left. + \big(\tilde{g}(I_\alpha\circ I_1\tilde{Z}, I_\beta t)\tilde{g}(I_\beta\circ I_1\tilde{Z},v) + \tilde{g}(I_\alpha\circ I_1\tilde{Z}, I_\beta v)\tilde{g}(I_\beta\circ I_1\tilde{Z},t)\big)I_\alpha u\Big)\right).
\end{aligned}
\tag{4.143}
$$

Making the replacement $I_\alpha \mapsto I_\beta\circ I_\alpha$ in some of the terms and swapping the labels α and β in others, we obtain

$$
\begin{aligned}
&S^{\mathrm{HQ}}_u \circ S^{\mathrm{HQ}}_v t \\
&= \frac{1}{4}\sum_{\alpha,\beta=0}\left(\frac{1}{\tilde{f}_{\mathrm{H}}^2}\tilde{\omega}_{\mathrm{H}}(I_\alpha u, \tilde{Z})\tilde{\omega}(I_\beta v,t)I_\beta\circ I_\alpha\tilde{Z}\right. \\
&\qquad + \frac{1}{\tilde{f}_1\tilde{f}_{\mathrm{H}}}\Big(\big(\tilde{\omega}_{\mathrm{H}}(I_\alpha u, t)\tilde{g}(I_\beta\circ I_1\tilde{Z},v) + \tilde{\omega}_{\mathrm{H}}(I_\alpha u, v)g(I_\beta\circ I_1\tilde{Z},t)\big)I_\beta\circ I_\alpha\tilde{Z} \\
&\qquad\quad + \tilde{g}(I_\alpha\circ I_1\tilde{Z},u)\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)I_\alpha\circ I_\beta\tilde{Z} + \tilde{g}(I_\alpha\circ I_1\tilde{Z}, I_\beta\tilde{Z})\tilde{\omega}_{\mathrm{H}}(I_\beta v,t)I_\alpha u\Big) \\
&\qquad + \frac{1}{\tilde{f}_1^2}\Big(\tilde{g}(I_\beta\circ I_1\tilde{Z},u)\big(\tilde{g}(I_\alpha\circ I_1\tilde{Z},v)I_\beta\circ I_\alpha t + \tilde{g}(I_\alpha\circ I_1\tilde{Z},t)I_\beta\circ I_\alpha v\big) \\
&\qquad\quad \left. + \big(\tilde{g}(I_\alpha\circ I_1\tilde{Z}, t)\tilde{g}(I_\beta\circ I_1\tilde{Z},v) + \tilde{g}(I_\alpha\circ I_1\tilde{Z}, v)\tilde{g}(I_\beta\circ I_1\tilde{Z},t)\big)I_\beta\circ I_\alpha u\Big)\right).
\end{aligned}
\tag{4.144}
$$

Note that $\tilde{g}(I_\alpha\circ I_1\tilde{Z}, I_\beta\tilde{Z})$ vanishes unless $\alpha = 1$. Moreover, certain pairs of terms are symmetric in u and v, as is $\omega_{\mathrm{H}}(I_\alpha u, v)$ whenever $\alpha \neq 0$. These terms drop out under antisymmetrisation, leaving us with

$$
\begin{aligned}
&\left[S_u^{\mathrm{HQ}}, S_v^{\mathrm{HQ}}\right] t \\
&= \frac{1}{4} \sum_{\alpha,\beta=0}^{3} \left(\frac{1}{\tilde{f}_{\mathrm{H}}^2} \big(\tilde{\omega}_{\mathrm{H}}(I_\mu u, \tilde{Z}) \tilde{\omega}_{\mathrm{H}}(I_\beta v, t) - \tilde{\omega}_{\mathrm{H}}(I_\alpha v, \tilde{Z}) \tilde{\omega}_{\mathrm{H}}(I_\beta u, t)\big) I_\beta \circ I_\alpha \tilde{Z} \right. \\
&\qquad + \frac{1}{\tilde{f}_1^2} \Big(\big(\tilde{g}(I_\alpha \circ I_1 \tilde{Z}, u) \tilde{g}(I_\beta \circ I_1 Z, v) - g(I_\alpha \circ I_1 \tilde{Z}, v) \tilde{g}(I_\beta \circ I_1 \tilde{Z}, u)\big) I_\alpha \circ I_\beta t \\
&\qquad \left. + \tilde{g}(I_\alpha \circ I_1 \tilde{Z}, v) g(I_\beta \circ I_1 \tilde{Z}, t) I_\beta \circ I_\alpha u - \tilde{g}(I_\alpha \circ I_1 \tilde{Z}, u) \tilde{g}(I_\beta \circ I_1 \tilde{Z}, t) I_\beta \circ I_\alpha v \Big) \right) \\
&\quad + \frac{1}{4 \tilde{f}_1 \tilde{f}_{\mathrm{H}}} \sum_{\alpha=0}^{3} \Big(2\, \tilde{\omega}_{\mathrm{H}}(u, v) \tilde{g}(I_\alpha \circ I_1 \tilde{Z}, t) I_\alpha \tilde{Z} \\
&\qquad - \tilde{g}(\tilde{Z}, \tilde{Z}) \big(\tilde{\omega}_{\mathrm{H}}(I_\alpha v, t) I_\alpha \circ I_1 u - \tilde{\omega}_{\mathrm{H}}(I_\alpha u, t) I_\alpha \circ I_1 v\big) \Big).
\end{aligned}
\tag{4.145}
$$

Equation (4.142) now follows by rewriting $\tilde{g}(\tilde{Z}, \tilde{Z}) = \tilde{k}^{-1} \tilde{f}_{\mathrm{H}} - \tilde{f}_1$. □

Lemma 4.A.3. *The endomorphism field S^{HQ} satisfies the equation*

$$
\begin{aligned}
&\nabla_t^{\tilde{g}} \tilde{Z} + S_{\tilde{Z}}^{\mathrm{HQ}} t \\
&= \frac{1}{2} \left(I_{\mathrm{H}} - \frac{1}{\tilde{k}} \frac{\tilde{f}_{\mathrm{H}}}{\tilde{f}_1} I_1 \right) t + \frac{1}{2} \sum_{\alpha=0}^{3} \left(\frac{1}{\tilde{f}_{\mathrm{H}}} \tilde{\omega}_{\mathrm{H}}(I_\alpha \tilde{Z}, t) + \frac{1}{\tilde{f}_1} \tilde{g}(I_\alpha \circ I_1 \tilde{Z}, t) \right) I_\alpha \tilde{Z}.
\end{aligned}
\tag{4.146}
$$

Proof. This follows from a short computation:

$$
\begin{aligned}
&\nabla_t^{\tilde{g}} \tilde{Z} + S_{\tilde{Z}}^{\mathrm{HQ}} t \\
&= \frac{1}{2} (I_{\mathrm{H}} - I_1) t \\
&\quad + \frac{1}{2} \sum_{\alpha=0}^{3} \left(\frac{1}{\tilde{f}_{\mathrm{H}}} \tilde{\omega}_{\mathrm{H}}(I_\alpha \tilde{Z}, t) I_\alpha \tilde{Z} + \frac{1}{\tilde{f}_1} \big(\tilde{g}(I_\alpha \circ I_1 \tilde{Z}, \tilde{Z}) I_\alpha t + \tilde{g}(I_\alpha \circ I_1 \tilde{Z}, t) I_\alpha \tilde{Z} \big) \right) \\
&= \frac{1}{2} (I_{\mathrm{H}} - I_1) t - \frac{\tilde{g}(\tilde{Z}, \tilde{Z})}{2 \tilde{f}_1} I_1 t \\
&\quad + \frac{1}{2} \sum_{\alpha=0}^{3} \left(\frac{1}{f_{\mathrm{H}}} \tilde{\omega}_{\mathrm{H}}(I_\alpha \tilde{Z}, t) + \frac{1}{\tilde{f}_1} \tilde{g}(I_\alpha \circ I_1 \tilde{Z}, t) \right) I_\alpha \tilde{Z} \\
&= \frac{1}{2} \left(I_{\mathrm{H}} - \frac{1}{\tilde{k}} \frac{\tilde{f}_{\mathrm{H}}}{\tilde{f}_1} I_1 \right) t + \frac{1}{2} \sum_{\alpha=0}^{3} \left(\frac{1}{\tilde{f}_{\mathrm{H}}} \tilde{\omega}_{\mathrm{H}}(I_\alpha \tilde{Z}, t) + \frac{1}{\tilde{f}_1} \tilde{g}(I_\alpha \circ I_1 \tilde{Z}, t) \right) I_\alpha \tilde{Z},
\end{aligned}
\tag{4.147}
$$

where in the final step, we have used the definition of $\tilde{f}_{\mathrm{H}}$ i.e. $\tilde{f}_{\mathrm{H}} = \tilde{k}(\tilde{f}_1 + \tilde{g}(\tilde{Z}, \tilde{Z})$. □

Finally, we put together all of the above results to get the main lemma of this section.

Lemma 4.A.4. *The endomorphism field S^{HQ} satisfies the equation*

$$
\begin{aligned}
&\left(\nabla^{\tilde{g}}_u S^{\mathrm{HQ}}\right)(v,t) - \left(\nabla^{\tilde{g}}_v S^{\mathrm{HQ}}\right)(u,t) \\
&\quad + \left[S^{\mathrm{HQ}}_u, S^{\mathrm{HQ}}_v\right] t - \frac{1}{\tilde{f}_{\mathrm{H}}}\,\tilde{\omega}_{\mathrm{H}}(u,v)\left(\nabla^{\tilde{g}}_t \tilde{Z} + S^{\mathrm{HQ}}_{\tilde{Z}} t\right) \\
&= -\frac{1}{2\tilde{f}_{\mathrm{H}}^2}\Bigg(\frac{1}{2}\sum_{\alpha,\beta=0}^{3}\left(\tilde{\omega}_{\mathrm{H}}(I_\alpha\tilde{Z},u)\omega_{\mathrm{H}}(I_\beta v,t) - \omega_{\mathrm{H}}(I_\alpha\tilde{Z},v)\tilde{\omega}_{\mathrm{H}}(I_\beta u,t)\right) I_\alpha\circ I_\beta\tilde{Z} \\
&\qquad\qquad + \sum_{\alpha=0}^{3}\tilde{\omega}_{\mathrm{H}}(u,v)\tilde{\omega}_{\mathrm{H}}(I_\alpha\tilde{Z},t)I_\alpha\tilde{Z}\Bigg) \\
&\quad + \frac{1}{2\tilde{f}_{\mathrm{H}}}\Bigg(\frac{1}{2}\sum_{\alpha=0}^{3}\Big(\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)I_\alpha\circ I_{\mathrm{H}}u - \tilde{\omega}_{\mathrm{H}}(I_\alpha u,t)I_\alpha\circ I_{\mathrm{H}}v + 4\tilde{k}\,\tilde{g}(I_\alpha\circ R^{\tilde{g}}(u,v)\tilde{Z},t)I_\alpha\tilde{Z}\Big) \\
&\qquad\qquad - \tilde{\omega}_{\mathrm{H}}(u,v)I_{\mathrm{H}}t\Bigg) \\
&\quad + \frac{1}{4}\sum_{\alpha=0}^{3}\Big(\tilde{g}_{\mathrm{H}}(I_\alpha u,t)I_\alpha v - \tilde{g}_{\mathrm{H}}(I_\alpha v,t)I_\alpha u + \big(\tilde{g}_{\mathrm{H}}(I_\alpha u,v) - g_{\mathrm{H}}(I_\alpha v,u)\big)I_\alpha t\Big).
\end{aligned}
\tag{4.148}
$$

Proof. Substituting (4.139), (4.142), and (4.142) into the left-hand side and cancelling terms yields

$$
\begin{aligned}
&\left(\nabla^{\tilde{g}}_u S^{\mathrm{HQ}}\right)(v,t) - \left(\nabla^{\tilde{g}}_v S^{\mathrm{HQ}}\right)(u,t) \\
&\quad + \left[S^{\mathrm{HQ}}_u, S^{\mathrm{HQ}}_v\right] t - \frac{1}{\tilde{f}_{\mathrm{H}}}\,\tilde{\omega}_{\mathrm{H}}(u,v)\left(\nabla^{\tilde{g}}_t \tilde{Z} + S^{\mathrm{HQ}}_{\tilde{Z}} t\right) \\
&\quad = \frac{1}{\tilde{f}_1} D_{1,1}(u,v,t) + \frac{1}{\tilde{f}_1^2} D_{1,2}(u,v,t) + \frac{1}{\tilde{f}_{\mathrm{H}}} D_{\mathrm{H},1}(u,v,t) + \frac{1}{\tilde{f}_{\mathrm{H}}^2} D_{\mathrm{H},2}(u,v,t),
\end{aligned}
\tag{4.149}
$$

where the tensor fields $D_{1,1}, D_{1,2}$ are given by

$$
\begin{aligned}
&D_{1,1}(u,v,t) \\
&= \frac{1}{4}\sum_{\alpha=0}^{3}\Big(\tilde{g}(I_\alpha u,t)I_\alpha v - \tilde{g}(I_\alpha v,t)I_\alpha u + \big(\tilde{g}(I_\alpha u,v) - \tilde{g}(I_\alpha v,u)\big)I_\alpha t\Big), \\
&D_{1,2}(u,v,t) \\
&= \frac{1}{2}\sum_{\alpha=0}^{3}\Big(\lambda_1(u)\big(\tilde{g}(I_\alpha\circ I_1\tilde{Z},v)I_\alpha t + \tilde{g}(I_\alpha\circ I_1\tilde{Z},t)I_\alpha v\big) \\
&\qquad\qquad - \lambda_1(v)\big(\tilde{g}(I_\alpha\circ I_1\tilde{Z},u)I_\alpha t + \tilde{g}(I_\alpha\circ I_1\tilde{Z},t)I_\alpha u\big)\Big) \\
&\quad + \frac{1}{4}\sum_{\alpha,\beta}\Big(\big(\tilde{g}(I_\alpha\circ I_1\tilde{Z},u)\tilde{g}(I_\beta\circ I_1\tilde{Z},v) - \tilde{g}(I_\alpha\circ I_1\tilde{Z},v)\tilde{g}(I_\beta\circ I_1\tilde{Z},u)\big)I_\alpha\circ I_\beta t \\
&\qquad\qquad + \tilde{g}(I_\beta\circ I_1\tilde{Z},v)\tilde{g}(I_\alpha\circ I_1\tilde{Z},t)I_\alpha\circ I_\beta u - \tilde{g}(I_\beta\circ I_1\tilde{Z},u)\tilde{g}(I_\alpha\circ I_1\tilde{Z},t)I_\alpha\circ I_\beta v\Big),
\end{aligned}
\tag{4.150}
$$

while the tensor fields $D_{\mathrm{H},1}, D_{\mathrm{H},2}$ are given by

$$\begin{aligned}
&D_{\mathrm{H},1}(u,v,t)\\
&= \frac{1}{4}\sum_{\alpha=0}^{3}\Big(\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t)I_\alpha\circ I_{\mathrm{H}}u - \tilde{\omega}_{\mathrm{H}}(I_\alpha u,t)I_\alpha\circ I_{\mathrm{H}}v + 4\tilde{k}\,\tilde{g}(I_\alpha\circ R^{\tilde{g}}(u,v)\tilde{Z},t)I_\alpha\tilde{Z}\Big)\\
&\qquad\qquad - \frac{1}{2}\tilde{\omega}_{\mathrm{H}}(u,v)I_{\mathrm{H}}t,\\
&D_{\mathrm{H},2}(u,v,t)\\
&= \frac{1}{2}\sum_{\alpha=0}^{3}\Big(\tilde{\omega}_{\mathrm{H}}(\tilde{Z},u)\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t) - \tilde{\omega}_{\mathrm{H}}(\tilde{Z},v)\tilde{\omega}_{\mathrm{H}}(I_\alpha u,t) - \tilde{\omega}_{\mathrm{H}}(u,v)\tilde{\omega}_{\mathrm{H}}(I_\alpha\tilde{Z},t)\Big)I_\alpha\tilde{Z}\\
&\qquad + \frac{1}{4}\sum_{\alpha,\beta=0}^{3}\big(\tilde{\omega}_{\mathrm{H}}(I_\beta u,\tilde{Z})\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t) - \tilde{\omega}_{\mathrm{H}}(I_\beta v,\tilde{Z})\tilde{\omega}_{\mathrm{H}}(I_\alpha u,t)\big)I_\alpha\circ I_\beta\tilde{Z}.
\end{aligned} \tag{4.151}$$

The expressions for $D_{1,2}$ and $D_{\mathrm{H},2}$ can be simplified further by absorbing some of the terms in the single summations into the double summation by rewriting

$$\begin{aligned}
\lambda_1(\cdot)I_\alpha &= \frac{1}{2}\sum_{\beta=0}^{3}(\tilde{g}(I_\beta\circ I_1\tilde{Z},\cdot) + \tilde{g}(I_1\tilde{Z},I_\beta\cdot))I_\alpha\circ I_\beta,\\
\tilde{\omega}_{\mathrm{H}}(\tilde{Z},\cdot)I_\alpha &= \frac{1}{2}\sum_{\beta=0}^{3}(\tilde{\omega}_{\mathrm{H}}(I_\beta\tilde{Z},\cdot) - \tilde{\omega}_{\mathrm{H}}(I_\beta\cdot,\tilde{Z}))I_\alpha\circ I_\beta.
\end{aligned} \tag{4.152}$$

This gives us the following expressions for $D_{1,2}$ and $D_{\mathrm{H},2}$:

$$\begin{aligned}
&D_{1,2}(u,v,t)\\
&= \frac{1}{4}\sum_{\alpha,\beta=0}^{3}\Big(\big(\tilde{g}(I_\alpha\circ I_1\tilde{Z},v)\tilde{g}(I_1\tilde{Z},I_\beta u) - \tilde{g}(I_\alpha\circ I_1\tilde{Z},u)\tilde{g}(I_1\tilde{Z},I_\beta v)\big)I_\alpha\circ I_\beta t\\
&\qquad + \tilde{g}(I_1\tilde{Z},I_\beta u)\tilde{g}(I_\alpha\circ I_1\tilde{Z},t)I_\alpha\circ I_\beta v - \tilde{g}(I_1\tilde{Z},I_\beta v)\tilde{g}(I_\alpha\circ I_1\tilde{Z},t)I_\alpha\circ I_\beta u\Big)\\
&= \frac{1}{4}\sum_{\alpha=0}^{3}\Big(\big(\tilde{g}_\lambda(I_\alpha u,v) - \tilde{g}_\lambda(I_\alpha v,u)\big)I_\alpha t + \tilde{g}_\lambda(I_\alpha u,t)I_\alpha v - \tilde{g}_\lambda(I_\alpha v,t)I_\alpha u\Big),\\
&D_{\mathrm{H},2}(u,v,t)\\
&= -\frac{1}{2}\sum_{\alpha=0}^{3}\tilde{\omega}_{\mathrm{H}}(u,v)\tilde{\omega}_{\mathrm{H}}(I_\alpha\tilde{Z},t)I_\alpha\tilde{Z}\\
&\quad + \frac{1}{4}\sum_{\alpha,\beta=0}^{3}\big(\tilde{\omega}_{\mathrm{H}}(I_\beta\tilde{Z},u)\tilde{\omega}_{\mathrm{H}}(I_\alpha v,t) - \tilde{\omega}_{\mathrm{H}}(I_\beta\tilde{Z},v)\tilde{\omega}_{\mathrm{H}}(I_\alpha u,t)\big)I_\alpha\circ I_\beta\tilde{Z},
\end{aligned} \tag{4.153}$$

where we have made use of the shorthand $\tilde{g}_\lambda$ introduced in (4.85). Plugging these simplified expressions into (4.149) and noting that the standard hyperkähler elementary deformation is

$$\tilde{g}_{\mathrm{H}} = \frac{\tilde{K}}{\tilde{f}_1}\tilde{g} + \frac{\tilde{K}}{\tilde{f}_1^2}\tilde{g}_\lambda \tag{4.154}$$

gives us the required expression (4.148). □

4.B Ferrara–Sabharwal curvature norm

In this section, we'll compute the curvature norm (4.128) of the quadratic prepotential Ferrara–Sabharwal metric. As preparation, we first introduce some notation and prove a general lemma about traces.

