Twists of quaternionic Kähler manifolds

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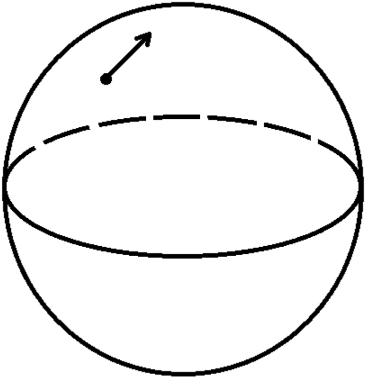
Overview

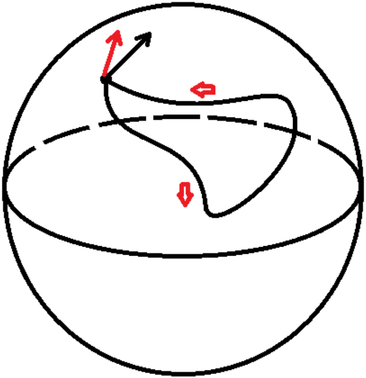
Parallel transport on a sphere I

Suppose we were given a tangent vector on a sphere.



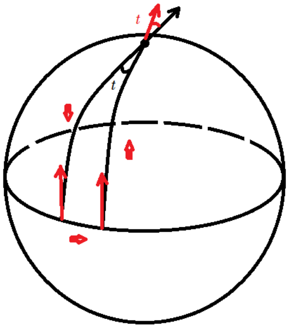
Parallel transport on a sphere II

Parallel-transporting the vector around a loop on the sphere doesn't bring it back to the original starting vector in general.



Parallel transport on a sphere III

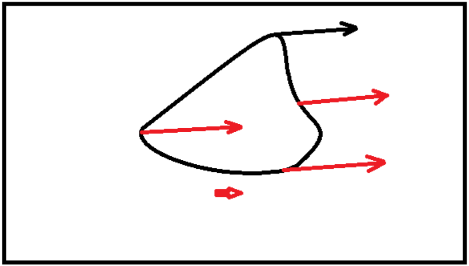
Can one transform a given vector at a point into any other vector at the same point by parallel-transporting it around some loop?



As long as length of the vector is unchanged, yes.

Parallel transport on a plane

On the other hand, we have parallel transport on a plane.



The vector undergoes no change at all!

Holonomy groups I

As loops based at a point can be composed, the transformations a vector undergoes under parallel-transport wrt any connection form a group—the *holonomy* group.

When the connection is Levi-Civita wrt a Riemannian metric on a manifold of dim m, this is a subgroup of $\mathrm{O}(m)$.

What are all the subgroups of $\mathrm{O}(m)$ that can actually occur as holonomy groups of Riemannian manifolds?

Holonomy groups II

A few observations:

- If M has noncontractible loops, then $\mathrm{Hol}(M)$ may not be connected.
- If M is isometric to $M_1 \times M_2$, then $\mathrm{Hol}(M) = \mathrm{Hol}(M_1) \times \mathrm{Hol}(M_2)$.
- If M is locally isometric to a symmetric space G/H, then $\mathrm{Hol}(M) = H$.

Holonomy groups III

Theorem (Berger)

Excluding the cases in the previous slide, only the following groups can arise as holonomy groups of a complete orientable Riemannian manifold.

$\dim(M)$	Holonomy group	Name
n	$\mathrm{SO}(n)$	orientable
$2n$	$\mathrm{U}(n)$	Kähler
$2n$	$\mathrm{SU}(n)$	Calabi–Yau
$4n > 4$	$\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$	quaternionic Kähler
$4n > 4$	$\mathrm{Sp}(n)$	hyperkähler
7	G_2	—
8	$\mathrm{Spin}(7)$	—

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A brief history

More generally, QK manifolds may be defined to have holonomy contained in $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ but not in $\mathrm{Sp}(n)$.

- **Wolf**: Classified symmetric spaces that are QK.
- **Alekseevsky, Cortés**: Classified homogeneous but nonsymmetric spaces that are QK.
- **LeBrun–Salamon**: Conjectured that all complete QK manifolds with positive scalar curvature are symmetric spaces.
- **LeBrun**: The moduli space of QK deformations of $\mathbb{H}\mathbf{H}^n$ is infinite-dimensional.

The goal is to describe some of these QK deformations as explicitly as possible. Ideas from physics have proved very helpful.

Almost quaternionic Hermitian manifolds

The definition in terms of holonomy groups is hard to work with, so we'll introduce an alternative definition.

Definition

An *almost quaternionic Hermitian (AQH) manifold* (M, g, Q) is a (pseudo-)Riemannian manifold (M, g) with a distinguished subbundle $Q \subset \mathrm{End}(TM)$ locally spanned by three almost Hermitian structures J_1, J_2, J_3 satisfying the following equation:

$$J_1 J_2 = J_3. \tag{1}$$

The quaternion algebra follows from

$$J_2 J_3 = -J_1^2 J_2 J_3 = -J_1(J_1 J_2) J_3 = -J_1 J_3^2 = J_1. \tag{2}$$

Quaternionic Kähler manifolds

Definition

A *quaternionic Kähler (QK) manifold* (M, g, Q) is a non-Ricci-flat AQH manifold (M, g, Q) such that the quaternionic bundle Q is parallel with respect to the Levi-Civita connection ∇^g associated to g and satisfies

$$[\mathscr{R}, Q^* \wedge \mathrm{id}_{T^*M}] = 0, \quad (3)$$

where $\mathscr{R} \in \Gamma(\mathrm{End}(\Lambda^2 T^*M))$ is the Riemann curvature map, and $Q^* \subset \mathrm{End}(T^*M)$ is the transpose of Q.

This extends the definition of QK manifolds to include 4-dimensional, pseudo-Riemannian, non-simply connected or incomplete manifolds.