Definition 4.B.1 (h-Kulkarni–Nomizu product)**.** Given a metric h, and two endomorphism fields B_1, B_2 self-adjoint with respect to it, their h-Kulkarni–Nomizu product $B_1 \owedge_h B_2 \in \Gamma(\mathrm{End}(\Lambda^2 TM))$ is defined by

$$h((B_1 \owedge_h B_2)(s \wedge t), u \wedge v) = (h \circ B_1) \owedge (h \circ B_2)(s, t, u, v). \tag{4.155}$$

Definition 4.B.2 (h-Riemann product)**.** Given a metric h, and two endomorphism fields B_1', B_2' skew-self-adjoint with respect to it, their h-Riemann product $B_1' \oplus_h B_2' \in \Gamma(\mathrm{End}(\Lambda^2 TM))$ is defined by

$$h((B_1' \oplus_h B_2')(s \wedge t), u \wedge v) = (h \circ B_1') \oplus (h \circ B_2')(s, t, u, v). \tag{4.156}$$

Lemma 4.B.3. *Given a metric h, self-adjoint endomorphism fields B_1, B_2, and skew-self-adjoint endomorphism fields B_1', B_2', we have*

$$\begin{aligned}
\mathrm{tr}((B_1 \owedge_h B_1) \circ (B_2 \owedge_h B_2)) &= 2(\mathrm{tr}(B_1 \circ B_2)^2 - \mathrm{tr}((B_1 \circ B_2)^2)),\\
\mathrm{tr}((B_1' \oplus_h B_1') \circ (B_2' \oplus_h B_2')) &= 6(\mathrm{tr}(B_1' \circ B_2')^2 + \mathrm{tr}((B_1' \circ B_2')^2))\\
\mathrm{tr}((B_1 \owedge_h B_1) \circ (B_2' \oplus_h B_2')) &= \mathrm{tr}((B_2' \oplus_h B_2') \circ (B_1 \owedge_h B_1))\\
&= 2(\mathrm{tr}(B_1 \circ B_2')^2 - 3\,\mathrm{tr}((B_1 \circ B_2')^2)).
\end{aligned} \tag{4.157}$$

Proof. Let $\{e_a\}$ be an orthonormal basis for h and let ε_a be the sign of $h(e_a, e_b)$. Now we compute:

$$\begin{aligned}
&\mathrm{tr}((B_1 \owedge_h B_1) \circ (B_2 \owedge_h B_2))\\
&= \frac{1}{4} \sum_{a,b,c,d} \varepsilon_a \varepsilon_b \varepsilon_c \varepsilon_d h((B_1 \owedge_h B_1) e_a \wedge e_b, e_c \wedge e_d) h((B_2 \owedge_h B_2) e_c \wedge e_d, e_a \wedge e_b)\\
&= \frac{1}{4} \sum_{a,b,c,d} \varepsilon_a \varepsilon_b \varepsilon_c \varepsilon_d (h \circ B_1) \owedge (h \circ B_1)(e_a, e_b, e_c, e_d)(h \circ B_2) \owedge (h \circ B_2)(e_c, e_d, e_a, e_b)\\
&= \sum_{a,b,c,d} \varepsilon_a \varepsilon_b \varepsilon_c \varepsilon_d (h(B_1 e_a, e_c) h(B_1 e_b, e_d) - h(B_1 e_a, e_d) h(B_1 e_b, e_c))\\
&\qquad\qquad (h(B_2 e_c, e_a) h(B_2 e_d, e_b) - h(B_2 e_c, e_b) h(B_2 e_d, e_a))\\
&= \sum_{a,b,c,d} \varepsilon_a \varepsilon_b \varepsilon_c \varepsilon_d (h(B_1 e_a, e_c) h(B_1 e_b, e_d) h(B_2 e_c, e_a) h(B_2 e_d, e_b)\\
&\qquad\qquad - h(B_1 e_a, e_d) h(B_1 e_b, e_c) h(B_2 e_c, e_a) h(B_2 e_d, e_b)\\
&\qquad\qquad - h(B_1 e_a, e_c) h(B_1 e_b, e_d) h(B_2 e_c, e_b) h(B_2 e_d, e_a)\\
&\qquad\qquad + h(B_1 e_a, e_d) h(B_1 e_b, e_c) h(B_2 e_c, e_b) h(B_2 e_d, e_a))\\
&= 2(\mathrm{tr}(B_1 \circ B_2)^2 - \mathrm{tr}((B_1 \circ B_2)^2)).
\end{aligned} \tag{4.158}$$

The next computation proceeds similarly, so we omit steps:

$$\begin{aligned}
&\operatorname{tr}((B_1' \obar_h B_1') \circ (B_2' \obar_h B_2')) \\
&= \sum_{a,b,c,d} \varepsilon_a\varepsilon_b\varepsilon_c\varepsilon_d (h(B_1'e_a, e_c)h(B_1'e_b, e_d) - h(B_1'e_a, e_d)h(B_1'e_b, e_c) + 2h(B_1'e_a, e_b)h(B_1'e_c, e_d)) \\
&\qquad\qquad (h(B_2'e_c, e_a)h(B_2'e_d, e_b) - h(B_2'e_c, e_b)h(B_2'e_d, e_a) + 2h(B_2'e_c, e_d)h(B_2'e_a, e_b)) \\
&= 2(\operatorname{tr}(B_1' \circ B_2')^2 - \operatorname{tr}((B_1' \circ B_2')^2 + 2\operatorname{tr}(B_1' \circ B_2' \circ B_1'^{\dagger h} \circ B_2'^{\dagger h}) \\
&\quad - \operatorname{tr}(B_1' \circ B_2' \circ B_1'^{\dagger h} \circ B_2') - \operatorname{tr}(B_1' \circ B_2' \circ B_1' \circ B_2'^{\dagger h}) + 2\operatorname{tr}(B_1' \circ B_2'^{\dagger h})^2) \\
&= 6(\operatorname{tr}(B_1' \circ B_2')^2 + \operatorname{tr}((B_1' \circ B_2')^2)),
\end{aligned} \tag{4.159}$$

where in the last step, we have used that B_1' and B_2' are skew-self-adjoint. Likewise, we have

$$\begin{aligned}
&\operatorname{tr}((B_1 \owedge_h B_1) \circ (B_2' \obar_h B_2')) \\
&= \sum_{a,b,c,d} \varepsilon_a\varepsilon_b\varepsilon_c\varepsilon_d (h(B_1e_a, e_c)h(B_1e_b, e_d) - h(B_1e_a, e_d)h(B_1e_b, e_c)) \\
&\qquad\qquad (h(B_2'e_c, e_a)h(B_2'e_d, e_b) - h(B_2'e_c, e_b)h(B_2'e_d, e_a) + 2h(B_2'e_c, e_d)h(B_2'e_a, e_b)) \\
&= 2(\operatorname{tr}(B_1 \circ B_2')^2 - \operatorname{tr}((B_1 \circ B_2')^2 + \operatorname{tr}(B_1 \circ B_2' \circ B_1^{\dagger h} \circ B_2'^{\dagger h}) - \operatorname{tr}(B_1 \circ B_2' \circ B_1^{\dagger h} \circ B_2') \\
&= 2(\operatorname{tr}(B_1 \circ B_2')^2 - 3\operatorname{tr}((B_1 \circ B_2')^2)),
\end{aligned} \tag{4.160}$$

where we have used that B_1 is self-adjoint and B_2' is skew-self-adjoint. □

Now we specialise to the case of the quadratic prepotential Ferrara–Sabharwal metrics. To apply the above lemma, we need to first rewrite the curvature norm in terms of traces of the above form.

Lemma 4.B.4. *The curvature norm in* (4.128) *may be written as*

$$\begin{aligned}
\operatorname{tr}(\mathscr{R}^2) = \operatorname{tr}\Bigg(\Bigg(\frac{1}{8\tilde{K}}\Bigg(\mathrm{id}_{TM} \owedge_{\tilde{g}_{\mathrm{H}}} \mathrm{id}_{TM} + \sum_{i=1}^{3} I_i \obar_{\tilde{g}_{\mathrm{H}}} I_i\Bigg) - \frac{\tilde{k}}{8\tilde{K}}\frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}}\big((A \circ I_{\mathrm{H}}) \obar_{\tilde{g}_{\mathrm{H}}} (A \circ I_{\mathrm{H}}) \\
+ \sum_{i=1}^{3}((A \circ I_{\mathrm{H}} \circ I_i) \owedge_{\tilde{g}_{\mathrm{H}}} (A \circ I_{\mathrm{H}} \circ I_i)\big)\Bigg)^2\Bigg),
\end{aligned} \tag{4.161}$$

where A is the endomorphism field introduced in (4.88), *namely*

$$A = \frac{\tilde{K}}{\tilde{f}_1}\tilde{g}_{\mathrm{H}}^{-1} \circ \tilde{g} = \mathrm{id}_{TM}|_{\mathbb{H}_H\tilde{Z}^{\perp}} + \tilde{k}\,\frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}}\,\mathrm{id}_{TM}|_{\mathbb{H}_H\tilde{Z}}. \tag{4.162}$$

Proof. Using the symmetries of the Riemann tensor, we may rewrite (4.128) as

$$\operatorname{tr}(\mathscr{R}^2) = \frac{1}{4}\sum_{a,b,c,d} C(E_a, E_b, E_c, E_d)C(E_c, E_d, E_a, E_b). \tag{4.163}$$

Equation (4.161) then follows from the definitions of the $\tilde{g}_{\mathrm{H}}$-Kulkarni–Nomizu and $\tilde{g}_{\mathrm{H}}$-Riemann products, and the following observations:

$$\tilde{g}_{\mathrm{H}} = \tilde{g}_{\mathrm{H}} \circ \mathrm{id}_{TM}, \quad \tilde{\omega}_{\mathrm{H}} = \tilde{k}\,\tilde{g} \circ I_{\mathrm{H}} = \frac{\tilde{k}}{\tilde{K}}\tilde{f}_1 \tilde{g}_{\mathrm{H}} \circ A \circ I_{\mathrm{H}}. \tag{4.164}$$

□

Next we collect the various trace computations we will need into a lemma.

Lemma 4.B.5. *The endomorphism fields* $\mathrm{id}_{TM}, I_i, A, I_{\mathrm{H}}$ *on the complex cotangent bundle* $T^*\mathbb{C}^n$ *satisfy the following trace formulae:*

$$\begin{aligned}
&\mathrm{tr}(\mathrm{id}_{TM}) = 4n, \quad \mathrm{tr}(I_i) = 0, \quad \mathrm{tr}(I_i \circ I_j) = -4n\delta_{ij}, \quad \mathrm{tr}((I_i \circ I_j)^2) = -(-1)^{\delta_{ij}}4n,\\
&\mathrm{tr}(A \circ I_{\mathrm{H}}) = 0, \quad \mathrm{tr}((A \circ I_{\mathrm{H}})^2) = -\mathrm{tr}(A^2) = -4\left(n - 1 + \tilde{k}^2\frac{\tilde{f}_1^2}{\tilde{f}_{\mathrm{H}}^2}\right),\\
&\mathrm{tr}((A \circ I_{\mathrm{H}})^4) = \mathrm{tr}(A^4) = 4\left(n - 1 + \tilde{k}^4\frac{\tilde{f}_1^4}{\tilde{f}_{\mathrm{H}}^4}\right), \quad \mathrm{tr}(A \circ I_{\mathrm{H}} \circ I_i) = 0,\\
&\mathrm{tr}((A \circ I_{\mathrm{H}} \circ I_i)^2) = \mathrm{tr}(A^2), \quad \mathrm{tr}(A \circ I_{\mathrm{H}} \circ A \circ I_{\mathrm{H}} \circ I_i) = -\mathrm{tr}(A^2 \circ I_i) = 0,\\
&\mathrm{tr}((A \circ I_{\mathrm{H}} \circ A \circ I_{\mathrm{H}} \circ I_i)^2) = -\mathrm{tr}(A^4), \quad \mathrm{tr}((A \circ I_{\mathrm{H}} \circ I_i \circ I_j)^2) = -(-1)^{\delta_{ij}}\mathrm{tr}(A^2),\\
&\mathrm{tr}(A \circ I_{\mathrm{H}} \circ I_i \circ I_j) = -\delta_{ij}\mathrm{tr}(A \circ I_{\mathrm{H}}) + \sum_{k=1}^{3}\epsilon_{ijk}\mathrm{tr}(A \circ I_{\mathrm{H}} \circ I_k) = 0,\\
&\mathrm{tr}(A \circ I_{\mathrm{H}} \circ I_i \circ A \circ I_{\mathrm{H}} \circ I_j) = \delta_{ij}\mathrm{tr}(A^2) - \sum_{k=1}^{3}\epsilon_{ijk}\mathrm{tr}(A^2 \circ I_k) = \delta_{ij}\mathrm{tr}(A^2),\\
&\mathrm{tr}((A \circ I_{\mathrm{H}} \circ I_i \circ A \circ I_{\mathrm{H}} \circ I_j)^2) = (-1)^{\delta_{ij}}\mathrm{tr}(A^4),
\end{aligned} \tag{4.165}$$

where δ_{ij} *is the Kronecker delta and* ϵ_{ijk} *is the Levi-Civita symbol.*

Proof. The trace of the identity endomorphism field on any vector bundle E is the rank of E. Thus, since we can write A as

$$A = \frac{\tilde{K}}{\tilde{f}_1}\tilde{g}_{\mathrm{H}}^{-1} \circ \tilde{g} = \mathrm{id}_{TM}|_{\mathbb{H}_H\tilde{Z}^\perp} + \tilde{k}\frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}}\mathrm{id}_{TM}|_{\mathbb{H}_H\tilde{Z}}, \tag{4.166}$$

it immediately follows that for all integers m, we have

$$\mathrm{tr}(A^m) = 4\left(n - 1 + \tilde{k}^m\frac{\tilde{f}_1^m}{\tilde{f}_{\mathrm{H}}^m}\right). \tag{4.167}$$

The explicit expression for A also makes it clear that A commutes with I_i. We also know that I_{H} commutes with I_i. Our next goal shall be to show that I_{H} and A commute as well.

As a matter of general fact, we have already noted that I_{H} is skew-self-adjoint with respect to $\tilde{g}$ by virtue of the Killing equation for $\tilde{Z}$. In the specific case of the complex cotangent bundle $M = T^*\mathbb{C}^n$, the explicit expression for I_{H} in (4.18), i.e.

$$I_{\mathrm{H}} = \mathrm{i}\left(-\mathrm{d}z_0 \wedge_{\tilde{g}} \partial_{z_0} + \mathrm{d}w_0 \wedge_{\tilde{g}} \partial_{w_0} + \sum_{a=1}^{n-1}(-\mathrm{d}z_a \wedge_{\tilde{g}} \partial_{z_a} + \mathrm{d}w_a \wedge_{\tilde{g}} \partial_{w_a})\right), \tag{4.168}$$

additionally tells us that $I_{\mathrm{H}}^2 = -\mathrm{id}_{TM}$, implying I_{H} is $\tilde{g}$-orthogonal, and that

$$I_{\mathrm{H}}\tilde{Z} = -I_1\tilde{Z}, \quad I_{\mathrm{H}} \circ I_i\tilde{Z} = I_i \circ I_{\mathrm{H}}\tilde{Z} = -I_i \circ I_1\tilde{Z}, \tag{4.169}$$

implying $\mathbb{H}_H\tilde{Z}$ and $\mathbb{H}_H\tilde{Z}^{\perp}$ are invariant subbundles of I_{H}. Thus, I_{H} commutes with A as well. This allows us to reduce the trace of any string consisting of A, I_{H}, I_i in some order to one of the following traces:

$$\mathrm{tr}(A^m), \quad \mathrm{tr}(A^m \circ I_i), \quad \mathrm{tr}(A^m \circ I_{\mathrm{H}}), \qquad \mathrm{tr}(A^m \circ I_{\mathrm{H}} \circ I_i). \tag{4.170}$$

The first trace we have already dealt with. The next two traces are traces of endomorphism fields that are skew-self-adjoint with respect to the metric $\tilde{g} \circ A^{-m}$, and so vanish. The last one also vanishes by virtue of the following chain of equalities:

$$\begin{aligned}\mathrm{tr}(A^m \circ I_{\mathrm{H}} \circ I_i) &= \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}\mathrm{tr}(A^m \circ I_{\mathrm{H}} \circ I_j \circ I_k) = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}\mathrm{tr}(I_j \circ A^m \circ I_{\mathrm{H}} \circ I_k)\\ &= \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}\mathrm{tr}(A^m \circ I_{\mathrm{H}} \circ I_k \circ I_j) = -\mathrm{tr}(A^m \circ I_{\mathrm{H}} \circ I_i),\end{aligned} \tag{4.171}$$

where in the first line we have used the fact that I_j commutes with A and I_{H}, and in the second line, we have used the cyclic invariance of traces. □

Now, we have all the ingredients we require to prove the main lemma of this section.