Properties and consequences

- Dimension is forced to be $4n$.
- The vanishing of $[\mathscr{R}, Q^* \wedge \mathrm{id}_{T^*M}]$ is automatic for $n > 1$.
- The manifold is Einstein i.e. has constant reduced scalar curvature $\nu := \mathrm{scal}_g / 4n(n+2)$.
- The curvature of the bundle Q is given by the following local expression:

$$(\nabla^2_{u,v} - \nabla^2_{v,u})A = -\frac{\nu}{2} \sum_i g(J_i u, v)[J_i, A]. \tag{4}$$

Hyperkähler manifolds

Ricci flatness $\Rightarrow$ Q is flat $\Rightarrow$ $\exists$ frame of local parallel sections

Definition

A *(locally) hyperkähler (HK) manifold* $(\tilde{M}, \tilde{g}, H)$ is an AQH manifold $(\tilde{M}, \tilde{g}, H)$ such that the quaternionic bundle H admits a (local) oriented orthonormal frame (I_1, I_2, I_3) of Kähler structures i.e. Hermitian structures parallel with respect to the Levi-Civita connection $\nabla^{\tilde{g}}$ associated to $\tilde{g}$.

Denote $\tilde{g}(I_i \cdot, \cdot) =: \varpi_i$.

Killing fields

Killing fields on (locally) HK manifolds are either *triholomorphic* or can be scaled to be *rotating*.

Definition

A Killing field $\tilde{Z}$ of a locally HK manifold $(\tilde{M}, \tilde{g}, H)$ is said to be *triholomorphic* if it preserves every local Kähler section of H.

Definition

A Killing field $\tilde{Z}$ of a locally HK manifold $(\tilde{M}, \tilde{g}, H)$ is said to be *rotating* if there is a local oriented orthonormal Kähler frame (I_1, I_2, I_3) such that

$$\mathscr{L}_{\tilde{Z}} I_1 = 0, \quad \mathscr{L}_{\tilde{Z}} I_2 = I_3, \quad \mathscr{L}_{\tilde{Z}} I_3 = -I_2. \tag{5}$$

QK/HK and HK/QK correspondence

Haydys: correspondence between 1-parameter families of quaternionic Kähler manifolds with a circle action having $\nu > 0$ and Riemannian hyperkähler manifolds of the same dimension with a rotating circle action.

Alekseevsky–Cortés–Mohaupt: correspondence between 1-parameter families of quaternionic Kähler manifolds with a circle action having $\nu < 0$ and pseudo-Riemannian hyperkähler manifolds of the same dimension with a rotating circle action.

Discovered in parallel by physicists **Neitzke–Pioline**. The 1-parameter family has interpretation as perturbative quantum correction coming from string theory.

Elementary deformation

Macia and Swann reformulated the HK/QK correspondence as a composition of two operations: *elementary deformations* and *twists*.

Definition

An *elementary deformation* of an AQH metric g by a vector field Z is a metric of the form

$$h_1 g_{\mathbb{H}_Q Z^\perp} + h_2 g_{\mathbb{H}_Q Z}, \tag{6}$$

where h_1, h_2 are nowhere vanishing functions.

An elementary deformation of an AQH metric is again AQH.

Twist construction I

Goal: Given a manifold with a circle action M, construct a new manifold with a circle action $\tilde{M}$ such that invariant tensor fields on M and $\tilde{M}$ are in a bijective correspondence.

$$M \leftarrow P \rightarrow \tilde{M}$$

Idea:

- Introduce a circle bundle P on M.
- Lift the given circle action on M to a circle action on P.
- Quotient out the lifted circle action on P to get $\tilde{M}$ with a circle action inherited from the principal circle action on P.

Twist data: a vector field Z, a closed 2-form ω, and a nowhere vanishing function f such that $\mathrm{d}f = -\iota_Z\omega$.

Twist construction II

Theorem (Swann)

Given twist data (Z, ω, f) on M such that Z generates a circle action on M and ω is integral, there exists a circle bundle $P \to M$ with connection $\hat{\eta}$ having curvature ω and fundamental vector field X_P such that the lift $\hat{Z} + fX_P$ of Z, where $\hat{Z}$ is the $\hat{\eta}$-horizontal lift of Z, acts properly on P and so defines a well-defined quotient manifold

$$\tilde{M} := P/\langle \hat{Z} + fX_P \rangle. \tag{7}$$

Furthermore, the $\hat{\eta}$-horizontal lift of any Z-invariant vector field on M and the pullback of any Z-invariant function on M to P descend to a well-defined vector field and function on $\tilde{M}$ respectively.

Note: Once P is fixed, only integer shifts of f are allowed.

Twist realisation of HK/QK

Theorem (Macia–Swann)

Let $(\tilde{M}, \tilde{g})$ be HK with rotating Killing field $\tilde{Z}$ Hamiltonian wrt Kähler form ϖ_1 with nowhere vanishing Hamiltonian $\tilde{f}_1$. Then the twist of the "standard hyperkähler elementary deformation"

$$\tilde{g}_{\mathrm{H}} = \frac{\tilde{K}}{\tilde{f}_1}\,\tilde{g}|_{\mathbb{H}_H\tilde{Z}^\perp} + \frac{\tilde{K}}{\tilde{k}}\frac{\tilde{f}_{\mathrm{H}}}{\tilde{f}_1^2}\,\tilde{g}|_{\mathbb{H}_H\tilde{Z}}, \tag{8}$$

with respect to twist data

$$(\tilde{Z}, \tilde{\omega}_{\mathrm{H}}, \tilde{f}_{\mathrm{H}}) := (\tilde{Z}, \tilde{k}(\varpi_1 + \mathrm{d} \circ \iota_{\tilde{Z}}\tilde{g}), \tilde{k}(\tilde{f}_1 + \tilde{g}(\tilde{Z}, \tilde{Z})), \tag{9}$$

is QK. Moreover, these are the only QK combinations of elementary deformations by Killing fields $\tilde{Z}$ and twists wrt $(\tilde{Z}, \ldots)$.

Today we will be discussing three classes of results:

- Reformulation of the twist construction
- Twist realisation of the QK/HK correspondence
- Symmetries and curvature under HK/QK

Local twist data

For the reformulation, we first augment the twist data to *local* twist data.