Lemma 4.B.6. *The curvature norm in (4.128) is given by*

$$\begin{aligned}&\mathrm{tr}(\mathscr{R}^2)\\ &= \frac{1}{\tilde{K}^2}\left(n(5n+1) + 3\left(\tilde{k}^3\frac{\tilde{f}_1^3}{\tilde{f}_{\mathrm{H}}^3} + (n-1)\tilde{k}\,\frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}}\right)^2 + 3\left(\tilde{k}^6\frac{\tilde{f}_1^6}{\tilde{f}_{\mathrm{H}}^6} + (n-1)\tilde{k}^2\frac{\tilde{f}_1^2}{\tilde{f}_{\mathrm{H}}^2}\right)\right).\end{aligned} \tag{4.172}$$

Proof. We begin by expanding the square inside the trace in (4.161) and using the cyclic invariance of the trace to write

$$
\begin{aligned}
&64\tilde{K}^2 \operatorname{tr}(\mathscr{R}^2)\\
&= \operatorname{tr}\Bigg((\mathrm{id}_{TM} \owedge_{\tilde{g}_{\mathrm{H}}} \mathrm{id}_{TM})^2 + \sum_{i,j=1}^{3} (I_i \obar_{\tilde{g}_{\mathrm{H}}} I_i) \circ (I_j \obar_{\tilde{g}_{\mathrm{H}}} I_j) + \tilde{k}^2 \frac{\tilde{f}_1^2}{\tilde{f}_{\mathrm{H}}^2} ((A \circ I_{\mathrm{H}}) \obar_{\tilde{g}_{\mathrm{H}}} (A \circ I_{\mathrm{H}}))^2 \\
&\quad + \tilde{k}^2 \frac{\tilde{f}_1^2}{\tilde{f}_{\mathrm{H}}^2} \sum_{i,j=1}^{3} ((A \circ I_{\mathrm{H}} \circ I_i) \owedge_{\tilde{g}_{\mathrm{H}}} (A \circ I_{\mathrm{H}} \circ I_i)) \circ ((A \circ I_{\mathrm{H}} \circ I_j) \owedge_{\tilde{g}_{\mathrm{H}}} (A \circ I_{\mathrm{H}} \circ I_j)) \\
&\quad + 2 \sum_{i=1}^{3} (\mathrm{id}_{TM} \owedge_{\tilde{g}_{\mathrm{H}}} \mathrm{id}_{TM}) \circ (I_i \obar_{\tilde{g}_{\mathrm{H}}} I_i) - 2\tilde{k} \frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}} (\mathrm{id}_{TM} \owedge_{\tilde{g}_{\mathrm{H}}} \mathrm{id}_{TM}) \circ ((A \circ I_{\mathrm{H}}) \obar_{\tilde{g}_{\mathrm{H}}} (A \circ I_{\mathrm{H}})) \\
&\quad - 2\tilde{k} \frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}} \sum_{i=1}^{3} (\mathrm{id}_{TM} \owedge_{\tilde{g}_{\mathrm{H}}} \mathrm{id}_{TM}) \circ ((A \circ I_{\mathrm{H}} \circ I_i) \owedge_{\tilde{g}_{\mathrm{H}}} (A \circ I_{\mathrm{H}} \circ I_i)) \\
&\quad - 2\tilde{k} \frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}} \sum_{i=1}^{3} (I_i \obar_{\tilde{g}_{\mathrm{H}}} I_i) \circ ((A \circ I_{\mathrm{H}}) \obar_{\tilde{g}_{\mathrm{H}}} (A \circ I_{\mathrm{H}})) \\
&\quad - 2\tilde{k} \frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}} \sum_{i,j=1}^{3} (I_i \obar_{\tilde{g}_{\mathrm{H}}} I_i) \circ ((A \circ I_{\mathrm{H}} \circ I_j) \owedge_{\tilde{g}_{\mathrm{H}}} (A \circ I_{\mathrm{H}} \circ I_j)) \\
&\quad + 2\tilde{k}^2 \frac{\tilde{f}_1^2}{\tilde{f}_{\mathrm{H}}^2} \sum_{i=1}^{3} ((A \circ I_{\mathrm{H}}) \obar_{\tilde{g}_{\mathrm{H}}} (A \circ I_{\mathrm{H}})) \circ ((A \circ I_{\mathrm{H}} \circ I_i) \owedge_{\tilde{g}_{\mathrm{H}}} (A \circ I_{\mathrm{H}} \circ I_i)) \Bigg).
\end{aligned}
\tag{4.173}
$$

Next, we use (4.157) to obtain

$$
\begin{aligned}
&64\tilde{K}^2 \operatorname{tr}(\mathscr{R}^2)\\
&= 2(\operatorname{tr}(\mathrm{id}_{TM})^2 - \operatorname{tr}(\mathrm{id}_{TM})) + 6 \sum_{i,j=1}^{3} (\operatorname{tr}(I_i \circ I_j)^2 + \operatorname{tr}((I_i \circ I_j)^2)) \\
&\quad + 6\tilde{k}^2 \frac{\tilde{f}_1^2}{\tilde{f}_{\mathrm{H}}^2} (\operatorname{tr}((A \circ I_{\mathrm{H}})^2)^2 + \operatorname{tr}((A \circ I_{\mathrm{H}})^4)) \\
&\quad + 2\tilde{k}^2 \frac{\tilde{f}_1^2}{\tilde{f}_{\mathrm{H}}^2} \sum_{i,j=1}^{3} (\operatorname{tr}(A \circ I_{\mathrm{H}} \circ I_i \circ A \circ I_{\mathrm{H}} \circ I_j)^2 - \operatorname{tr}((A \circ I_{\mathrm{H}} \circ I_i \circ A \circ I_{\mathrm{H}} \circ I_j)^2)) \\
&\quad + 4 \sum_{i=1}^{3} (\operatorname{tr}(I_i)^2 + 3 \operatorname{tr}(\mathrm{id}_{TM})) - 4\tilde{k} \frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}} (\operatorname{tr}(A \circ I_{\mathrm{H}})^2 - 3 \operatorname{tr}((A \circ I_{\mathrm{H}})^2) \\
&\quad - 4\tilde{k} \frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}} \sum_{i=1}^{3} (\operatorname{tr}(A \circ I_{\mathrm{H}} \circ I_i)^2 - \operatorname{tr}((A \circ I_{\mathrm{H}} \circ I_i)^2) + 3(\operatorname{tr}(A \circ I_{\mathrm{H}} \circ I_i)^2 + \operatorname{tr}((A \circ I_{\mathrm{H}} \circ I_i)^2))) \\
&\quad - 4\tilde{k} \frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}} \sum_{i,j=1}^{3} (\operatorname{tr}(A \circ I_{\mathrm{H}} \circ I_i \circ I_j)^2 - 3 \operatorname{tr}((A \circ I_{\mathrm{H}} \circ I_i \circ I_j)^2) \\
&\quad + 4\tilde{k}^2 \frac{\tilde{f}_1^2}{\tilde{f}_{\mathrm{H}}^2} \sum_{i=1}^{3} (\operatorname{tr}(A \circ I_{\mathrm{H}} \circ A \circ I_{\mathrm{H}} \circ I_i)^2 - 3 \operatorname{tr}((A \circ I_{\mathrm{H}} \circ A \circ I_{\mathrm{H}} \circ I_i)^2)).
\end{aligned}
\tag{4.174}
$$

Finally, substituting the trace formulae (4.165) into the above gives us

$$
\begin{aligned}
&64\tilde{K}^2\mathrm{tr}(\mathscr{R}^2)\\
&= 2((4n)^2 - 4n) + 6\sum_{i,j=1}^{3}((4n)^2\delta_{ij} + (-1)^{\delta_{ij}}4n) + 6\tilde{k}^2\frac{\tilde{f}_1^2}{\tilde{f}_{\mathrm{H}}^2}\left(\mathrm{tr}(A^2)^2 + \mathrm{tr}(A^4)\right)\\
&\quad + 2\tilde{k}^2\frac{\tilde{f}_1^2}{\tilde{f}_{\mathrm{H}}^2}\sum_{i,j=1}^{3}\left(\delta_{ij}\mathrm{tr}(A^2)^2 - (-1)^{\delta_{ij}}\mathrm{tr}(A^4)\right) + 4\sum_{i=1}^{3}12n - 4\tilde{k}\frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}}\left(3\,\mathrm{tr}(A^2)\right)\\
&\quad - 4\tilde{k}\frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}}\sum_{i=1}^{3}(-\mathrm{tr}(A^2) + 3\,\mathrm{tr}(A^2)) - 4\tilde{k}\frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}}\sum_{i,j=1}^{3}(-1)^{\delta_{ij}}3\,\mathrm{tr}(A^2) + 4\tilde{k}^2\frac{\tilde{f}_1^2}{\tilde{f}_{\mathrm{H}}^2}\sum_{i=1}^{3}(3\,\mathrm{tr}(A^4))\\
&= 64\left(n(5n+1) + 3\left(\tilde{k}^3\frac{\tilde{f}_1^3}{\tilde{f}_{\mathrm{H}}^3} + (n-1)\tilde{k}\frac{\tilde{f}_1}{\tilde{f}_{\mathrm{H}}}\right)^2 + 3\left(\tilde{k}^6\frac{\tilde{f}_1^6}{\tilde{f}_{\mathrm{H}}^6} + (n-1)\tilde{k}^2\frac{\tilde{f}_1^2}{\tilde{f}_{\mathrm{H}}^2}\right)\right).
\end{aligned}
\tag{4.175}
$$

□

Chapter 5

Deformations of quaternionic Kähler structures

In this chapter, we revisit the local twist and prove two useful lemmata about it. Lemma 5.1.1 is a generalisation of Proposition 3.1.13 that lets us write the composition of two twists as a single twist, while Lemma 5.1.8 introduces a notion of differentiating twists with respect to twist data.

We then use Lemma 5.1.1 to write the one-loop deformation of a quaternionic Kähler manifold as a twist of an elementary deformation of the quaternionic Kähler manifold (Theorem 5.2.1 and Definition 5.2.2). Note that this differs from Macia and Swann [MS14] in that we are taking the elementary deformation and twist of the quaternionic Kähler manifold directly rather than its QK/HK dual. In fact, as we show in Proposition 5.2.11, the QK/HK correspondence may be thought of as a certain limit of the one-loop deformation.

Meanwhile, Lemma 5.1.8 is used to derive three sets of geometric evolution equations on the space of quaternionic Kähler metrics on a contractible open set U, namely the naïve, reparametrised, and rescaled one-loop flow equations (Propositions 5.3.2, 5.3.5, and 5.3.7 respectively).

Finally, in Subsection 5.3.4, we conclude with an outline of speculative directions to be explored in the future.

5.1 Local twists revisited

In Chapter 4, we saw that applying the HK/QK correspondence to a hyperkähler metric with a rotating Killing field followed by the QK/HK correspondence gives us back the same hyperkähler metric up to an overall scaling. However, applying the QK/HK correspondence to a quaternionic Kähler metric with a Killing field followed by the HK/QK correspondence gives us back the same quaternionic Kähler manifold up to an overall scaling for *certain* choices of $\tilde{f}_{\mathrm{H}}, \tilde{K}, \tilde{k}$ as described in (4.72).

In general, such a procedure will give an honest deformation of the original quaternionic Kähler metric, and since our goal is to construct interesting examples of quaternionic Kähler manifolds, we are obviously interested in describing these deformations. But in order to do so, we first need to establish two additional results regarding the local twist, one regarding the compositions of the local twist, and one regarding derivatives of the local twist map with respect to twist data.

Lemma 5.1.1. *If (U, Z, ω, f, η) and $(U, a\tilde{Z}, \tilde{\omega}', a\tilde{f}', \tilde{\eta}')$ are local twist data on some manifold M with a being a nonzero constant and*

$$\tilde{Z} = -\frac{1}{f}\operatorname{tw}_{Z,f,\eta}(Z) = -\frac{1}{f-\eta(Z)}Z, \tag{5.1}$$

then so is

$$(U, Z, \omega'', f'', \eta'') := (U, Z, a'\mathrm{d}(\tilde{f}'\eta - \tilde{\eta}'), a' f \tilde{f}', a'(\tilde{f}'\eta - \tilde{\eta}')), \tag{5.2}$$

for any nonzero constant a'. *Moreover, the local twist maps with respect to the above tuples of local twist data satisfy*

$$\mathrm{tw}_{a\tilde{Z},\tilde{f}',a\tilde{\eta}'} \circ \mathrm{tw}_{Z,f,\eta} = \mathrm{tw}_{Z,f'',\eta''}. \tag{5.3}$$

Proof. Since (U, Z, ω, f, η) and $(U, a\tilde{Z}, \tilde{\omega}', a\tilde{f}', \tilde{\eta}')$ constitute local twist data, we must have

$$\iota_Z \omega = \iota_Z \mathrm{d}\eta = -\mathrm{d}f, \quad \iota_{a\tilde{Z}} \tilde{\omega}' = -\frac{a\, \iota_Z \mathrm{d}\tilde{\eta}'}{f - \eta(Z)} = -a\, \mathrm{d}\tilde{f}'. \tag{5.4}$$

It then follows that

$$\begin{aligned} \iota_Z \omega'' &= a'\iota_Z \mathrm{d}(\tilde{f}'\eta - \tilde{\eta}') = a'\iota_Z(\mathrm{d}\tilde{f}' \wedge \eta + \tilde{f}'\mathrm{d}\eta - \mathrm{d}\tilde{\eta}') \\ &= a'(-\eta(Z)\mathrm{d}\tilde{f}' - \tilde{f}'\,\mathrm{d}f - (f - \eta(Z)\mathrm{d}\tilde{f}') \\ &= -a'(\tilde{f}'\,\mathrm{d}f + f\,\mathrm{d}\tilde{f}') = -\mathrm{d}(a' f \tilde{f}') = -\mathrm{d}f''. \end{aligned} \tag{5.5}$$

In addition, since $f - \eta(Z)$ and $a\tilde{f}' - \tilde{\eta}'(a\tilde{Z})$, and so $\tilde{f}' - \tilde{\eta}'(Z)$, are nowhere vanishing,

$$\begin{aligned} f'' - \eta''(Z) &= a' f \tilde{f}' - a'(\tilde{f}'\eta(Z) - \tilde{\eta}'(Z)) \\ &= a'(f\tilde{f}' - \tilde{f}'\eta(Z) - (f - \eta(Z))\tilde{\eta}'(\tilde{Z})) \\ &= a'(f - \eta(Z))(\tilde{f}' - \tilde{\eta}'(Z)) \end{aligned} \tag{5.6}$$

is nowhere vanishing as well. So, $(U, Z, \omega'', f'', \eta'')$ indeed constitutes local twist data.

To prove (5.3), it's enough to show that it holds for functions and 1-forms since local twists preserve contractions and tensor products by definition. That it holds for functions is clear, since the local twist is just the identity in that case. For a 1-form α, we introduce the shorthand $\alpha' = \mathrm{tw}_{Z,f,\eta}(\alpha)$ and compute

$$\begin{aligned} \mathrm{tw}_{a\tilde{Z},\tilde{f}',a\tilde{\eta}'} \circ \mathrm{tw}_{Z,f,\eta}(\alpha) &= \mathrm{tw}_{a\tilde{Z},\tilde{f}',a\tilde{\eta}'}(\alpha') = \alpha' - \frac{\alpha'(a\tilde{Z})}{a\tilde{f}'}\,\tilde{\eta}' \\ &= \alpha' + \frac{\alpha(Z)}{f\tilde{f}'}\,\tilde{\eta}' = \alpha - \frac{\alpha(Z)}{f}\,\eta + \frac{\alpha(Z)}{f\tilde{f}'}\,\tilde{\eta}' \\ &= \alpha - \frac{\alpha(Z)}{a' f \tilde{f}'}\, a'(\tilde{f}'\eta - \tilde{\eta}') = \mathrm{tw}_{Z,f'',\omega''}(\alpha). \end{aligned} \tag{5.7}$$

□

Corollary 5.1.2. *If* (U, Z, ω, f, η) *is local twist data on some manifold M and* a_1, a_2, a_3 *are nonzero constants such that*

$$\frac{a_2}{f} + a_3 \tag{5.8}$$

is nowhere vanishing, and if $(U, \tilde{Z}, \tilde{\omega}, \tilde{f}, \tilde{\eta})$ *are the local twist data dual to* (U, Z, ω, f, η), *then we have the following relation of local twist maps:*

$$\mathrm{tw}_{a_1\tilde{Z},a_1(a_2\tilde{f}+a_3),a_2\tilde{\eta}} \circ \mathrm{tw}_{Z,f,\eta} = \mathrm{tw}_{Z,f+\frac{a_2}{a_3},\eta}. \tag{5.9}$$

Proof. This is just a specialisation of Lemma 5.1.1 to the case

$$\tilde{\eta}' = a_2\tilde{\eta}, \quad \tilde{f}' = a_2\tilde{f} + a_3, \quad a = a_1, \quad a' = \frac{1}{a_3}. \tag{5.10}$$

□

Remark 5.1.3. The limit $a_3 \to 0$ is well-defined and reproduces (3.30), i.e. local twists with respect to tuples of twist data dual to each other are the inverses of each other.

In order to state the other lemma, we need two new definitions.

Definition 5.1.4 (Local log-derivative twist)**.** The local log-derivative twist $\mathrm{dltw}_{Z,f,\eta}$ with respect to local twist data (U, Z, ω, f, η) is a graded $C^\infty(U)$-linear map

$$\mathrm{dltw}_{Z,f,\eta} : \Gamma(\mathbb{T}^{\bullet,\bullet}U) \to \Gamma(\mathbb{T}^{\bullet,\bullet}U) \tag{5.11}$$

of tensor fields, satisfying the Leibniz rule over tensor products and contractions, whose action on an arbitrary 1-form α is given by

$$\mathrm{dltw}_{Z,f,\eta}(\alpha) = \frac{\alpha(Z)}{f(f - \eta(Z))}\,\eta. \tag{5.12}$$

Remark 5.1.5. The fact that $\mathrm{dltw}_{Z,f,\eta}$ is $C^\infty(U)$-linear and satisfies the Leibniz property with respect to tensor products forces it to vanish on 0-forms.