Definition

Local twist data is a tuple (U, Z, ω, f, η) where

- Z is a nowhere vanishing vector field,
- ω is a closed 2-form,
- f is a nowhere vanishing function such that $\mathrm{d}f = -\iota_Z\omega$,
- U is an open set on which ω is exact,
- η is a 1-form on U such that $\mathrm{d}\eta = \omega$ and $f - \eta(Z)$ is nowhere vanishing.

Local twist map

Definition

The *local twist map* $\mathrm{tw}_{Z,f,\eta}$ with respect to local twist data (U, Z, ω, f, η) is a graded $C^\infty(U)$-linear map of tensor fields, compatible with tensor products and contractions, whose action on an arbitrary function h and 1-form α is given by

$$\mathrm{tw}_{Z,f,\eta}(h) = h, \quad \mathrm{tw}_{Z,f,\eta}(\alpha) = \alpha - \frac{\alpha(Z)}{f}\,\eta. \tag{10}$$

Example

For u a vector field, we have

$$\mathrm{tw}_{Z,f,\eta}(u) = u + \frac{\eta(u)}{f - \eta(Z)}\,Z. \tag{11}$$

From Swann twist to local twist

Choice of 1-form η on U
$\Rightarrow$ choice of closed 1-form $\hat{\eta} - \pi^*\eta$ on $P|_U$
$\Rightarrow$ choice of submanifold transverse to $P \to M$ and $P \to \tilde{M}$
$\Rightarrow$ identification of $U \subseteq M$ with open set $\tilde{U} \subseteq \tilde{M}$.

Local twist map then is simply the twist correspondence under the identification $U \cong \tilde{U}$.

Note: Tranversality to $P \to \tilde{M}$ follows from nonvanishing of $f - \eta(Z)$.

From local twist to Swann twist I

Proposition (AS)

When restricted to Z-invariant tensor fields in an open set U around a given point p, the local twists with respect to two different choices of auxiliary 1-forms η_0 and η_1 such that $f - \eta_0(Z)$ and $f - \eta_1(Z)$ have the same sign, are related by a diffeomorphism of local neighbourhoods $V_0, V_1 \subseteq U$ of p.

In fact, if ω is integral, we can always find an open neighbourhood U of any point so that at least for certain η_0, η_1, the sets V_0, V_1 can be taken to be U itself.

Idea: Glue open sets U_Λ with modified transition functions to get a new manifold $\tilde{M}$ such that local twist maps agree on overlaps, giving a global twist map T.

From local twist to Swann twist II

Definition

Given twist data (Z, ω, f) on M, the *global twist* $(\tilde{M}, \tilde{Z}, T)$ is a tuple consisting of

- a manifold $\tilde{M}$ with circle action generated by a vector field $\tilde{Z}$,
- a linear map T sending Z-invariant tensor fields on M to $\tilde{Z}$-invariant tensor fields on $\tilde{M}$ and in particular, $-\frac{1}{f}Z$ to $\tilde{Z}$,

such that there exist

- open covers $\{U_\Lambda\}$ of M and $\{\tilde{U}_\Lambda\}$ of $\tilde{M}$ and diffeomorphisms $\{\psi_\Lambda : U_\Lambda \to \tilde{U}_\Lambda\}$, all indexed by the same set $\{\Lambda\}$,
- local twist data $(U_\Lambda, Z, \omega, f, \eta_\Lambda)$ satisfying for any Z-invariant differential form α

$$\psi_\Lambda^* T(\alpha)|_{\tilde{U}_\Lambda} = \mathrm{tw}_{Z,f,\eta_\Lambda}(\alpha|_{U_\Lambda}). \tag{12}$$

From local twist to Swann twist III

Proposition (AS)

Given a manifold M with twist data (Z, ω, f) and a circle action generated by Z, then local twist maps on M may be consistently glued together into a global twist map T only if there is a double surjection $M \leftarrow P \rightarrow \tilde{M}$ having the properties of the Swann twist.

Idea behind proof: Assume $\tilde{M}$ exists. Then P is the set of pairs of points $(p, \tilde{p}) \in M \times \tilde{M}$ such that for all Z-invariant functions h on M, we have $h(p) = T(h)(\tilde{p})$.

Technical advantages I

Local twists are globally equivalent to Swann twists. However...

- We get to work directly with open sets on M without having to first lift tensor fields to P.
- Technical difficulties associated with ensuring properness of group actions on P can be entirely avoided. In particular, f can be shifted by any real constant as long as it is still nowhere vanishing.
- We can work with tensor fields which are not Z-invariant to verify local properties that the twists of certain tensor fields need to satisfy.

Technical advantages II

The following algebraic structure on the space of (local) twist data is much clearer.

Proposition (AS)

Let (U, Z, ω, f, η) and $(U, \tilde{Z}, \tilde{\omega}', \tilde{f}', \tilde{\eta}')$ be local twist data where

$$\tilde{Z} = -\frac{1}{f}\,\mathrm{tw}_{Z,f,\eta}(Z). \tag{13}$$

Then the composition of local twist maps $\mathrm{tw}_{\tilde{Z},\tilde{f}',\tilde{\eta}'} \circ \mathrm{tw}_{Z,f,\eta}$ is itself a local twist map with respect to the local twist data

$$(U, Z, \omega'', f'', \eta'') := (U, Z, \mathrm{d}(\tilde{f}'\eta - \tilde{\eta}'), f\tilde{f}', \tilde{f}'\eta - \tilde{\eta}').$$

Dual twist data

Definition

Given local twist data (U, Z, ω, f, η), its *dual twist data* is the local twist data

$$(U, \tilde{Z}, \tilde{\omega}, \tilde{f}, \tilde{\eta}) := \left(U, -\frac{1}{f} \operatorname{tw}_{Z,f,\eta}(Z), \frac{1}{f} \operatorname{tw}_{Z,f,\eta}(\omega), \frac{1}{f}, \frac{\eta}{f} \right).$$

The two sets of twist data satisfy

$$\operatorname{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}} \circ \operatorname{tw}_{Z,f,\eta} = \operatorname{tw}_{Z,f,\eta} \circ \operatorname{tw}_{\tilde{Z},\tilde{f},\tilde{\eta}} = \mathrm{id}. \quad (14)$$

Quaternionic moment map

In order to introduce twist data on QK manifolds, we first recall...