Definition 5.1.6 (Local derivative twist)**.** The local derivative twist map $\mathrm{dtw}_{Z,f,\eta}$ with respect to local twist data (U, Z, ω, f, η) is the composition

$$\mathrm{dtw}_{Z,f,\eta} = \mathrm{tw}_{Z,f,\eta} \circ \mathrm{dltw}_{Z,f,\eta}. \tag{5.13}$$

Example 5.1.7. The local log-derivative twists of a vector field u, a symmetric bilinear form g, and an endomorphism field A may be computed to be

$$\begin{aligned} \mathrm{dltw}_{Z,f,\eta}(u) &= -\frac{\eta(u)}{f(f-\eta(Z))}\,Z, \quad \mathrm{dltw}_{Z,f,\eta}(g) = \frac{2}{f(f-\eta(Z))}\,\eta\,\iota_Z g, \\ \mathrm{dltw}_{Z,f,\eta}(A) &= \frac{1}{f(f-\eta(Z))}\,[A, \eta\otimes Z]. \end{aligned} \tag{5.14}$$

Thus, their local derivative twists are given by

$$\begin{aligned} \mathrm{dtw}_{Z,f,\eta}(u) &= -\frac{\eta(u)}{(f-\eta(Z))^2}\,Z, \quad \mathrm{dtw}_{Z,f,\eta}(g) = \frac{2}{f^2}\,\eta\,\mathrm{tw}_{Z,f,\eta}(\iota_Z g), \\ \mathrm{dtw}_{Z,f,\eta}(A) &= \frac{1}{f(f-\eta(Z))}\,[\mathrm{tw}_{Z,f,\eta}(A), \eta\otimes Z]. \end{aligned} \tag{5.15}$$

Lemma 5.1.8. *Given local twist data (U, Z, ω, f, η) on some manifold M, we have for all tensor fields S on M, the following identity:*

$$\frac{\mathrm{d}}{\mathrm{d}a}(\mathrm{tw}_{Z,f+a,\eta}(S))\bigg|_{a=0} = \mathrm{dtw}_{Z,f,\eta}(S). \tag{5.16}$$

Proof. Let $(U, \tilde{Z}, \tilde{\omega}, \tilde{f}, \tilde{\eta})$ denote the local twist data dual to (U, Z, ω, f, η), as usual. We begin by showing that the $C^\infty(U)$-linear map

$$\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}\left(\frac{\mathrm{d}}{\mathrm{d}a}(\mathrm{tw}_{Z,f+a,\eta}(\cdot))\Big|_{a=0}\right) \tag{5.17}$$

satisfies the Leibniz property over tensor products. Let α be an arbitrary 1-form and β an arbitrary k-form. Then we have

$$\begin{aligned}
&\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}\left(\frac{\mathrm{d}}{\mathrm{d}a}(\mathrm{tw}_{Z,f+a,\eta}(\alpha\otimes\beta))\Big|_{a=0}\right)\\
&= \mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}\left(\frac{\mathrm{d}}{\mathrm{d}a}(\mathrm{tw}_{Z,f+a,\eta}(\alpha)\otimes\mathrm{tw}_{Z,f+a,\eta}(\beta))\Big|_{a=0}\right)\\
&= \mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}\left(\frac{\mathrm{d}}{\mathrm{d}a}(\mathrm{tw}_{Z,f+a,\eta}(\alpha))\Big|_{a=0}\otimes\mathrm{tw}_{Z,f,\eta}(\beta) + \mathrm{tw}_{Z,f,\eta}(\alpha)\otimes\frac{\mathrm{d}}{\mathrm{d}a}(\mathrm{tw}_{Z,f+a,\eta}(\beta))\Big|_{a=0}\right)\\
&= \mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}\left(\frac{\mathrm{d}}{\mathrm{d}a}(\mathrm{tw}_{Z,f+a,\eta}(\alpha))\Big|_{a=0}\right)\otimes\beta + \alpha\otimes\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}\left(\frac{\mathrm{d}}{\mathrm{d}a}(\mathrm{tw}_{Z,f+a,\eta}(\beta))\Big|_{a=0}\right).
\end{aligned} \tag{5.18}$$

A similar chain of equalities shows that the map (5.17) satisfies the Leibniz property over contractions as well. All that now remains to be shown in order to prove the lemma is that (5.17) coincides with the local log-derivative twist for functions and 1-forms. Letting α be an arbitrary 1-form, we have

$$\begin{aligned}
\mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}\left(\frac{\mathrm{d}}{\mathrm{d}a}(\mathrm{tw}_{Z,f+a,\eta}(\alpha))\Big|_{a=0}\right) &= \mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}\left(\frac{\mathrm{d}}{\mathrm{d}a}\left(\alpha - \frac{\alpha(Z)}{f+a}\eta\right)\Big|_{a=0}\right)\\
&= \mathrm{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}}\left(\frac{\alpha(Z)}{f^2}\eta\right) = \frac{\alpha(Z)}{f^2}\left(\eta - \frac{\eta(\tilde{Z})}{\tilde{f}}\tilde{\eta}\right)\\
&= \frac{\alpha(Z)}{f^2}\left(1 + \frac{\eta(Z)}{f-\eta(Z)}\tilde{\eta}\right)\eta = \frac{\alpha(Z)}{f(f-\eta(Z))}\eta.
\end{aligned} \tag{5.19}$$

□

5.2 One-loop deformations

5.2.1 Definition and characterisation

Now that we know how to compose twists that are not dual to each other, we can enhance this to a statement about composing a QK/HK correspondence with an HK/QK correspondence.

Theorem 5.2.1. *Let $(M, g, Q, Z, \omega_Q, f_Q, g_Q)$ constitute QK/HK data with hyperkähler dual $(\tilde{M}, \tilde{g}, H, \tilde{Z}, I_1, \tilde{f}_1, \tilde{\omega}_H, \tilde{f}_H, \tilde{g}_H)$ such that*

$$\tilde{f}_1^c := \tilde{f}_1 + 4Kc = -\frac{K}{\|\mu^Z\|} + 4Kc, \quad \tilde{f}_H^c := \tilde{f}_H + 4Kc, \tag{5.20}$$

are nowhere vanishing. Let $(M^c, g^c, Q^c, Z^c, \omega^c_{\mathrm{Q}}, f^c_{\mathrm{Q}}, g^c_{\mathrm{Q}})$ *be a quaternionic dual of the HK/QK data* $(\tilde{M}, \tilde{g}, H, \tilde{Z}, I_1, \tilde{f}^c_1, \tilde{\omega}_{\mathrm{H}}, \tilde{f}^c_{\mathrm{H}}, \tilde{g}_{\mathrm{H}})$*. Then*

$$\left(M^c, -4\tilde{k}Kc\, Z^c, 4\tilde{k}Kc\, \omega^c, \frac{4c}{\nu}\left(1 - 4\tilde{k}Kc\, f^c_{\mathrm{Q}}\right)\right) \tag{5.21}$$

is a twist of $(M, Z, \omega_{\mathrm{Q}}, f_{\mathrm{Q}} + \frac{\nu}{4c})$ *and the metric* g^c *is given by*

$$g^c = -\tilde{K}\, T_c \left(\frac{\nu}{1 - 4c\|\mu^Z\|}\, g|_{\mathbb{H}_{\mathrm{Q}}Z^\perp} + \frac{\nu + 4cf_{\mathrm{Q}}}{(1 - 4c\|\mu^Z\|)^2}\, g|_{\mathbb{H}_{\mathrm{Q}}Z}\right), \tag{5.22}$$

where T_c *is a global twist map,* ν *is the reduced scalar curvature of the quaternionic Kähler manifold* (M, g, Q) *and* $\|\mu^Z\|$ *its quaternionic moment map.*

Proof. As usual, we work locally on appropriate contractible open sets $U, \tilde{U}, U^c$ of $M, \tilde{M}, M^c$ that we all identify via the local diffeomorphisms underlying the global twist maps T and $\tilde{T}$ that realise the QK/HK correspondence from M to $\tilde{M}$ and the HK/QK corespondence from $\tilde{M}$ to M^c. Then the quaternionic Kähler metric g^c on $U^c \cong U$ is given by

$$\begin{aligned} g^c &= \mathrm{tw}_{\tilde{Z}, \tilde{f}^c_{\mathrm{H}}, \tilde{\eta}_{\mathrm{H}}}(\tilde{g}_{\mathrm{H}}) = \mathrm{tw}_{\tilde{Z}, \tilde{f}^c_{\mathrm{H}}, \tilde{\eta}_{\mathrm{H}}}\left(\frac{\tilde{K}}{\tilde{f}^c_1}\, \tilde{g}|_{\mathbb{H}_H \tilde{Z}^\perp} + \frac{\tilde{K}}{\tilde{k}} \frac{\tilde{f}_{\mathrm{H}}}{(\tilde{f}^c_1)^2}\, \tilde{g}|_{\mathbb{H}_H \tilde{Z}}\right) \\ &= \mathrm{tw}_{\tilde{Z}, \tilde{f}^c_{\mathrm{H}}, \tilde{\eta}_{\mathrm{H}}} \circ \mathrm{tw}_{Z, f_{\mathrm{Q}}, \eta_{\mathrm{Q}}}\left(\frac{\tilde{K}}{\tilde{f}^c_1}\, g_{\mathrm{Q}}|_{\mathbb{H}_{\mathrm{Q}}Z^\perp} + \frac{\tilde{K}}{\tilde{k}} \frac{\tilde{f}^c_{\mathrm{H}}}{(\tilde{f}^c_1)^2}\, g_{\mathrm{Q}}|_{\mathbb{H}_{\mathrm{Q}}Z}\right) \\ &\overset{(2.61)}{=} \mathrm{tw}_{\tilde{Z}, \tilde{f}^c_{\mathrm{H}}, \tilde{\eta}_{\mathrm{H}}} \circ \mathrm{tw}_{Z, f_{\mathrm{Q}}, \eta_{\mathrm{Q}}}\left(\frac{\tilde{K}}{\tilde{f}^c_1} \frac{K\nu}{\|\mu^Z\|}\, g|_{\mathbb{H}_{\mathrm{Q}}Z^\perp} - \frac{\tilde{K}}{\tilde{k}} \frac{\tilde{f}^c_{\mathrm{H}}}{(\tilde{f}^c_1)^2} \frac{Kf_{\mathrm{Q}}}{\|\mu^Z\|^2}\, g|_{\mathbb{H}_{\mathrm{Q}}Z}\right). \end{aligned} \tag{5.23}$$

Substituting into the above

$$\begin{aligned} \tilde{f}^c_1 &= -\frac{K}{\|\mu^Z\|} + 4Kc, \\ \tilde{f}^c_{\mathrm{H}} &= \tilde{k}(\tilde{f}_1 + \tilde{g}(\tilde{Z}, \tilde{Z})) = \tilde{k}\left(-\frac{K}{\|\mu^Z\|} + \frac{1}{f^2_{\mathrm{Q}}}\, g_{\mathrm{Q}}(Z, Z) + 4Kc\right) \\ &\overset{(2.61)}{=} \tilde{k}\left(-\frac{K}{\|\mu^Z\|} - \frac{K}{\|\mu^Z\|^2} \frac{g(Z, Z)}{f_{\mathrm{Q}}} + 4Kc\right) \\ &\overset{(2.57)}{=} -\frac{\tilde{k}K}{\|\mu^Z\|}\left(1 - \frac{g(Z, Z)}{g(Z, Z) + \nu\|\mu^Z\|^2}\right) + 4\tilde{k}Kc \overset{(2.57)}{=} \frac{\tilde{k}K\nu}{f_{\mathrm{Q}}} + 4\tilde{k}Kc, \end{aligned} \tag{5.24}$$

and using (5.9) along with an appropriate choice of $\tilde{\eta}_{\mathrm{H}}$, we get

$$g^c = -\tilde{K}\, \mathrm{tw}_{Z, f_{\mathrm{Q}} + \frac{\nu}{4c}, \eta_{\mathrm{Q}}}\left(\frac{\nu}{1 - 4c\|\mu^Z\|}\, g|_{\mathbb{H}_{\mathrm{Q}}Z^\perp} + \frac{\nu + 4cf_{\mathrm{Q}}}{(1 - 4c\|\mu^Z\|)^2}\, g|_{\mathbb{H}_{\mathrm{Q}}Z}\right). \tag{5.25}$$

By similiar reasoning, we find that the accompanying data of the quaternionic bundle Q^c and Killing field Z^c, along with the rest of the twist data, ω^c_{Q} and f^c_{Q}, are given by

$$\begin{aligned} Q^c &= \mathrm{tw}_{\tilde{Z},\tilde{f}^c_{\mathrm{H}},\tilde{\eta}_{\mathrm{H}}} \circ \mathrm{tw}_{Z,f_{\mathrm{Q}},\eta_{\mathrm{Q}}}(Q) = \mathrm{tw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}(Q) \\ Z^c &= -\frac{1}{\tilde{f}^c_{\mathrm{H}}} \mathrm{tw}_{\tilde{Z},\tilde{f}^c_{\mathrm{H}},\tilde{\eta}_{\mathrm{H}}} \left(-\frac{1}{f_{\mathrm{Q}}} \mathrm{tw}_{Z,f_{\mathrm{Q}},\eta_{\mathrm{Q}}}(Z) \right) \\ &= \frac{(\tilde{k}K)^{-1}}{\nu + 4cf_{\mathrm{Q}}} \mathrm{tw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}(Z), \\ \omega^c_{\mathrm{Q}} &= \frac{1}{\tilde{f}^c_{\mathrm{H}}} \mathrm{tw}_{\tilde{Z},\tilde{f}^c_{\mathrm{H}},\tilde{\eta}_{\mathrm{H}}} \left(\frac{1}{f_{\mathrm{Q}}} \mathrm{tw}_{Z,f_{\mathrm{Q}},\eta_{\mathrm{Q}}}(\omega_{\mathrm{Q}}) \right) \\ &= \frac{(\tilde{k}K)^{-1}}{\nu + 4cf_{\mathrm{Q}}} \mathrm{tw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}(\omega_{\mathrm{Q}}), \\ f^c_{\mathrm{Q}} &= \frac{1}{\tilde{f}^c_{\mathrm{H}}} = \frac{(\tilde{k}K)^{-1} f_{\mathrm{Q}}}{\nu + 4cf_{\mathrm{Q}}}. \end{aligned} \tag{5.26}$$

The twist data in (5.21) may then be obtained by writing the twist data

$$\left(-\frac{1}{f_{\mathrm{Q}} + \frac{\nu}{4c}} \mathrm{tw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}(Z), \frac{1}{f_{\mathrm{Q}} + \frac{\nu}{c}} \mathrm{tw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}(\omega_{\mathrm{Q}}), \frac{1}{f_{\mathrm{Q}} + \frac{\nu}{4c}} \right) \tag{5.27}$$

dual to $(Z, \omega_{\mathrm{Q}}, f_{\mathrm{Q}} + \frac{\nu}{4c})$ in terms of the usual quaternionic twist data (Z^c, ω^c, f^c). □

A notable aspect of the quaternionic Kähler manifold (M^c, g^c, Q^c) is that it depends only on the parameters c and $\tilde{K}$, of which $\tilde{K}$ is an overall scaling factor. The Killing field Z^c does depend on $\tilde{k}$ and K, but these enter only as overall scaling factors. There is thus no essential loss of generality if we impose the same relations on these as in (4.72), namely

$$\tilde{K} = -\frac{1}{\nu}, \quad \tilde{k} = \frac{1}{K\nu}. \tag{5.28}$$

This gives us a general notion of a 1-loop deformation of quaternionic Kähler manifolds with a Killing field.

Definition 5.2.2 (1-loop deformation)**.** Let (M, g, Q) be a quaternionic Kähler manifold equipped with a Killing field Z with nowhere vanishing quaternionic moment map μ^Z and so well-defined quaternionic twist data $(Z, \omega_{\mathrm{Q}}, f_{\mathrm{Q}})$ and let $c \neq 0$ be a constant such that

$$1 - 4c\|\mu^Z\|, \quad \nu + 4cf_{\mathrm{Q}} \tag{5.29}$$

are nowhere vanishing. Then its 1-loop c-deformation (M^c, g^c, Q^c, Z^c) by Z is a quaternionic Kähler manifold (M^c, g^c, Q^c) equipped with a Killing vector field Z^c with nowhere vanishing moment map μ^{Z^c} and so well-defined quaternionic twist data $(Z^c, \omega^c_{\mathrm{Q}}, f^c_{\mathrm{Q}})$, such that

$$(M^c, \tilde{Z}_c, \tilde{\omega}_c, \tilde{f}_c) := \left(M^c, -\frac{4c}{\nu} Z^c, \frac{4c}{\nu} \omega^c, -\frac{4c}{\nu} \left(\frac{4c}{\nu} f^c_{\mathrm{Q}} - 1 \right) \right) \tag{5.30}$$

is a twist of $(M, Z, \omega_{\mathrm{Q}}, f_{\mathrm{Q}} + \frac{\nu}{4c})$ and a global twist map

$$T_c : \Gamma(\mathbb{T}^{\bullet,\bullet} M)^Z \to \Gamma(\mathbb{T}^{\bullet,\bullet} M^c)^{Z^c} \tag{5.31}$$

realising this satisfies

$$g^c = T_c(g_{\mathrm{L}}) := T_c\left(\frac{1}{1-4c\|\mu^Z\|}\, g|_{\mathbb{H}_{\mathrm{Q}}Z^\perp} + \frac{1+\frac{4c}{\nu}f_{\mathrm{Q}}}{(1-4c\|\mu^Z\|)^2}\, g|_{\mathbb{H}_{\mathrm{Q}}Z}\right). \tag{5.32}$$

Example 5.2.3. We consider the Przanowski–Tod Ansatz metric g given in (2.75), with (ρ,ζ,τ) coordinatising

$$M = \mathbb{R}_{>0}\times\mathbb{C}\times\mathbb{R}. \tag{5.33}$$

Note that, strictly speaking, this metric does not admit a 1-loop c-deformation by ∂_τ for any $c>0$ since

$$1-4c\|\mu^Z\| = 1-\frac{c}{\rho} \tag{5.34}$$

vanishes when $\rho=c$. One way to get around this is to restrict to the region $\rho>c$ on M. We adopt here an alternative approach where we instead look at the pullback ϕ_c^*g of g under the smooth map $\phi_c : M\to M$ given by

$$(\rho,\zeta,\tau)\mapsto(\rho+c,\zeta,\tau). \tag{5.35}$$

If we denote by u', P', Θ' the pullbacks of u, P, Θ under ϕ_c, then we explictly have

$$g' := \phi_c^*g = \frac{1}{4(\rho+c)^2}\left(P'\,\mathrm{d}\rho^2 + 2P'e^{u'}|\mathrm{d}\zeta|^2 + \frac{1}{P'}\,(\mathrm{d}\tau+\Theta')^2\right), \tag{5.36}$$

with the norm $\phi_c^*\|\mu^Z\|$ of the quaternionic moment map for $Z=\partial_\tau$ satisfying

$$1-4c\,\phi_c^*\|\mu^Z\| = 1-\frac{c}{\rho+c} = \frac{\rho}{\rho+c}. \tag{5.37}$$

This is now nowhere vanishing. If we additionally have that $\nu+4cf'_{\mathrm{Q}}$ is nowhere vanishing, with

$$f'_{\mathrm{Q}} := \phi_c^*f_{\mathrm{Q}} = -\frac{\partial_\rho u'}{P'}, \tag{5.38}$$

then g' does admit a 1-loop c-deformation by $Z=\partial_\tau$. To determine it, we first compute the elementary deformation g'_{L} to be

$$g'_{\mathrm{L}} = \frac{1}{4\rho^2}\left(1-\frac{2c}{\nu}\frac{\partial_\rho u'}{P'}\right)\left(P'\,\mathrm{d}\rho^2 + 2P'e^{u'}|\mathrm{d}\zeta|^2 + \frac{1}{P'}\,(\mathrm{d}\tau+\Theta')^2\right). \tag{5.39}$$

Meanwhile, if we choose the auxiliary 1-form η'_{Q} to be

$$\eta'_{\mathrm{Q}} = -\frac{1}{2}\left(\frac{1}{P'}\,(\partial_\rho u')(\mathrm{d}\tau+\Theta') + \partial_y u'\,\mathrm{d}x - \partial_x u'\,\mathrm{d}y\right), \tag{5.40}$$

the local twists of $\phi_c^*(\iota_Z g) = \iota_Z g'$ and $\phi_c^*(\iota_Z g_{\mathrm{L}}) = \iota_Z g'_{\mathrm{L}}$ for the choice of Hamiltonian function $f'_{\mathrm{Q}} + \nu/4c$ is

$$\begin{aligned} &\mathrm{tw}_{Z,f'_{\mathrm{Q}}+\frac{\nu}{4c},\eta'_q}(\iota_Z g') \\ &= \frac{1}{4(\rho+c)^2 P'}\left(1-\frac{2c}{\nu}\frac{\partial_\rho u'}{P'}\right)^{-1}\left(\mathrm{d}\tau + \Theta' - \frac{4c}{\nu}\left(\partial_y u'\,\mathrm{d}x - \partial_x u'\,\mathrm{d}y\right)\right), \\ &\mathrm{tw}_{Z,f'_{\mathrm{Q}}+\frac{\nu}{4c},\eta'_q}(\iota_Z g'_{\mathrm{L}}) \\ &= \frac{1}{4\rho^2 P'}\left(\mathrm{d}\tau + \Theta' - \frac{4c}{\nu}\left(\partial_y u'\,\mathrm{d}x - \partial_x u'\,\mathrm{d}y\right)\right). \end{aligned} \tag{5.41}$$

Thus, the 1-loop correction g'^c is given by

$$\begin{aligned} g'^c &= \mathrm{tw}_{Z,f'_{\mathrm{Q}}+\frac{\nu}{4c},\eta'_q}(g'_{\mathrm{L}}) \\ &= g'_{\mathrm{L}} - \frac{1}{g_{\mathrm{L}}(Z,Z)}\left((\iota_Z g'_{\mathrm{L}})^2 - (\mathrm{tw}_{Z,f'_{\mathrm{Q}}+\frac{\nu}{4c},\eta'_q}(\iota_Z g'_{\mathrm{L}}))^2\right) \\ &= \frac{1}{4\rho^2}\left(P^c\,\mathrm{d}\rho^2 + 2P^c e^{u^c}|\mathrm{d}\zeta|^2 + \frac{1}{P^c}\left(\mathrm{d}\tau + \Theta^c\right)^2\right), \end{aligned} \tag{5.42}$$

where u^c, P^c, Θ^c (with c *not* an exponent) are given by

$$\begin{aligned} u^c &= u', \quad P^c = \left(1-\frac{2c}{\nu}\frac{\partial_\rho u'}{P'}\right)P' = P' - \frac{2c}{\nu}\,\partial_\rho u^c, \\ \Theta^c &= \Theta' - \frac{4c}{\nu}\left(\partial_y u'\,\mathrm{d}x - \partial_x u'\,\mathrm{d}y\right) = \Theta' - \frac{4c\mathrm{i}}{\nu}\left(\partial_\zeta u^c\,\mathrm{d}\zeta - \partial_{\overline{\zeta}} u^c\,\mathrm{d}\overline{\zeta}\right). \end{aligned} \tag{5.43}$$

Note that these satisfy

$$\begin{aligned} &\partial_\zeta\partial_{\overline{\zeta}} u^c = -\frac{1}{2}\,\partial_\rho^2(e^{u^c}), \quad P^c = \frac{2}{\nu}\left(\rho\,\partial_\rho u^c - 2\right) \\ &\mathrm{d}\Theta^c = \mathrm{i}\left(\left(\partial_\zeta P^c\,\mathrm{d}\zeta - \partial_{\overline{\zeta}} P^c\,\mathrm{d}\overline{\zeta}\right)\wedge\mathrm{d}\rho - \partial_\rho(P^c e^{u^c})\,\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}\right). \end{aligned} \tag{5.44}$$

Thus, we again obtain the Przanowski–Tod Ansatz, but with the Toda potential u replaced by $u^c = \phi_c^* u$. Note in particular that the quaternionic moment map is unchanged owing to what amounted to a coordinate reparametrisation $\rho \mapsto \rho + c$.

As it turns out, we can use Theorem 4.2.5 due to Swann and Macia to deduce that 1-loop deformations are the *only* way the twist of an elementary deformation of a quaternionic Kähler metric by a Killing field, with respect to twist data featuring the same Killing field, can produce another quaternionic Kähler metric. This therefore characterises 1-loop deformations.

Theorem 5.2.4. *Let (M, g, Q) be a quaternionic Kähler manifold equipped with a nowhere vanishing Killing field Z, an arbitrary 2-form ω, and an arbitrary nowhere vanishing function f satisfying $\iota_Z\omega = -\mathrm{d}f$. If the twist (M', g', Q') of any elementary deformation of (M, g, Q) by Z, with respect to the twist data (Z, ω, f) is a quaternionic Kähler manifold different from (M, g, Q), then (M', g', Q') is necessarily a 1-loop c-deformation of (M, g, Q) by Z for some parameter c.*

Proof. We know that the quaternionic Kähler manifold (M, g, Q) is the twist of an elementary deformation of a locally hyperkähler manifold $(\tilde{M}, \tilde{g}, H)$ by some rotating Killing field $\tilde{Z}$ given by the QK/HK correspondence. By virtue of Lemma 5.1.1, (M', g', Q') would then be the twist of some elementary deformation of $(\tilde{M}, \tilde{g}, H)$

by $\tilde{Z}$. Theorem 4.2.5 then implies that this is necessarily a 1-loop c-deformation of (M, g, Q) by Z for some parameter c. □

5.2.2 Basic results

Given the interpretation of the 1-loop c-deformation of a quaternionic Kähler manifold in terms of shifts in the Hamiltonian function $\tilde{f}_{\mathrm{H}}$ associated with its hyperkähler dual by c, the following facts are immediately evident.

Proposition 5.2.5. *Let (M, g, Q) be a quaternionic Kähler manifold equipped with a Killing field Z with nowhere vanishing quaternionic moment map μ^Z admitting a 1-loop c-deformation (M^c, g^c, Q^c, Z^c) with respect to Z. Then (M^c, g^c, Q^c) has a nowhere vanishing quaternionic moment map with norm $\|\mu^{Z^c}\|$ given by*

$$\frac{1}{\|\mu^{Z^c}\|} = \frac{1}{\|\mu^Z\|} - 4c. \tag{5.45}$$

Proposition 5.2.6. *Let (M, g, Q) be a quaternionic Kähler manifold equipped with a Killing field Z with nowhere vanishing quaternionic moment map μ^Z admitting a 1-loop c-deformation (M^c, g^c, Q^c, Z^c) with respect to Z, such that (M^c, g^c, Q^c) itself admits a 1-loop c'-deformation*

$$((M^c)^{c'}, (g^c)^{c'}, (Q^c)^{c'}, (Z^c)^{c'}) \tag{5.46}$$

with respect to Z^c. Then this is locally isometric to a 1-loop $(c + c')$-deformation of (M, g, Q) with respect to Z when $c' \neq -c$ and to (M, g) itself otherwise.

A slightly less obvious fact is the following relation between the Levi-Civita connections ∇^g and ∇^{g^c} of a quaternionic Kähler metric g and its 1-loop c-deformation g^c.

Proposition 5.2.7. *Let (M, g, Q) be a quaternionic Kähler manifold equipped with a Killing field Z with nowhere vanishing quaternionic moment map μ^Z admitting a 1-loop c-deformation (M^c, g^c, Q^c, Z^c) with respect to Z, realised by a global twist map T_c. Then, for any Z-invariant vector fields u, v on M, the Levi-Civita connections ∇^g and ∇^{g^c} satisfy*

$$\nabla^{g^c}_{T_c(u)} \circ T_c(v) = T_c\left(\nabla^g_u v + S^c_u v\right), \tag{5.47}$$

where $S^c \in \mathbb{T}^{1,2}M$ is given by

$$\begin{aligned} S^c_{T_c(u)} \circ T_c(v) = \frac{1}{2}\sum_{\alpha=0}^{3} \Bigg(&\frac{1}{f_Q + \frac{\nu}{4c}}\, \omega_Q(J_\alpha u, v) J_\alpha Z \\ &- \frac{1}{\|\mu^Z\| - \frac{1}{4c}} \left(g(J_\alpha Z, u) J_\alpha \circ J_1 v + g(J_\alpha Z, v) J_\alpha \circ J_1 u\right)\Bigg). \end{aligned} \tag{5.48}$$

Proof. We begin by rewriting Proposition 4.2.8 so that the Levi-Civita connection $\nabla^{\tilde{g}}$ on the hyperkähler side is expressed in terms of the Levi-Civita connection ∇^g on the quaternionic Kähler side:

$$\nabla^{\tilde{g}}_{T(u)} \circ T(v) = T\left(\nabla^g_u v + S^{\mathrm{QH}}_u v\right), \tag{5.49}$$

where $T = \tilde{T}^{-1}$ is a global twist map realising the QK/HK correspondence and $S^{\mathrm{QH}} \in \Gamma(\mathbb{T}^{1,2}M)$ is a tensor field given by

$$\begin{aligned} S^{\mathrm{QH}}_u v &= -\tilde{T}\left(S^{\mathrm{HQ}}_{T(u)} \circ T(v)\right) \\ &= \frac{1}{2}\sum_{\alpha=0}^{3}\left(\frac{1}{f_{\mathrm{Q}}}\,\omega_{\mathrm{Q}}(J_\alpha u, v)J_\alpha Z - \frac{1}{\|\mu^Z\|}\left(g(J_\alpha Z, u)J_\alpha \circ J_1 v + g(J_\alpha Z, v)J_\alpha \circ J_1 u\right)\right), \end{aligned} \tag{5.50}$$

with J_0 denoting the identity endomorphism field. Since the 1-loop deformation on the quaternionic Kähler side leaves the hyperkähler metric (and so its Levi-Civita connection unchanged, we have

$$\nabla^{g^c}_{T_c(u)} \circ T_c(v) + S^{\mathrm{QH},c}_{T_c(u)} \circ T_c(v) = T_c\left(\nabla^g_u v + S^{\mathrm{QH}}_u v\right), \tag{5.51}$$

where $S^{\mathrm{QH},c}$ is given by

$$\begin{aligned} S^{\mathrm{QH},c}_{u'} v' = \frac{1}{2}\sum_{\alpha=0}^{3}\Bigg(&\frac{1}{f^c_{\mathrm{Q}}}\,\omega^c_{\mathrm{Q}}(J^c_\alpha u', v')J^c_\alpha Z^c \\ &- \frac{1}{\|\mu^{Z^c}\|}\left(g^c(J^c_\alpha Z^c, u')J^c_\alpha \circ J^c_1 v' + g^c(J^c_\alpha Z^c, v')J^c_\alpha \circ J^c_1 u'\right)\Bigg). \end{aligned} \tag{5.52}$$

We choose J_α to be Z-invariant (this is always possible due to Lemma 2.2.11), so that it makes sense to say

$$J^c_\alpha = T_c(J_\alpha), \quad \iota_{J^c_\alpha Z^c} g^c = \frac{T_c(\iota_{J_i Z} g)}{(1 - 4c\|\mu^Z\|)^2}, \tag{5.53}$$

in addition to the following:

$$\begin{aligned} &Z^c = \frac{T_c(Z)}{1 + \frac{4c}{\nu} f_{\mathrm{Q}}}, \quad \omega^c_{\mathrm{Q}} = \frac{T_c(\omega_{\mathrm{Q}})}{1 + \frac{4c}{\nu} f_{\mathrm{Q}}}, \quad f^c_{\mathrm{Q}} = \frac{f_{\mathrm{Q}}}{1 + \frac{4c}{\nu} f_{\mathrm{Q}}}, \\ &\frac{1}{\|\mu^{Z^c}\|} = \frac{1 - 4c\|\mu^Z\|}{\|\mu^Z\|}, \quad \iota_{J^c_\alpha Z^c} g^c = \frac{T_c(\iota_Z g)}{(1 - 4c\|\mu^Z\|)^2}. \end{aligned} \tag{5.54}$$

Substituting these gives us

$$\begin{aligned} S^{\mathrm{QH},c}_{T_c(u)} \circ T_c(v) = \frac{1}{2}\sum_{\alpha=0}^{3} T_c\Bigg(&\frac{1}{1 + \frac{4c}{\nu} f_{\mathrm{Q}}}\frac{1}{f_{\mathrm{Q}}}\,\omega_{\mathrm{Q}}(J_\alpha u, v)J_\alpha Z \\ &- \frac{1}{1 - 4c\|\mu^Z\|}\frac{1}{\|\mu^Z\|}\left(g(J_\alpha Z, u)J_\alpha \circ J_1 v + g(J_\alpha Z, v)J_\alpha \circ J_1 u\right)\Bigg). \end{aligned} \tag{5.55}$$

Plugging this into (5.51) and rearranging gives us (5.48). Note that the expression inside the big brackets is independent of the particular choice of J_α, so they don't necessarily have to be individually Z-invariant for the expression to make sense. □

Remark 5.2.8. Analogous to the comments made Remark 4.2.9, the above statement can also be easily generalised to vector fields u and v which are not Z-invariant if we are given auxiliary local twist data (U, η_Q). This is given by

$$\nabla^{g^c}_{u'} v' = \mathrm{tw}_{Z, f_Q + \frac{\nu}{4c}, \eta_Q} \left(\nabla^g_u v + S^c_u v + \frac{\mathscr{L}_Z v}{f_Q + \frac{\nu}{4c} - \eta_Q(Z)} \eta_Q \right), \tag{5.56}$$

where u' and v' are the local twists of u and v. In particular, if J^c_i are the local twists of the sections J_i constituting a local oriented orthonormal frame of Q, we have

$$\nabla^{g^c}_{u'} J^c_i = \mathrm{tw}_{Z, f_Q + \frac{\nu}{4c}, \eta_Q} \left(\nabla^g_u J_i + [S^c_u, J_i] + \frac{\eta_Q(u)}{\tilde{f}_Q + \frac{\nu}{4c} - \eta_Q(Z)} \mathscr{L}_Z J_i \right). \tag{5.57}$$

By making the replacement $J_\alpha \mapsto J_i \circ J_\alpha$, we see that

$$\begin{aligned} &\frac{1}{2} \sum_{\alpha=0}^{3} \left(\frac{1}{f_Q + \frac{\nu}{4c}} \omega_Q(J_\alpha u, J_i v) J_\alpha Z - \frac{1}{\|\mu^Z\| - \frac{1}{4c}} g(J_\alpha Z, J_i v) J_\alpha \circ J_1 u \right) \\ &= \frac{1}{2} \sum_{\alpha=0}^{3} J_i \left(\frac{1}{f_Q + \frac{\nu}{4c}} \omega_Q(J_\alpha u, v) J_\alpha Z - \frac{1}{\|\mu^Z\| - \frac{1}{4c}} g(J_\alpha Z, v) J_\alpha \circ J_1 u \right). \end{aligned} \tag{5.58}$$

This implies that the commutators of S^c with the Hermitian structures J_i are given by

$$[S^c_u, J_i] = -\frac{1}{2\left(\|\mu^Z\| - \frac{1}{4c}\right)} \sum_{\alpha=0}^{3} g(J_\alpha Z, u) [J_\alpha \circ J_1, J_i]. \tag{5.59}$$

Written out more explicitly, this becomes

$$\begin{aligned} [S^c_u, J_1] &= \frac{1}{\|\mu^Z\| - \frac{1}{4c}} (\omega_2(Z, u) J_2 + \omega_3(Z, u) J_3), \\ [S^c_u, J_2] &= -\frac{1}{\|\mu^Z\| - \frac{1}{4c}} (g(Z, u) J_3 + \omega_2(Z, u) J_1), \\ [S^c_u, J_3] &= \frac{1}{\|\mu^Z\| - \frac{1}{4c}} (g(Z, u) J_2 - \omega_3(Z, u) J_1), \end{aligned} \tag{5.60}$$

which gives us

$$\begin{aligned} \nabla^{g^c} J^c_1 &= \mathrm{tw}_{Z, f_Q + \frac{\nu}{4c}, \eta_Q} \Bigg(\nabla^g J_1 + \frac{1}{\|\mu^Z\| - \frac{1}{4c}} (\iota_Z \omega_2 \otimes J_2 + \iota_Z \omega_3 \otimes J_3) \\ &\qquad + \frac{\eta_Q \otimes \mathscr{L}_Z J_1}{f_Q + \frac{\nu}{4c} - \eta_Q(Z)} \Bigg), \\ \nabla^{g^c} J^c_2 &= \mathrm{tw}_{Z, f_Q + \frac{\nu}{4c}, \eta_Q} \Bigg(\nabla^g J_2 - \frac{1}{\|\mu^Z\| - \frac{1}{4c}} (\iota_Z g \otimes J_3 + \iota_Z \omega_2 \otimes J_1) \\ &\qquad + \frac{\eta_Q \otimes \mathscr{L}_Z J_2}{f_Q + \frac{\nu}{4c} - \eta_Q(Z)} \Bigg), \\ \nabla^{g^c} J^c_3 &= \mathrm{tw}_{Z, f_Q + \frac{\nu}{4c}, \eta_Q} \Bigg(\nabla^g J_3 + \frac{1}{\|\mu^Z\| - \frac{1}{4c}} (\iota_Z g \otimes J_2 - \iota_Z \omega_3 \otimes J_1) \\ &\qquad + \frac{\eta_Q \otimes \mathscr{L}_Z J_3}{f_Q + \frac{\nu}{4c} - \eta_Q(Z)} \Bigg). \end{aligned} \tag{5.61}$$

The above remark can be used to derive a useful consequence.