Definition (Galicki–Lawson)

The *quaternionic moment map* μ^Z of a quaternionic Kähler manifold (M, g, Q) wrt a Killing field Z is a section of Q satisfying for any vector field u

$$\nabla^g_u \mu^Z = -\sum_i g(J_i Z, u) J_i. \qquad (15)$$

This exists and is given by the explicit expression

$$\mu^Z = -\frac{2}{\nu} \mathrm{pr}_Q(\nabla^g Z) = \frac{1}{2\nu} \sum_{i,j,k} \epsilon_{ijk} \langle J_i, (\nabla^g_Z - \mathscr{L}_Z) J_j \rangle J_k. \qquad (16)$$

Let $b \in \mathbb{R}$ a constant. When μ^Z is nowhere vanishing, we can choose an oriented orthonormal frame (J_1, J_2, J_3) of Q such that $\mu^Z = \|\mu^Z\| J_1$ and $\langle J_2, \mathscr{L}_Z J_3 \rangle + b$ is nowhere vanishing.

In fact, it can be chosen to be any (nonzero) constant $a' := b - a \in \mathbb{R}$. This means that we have

$$\mathscr{L}_Z J_1 = 0, \quad \mathscr{L}_Z J_2 = aJ_3, \quad \mathscr{L}_Z J_3 = -aJ_2. \tag{17}$$

Quaternionic local twist data

Proposition (AS)

Let $\omega_{\mathrm{Q}}, \eta_{\mathrm{Q}}, f_{\mathrm{Q}}$ be given by

$$\begin{aligned} \omega_{\mathrm{Q}}(u,v) &= -\mathrm{d}\left(\frac{\iota_Z g}{\|\mu^Z\|}\right)(u,v) - \nu g(J_1 u, v) + \langle \nabla^g_u J_1, J_1 \circ \nabla^g_v J_1 \rangle, \\ \eta_{\mathrm{Q}}(u) &= -\frac{g(Z,u)}{\|\mu^Z\|} - \langle J_2, \nabla^g_u J_3 \rangle, \\ f_{\mathrm{Q}} &= -\frac{g(Z,Z)}{\|\mu^Z\|} - \nu \|\mu^Z\|. \end{aligned} \tag{18}$$

When $f_{\mathrm{Q}} + b$ is nowhere vanishing, $(U, Z, \omega_{\mathrm{Q}}, f_{\mathrm{Q}} + b, \eta_{\mathrm{Q}})$ constitutes local twist data on a QK manifold.

Note: $f_{\mathrm{Q}} + b - \eta_{\mathrm{Q}}(Z) = \langle J_2, \mathscr{L}_Z J_3 \rangle + b \neq 0$.

Twist realisation of QK/HK and one-loop deformation

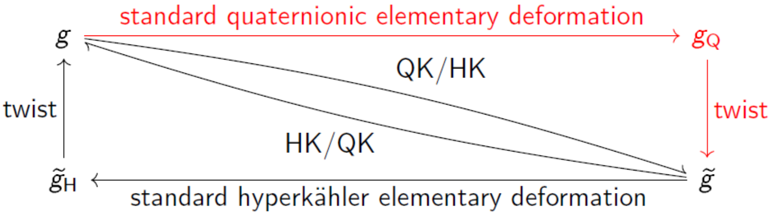
Theorem (AS)

The metric

$$\mathrm{tw}_{Z,f_{\mathrm{Q}}+b,\eta_{\mathrm{Q}}}\left(\frac{1}{\nu\|\mu^Z\|-b}\,g|_{\mathbb{H}_Q Z^\perp}-\frac{f_{\mathrm{Q}}+b}{(\nu\|\mu^Z\|-b)^2}\,g|_{\mathbb{H}_Q Z}\right) \quad (19)$$

is locally hyperkähler with rotating Killing field when $b=0$ and quaternionic Kähler otherwise. Moreover, these are up to an overall scaling the only combinations of elementary deformations by Killing fields Z and twists with respect to local twist data of the form (U,Z,ω,f,η) that produce other quaternionic Kähler metrics.

Comparison with Macia–Swann



$$\nu = -\frac{1}{8\tilde{K}}, \quad \mu^Z = \mathrm{tw}_{\tilde{Z},\tilde{\omega}_H,\tilde{f}_H}\left(\frac{\tilde{K}}{\tilde{k}\tilde{f}_1} I_1\right), \quad \|\mu^Z\| = \frac{\tilde{K}}{\tilde{k}\tilde{f}_1}, \quad f_Q = \frac{1}{\tilde{f}_H}$$

HK manifolds are easier to study than QK ones.

We can use the twist realisation of HK/QK to express symmetries and curvature on the QK side in terms of those on the HK side.

Basic idea: The twist map preserves algebraic relations but not differential ones. Knowing how differential relations are affected can give us control over Killing fields and curvature.

$$\mathrm{d} \circ \mathrm{tw}_{Z,f,\eta}(\alpha) = \mathrm{tw}_{Z,f,\eta}\left(\mathrm{d}\alpha - \frac{1}{f}\,\omega \wedge \iota_Z \alpha\right) + \frac{1}{f}\,\eta \wedge \mathscr{L}_Z \alpha. \quad (20)$$

Killing vector fields I

Proposition (AS)

Let (Z, ω, f) be twist data with dual twist data $(\tilde{Z}, \tilde{\omega}, \tilde{f})$. If S is a Z-invariant tensor field annihilated by a ω-Hamiltonian vector field v with Z-invariant Hamiltonian function f_v, then the twist of S is annihilated by the twist $\tilde{v}$ of

$$v - \frac{f_v + 1}{f} Z. \tag{21}$$

Furthermore, $\tilde{v}$ is Hamiltonian with respect to $\tilde{\omega}$ with $\tilde{Z}$-invariant Hamiltonian function

$$\tilde{f}_{\tilde{v}} := \frac{f_v + 1}{f} - 1. \tag{22}$$

Killing vector fields II

Theorem (Cortés–AS–Thung)

Let (M, g, Q) be a quaternionic Kähler manifold that is the HK/QK dual of a locally hyperkähler manifold $(\tilde{M}, \tilde{g}, H)$ with associated data as in Macia–Swann. If $\tilde{v}$ is a ϖ_1-Hamiltonian triholomorphic Killing field of $\tilde{g}$ that preserves $\tilde{f}_{\mathrm{H}}$. Then, $\tilde{v}$ admits a $\tilde{Z}$-invariant Hamiltonian function $\tilde{f}_{\tilde{v}}$ wrt $\tilde{\omega}_{\mathrm{H}}$, and the twist v of

$$\tilde{v} - \frac{f_{\tilde{v}} + 1}{\tilde{f}_{\mathrm{H}}} \tilde{Z} \tag{23}$$

with respect to the twist data $(\tilde{Z}, \tilde{\omega}_{\mathrm{H}}, \tilde{f}_{\mathrm{H}})$ is a Killing field of g.