Proposition 5.2.9. *Let (M, g, Q) be a quaternionic Kähler manifold equipped with a Killing field Z with nowhere vanishing quaternionic moment map μ^Z with normalisation J^Z admitting a 1-loop c-deformation (M^c, g^c, Q^c, Z^c) with respect to Z realised by a global twist map T_c. Let $(J_1 = J^Z, J_2, J_3)$ be a local oriented orthonormal frame for Q over some contractible open set U such that the 1-form*

$$\eta_Q = -\frac{\iota_Z g}{\|\mu^Z\|} - \langle J_2, \nabla^g J_3\rangle, \tag{5.62}$$

is a valid choice of an auxiliary local 1-form for the twist data associated to the global twist T_c. Then

$$\eta_Q^c := -\frac{\iota_{Z^c} g^c}{\|\mu^{Z^c}\|} - \langle J_2^c, \nabla^{g^c} J_3^c\rangle = \frac{\eta_Q}{\frac{4c}{\nu} f_Q + 1}, \tag{5.63}$$

where J_2^c and J_3^c are the local twists of J_2 and J_3 with respect to local twist data

$$\left(U, Z, \omega_Q, f_Q + \frac{\nu}{4c}, \eta_Q\right). \tag{5.64}$$

Proof. Substituting the local realisations of (5.54) and (5.61) into the definition of η_Q^c yields

$$\begin{aligned}
\eta_Q^c &= \mathrm{tw}_{Z, f_Q + \frac{\nu}{4c}, \eta_Q}\left(-\left(\frac{1}{\|\mu^Z\|} - 4c\right)\frac{\iota_Z g}{(1-4c\|\mu^Z\|)^2} - \frac{\iota_Z g}{\|\mu^Z\| - \frac{1}{4c}}\right.\\
&\qquad \left. - \langle J_2, \nabla^g J_3\rangle - \frac{\langle J_2, \mathscr{L}_Z J_3\rangle}{f_Q + \frac{\nu}{4c} - \eta_Q(Z)}\eta_Q\right)\\
&= \mathrm{tw}_{Z, f_Q + \frac{\nu}{4c}, \eta_Q}\left(-\frac{\iota_Z g}{\|\mu^Z\|} - \langle J_2, \nabla^g J_3\rangle - \frac{\langle J_2, \mathscr{L}_Z J_3\rangle}{f_Q + \frac{\nu}{4c} - \eta_Q(Z)}\eta_Q\right)\\
&= \frac{f_Q + \frac{\nu}{4c} - \eta_Q(Z) - \langle J_2, \mathscr{L}_Z J_3\rangle}{f_Q + \frac{\nu}{4c}}\eta_Q \overset{(3.7)}{=} \frac{\eta_Q}{\frac{4c}{\nu} f_Q + 1}.
\end{aligned} \tag{5.65}$$

□

Proposition 5.2.10. *Let (M, g, Q) be a quaternionic Kähler manifold equipped with a Killing field Z with nowhere vanishing quaternionic moment map μ^Z with normalisation J^Z admitting a 1-loop c-deformation (M^c, g^c, Q^c, Z^c) with respect to Z realised by a global twist map T_c. Let $(J_1 = J^Z, J_2, J_3)$ be a local oriented orthonormal frame for Q over some contractible open set U such that the 1-form*

$$\eta_Q = -\frac{\iota_Z g}{\|\mu^Z\|} - \langle J_2, \nabla^g J_3\rangle, \tag{5.66}$$

is a valid choice of an auxiliary local 1-form for the twist data associated to the global twist T_c. Then

$$\eta_Q^c = \frac{\eta_Q}{\frac{\nu}{4c} f_Q + 1} \tag{5.67}$$

satisfies the following equation:

$$f_Q^c - \eta_Q^c(Z^c) = \frac{f_Q - \eta_Q(Z)}{1 + \frac{4c}{\nu}(f_Q - \eta_Q(Z))}. \tag{5.68}$$

Proof. Applying (3.7) to the local twist data $(U, Z^c, \omega_Q^c, f_Q^c, \eta_Q^c)$, we have

$$f_Q^c - \eta^c(Z^c) = \langle J_2^c, \mathscr{L}_{Z^c} J_3^c \rangle \overset{(5.30)}{=} -\frac{\nu}{4c} \langle J_2^c, \mathscr{L}_{\tilde{Z}_c} J_3^c \rangle. \tag{5.69}$$

Now we may use (3.44) to deduce

$$\begin{aligned} \langle J_2^c, \mathscr{L}_{\tilde{Z}_c} J_3^c \rangle &= \langle J_2, \mathscr{L}_Z J_3 \rangle - \frac{f_Q + \frac{\nu}{4c} - \eta_Q(Z) + 1}{f_Q + \frac{\nu}{4c} - \eta_Q(Z)} \langle J_2, \mathscr{L}_Z J_3 \rangle \\ &= -\frac{\langle J_2, \mathscr{L}_Z J_3 \rangle}{f_Q + \frac{\nu}{4c} - \eta_Q(Z)} = -\frac{f_Q - \eta_Q(Z)}{f_Q + \frac{\nu}{4c} - \eta_Q(Z)}, \end{aligned} \tag{5.70}$$

giving us the required result. □

Note that the reason we didn't pass through the hyperkähler side to deduce the two propositions above as we did in the case of the rest of the twist data is that then the argument would have held only for those η_Q for which $f_Q - \eta_Q(Z)$ is nowhere vanishing, i.e. those η_Q which are necessarily valid choices of auxiliary 1-forms for the twist data (Z, ω_Q, f_Q).

This is however not required for the above argument to work; we only need that $f_Q + \frac{\nu}{4c} - \eta_Q(Z)$ is nowhere vanishing i.e. η_Q which are necessarily valid choices of auxiliary 1-forms for the twist data $(Z, \omega_Q, f_Q + \frac{\nu}{4c})$.

However, if we do choose η_Q such that these conditions simultaneously hold, we obtain the following result regarding the $c \to \pm\infty$ limit of the 1-loop c-deformation.

Proposition 5.2.11. *Let (M, g, Q) be a quaternionic Kähler manifold of reduced scalar curvature ν equipped with a Killing field Z with nowhere vanishing quaternionic moment map μ^Z admitting quaternionic twist data (Z, ω_Q, f_Q) and a locally hyperkähler manifold $(\tilde{M}, \tilde{g}, H)$ as its image under the QK/HK correspondence locally realised by a local twist with respect to the local twist data $(U, Z, \omega_Q, f_Q, \eta_Q)$. Let (M, g, Q) also admit a 1-loop c-deformation (M^c, g^c, Q^c, Z^c) with respect to Z for all $c > c_0 > 0$, respectively $c < c_0 < 0$, for some c_0, locally realised by a local twist with respect to local twist data*

$$\left(U, Z, \omega_Q, f_Q + \frac{\nu}{4c}, \eta_Q\right). \tag{5.71}$$

Then, we have on U the well-defined limit

$$\lim_{c \to \infty} 4cK\nu\, g^c = -\tilde{g}, \quad \textit{respectively} \quad \lim_{c \to -\infty} 4cK\nu\, g^c = -\tilde{g}. \tag{5.72}$$

Proof. We prove this only in the $c > c_0$ case since the other case is similar. By hypothesis, the expression

$$\begin{aligned} &4cK\nu\, g^c \\ &= 4cK\nu\, \mathrm{tw}_{Z, f_Q + \frac{\nu}{4c}, \eta_Q}(g_L) \\ &= 4cK\nu\, \mathrm{tw}_{Z, f_Q + \frac{\nu}{4c}, \eta_Q} \left(\frac{1}{1 - 4c\|\mu^Z\|} g|_{\mathbb{H}_Q Z^\perp} + \frac{1 + \frac{4c}{\nu} f_Q}{(1 - 4c\|\mu^Z\|)^2} g|_{\mathbb{H}_Q Z} \right) \\ &= -\mathrm{tw}_{Z, f_Q + \frac{\nu}{4c}, \eta_Q} \left(\frac{K\nu}{\|\mu^Z\| - \frac{1}{4c}} g|_{\mathbb{H}_Q Z^\perp} - K \frac{f_Q + \frac{\nu}{4c}}{(\|\mu^Z\| - \frac{1}{4c})^2} g|_{\mathbb{H}_Q Z} \right) \end{aligned} \tag{5.73}$$

is well-defined for all $c > c_0$. The limit $c \to \infty$ then clearly exists and is given by

$$\lim_{c\to\infty} 4cK\nu\, g^c = -\mathrm{tw}_{Z,f_Q,\eta_Q}\left(\frac{K\nu}{\|\mu^Z\|}\, g|_{\mathbb{H}_Q Z^\perp} - \frac{Kf_Q}{\|\mu^Z\|^2}\, g|_{\mathbb{H}_Q Z}\right) = -\tilde{g}. \tag{5.74}$$

□

Example 5.2.12. We once again consider the $\mathbb{C}\mathbf{H}^2$ metric from Example 3.1.11, namely

$$g = \frac{1}{4\rho^2}\left(\mathrm{d}\rho^2 + 2\rho\,|\mathrm{d}\zeta|^2 + (\mathrm{d}\tau + \mathrm{Im}(\zeta\,\mathrm{d}\bar{\zeta}))^2\right), \tag{5.75}$$

but restricted to $0 < \rho < c_0$ so that $\|\mu^Z\| - \frac{1}{c}$ is nowhere vanishing for all $c > c_0$. We have already seen that the auxiliary 1-form η_Q may be chosen to be of the form

$$\eta_Q = -\vartheta^4 + \kappa\,\mathrm{d}\tau = -\frac{1}{2\rho}\left(\mathrm{d}\tau + \mathrm{Im}(\zeta\,\mathrm{d}\bar{\zeta})\right) + \kappa\,\mathrm{d}\tau, \tag{5.76}$$

so that we have

$$f_Q + \frac{\nu}{4c} - \eta_Q(Z) = -\kappa - \frac{\nu}{4c} = -\kappa + \frac{1}{2c}. \tag{5.77}$$

Thus, we can ensure that this is nowhere vanishing as well for all $c > c_0$ by making the choice

$$\kappa = \frac{1}{2c_0}. \tag{5.78}$$

We compute the elementary deformation to be

$$g_L = \frac{1}{4\rho}\frac{\rho + c}{(\rho - c)^2}\left(\mathrm{d}\rho^2 + 2\rho\,|\mathrm{d}\zeta|^2 + (\mathrm{d}\tau + \mathrm{Im}(\zeta\,\mathrm{d}\bar{\zeta}))^2\right), \tag{5.79}$$

and the local twist of $\iota_Z g_L$ to be

$$\mathrm{tw}_{Z,f_Q+\frac{\nu}{4c},\eta_Q}(\iota_Z g_L) = \frac{c}{4}\frac{1}{(\rho - c)^2}\left(\frac{\mathrm{d}\tau}{c_0} - \frac{1}{c}\left(\mathrm{d}\tau + \mathrm{Im}(\zeta\,\mathrm{d}\bar{\zeta})\right)\right). \tag{5.80}$$

So all in all, we have have following expression for g^c:

$$\begin{aligned} g^c &= \mathrm{tw}_{Z,f_Q+\frac{\nu}{4c},\eta_Q}(g_L) \\ &= \frac{1}{(\rho - c)^2}\left(\frac{\rho + c}{\rho}\,\mathrm{d}\rho^2 + 2(\rho + c)|\mathrm{d}\zeta|^2 + \frac{c^2\rho}{\rho + c}\left(\frac{\mathrm{d}\tau}{c_0} - \frac{1}{c}\left(\mathrm{d}\tau + \mathrm{Im}(\zeta\,\mathrm{d}\bar{\zeta})\right)\right)^2\right). \end{aligned} \tag{5.81}$$

Finally, we compute the limit

$$\lim_{c\to\infty} 4cK\nu\, g^c = \lim_{c\to\infty} -8cK\, g^c = -8K\left(\frac{\mathrm{d}\rho^2}{\rho^2} + 2|\mathrm{d}\zeta|^2 + \frac{\rho}{c_0^2}\,\mathrm{d}\tau^2\right), \tag{5.82}$$

which after a change of coordinates

$$\rho = r^2, \quad \zeta = \sqrt{2}\,w, \quad \tau = c_0\tau', \tag{5.83}$$

yields the QK/HK dual metric

$$\tilde{g} = 8K(\mathrm{d}r^2 + r^2\mathrm{d}\tau'^2 + |\mathrm{d}w|^2), \tag{5.84}$$

as we had found in Example 4.1.13.

5.3 Flows of quaternionic Kähler structures

5.3.1 Naïve one-loop flow

Our final task in this dissertation will be to cast deformations of quaternionic Kähler structures as solutions to certain geometric flow equations defined on the underlying manifolds. We begin by establishing an easy lemma.

Lemma 5.3.1. *The c-dependent elementary deformation*

$$g_{\mathrm{L}} = \frac{1}{1-4c\|\mu^Z\|}\, g|_{\mathbb{H}_Q Z^\perp} + \frac{1+\frac{4c}{\nu} f_{\mathrm{Q}}}{(1-4c\|\mu^Z\|)^2}\, g|_{\mathbb{H}_Q Z} \tag{5.85}$$

introduced in (5.32) *satisfies the following differential equation:*

$$\frac{\mathrm{d}g_{\mathrm{L}}}{\mathrm{d}c} = 4\|\mu^{Z^c}\|\, g_{\mathrm{L}} - \frac{4}{\nu}\frac{g^c(Z^c,Z^c)}{\|\mu^{Z^c}\|}\, g_{\mathrm{L}}|_{\mathbb{H}_Q Z}. \tag{5.86}$$

Proof. This follows from a short computation:

$$\begin{aligned}
\frac{\mathrm{d}g_{\mathrm{L}}}{\mathrm{d}c} &= \frac{4\|\mu^Z\|}{(1-4c\|\mu^Z\|)^2}\, g|_{\mathbb{H}_Q Z^\perp} + \left(\frac{4f_{\mathrm{Q}}}{\nu(1-4c\|\mu^Z\|)^2} + \frac{8\|\mu^Z\|\,(1+\frac{4c}{\nu}f_{\mathrm{Q}})}{(1-4c\|\mu^Z\|)^3}\right) g|_{\mathbb{H}_Q Z} \\
&= \frac{4\|\mu^Z\|}{1-4c\|\mu^Z\|}\, g_{\mathrm{L}}|_{\mathbb{H}_Q Z^\perp} + \left(\frac{4f_{\mathrm{Q}}}{\nu+4c\,f_{\mathrm{Q}}} + \frac{8\|\mu^Z\|}{1-4c\|\mu^Z\|}\right) g_{\mathrm{L}}|_{\mathbb{H}_Q Z} \\
&= 4\|\mu^{Z^c}\|\, g_{\mathrm{L}}|_{\mathbb{H}_Q Z^\perp} + \left(\frac{4}{\nu} f_{\mathrm{Q}}^c + 8\|\mu^{Z^c}\|\right) g_{\mathrm{L}}|_{\mathbb{H}_Q Z} \\
&= 4\|\mu^{Z^c}\|\, g_{\mathrm{L}} + \frac{4}{\nu}\left(f_{\mathrm{Q}}^c + \nu\|\mu^{Z^c}\|\right) g_{\mathrm{L}}|_{\mathbb{H}_Q Z} \overset{(2.57)}{=} 4\|\mu^{Z^c}\|\, g_{\mathrm{L}} - \frac{4}{\nu}\frac{g^c(Z^c,Z^c)}{\|\mu^{Z^c}\|}\, g_{\mathrm{L}}|_{\mathbb{H}_Q Z}.
\end{aligned} \tag{5.87}$$

□

This along with Lemma 5.1.8 regarding the local derivative twist gives us the following result.

Proposition 5.3.2. *The local* 1*-loop c-deformation* (U, g^c, Q^c, Z^c) *of a quaternionic Kähler manifold* (M, g, Q) *with respect to a Killing field* Z *and auxiliary local twist data* (U, η_{Q}) *satisfies the "naïve* 1*-loop flow" equations:*

$$\frac{\mathrm{d}g^c}{\mathrm{d}c} = -\frac{8}{\nu}\,\eta_{\mathrm{Q}}^c\,\iota_{Z^c} g^c + 4\|\mu^{Z^c}\|\, g^c - \frac{4}{\nu}\frac{g^c(Z^c,Z^c)}{\|\mu^{Z^c}\|}\, g^c|_{\mathbb{H}_{Q^c} Z^c}, \tag{5.88a}$$

$$\frac{\mathrm{d}Z^c}{\mathrm{d}c} = -\frac{4}{\nu}\,(f_{\mathrm{Q}}^c - \eta_{\mathrm{Q}}^c(Z^c))Z^c, \tag{5.88b}$$

$$\frac{\mathrm{d}Q^c}{\mathrm{d}c} = -\frac{4}{\nu}\,[Q^c, \eta_{\mathrm{Q}}^c \otimes Z^c], \tag{5.88c}$$

$$\frac{\mathrm{d}\eta_{\mathrm{Q}}^c}{\mathrm{d}c} = -\frac{4}{\nu}\, f_{\mathrm{Q}}^c\,\eta_{\mathrm{Q}}^c, \tag{5.88d}$$

subject to the intial conditions

$$(g^c, Z^c, Q^c, \eta_{\mathrm{Q}}^c)|_{c=0} = (g, Z, Q, \eta_{\mathrm{Q}}). \tag{5.89}$$

Proof. The Leibniz rule together with (5.16) and the chain rule yields the following equation for g^c:

$$\begin{aligned}\frac{\mathrm{d}g^c}{\mathrm{d}c} &= \frac{\mathrm{d}}{\mathrm{d}c}\left(\mathrm{tw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}(g_{\mathrm{L}})\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{\nu}{4c}\right)\mathrm{dtw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}(g_{\mathrm{L}}) + \mathrm{tw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}\left(\frac{\mathrm{d}g_{\mathrm{L}}}{\mathrm{d}c}\right)\\ &= -\frac{\nu}{4c^2}\frac{2}{(f_{\mathrm{Q}}+\frac{\nu}{4c})^2}\eta_{\mathrm{Q}}\,\mathrm{tw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}(\iota_Z g_{\mathrm{L}}) + \mathrm{tw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}\left(\frac{\mathrm{d}g_{\mathrm{L}}}{\mathrm{d}c}\right)\\ &= -\frac{8}{\nu}\eta^c_{\mathrm{Q}}\,\iota_{Z^c}g^c + \mathrm{tw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}\left(\frac{\mathrm{d}g_{\mathrm{L}}}{\mathrm{d}c}\right).\end{aligned} \tag{5.90}$$

Substituting (5.85) into the above then results in (5.88a). Likewise, (5.88b) follows from the following computation:

$$\begin{aligned}\frac{\mathrm{d}Z^c}{\mathrm{d}c} &\overset{(5.30)}{=} \frac{\mathrm{d}}{\mathrm{d}c}\left(-\frac{\nu}{4c}\tilde{Z}_c\right) = \frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{Z}{1+\frac{4c}{\nu}(f_{\mathrm{Q}}-\eta_{\mathrm{Q}}(Z))}\right)\\ &= -\frac{4}{\nu}\frac{f_{\mathrm{Q}}-\eta_{\mathrm{Q}}(Z)}{\left(1+\frac{4c}{\nu}(f_{\mathrm{Q}}-\eta_{\mathrm{Q}}(Z))\right)^2}Z = -\frac{4}{\nu}\frac{f_{\mathrm{Q}}-\eta_{\mathrm{Q}}(Z)}{1+\frac{4c}{\nu}(f_{\mathrm{Q}}-\eta_{\mathrm{Q}}(Z))}Z^c\\ &\overset{(5.68)}{=} -\frac{4}{\nu}(f^c_{\mathrm{Q}}-\eta^c_{\mathrm{Q}}(Z^c))Z^c.\end{aligned} \tag{5.91}$$

For deriving (5.88c), we once again use (5.16):

$$\begin{aligned}\frac{\mathrm{d}Q^c}{\mathrm{d}c} &= \frac{\mathrm{d}}{\mathrm{d}c}\left(\mathrm{dtw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}(Q)\right) = \frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{\nu}{4c}\right)\mathrm{dtw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}(Q)\\ &= -\frac{\nu}{4c^2}\frac{\left[\mathrm{tw}_{Z,f_{\mathrm{Q}}+\frac{\nu}{4c},\eta_{\mathrm{Q}}}(Q),\eta_{\mathrm{Q}}\otimes Z\right]}{(f_{\mathrm{Q}}+\frac{\nu}{4c})\left(f_{\mathrm{Q}}+\frac{\nu}{4c}-\eta_{\mathrm{Q}}(Z)\right)}\\ &= -\frac{4}{\nu}\left[Q^c,\frac{\eta_{\mathrm{Q}}}{1+\frac{4c}{\nu}f_{\mathrm{Q}}}\otimes\frac{Z}{1+\frac{4c}{\nu}(f_{\mathrm{Q}}-\eta_{\mathrm{Q}}(Z))}\right] = -\frac{4}{\nu}[Q^c,\eta^c_{\mathrm{Q}}\otimes Z^c].\end{aligned} \tag{5.92}$$

Equation (5.88d) meanwhile follows from the following computation:

$$\begin{aligned}\frac{\mathrm{d}\eta^c_{\mathrm{Q}}}{\mathrm{d}c} &= \frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{\eta_{\mathrm{Q}}}{1+\frac{4c}{\nu}f_{\mathrm{Q}}}\right) = -\frac{4}{\nu}\frac{f_{\mathrm{Q}}\,\eta_{\mathrm{Q}}}{(1+\frac{4c}{\nu}f_{\mathrm{Q}})^2}\\ &= -\frac{4}{\nu}f^c_{\mathrm{Q}}\,\eta^c_{\mathrm{Q}}.\end{aligned} \tag{5.93}$$

Finally, (5.89) follows from the fact that the $c\to 0$ limits of $g^c, Z^c, Q^c, \eta^c_{\mathrm{Q}}$ are well-defined and given by $g, Z, Q, \eta_{\mathrm{Q}}$ respectively. □

5.3.2 Reparametrised one-loop flow

As we saw in Example 5.2.3, sometimes we may have to work with a c-dependent pullback $g' = \phi_c^* g$ of a quaternionic Kähler metric in order to ensure that the 1-loop c-deformation is well-defined. In the case of Example 5.2.12, this boiled down to choosing a pullback so that the norm of quaternionic moment map

$$\|\mu^{Z'^c}\| := \phi_c^*\|\mu^{Z^c}\| \tag{5.94}$$

of the 1-loop deformation of pulled back data is independent of c. We generalise this to an arbitrary quaternionic Kähler manifold (M, g, Q) with nowhere vanishing Killing field Z, nowhere vanishing quaternionic moment map μ^Z, and well-defined quaternionic twist data (Z, ω_Q, f_Q), by constructing a c-dependent map ϕ_c^* such that $\phi_c^* \|\mu^{Z^c}\|$, if well-defined, is equal to $\|\mu^Z\|$.