Riemann curvature

Theorem (Cortés–AS–Thung)

Let (M, g, Q) be a quaternionic Kähler manifold that is the HK/QK dual of a locally hyperkähler metric $\tilde{g}$ with associated data as in Macia–Swann. Then its Riemann curvature $g \circ R^g$ is the twist of

$$\begin{aligned} &\frac{\tilde{K}}{\tilde{f}_1}\tilde{g} \circ R^{\tilde{g}} + \frac{1}{8\tilde{K}}\left(\tilde{g}_{\mathrm{H}} \owedge \tilde{g}_{\mathrm{H}} + \sum_{i=1}^{3}(\tilde{g}_{\mathrm{H}} \circ I_i) \varowedge (\tilde{g}_{\mathrm{H}} \circ I_i)\right) \\ &\quad - \frac{\tilde{K}}{8\tilde{k}}\frac{1}{\tilde{f}_1 \tilde{f}_{\mathrm{H}}}\left(\tilde{\omega}_{\mathrm{H}} \varowedge \tilde{\omega}_{\mathrm{H}} + \sum_{i=1}^{3}(\tilde{\omega}_{\mathrm{H}} \circ I_i) \owedge (\tilde{\omega}_{\mathrm{H}} \circ I_i)\right), \end{aligned} \tag{24}$$

with respect to the twist data $(\tilde{Z}, \tilde{\omega}_{\mathrm{H}}, \tilde{f}_{\mathrm{H}})$.

Alekseevsky decomposition

$\owedge$ and $\obar$ are projections of $(S^2(T^*M))^{\otimes 2}$ and $(\Lambda^2(T^*M))^{\otimes 2}$ respectively to the space of abstract curvature tensor fields.

In particular, we have a refinement of Alekseevsky decomposition for QK metrics that arise as HK/QK duals:

$$\overbrace{-\frac{1}{8\tilde{K}}}^{\nu}\overbrace{\mathrm{tw}\left(-\tilde{g}_{\mathrm{H}} \owedge \tilde{g}_{\mathrm{H}} - \sum_{i=1}^{3}(\tilde{g}_{\mathrm{H}} \circ I_i) \obar (\tilde{g}_{\mathrm{H}} \circ I_i)\right)}^{\text{curvature of } \mathbb{H}\mathbf{P}^n \ (\propto \mathfrak{sp}(1) \text{ part})}$$
$$\underbrace{+\,\mathrm{tw}\left(\frac{\tilde{K}}{\tilde{f}_1}\tilde{g} \circ R^{\tilde{g}} - \frac{\tilde{K}}{8\tilde{k}}\frac{1}{\tilde{f}_1\tilde{f}_{\mathrm{H}}}\left(\tilde{\omega}_{\mathrm{H}} \obar \tilde{\omega}_{\mathrm{H}} + \sum_{i=1}^{3}(\tilde{\omega}_{\mathrm{H}} \circ I_i) \owedge (\tilde{\omega}_{\mathrm{H}} \circ I_i)\right)\right)}_{\text{curvature of hyperkähler type } (\mathfrak{sp}(n) \text{ part})}$$

Example: Quadratic-prepotential Ferrara–Sabharwal

Consider $\mathrm{SU}(n,2)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))$ with coordinates
$\rho > 0, \quad \tau \in \mathbb{R}/2\pi\mathbb{Z}, \quad X_a, \zeta_\mu \in \mathbb{C}, \quad \sum_{b=1}^{n-1}|X_b|^2 < 1$, and metric

$$2\tilde{K}\left(\frac{\mathrm{d}\rho^2}{4\rho^2} + \left(\sum_{a=1}^{n-1}\frac{|\mathrm{d}X_a|^2}{1-\sum_{b=1}^{n-1}|X_b|^2} + \frac{\left|\sum_{a=1}^{n-1}X_a\mathrm{d}\overline{X}_a\right|^2}{\left(1-\sum_{b=1}^{n-1}|X_b|^2\right)^2}\right)\right.$$
$$+\frac{1}{2\rho}\left(-|\mathrm{d}\zeta_0|^2 + \sum_{a=1}^{n-1}|\mathrm{d}\zeta_a|^2\right) + \frac{1}{\rho}\frac{\left|\mathrm{d}\zeta_0 + \sum_{a=1}^{n-1}X_a\mathrm{d}\zeta_a\right|^2}{1-\sum_{b=1}^{n-1}|X_b|^2}$$
$$\left.+\frac{1}{4\rho^2}\left(\frac{\mathrm{d}\tau}{2\tilde{K}} + \mathrm{Im}\left(\zeta_0\mathrm{d}\overline{\zeta}_0 - \sum_{a=1}^{n-1}\zeta_a\mathrm{d}\overline{\zeta}_a\right)\right)^2\right).$$

Example: One-loop-deformed quadratic-prepotential FS

Performing the one-loop deformation by ∂_τ with $b = \nu/4c$ on the region $\rho > c$ and then redefining $\rho \mapsto \rho - c$ gives

$$2\tilde{K}\left(\frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}\,\mathrm{d}\rho^2 + \frac{\rho+c}{\rho}\left(\sum_{a=1}^{n-1}\frac{|\mathrm{d}X_a|^2}{1-\sum_{b=1}^{n-1}|X_b|^2} + \frac{\left|\sum_{a=1}^{n-1}X_a\mathrm{d}\overline{X}_a\right|^2}{\left(1-\sum_{b=1}^{n-1}|X_b|^2\right)^2}\right)\right.$$
$$+\frac{1}{2\rho}\left(-|\mathrm{d}\zeta_0|^2 + \sum_{a=1}^{n-1}|\mathrm{d}\zeta_a|^2\right) + \frac{\rho+c}{\rho^2}\frac{\left|\mathrm{d}\zeta_0 + \sum_{a=1}^{n-1}X_a\mathrm{d}\zeta_a\right|^2}{1-\sum_{b=1}^{n-1}|X_b|^2}$$
$$\left.+\frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}\left(\frac{\mathrm{d}\tau}{2\tilde{K}} - \sum_{a=1}^{n-1}\frac{2c\,\mathrm{Im}(X_a\mathrm{d}\overline{X}_a)}{1-\sum_{b=1}^{n-1}|X_b|^2} + \mathrm{Im}\left(\zeta_0\mathrm{d}\overline{\zeta}_0 - \sum_{a=1}^{n-1}\zeta_a\mathrm{d}\overline{\zeta}_a\right)\right)^2\right).$$

Example: 1ld qp FS as HK/QK dual of $T^*\mathbb{C}^n$ I

This is the HK/QK dual of an open subset of $T^*\mathbb{C}^n$ given by $|z_0|^2 - \sum_{a=1}^{n-1} |z_a|^2 > c$, with standard HK structure:

$$\tilde{g} = -(|\mathrm{d}z_0|^2 + |\mathrm{d}w_0|^2) + \sum_{a=1}^{n-1}(|\mathrm{d}z_a|^2 + |\mathrm{d}w_a|^2),$$

$$\varpi_1 = \frac{\mathrm{i}}{2}\left(-\mathrm{d}z_0 \wedge \mathrm{d}\overline{z}_0 - \mathrm{d}w_0 \wedge \mathrm{d}\overline{w}_0 + \sum_{a=1}^{n-1}(\mathrm{d}z_a \wedge \mathrm{d}\overline{z}_a + \mathrm{d}w_a \wedge \mathrm{d}\overline{w}_a)\right),$$

$$\varpi_2 = \frac{\mathrm{i}}{2}\left(\mathrm{d}z_0 \wedge \mathrm{d}w_0 - \mathrm{d}\overline{z}_0 \wedge \mathrm{d}\overline{w}_0 + \sum_{a=1}^{n-1}(\mathrm{d}z_a \wedge \mathrm{d}w_a - \mathrm{d}\overline{z}_a \wedge \mathrm{d}\overline{w}_a)\right),$$

$$\varpi_3 = \frac{1}{2}\left(\mathrm{d}z_0 \wedge \mathrm{d}w_0 + \mathrm{d}\overline{z}_0 \wedge \mathrm{d}\overline{w}_0 + \sum_{a=1}^{n-1}(\mathrm{d}z_a \wedge \mathrm{d}w_a + \mathrm{d}\overline{z}_a \wedge \mathrm{d}\overline{w}_a)\right).$$

Example: 1ld qp FS as HK/QK dual of $T^*\mathbb{C}^n$ II

Rotating Killing vector field:

$$\tilde{Z} = -\mathrm{i}\left(z_0\partial_{z_0} - \overline{z}_0\partial_{\overline{z}_0} + \sum_{a=1}^{n-1}(z_a\partial_{z_a} - \overline{z}_a\partial_{\overline{z}_a})\right).$$

Choice of ϖ_1-Hamiltonian:

$$\tilde{f}_1 = \frac{1}{2}\left(|z_0|^2 - \sum_{b=1}^{n-1}|z_b|^2 - c\right) > 0.$$

Change of coordinates:

$$z_0 = \sqrt{\frac{\rho + c}{1 - \sum_{b=1}^{n-1}|X_b|^2}}\, e^{-\mathrm{i}\tau}, \quad z_a = X_a\sqrt{\frac{\rho + c}{1 - \sum_{b=1}^{n-1}|X_b|^2}}\, e^{-\mathrm{i}\tau}, \quad w_\mu = \frac{\zeta_\mu}{\sqrt{2}}.$$

Example: Curvature norm of 1ld qp FS

The curvature of $T^*\mathbb{C}^n$ identically vanishes, so the curvature of 1ld qp FS is just the twist of

$$\frac{1}{8\tilde{K}}\left(\tilde{g}_{\mathrm{H}} \owedge \tilde{g}_{\mathrm{H}} + \sum_{i=1}^{3}(\tilde{g}_{\mathrm{H}} \circ I_i) \varobar (\tilde{g}_{\mathrm{H}} \circ I_i)\right) - \frac{\tilde{K}}{8\tilde{k}}\frac{1}{\tilde{f}_1\tilde{f}_{\mathrm{H}}}\left(\tilde{\omega}_{\mathrm{H}} \varobar \tilde{\omega}_{\mathrm{H}} + \sum_{i=1}^{3}(\tilde{\omega}_{\mathrm{H}} \circ I_i) \owedge (\tilde{\omega}_{\mathrm{H}} \circ I_i)\right).$$

Curvature norm of 1ld qp FS is just the curvature norm of the above. This gives us an injective function of ρ for $\rho, c > 0$:

$$\nu^2\left(n(5n+1) + 3\left(\frac{\rho^3}{(\rho+2c)^3} + \frac{(n-1)\rho}{(\rho+2c)}\right)^2 + 3\left(\frac{\rho^6}{(\rho+2c)^6} + \frac{(n-1)\rho^2}{(\rho+2c)^2}\right)\right).$$