Lemma 5.3.3. *Let (M, g, Q) be a quaternionic Kähler manifold of reduced scalar curvature ν equipped with a Killing field Z with nowhere vanishing quaternionic moment map μ^Z, admitting quaternionic twist data (Z, ω_Q, f_Q) and auxiliary local twist data (U, η_Q). Let U_c be open subsets of U on which the 1-loop c-deformation of the restriction of (M, g, Q) with respect to Z is well-defined and let $\phi_c : U \to U_c$ be a 1-parameter family of diffeomorphisms satisfying for all $p \in U_c$*

$$\phi_0 = \mathrm{id}_U, \qquad \left.\frac{\mathrm{d}\phi_b}{\mathrm{d}b}\right|_{b=c} (\phi_c^{-1}(p)) = W_{c,p}, \tag{5.95}$$

where W_c is a vector field given by

$$W_c = \frac{4\|\mu^{Z^c}\|}{g^c(Z^c, Z^c)}\, \mu^{Z^c} Z^c. \tag{5.96}$$

Then we have on U, the following equation:

$$\phi_c^* \|\mu^{Z^c}\| = \|\mu^Z\|. \tag{5.97}$$

Proof. It suffices to show that the c-derivative of $\phi_c^* \|\mu^{Z^c}\|^{-1}$ vanishes. Applying the Leibniz rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}c}(\phi_c^* \|\mu^{Z^c}\|^{-1}) = \phi_c^* \left(W_c \left(\|\mu^{Z^c}\|^{-1} \right) + \frac{\mathrm{d}\|\mu^{Z^c}\|}{\mathrm{d}c} \right) \overset{(5.45)}{=} \phi_c^* \left(-\frac{W_c(\|\mu^{Z^c}\|)}{\|\mu^{Z^c}\|^2} \right) - 4. \tag{5.98}$$

The hypothesis that the 1-loop c-deformation is well-defined on U_c entails that μ^{Z^c} is nowhere vanishing on U_c and its normalisation J^{Z^c} is well-defined, so that we have

$$W_c = \frac{4\|\mu^{Z^c}\|}{g^c(Z^c, Z^c)}\, \mu^{Z^c} Z^c = \frac{4\|\mu^{Z^c}\|^2}{g^c(Z^c, Z^c)}\, J^{Z^c} Z^c. \tag{5.99}$$

Furthermore, (2.49) applies to this context and we have

$$\begin{aligned} W_c(\|\mu^{Z^c}\|) &= \mathrm{d}\|\mu^{Z^c}\|(W_c) = -g(J^{Z^c} Z^c, W_c) \\ &= -\frac{4\|\mu^{Z^c}\|^2}{g^c(Z^c, Z^c)}\, g^c(J^{Z^c} Z^c, J^{Z^c} Z^c) = -4\|\mu^{Z^c}\|^2. \end{aligned} \tag{5.100}$$

This gives us the required result

$$\frac{\mathrm{d}}{\mathrm{d}c}(\phi_c^* \|\mu^{Z^c}\|^{-1}) = \phi_c^* \left(-\frac{W_c(\|\mu^{Z^c}\|)}{\|\mu^{Z^c}\|^2} \right) - 4 = 4 - 4 = 0. \tag{5.101}$$

□

Example 5.3.4. In the case of the Przanowski–Tod Ansatz, the *naïve* 1-loop-deformed quantities may be read off from our computations in Example 5.2.3 by undoing the action of the diffeomorphism ϕ_c. In particular, we have

$$\mu^{Z^c} = \frac{1}{4(\rho - c)} J_1^c, \quad J_1^c Z^c = \frac{1}{P - \frac{2c}{\nu}\partial_\rho u}\partial_\rho, \quad g^c(Z^c, Z^c) = \frac{1}{4}\frac{(\rho-c)^{-2}}{P - \frac{2c}{\nu}\partial_\rho u}. \tag{5.102}$$

So, the vector field W_c is given by

$$W_c = \frac{4\|\mu^{Z^c}\|}{g^c(Z^c, Z^c)}\mu^{Z^c} Z^c = \frac{4\|\mu^{Z^c}\|^2}{g^c(Z^c, Z^c)} J_1^c Z^c = \partial_\rho. \tag{5.103}$$

This generates precisely the diffeomorphism ϕ_c i.e. $(\rho, \zeta, \tau) \mapsto (\rho + c, \zeta, \tau)$.

Proposition 5.3.5. *Let (U, g^c, Q^c, Z^c) be the local* 1*-loop c-deformation of a quaternionic Kähler manifold (M, g, Q) with respect to a Killing field Z and auxiliary local twist data (U, η_Q) and let*

$$(g'^c, Q'^c, Z'^c, f'^c_Q, \eta'^c_Q) := (\phi_c^* g^c, \phi_c^* Q^c, \phi_c^* Z^c, \phi_c^* f^c_Q, \phi_c^* \eta^c_Q) \tag{5.104}$$

denote the action via Lie-dragging of the 1*-parameter family of diffeomorphisms $\phi_c : U \to U_c \subseteq U$ satisfying for all $p \in U_c$*

$$\phi_0 = \mathrm{id}_U, \quad \left.\frac{\mathrm{d}\phi_b}{\mathrm{d}b}\right|_{b=c}(\phi_c^{-1}(p)) = W_{c,p}, \tag{5.105}$$

where W_c is a vector field given by

$$W_c = \frac{4\|\mu^{Z^c}\|}{g^c(Z^c, Z^c)}\mu^{Z^c} Z^c. \tag{5.106}$$

Furthermore, let W'_c denote $\pi_c^* W_c$. Then $(g'^c, Q'^c, Z'^c, \eta'^c_Q)$ satisfies the "reparametrised 1*-loop flow" equations:*

$$\begin{aligned}
\frac{\mathrm{d}g'^c}{\mathrm{d}c} &= \mathscr{L}_{W'_c} g'^c - \frac{8}{\nu}\eta'^c_Q \,\iota_{Z'^c} g'^c + 4\|\mu^{Z'^c}\|\, g'^c - \frac{4}{\nu}\frac{g'^c(Z'^c, Z'^c)}{\|\mu^{Z'^c}\|}\, g'^c|_{\mathbb{H}_{Q'^c}Z'^c},\\
\frac{\mathrm{d}Z'^c}{\mathrm{d}c} &= \mathscr{L}_{W'_c} Z'^c - \frac{4}{\nu}(f'^c_Q - \eta'^c_Q(Z'^c))Z'^c,\\
\frac{\mathrm{d}Q'^c}{\mathrm{d}c} &= \mathscr{L}_{W'_c} Q'^c - \frac{4}{\nu}[Q'^c, \eta'^c_Q \otimes Z'^c],\\
\frac{\mathrm{d}\eta'^c_Q}{\mathrm{d}c} &= \mathscr{L}_{W'_c}\eta'^c_Q - \frac{4}{\nu} f'^c_Q\, \eta'^c_Q,
\end{aligned} \tag{5.107}$$

subject to the intial conditions

$$(g'^c, Z'^c, Q'^c, \eta'^c_Q)|_{c=0} = (g, Z, Q, \eta_Q). \tag{5.108}$$

Proof. This follows from the general observation that if S_1 and S_2 are c-dependent tensor fields satisfying

$$\frac{\mathrm{d}S_1}{\mathrm{d}c} = S_2, \tag{5.109}$$

then, by the Leibniz rule and naturality of the Lie derivative

$$\frac{\mathrm{d}}{\mathrm{d}c}(\phi_c^* S_1) = \phi_c^*\left(\mathscr{L}_{W_c} S_1 + \frac{\mathrm{d}S_1}{\mathrm{d}c}\right) = \mathscr{L}_{W'_c}(\phi_c^* S_1) + \phi_c^* S_2. \tag{5.110}$$

□

5.3.3 Rescaled one-loop flow

There is yet another way we might consider modifying the naïve 1-loop flow. We may rescale the metric g^c by a factor Lc with L a fixed constant, which, as we saw in Proposition 5.2.11, is expected to have a sensible large c limit as the QK/HK dual metric $\tilde{g}$, up to an overall factor. For this we need a lemma regarding how the QK/HK data changes when the quaternionic Kähler metric g is rescaled to ag.

Lemma 5.3.6. *Let (M, g, Q) be a quaternionic Kähler metric equipped with a Killing field Z with nowhere vanishing quaternionic moment map μ^Z with normalisation J^Z and quaternionic twist data (Z, ω_Q, f_Q). Then, for any nonzero constant a, (M, ag, Q) is a quaternionic Kähler metric with Z being a Killing field thereof with nowhere vanishing quaternionic moment map $a\mu^Z$ and quaternionic twist data (Z, ω_Q, f_Q).*

Proof. We make use of the well-known fact that the Levi-Civita connection ∇^{ag} associated to the rescaled metric ag and inner product induced on the endomorphism bundle $\mathrm{End}(TM)$ by ag are the same as the Levi-Civita connection ∇^g and the inner product $\langle\cdot,\cdot\rangle$ induced on $\mathrm{End}(TM)$ by g. As a consequence, when g is Einstein with reduced scalar curvature ν, the metric ag has reduced scalar curvature $a^{-1}\nu$, since there is a contraction with the inverse metric $a^{-1}g^{-1}$ involved. Thus, the quaternionic moment map for the new metric ag is given by

$$-\frac{2}{a^{-1}\nu}\,\mathrm{pr}_Q(\nabla^{ag}Z) = a\mu^Z. \tag{5.111}$$

Choosing an appropriate local oriented orthonormal frame $J_1 = J^Z, J_2, J_3)$ for Q, we additionally see that we have the auxiliary local 1-form

$$-\frac{\iota_Z(ag)}{a\|\mu^Z\|} - \langle J_2, \nabla^{ag}J_3\rangle = -\frac{\iota_Z g}{\|\mu^Z\|} - \langle J_2, \nabla^g J_3\rangle = \eta_Q. \tag{5.112}$$

Since this is unchanged under the rescaling $g \mapsto ag$ in addition to Z, so must be the quaternionic twist data (Z, ω_Q, f_Q), given that $\omega_Q = \mathrm{d}\eta_Q$ and

$$-\frac{ag(Z,Z)}{a\|\mu^Z\|} - a^{-1}\nu a\|\mu^Z\| = -\frac{g(Z,Z)}{\|\mu^Z\|} - \nu\|\mu^Z\| = f_Q. \tag{5.113}$$

□

Now direct substitution of the above into the naïve 1-loop flow equations has the following result.

Proposition 5.3.7. *Let (U, g^c, Q^c, Z^c) be the local 1-loop c-deformation of a quaternionic Kähler manifold (M, g, Q) with respect to a Killing field Z and auxiliary local twist data (U, η_Q). Let the family*

$$\begin{aligned}&\left(g^{(t)}, Q^{(t)}, Z^{(t)}, \nu^{(t)}, \mu^{Z^{(t)}}, f_Q^{(t)}, \eta_Q^{(t)}\right)\\ &:= (4Lc\, g^c, Q^c, Z^c, (4Lc)^{-1}\nu, 4Lc\,\mu^{Z^c}, f_Q^c, \eta_Q^c),\end{aligned} \tag{5.114}$$

be parametrised by a parameter t related to c restricted to be either positive or negative by

$$e^{\ell t} = 4Lc, \tag{5.115}$$

for some fixed nonzero constants L and ℓ such that Lc and ℓ are positive. Then it satisfies the "the rescaled 1-loop flow" equations:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}g^{(t)} &= -\frac{2\ell}{L\nu^{(t)}}\,\eta_Q^{(t)}\,\iota_{Z^{(t)}}g^{(t)} + \left(\frac{\ell}{L}\,\|\mu^{Z^{(t)}}\| + \ell\right)g^{(t)} \\ &\quad - \frac{\ell}{L\nu^{(t)}}\frac{g^{(t)}(Z^{(t)},Z^{(t)})}{\|\mu^{Z^{(t)}}\|}\;g^{(t)}\Big|_{\mathbb{H}_{Q^{(t)}}Z^{(t)}}, \\ \frac{\mathrm{d}}{\mathrm{d}t}Z^{(t)} &= -\frac{\ell}{L\nu^{(t)}}\,(f_Q^{(t)} - \eta_Q^{(t)}(Z^{(t)}))Z^{(t)}, \\ \frac{\mathrm{d}}{\mathrm{d}t}Q^{(t)} &= -\frac{\ell}{L\nu^{(t)}}\,\left[Q^{(t)}, \eta_Q^{(t)}\otimes Z^{(t)}\right], \\ \frac{\mathrm{d}}{\mathrm{d}t}\eta_Q^{(t)} &= -\frac{\ell}{L\nu^{(t)}}\,f_Q^{(t)}\,\eta_Q^{(t)},\end{aligned} \tag{5.116}$$

subject to the initial conditions

$$\lim_{t\to-\infty}\left(e^{-\ell t}g^{(t)}, Z^{(t)}, Q^{(t)}, \eta_Q^{(t)}\right) = (g, Z, Q, \eta_Q). \tag{5.117}$$

5.3.4 Towards instanton corrections

A noteworthy aspect of the (naïve) 1-loop flow equations is that they may be formally extended to AQH manifolds. This is due to the fact that the ingredients that go into defining these flow equations, namely the quaternionic moment map μ^Z, the auxiliary 1-form η_Q, and the Hamiltonian function f_Q, all have rather explicit expressions in terms of the data (g, Q, Z). Such a formal extension is by no means canonical and there are a whole lot of ways one could go about it.

As an example, consider a family of AQH structures (g^c, Q^c) on a fixed contractible open set U of dimension $4n$, each member of which is equipped with a vector field Z^c, and a choice of section $J_2^c \in \Gamma(Q^c)$ orthogonal to the endomorphism field $\nabla^{g^c}Z^c$. Away from the points where this endomorphism field is orthogonal to Q^c, such a choice amounts to a choice of a local oriented orthonormal frame

$$(J_1^c, J_2^c, J_3^c) := \left(\frac{\mathrm{pr}_{Q^c}(\nabla^{g^c}Z^c)}{\|\mathrm{pr}_{Q^c}(\nabla^{g^c}Z^c)\|}, J_2^c, \frac{\mathrm{pr}_{Q^c}(\nabla^{g^c}Z^c)}{\|\mathrm{pr}_{Q^c}(\nabla^{g^c}Z^c)\|}\circ J_2^c\right) \tag{5.118}$$

for Q^c and so a connection 1-form α_{23}^c given by

$$\alpha_{23}^c(u) = \langle J_2^c, \nabla_u^{g^c} J_3^c\rangle. \tag{5.119}$$

Thus, it makes sense to consider the following system of differential equations:

$$\begin{aligned}\frac{\mathrm{d}g^c}{\mathrm{d}c} &= 2(n+2)(-\|\mathrm{pr}_{Q^c}(\nabla^{g^c}Z^c)\|g^c + \alpha_{23}^c\,\iota_{Z^c}g^c) \\ &\quad + \frac{\mathrm{Ric}^c(Z^c,Z^c)}{2\|\mathrm{pr}_{Q^c}(\nabla^{g^c}Z^c)\|}\left(-(\iota_{Z^c}g^c)^2 + \sum_{i=1}^{c}(\iota_{Z^c}g^c\circ J_i^c)^2\right), \\ \frac{\mathrm{d}Z^c}{\mathrm{d}c} &= -(n+2)\langle J_2^c, \mathscr{L}_{Z^c}J_3^c\rangle Z^c, \\ \frac{\mathrm{d}J_2^c}{\mathrm{d}c} &= \left[J_2^c, \left(\frac{\iota_{Z^c}\mathrm{Ric}^c}{2\|\mathrm{pr}_{Q^c}(\nabla^{g^c}Z^c)\|} + (n+2)\alpha_{23}^c\right)\otimes Z^c\right],\end{aligned} \tag{5.120}$$

where Ric^c denotes the Ricci curvature of g^c. Up to absorption of constant factors into the parameter c, this system of differential equations reduces to the naïve 1-loop flow equations in the case when the almost quaternionic Hermitian structures are quaternionic Kähler, with the auxiliary 1-forms η^c_Q given by (5.63).

As mentioned in Folklore 1.A.3, the quaternionic Kähler property of the hypermultiplet moduli space is a consequence of $\mathcal{N} = 2$ supersymmetry. So, a deformation away from quaternionic Kähler structures has a physical interpretation as a supersymmetry-breaking deformation of the theory. Now, while an order-by-order perturbative expansion of the quantum corrections to the correlators of a supersymmetric theory often converges or even truncates, the same isn't the case for non-supersymmetric theories, where perturbative expansions generically diverge and have to be interpreted as asymptotic expansions of analytic functions. Based on the observation that such asymptotic expansions contain a great deal of information about the nonperturbative sectors of a theory, recent works in physics [DÜ16; Koz+18; DG18] have fruitfully made use of supersymmetry-breaking deformations to determine instanton contributions to expectation values of observables in supersymmetric theories.

Taking cue from this, one can hope that an alternative description of the twistor-based construction of instanton corrections to quaternionic Kähler metrics in [APP11] might emerge from investigating the behaviour of solutions of a suitable generalisation of the 1-loop flow equations such as (5.120) around singularities due to $\nabla^{g^c} Z^c$ becoming orthogonal to Q^c. In particular, it is hoped that by introducing a small deformation away from quaternionic Kähler structure and taking the limit of this deformation going to zero only at the end, we will be able to construct nonanalytic 1-parameter families of quaternionic Kähler metrics that solve the 1-loop flow equations but are different from the 1-loop deformation. This failure of uniqueness is consistent with the Cauchy–Kovaleskaya theorem, which applies only to analytic solutions.