Example: Killing fields of 1ld qp FS I

Start with the following Killing fields on the HK side:

$$\tilde{u}_a^+ = \mathrm{Re}(z_a\partial_{z_0} + \overline{z}_0\partial_{\overline{z}_a} - w_0\partial_{w_a} - \overline{w}_a\partial_{\overline{w}_o}), \quad \tilde{v}_0^+ = \mathrm{Re}(\partial_{w_0}), \quad \tilde{v}_a^+ = \mathrm{Re}(\partial_{w_a}),$$
$$\tilde{u}_a^- = \mathrm{Im}(z_a\partial_{z_0} + \overline{z}_0\partial_{\overline{z}_a} - w_0\partial_{w_a} - \overline{w}_a\partial_{\overline{w}_o}), \quad \tilde{v}_0^- = \mathrm{Im}(\partial_{w_0}), \quad \tilde{v}_a^- = \mathrm{Im}(\partial_{w_a}).$$

They are $\tilde{\omega}_{\mathrm{H}}$-Hamiltonian. Choose the Hamiltonians to be

$$\tilde{f}_{\tilde{u}_a^+} = \frac{\tilde{k}}{2}\mathrm{Re}(\mathrm{i}(\overline{w}_a w_0 - z_a\overline{z}_0)) - 1, \quad \tilde{f}_{\tilde{u}_a^-} = \frac{\tilde{k}}{2}\mathrm{Im}(\mathrm{i}(\overline{w}_a w_0 - z_a\overline{z}_0)) - 1,$$
$$\tilde{f}_{\tilde{v}_0^+} = \frac{\tilde{k}}{2}\mathrm{Re}(\mathrm{i}\overline{w}_0) - 1, \quad \tilde{f}_{\tilde{v}_a^+} = \frac{\tilde{k}}{2}\mathrm{Re}(-\mathrm{i}\overline{w}_a) - 1,$$
$$\tilde{f}_{\tilde{v}_0^-} = \frac{\tilde{k}}{2}\mathrm{Im}(\mathrm{i}\overline{w}_0) - 1, \quad \tilde{f}_{\tilde{v}_a^-} = \frac{\tilde{k}}{2}\mathrm{Im}(-\mathrm{i}\overline{w}_a) - 1.$$

Example: Killing fields of 1ld qp FS II

Following the prescription given earlier, we obtain the following Killing fields on the QK side:

$$u_a^+ = \mathrm{Re}\left(-\sum_{b=1}^{n-1} X_a X_b \partial_{X_b} + \partial_{\overline{X}_a} - \zeta_0 \partial_{\zeta_a} - \overline{\zeta}_a \partial_{\overline{\zeta}_0} + 2\mathrm{i}\tilde{K} c X^a \partial_\tau\right),$$

$$v_0^+ = \sqrt{2}\,\mathrm{Re}(\partial_{\zeta_0} + \mathrm{i}\tilde{K}\overline{\zeta}_0 \partial_\tau), \quad v_a^+ = \sqrt{2}\,\mathrm{Re}(\partial_{\zeta_a} - \mathrm{i}\tilde{K}\overline{\zeta}_a \partial_\tau),$$

$$u_a^- = \mathrm{Im}\left(-\sum_{b=1}^{n-1} X_a X_b \partial_{X_b} + \partial_{\overline{X}_a} - \zeta_0 \partial_{\zeta_a} - \overline{\zeta}_a \partial_{\overline{\zeta}_0} + 2\mathrm{i}\tilde{K} c X^a \partial_\tau\right),$$

$$v_0^- = \sqrt{2}\,\mathrm{Im}(\partial_{\zeta_0} + \mathrm{i}\tilde{K}\overline{\zeta}_0 \partial_\tau), \quad v_a^- = \sqrt{2}\,\mathrm{Im}(\partial_{\zeta_a} - \mathrm{i}\tilde{K}\overline{\zeta}_a \partial_\tau).$$

Along with ∂_τ, these act transitively on the constant ρ hypersurfaces.

Conclusion: 1ld qp FS metrics have cohomogeneity exactly 1.

Publications and preprints

- Vicente Cortés, Arpan Saha. "Quarter-pinched Einstein metrics interpolating between real and complex hyperbolic metrics". In: *Mathematische Zeitschrift* (May 2017). DOI: 10.1007/s00209-017-2013-x.
- Vicente Cortés, Arpan Saha, Danu Thung. "Symmetries of quaternionic Kähler manifolds with S^1-symmetry". 2020. arXiv: 2001.10026 [math.DG].
- Vicente Cortés, Arpan Saha, Danu Thung. "Curvature of quaternionic Kähler manifolds with S^1-symmetry". 2020. arXiv: 2001.10032 [math.DG].

Details: From Swann twists to local twists

Let $\tilde{\pi} : P \to \tilde{M}$ denote the quotient of P by $\hat{Z} + fX_P$. The twist of a Z-invariant 1-form α on M is a 1-form α' on $\tilde{M}$ such that the pullbacks $\pi^*\alpha$ and $\tilde{\pi}^*\alpha'$ on P agree on $\hat{\eta}$-horizontal vector fields. So if $\hat{u}$ is the horizontal lift of u, we have

$$\begin{aligned} \tilde{\pi}^*\alpha'(\hat{u}) &= \pi^*\alpha(\hat{u}) = \alpha(u), \\ \tilde{\pi}^*\alpha'(X_P) &= \frac{1}{f}\,\tilde{\pi}^*\alpha'(\hat{Z} + fX_P) - \frac{1}{f}\,\tilde{\pi}^*\alpha'(\hat{Z}) = -\frac{1}{f}\,\alpha(Z). \end{aligned} \tag{25}$$

Let $\theta = \hat{\eta} - \pi^*\eta$. This is a closed 1-form defining integral submanifolds in P. We identify one such submanifold U_P with U.

Details: From Swann twists to local twists

The tangent vectors to U_P are elements of the kernel of θ. We may check

$$\theta(\hat{u} + \eta(u)X_P) = \eta(u)\hat{\eta}(X_P) - \pi^*\eta(\hat{u}) = 0. \tag{26}$$

The vector fields $\hat{u} + \eta(u)X_P$ are thus tangent to U_P and may be identified with u on U.