Bibliography

[Aal11] T. A. F. van der Aalst. "A geometric interpretation of the c-map". 2011.

[ACM13] Dmitri V. Alekseevsky, Vicente Cortés, and Thomas Mohaupt. "Conification of Kähler and Hyper-Kähler Manifolds". In: *Communications in Mathematical Physics* 324.2 (Dec. 2013), 637–655. ISSN: 1432-0916. DOI: 10.1007/s00220-013-1812-0. URL: https://doi.org/10.1007/s00220-013-1812-0.

[Ale+03] Dmitri V. Alekseevsky, Vicente Cortés, Chandrashekar Devchand, and Antoine Van Proeyen. "Flows on Quaternionic-Kähler and Very Special Real Manifolds". In: *Communications in Mathematical Physics* 238.3 (July 2003), 525–543. ISSN: 1432-0916. DOI: 10.1007/s00220-003-0850-4. URL: https://doi.org/10.1007/s00220-003-0850-4.

[Ale13] Sergei Alexandrov. "Twistor Approach to String Compactifications: a Review". In: *Phys. Rept.* 522 (2013), pp. 1–57. DOI: 10.1016/j.physrep.2012.09.005. arXiv: 1111.2892 [hep-th].

[Ale+15] Dmitri V. Alekseevsky, Vicente Cortés, Malte Dyckmanns, and Thomas Mohaupt. "Quaternionic Kähler metrics associated with special Kähler manifolds". In: *Journal of Geometry and Physics* 92 (2015), 271–287. ISSN: 0393-0440. DOI: 10.1016/j.geomphys.2014.12.012. URL: http://www.sciencedirect.com/science/article/pii/S0393044015000571.

[Ale68] Dmitriĭ V. Alekseevskiĭ. "Riemannian spaces with exceptional holonomy groups". In: *Functional Analysis and Its Applications* 2.2 (Apr. 1968), 97–105. ISSN: 1573-8485. DOI: 10.1007/BF01075943. URL: https://doi.org/10.1007/BF01075943.

[Ale75] Dmitriĭ V. Alekseevskiĭ. "Classification of quaternionic spaces with a transitive solvable group of motions". In: *Mathematics of the USSR-Izvestiya* 9.2 (Apr. 1975), 297–339. DOI: 10.1070/im1975v009n02abeh001479. URL: https://doi.org/10.1070%2Fim1975v009n02abeh001479.

[APP11] Sergei Alexandrov, Daniel Persson, and Boris Pioline. "Wall-crossing, Rogers dilogarithm, and the QK/HK correspondence". In: *Journal of High Energy Physics* 2011.12 (Dec. 2011), p. 27. ISSN: 1029-8479. DOI: 10.1007/JHEP12(2011)027. URL: https://doi.org/10.1007/JHEP12(2011)027.

[Bat79] Marjorie Batchelor. "The Structure of Supermanifolds". In: *Transactions of the American Mathematical Society* 253 (1979), pp. 329–338. ISSN: 00029947. URL: http://www.jstor.org/stable/1998201.

[Ber55] Marcel Berger. "Sur les groupes d'holonomie homogènes de variétés à connexion affine et des variétés riemanniennes". fr. In: *Bulletin de la Société Mathématique de France* 83 (1955), pp. 279–330. DOI: 10.24033/bsmf.1464. URL: http://www.numdam.org/item/BSMF_1955__83__279_0.

[Ber66] Marcel Berger. “Remarques sur les groupes d’holonomie des variétés riemanniennes.” French. In: *C. R. Acad. Sci., Paris, Sér. A* 262 (1966), 1316–1318. ISSN: 0366-6034; 0302-8429.

[BF82] Charles P. Boyer and J. Daniel Finley III. “Killing vectors in self-dual, Euclidean Einstein spaces”. In: *Journal of Mathematical Physics* 23.6 (June 1982), pp. 1126–1130. DOI: 10.1063/1.525479.

[BG72] Robert B. Brown and Alfred Gray. “Riemannian manifolds with holonomy group Spin (9)”. In: *Differential Geometry (in honor of Kentaro Yano), Kinokunia, Tokyo* (1972), pp. 41–59.

[Boo62] William M. Boothby. “A Note on Homogeneous Complex Contact Manifolds”. In: *Proceedings of the American Mathematical Society* 13.2 (1962), pp. 276–280. ISSN: 00029939, 10886826. URL: http://www.jstor.org/stable/2034482.

[BW83] Jonathan Bagger and Edward Witten. “Matter couplings in N = 2 supergravity”. In: *Nuclear Physics B* 222.1 (1983), 1–10. ISSN: 0550-3213. DOI: 10.1016/0550-3213(83)90605-3. URL: http://www.sciencedirect.com/science/article/pii/0550321383906053.

[CDS17] Vicente Cortés, Malte Dyckmanns, and Stefan Suhr. “Completeness of Projective Special Kähler and Quaternionic Kähler Manifolds”. In: *Special Metrics and Group Actions in Geometry*. Ed. by Simon G. Chiossi, Anna Fino, Emilio Musso, Fabio Podestà, and Luigi Vezzoni. Cham: Springer International Publishing, 2017, 81–106. ISBN: 978-3-319-67519-0. DOI: 10.1007/978-3-319-67519-0_4. URL: https://doi.org/10.1007/978-3-319-67519-0_4.

[Cec09] Sergio Cecotti. *Geometrical Structures in Supersymmetric (Q)FT*. 2009. URL: https://people.sissa.it/~cecotti/partsI-IIv4.pdf.

[CFG89] Sergio Cecotti, Sergio Ferrara, and Luciano Girardello. “Geometry of Type II superstrings and the moduli of superconformal field theories”. In: *International Journal of Modern Physics A* 04.10 (1989), 2475–2529. DOI: 10.1142/S0217751X89000972. eprint: https://doi.org/10.1142/S0217751X89000972. URL: https://doi.org/10.1142/S0217751X89000972.

[CHM12] Vicente Cortés, Xu Han, and Thomas Mohaupt. “Completeness in Supergravity Constructions”. In: *Communications in Mathematical Physics* 311.1 (Apr. 2012), 191–213. ISSN: 1432-0916. DOI: 10.1007/s00220-012-1443-x. URL: https://doi.org/10.1007/s00220-012-1443-x.

[Cor+17] Vicente Cortés, Malte Dyckmanns, Michel Jüngling, and David Lindemann. *A class of cubic hypersurfaces and quaternionic Kähler manifolds of cohomogeneity one*. 2017. arXiv: 1701.07882 [math.DG].

[Cor96] Vicente Cortés. “Alekseevskian spaces”. In: *Differential Geometry and its Applications* 6.2 (1996), 129–168. ISSN: 0926-2245. DOI: 10.1016/0926-2245(96)89146-7. URL: http://www.sciencedirect.com/science/article/pii/0926224596891467.

[CS17] Vicente Cortés and Arpan Saha. “Quarter-pinched Einstein metrics interpolating between real and complex hyperbolic metrics”. In: *Mathematische Zeitschrift* (May 2017). DOI: 10.1007/s00209-017-2013-x.

[CST20a] Vicente Cortés, Arpan Saha, and Danu Thung. *Curvature of quaternionic Kähler manifolds with S^1-symmetry*. 2020. arXiv: 2001.10032 [math.DG].

[CST20b] Vicente Cortés, Arpan Saha, and Danu Thung. *Symmetries of quaternionic Kähler manifolds with S^1-symmetry*. 2020. arXiv: 2001.10026 [math.DG].

[DG18] Daniele Dorigoni and Philip Glass. "The grin of Cheshire cat resurgence from supersymmetric localization". In: *SciPost Phys.* 4 (2 2018), p. 012. DOI: 10.21468/SciPostPhys.4.2.012. URL: https://scipost.org/10.21468/SciPostPhys.4.2.012.

[DS79] John Dell and Lee Smolin. "Graded manifold theory as the geometry of supersymmetry". In: *Communications in Mathematical Physics* 66.3 (Aug. 1979), 197–221. ISSN: 1432-0916. DOI: 10.1007/BF01197186. URL: https://doi.org/10.1007/BF01197186.

[DÜ16] Gerald V. Dunne and Mithat Ünsal. "Deconstructing zero: resurgence, supersymmetry and complex saddles". In: *Journal of High Energy Physics* 2016.12 (Dec. 2016), p. 2. ISSN: 1029-8479. DOI: 10.1007/JHEP12(2016)002. URL: https://doi.org/10.1007/JHEP12(2016)002.

[Dyc15] Malte Dyckmanns. "The hyper-Kähler/quaternionic Kähler correspondence and the geometry of the c-map". PhD thesis. Universität Hamburg, 2015. URL: http://ediss.sub.uni-hamburg.de/volltexte/2015/7542.

[FK82] Thomas Friedrich and Herbert Kurke. "Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature". In: *Mathematische Nachrichten* 106.1 (1982), 271–299. DOI: 10.1002/mana.19821060124. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/mana.19821060124. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/mana.19821060124.

[Fre99] Daniel S. Freed. "Special Kahler manifolds". In: *Commun. Math. Phys.* 203 (1999), pp. 31–52. DOI: 10.1007/s002200050604. arXiv: hep-th/9712042 [hep-th].

[FS90] Sergio Ferrara and Sarabjit Sabharwal. "Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces". In: *Nuclear Physics B* 332.2 (1990), 317–332. ISSN: 0550-3213. DOI: 10.1016/0550-3213(90)90097-W. URL: http://www.sciencedirect.com/science/article/pii/055032139090097W.

[FVP12] Daniel Z. Freedman and Antoine Van Proeyen. *Supergravity*. Cambridge University Press, 2012. DOI: 10.1017/CBO9781139026833.

[GL88] Krzysytof Galicki and Jr. Lawson H. Blaine. "Quaternionic Reduction and Quaternionic Orbifolds." In: *Mathematische Annalen* 282.1 (1988), pp. 1–22. URL: http://eudml.org/doc/164446.

[GMS11] Paul Gauduchon, Andrei Moroianu, and Uwe Semmelmann. "Almost complex structures on quaternion-Kähler manifolds and inner symmetric spaces". In: *Inventiones mathematicae* 184.2 (Oct. 2011), pp. 389–403. DOI: 10.1007/s00222-010-0291-6. URL: https://doi.org/10.1007/s00222-010-0291-6.

[Hay06] Andriy Haydys. "Generalized Seiberg-Witten equations and hyperKähler geometry". PhD thesis. University of Göttingen, 2006. URL: https://ediss.uni-goettingen.de/bitstream/handle/11858/00-1735-0000-0006-B381-C/haydys.pdf.

[Hay08] Andriy Haydys. “HyperKähler and quaternionic Kähler manifolds with S^1-symmetries”. In: *Journal of Geometry and Physics* 58.3 (2008), 293–306. ISSN: 0393-0440. DOI: 10.1016/j.geomphys.2007.11.004. URL: http://www.sciencedirect.com/science/article/pii/S0393044007001453.

[Hir54] Friedrich Hirzebruch. “Über die quaternionalen projektiven Räume”. In: Sitzungsberichte 1953,27 (1954). URL: http://publikationen.badw.de/de/003383776.

[Hit09] Nigel J. Hitchin. “Quaternionic Kähler Moduli Spaces”. In: *Riemannian Topology and Geometric Structures on Manifolds*. Ed. by Krzysztof Galicki and Santiago R. Simanca. Boston, MA: Birkhäuser Boston, 2009, 49–61. ISBN: 978-0-8176-4743-8. DOI: 10.1007/978-0-8176-4743-8_3. URL: https://doi.org/10.1007/978-0-8176-4743-8_3.

[Hit13] Nigel J. Hitchin. “On the Hyperkähler/Quaternion Kähler Correspondence”. In: *Communications in Mathematical Physics* 324.1 (Nov. 2013), 77–106. ISSN: 1432-0916. DOI: 10.1007/s00220-013-1689-y. URL: https://doi.org/10.1007/s00220-013-1689-y.

[Hit81] Nigel J. Hitchin. “Kählerian Twistor Spaces”. In: *Proceedings of the London Mathematical Society* s3-43.1 (1981), 133–150. DOI: 10.1112/plms/s3-43.1.133. eprint: https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/plms/s3-43.1.133. URL: https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/plms/s3-43.1.133.

[Hit87] N. J. Hitchin. “The self-duality equations on a Riemann surface”. In: *Proc. Lond. Math. Soc., III. Ser* (1987).

[Jos+17] Jürgen Jost, Enno Keßler, Jürgen Tolksdorf, Ruijun Wu, and Miaomiao Zhu. “From harmonic maps to the nonlinear supersymmetric sigma model of quantum field theory. At the interface of theoretical physics, Riemannian geometry and nonlinear analysis”. In: (2017). arXiv: 1710.01519 [math.DG].

[Ket01] Sergei V. Ketov. “Universal hypermultiplet metrics”. In: *Nucl. Phys.* B604 (2001), pp. 256–280. DOI: 10.1016/S0550-3213(01)00184-5. arXiv: hep-th/0102099 [hep-th].

[Kos77] Bertram Kostant. “Graded manifolds, graded Lie theory, and prequantization”. In: *Differential Geometrical Methods in Mathematical Physics*. Ed. by Konrad Bleuler and Axel Reetz. Berlin, Heidelberg: Springer Berlin Heidelberg, 1977, 177–306. ISBN: 978-3-540-37498-5.

[Koz+18] Can Kozçaz, Tin Sulejmanpasic, Yuya Tanizaki, and Mithat Ünsal. “Cheshire Cat Resurgence, Self-Resurgence and Quasi-Exact Solvable Systems”. In: *Communications in Mathematical Physics* 364.3 (Dec. 2018), 835–878. ISSN: 1432-0916. DOI: 10.1007/s00220-018-3281-y. URL: https://doi.org/10.1007/s00220-018-3281-y.

[LeB91] Claude LeBrun. “On complete quaternionic-Kähler manifolds”. In: *Duke Math. J.* 63.3 (Aug. 1991), pp. 723–743. DOI: 10.1215/S0012-7094-91-06331-3. URL: https://doi.org/10.1215/S0012-7094-91-06331-3.

[LS94] Claude LeBrun and Simon M. Salamon. “Strong rigidity of positive quaternion Kähler manifolds.” In: *Inventiones mathematicae* 118.1 (1994), pp. 109–132. URL: http://eudml.org/doc/144231.

[MS14] Oscar Macia and Andrew Swann. "Elementary Deformations and the HyperKähler-Quaternionic Kähler Correspondence". In: *Real and Complex Submanifolds*. Ed. by Young Jin Suh, Jürgen Berndt, Yoshihiro Ohnita, Byung Hak Kim, and Hyunjin Lee. Tokyo: Springer Japan, 2014, 339–347. ISBN: 978-4-431-55215-4.

[Ped86] Henrik Pedersen. "Einstein metrics, spinning top motions and monopoles". In: *Mathematische Annalen* 274.1 (Mar. 1986), 35–59. ISSN: 1432-1807. DOI: 10.1007/BF01458016. URL: https://doi.org/10.1007/BF01458016.

[Pen67] Roger Penrose. "Twistor Algebra". In: *Journal of Mathematical Physics* 8.2 (1967), 345–366. DOI: 10.1063/1.1705200. eprint: https://doi.org/10.1063/1.1705200. URL: https://doi.org/10.1063/1.1705200.

[Pol07a] Joseph Polchinski. *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007. ISBN: 9780511252273, 9780521672276, 9780521633031. DOI: 10.1017/CBO9780511816079.

[Pol07b] Joseph Polchinski. *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007. ISBN: 9780511252280, 9780521633048, 9780521672283. DOI: 10.1017/CBO9780511618123.

[Prz91] Maciej Przanowski. "Killing vector fields in self-dual, Euclidean Einstein spaces with $\Lambda \neq 0$". In: *Journal of Mathematical Physics* 32.4 (1991), 1004–1010. DOI: 10.1063/1.529375. eprint: https://doi.org/10.1063/1.529375. URL: https://doi.org/10.1063/1.529375.

[PS91] Yat Sun Poon and Simon M. Salamon. "Quaternionic Kähler 8-manifolds with positive scalar curvature". In: *J. Differential Geom.* 33.2 (1991), pp. 363–378. DOI: 10.4310/jdg/1214446322. URL: https://doi.org/10.4310/jdg/1214446322.

[RSV06] Daniel Robles-Llana, Frank Saueressig, and Stefan Vandoren. "String loop corrected hypermultiplet moduli spaces". In: *Journal of High Energy Physics* 2006.03 (Mar. 2006), 081–081. DOI: 10.1088/1126-6708/2006/03/081. URL: https://doi.org/10.1088%2F1126-6708%2F2006%2F03%2F081.

[RVV06] Martin Roček, Cumrun Vafa, and Stefan Vandoren. "Quaternion-Kähler spaces, hyperkahler cones, and the c-map". In: (2006). arXiv: math/0603048 [math-dg].

[Sal82] Simon M. Salamon. "Quaternionic Kähler manifolds". In: *Inventiones mathematicae* 67.1 (Feb. 1982), 143–171. ISSN: 1432-1297. DOI: 10.1007/BF01393378. URL: https://doi.org/10.1007/BF01393378.

[Soh85] Martin F. Sohnius. "Introducing supersymmetry". In: *Physics Reports* 128.2 (1985), 39–204. ISSN: 0370-1573. DOI: 10.1016/0370-1573(85)90023-7. URL: http://www.sciencedirect.com/science/article/pii/0370157385900237.

[Swa10] Andrew F. Swann. "Twisting Hermitian and hypercomplex geometries". In: *Duke Math. J.* 155.2 (Nov. 2010), pp. 403–431. DOI: 10.1215/00127094-2010-059. URL: https://doi.org/10.1215/00127094-2010-059.

[Swa90] Andrew F. Swann. "HyperKähler and quaternionic Kähler geometry". PhD thesis. University of Oxford, 1990. URL: https://users-math.au.dk/~swann/thesisafs.pdf.

[Swa91] Andrew F. Swann. “HyperKähler and quaternionic Kähler geometry”. In: *Mathematische Annalen* 289.1 (Mar. 1991), 421–450. ISSN: 1432-1807. DOI: 10.1007/BF01446581. URL: https://doi.org/10.1007/BF01446581.

[Tod95] K. Paul Tod. “The SU(∞) Toda field equation and special four-dimensional metrics”. In: *Geometry and physics. Proceedings, Conference, Aarhus, Denmark, July 18-27, 1995*. 1995, pp. 307–312.

[Ton09] David Tong. *Lectures on String Theory*. 2009. arXiv: 0908.0333 [hep-th].

[Wol65] Joseph A. Wolf. “Complex Homogeneous Contact Manifolds and Quaternionic Symmetric Spaces”. In: *Journal of Mathematics and Mechanics* 14.6 (1965), pp. 1033–1047. ISSN: 00959057, 19435274. URL: http://www.jstor.org/stable/24901319.

[WVP92] Bernard de Wit and Antoine Van Proeyen. “Special geometry, cubic polynomials and homogeneous quaternionic spaces”. In: *Commun. Math. Phys.* 149 (1992), pp. 307–334. DOI: 10.1007/BF02097627. arXiv: hep-th/9112027 [hep-th].