To obtain the local twist of α, we need to pull back $\tilde{\pi}^*\alpha'$ on P to U_P along the inclusion. We now see that

$$\tilde{\pi}^*\alpha'(\hat{u} + \eta(u)X_P) = \alpha(u) - \frac{1}{f}\,\alpha(Z)\eta(u) = (\mathrm{tw}_{Z,f,\eta}(\alpha))(u). \tag{27}$$

Details: η-independence

We use Moser's trick. On a contractible open set in U introduce interpolating 1-forms

$$\eta_t = (1-t)\eta_0 + t\eta_1 =: \eta_0 + t\,\mathrm{d}h, \tag{28}$$

let $\alpha'_t = \mathrm{tw}_{Z,f,\eta_t}(\alpha)$ and $\tilde{Z}_t = -\frac{1}{f}\mathrm{tw}_{Z,f,\eta_t}(Z)$. When α is Z-invariant, it satisfies

$$\mathscr{L}_{h\tilde{Z}_t}\alpha'_t = \frac{\mathrm{d}\alpha'_t}{\mathrm{d}t}. \tag{29}$$

The diffeomorphism ϕ_1 s.t. $\phi_1^*\alpha'_0 = \alpha'_1$ is a solution to the following differential equation for $\phi_t : V_0 \to V_t$:

$$\phi_0 = \mathrm{id}_{V_0}, \quad \left.\frac{\mathrm{d}\phi_s}{\mathrm{d}s}\right|_{s=t}(\phi_t^{-1}(p)) = h(p)\tilde{Z}_{t,p}. \tag{30}$$

Details: Gluing local twists

Philosophy: $M \to B \leftarrow \tilde{M}$ instead of $M \leftarrow P \to \tilde{M}$.

Let $\pi_B : M \to B$ denote the quotient by the Z-action and let $\tau \in \mathbb{R}/\mathbb{Z}$ be the fibre coordinate so that

$$\omega =: \mathrm{d}f \wedge \mathrm{d}\tau + \pi_B^* \omega^B. \tag{31}$$

ω^B is closed. Because $Z = \partial_\tau$ is Hamiltonian, ω is integral iff ω^B is integral. Choose open sets $U_\Lambda = \pi_B^{-1}(U_\Lambda^B)$ with U_Λ^B contractible and

$$\eta_\Lambda =: (f + 1)\,\mathrm{d}\tau + \pi_B^* \eta_\Lambda^B. \tag{32}$$

Note that in this case $\tilde{Z} = Z = \partial_\tau$.

Details: Gluing local twists

Let $h_{\Lambda\Sigma}$ be functions defined on $U^B_\Lambda \cap U^B_\Sigma$ (and hence by pullback on $U_\Lambda \cap U_\Sigma$) such that

$$\eta^B_\Lambda - \eta^B_\Sigma = \mathrm{d}h_{\Lambda\Sigma}. \tag{33}$$

Then for α a Z-invariant form on M, we have on $U_\Lambda \cap U_\Sigma$

$$\mathrm{tw}_{Z,f,\eta_\Lambda}(\alpha) = \phi^*_{\Lambda\Sigma}\mathrm{tw}_{Z,f,\eta_\Sigma}(\alpha), \tag{34}$$

where the diffeomorphism $\phi_{\Lambda\Sigma}$ sends (τ, p^B) to $(\tau + h_{\Lambda\Sigma}(p^B), p^B)$ for all $p^B \in U^B_\Lambda \cap U^B_\Sigma$. Integrality of ω^B implies that we can always choose $h_{\Sigma\Lambda}$ so that they satisfy the cocycle condition on the triple overlap $U^B_\Lambda \cap U^B_\Sigma \cap U^B_\Pi$:

$$h_{\Sigma\Pi} - h_{\Lambda\Pi} + h_{\Lambda\Sigma} \equiv 0 \pmod 1. \tag{35}$$

Details: $P \subset M \times \tilde{M}$ is a submanifold

Lemma. If $(p, \tilde{p}) \in P$ and h_0 is a Z-invariant function on M that is identically zero in some open nbhd $V \ni p$, then $T(h_0)$ is identically zero in some open nbhd $\tilde{V} \ni \tilde{p}$ that is independent of what h_0 is outside V.

Proof. Assume wlog that V is Z-invariant. Then $\exists$ a Z-invariant function h_1 s.t. $h_1(p) \neq 0$ and $\mathrm{supp}(h_1) \subset V$. Thus $h_0 h_1$ is identically zero on M, and so $T(h_0)T(h_1) = T(h_0 h_1)$ is identically zero on $\tilde{M}$. But $T(h_1)(\tilde{p}) = h_1(p) \neq 0$ by hypothesis. Since $T(h_1)$ is continuous, $\exists$ a nbhd $\tilde{V} \ni \tilde{p}$ on which $T(h_1)$ is nowhere vanishing. But since $T(h_0)T(h_1)$ is identically zero on $\tilde{V}$, so must be $T(h_0)$.

Details: $P \subset M \times \tilde{M}$ is a submanifold

Corollary. If $(p, \tilde{p}) \in P$ and h, h' are Z-invariant functions on M that coincide on some open nbhd $V \ni p$, then $T(h), T(h')$ coincide on some open nbhd $\tilde{V} \ni \tilde{p}$ independent of h, h'.

Let $N' \subset M$ be a submfld of dim $n-1$ that contains p and is transversal to Z so that every Z-flowline intersects it at most once. Let N be an open set of N' containing p. Let $h_a|_N$ be $n-1$ coordinate functions on N. These can be extended to $n-1$ functions $h_a|_{N'}$ with compact support on N'. Then we define the value of the function h_a at any other point $p' \in M$ to be equal to the value of the function $h_a|_{N'}$ at p'' if the Z-flowline through p' intersects N' in some point p'' and equal to zero if the Z-flowline through p' does not intersect N'. This extends the coordinate functions $h_a|_N$ on N to $n-1$ Z-invariant functions h_a on M.

Details: $P \subset M \times \tilde{M}$ is a submanifold

Wlog, take (h_a) to yield a diffeomorphism $\phi : N \to \mathbb{R}^{n-1}$. Given any Z-invariant function h on M, this yields a map $F_h : \mathbb{R}^{n-1} \to \mathbb{R}$, given by $F_h = h|_N \circ \phi^{-1}$.

Construct a new Z-invariant function $h' = F_h(h_a) := F_h(h_1, h_2, \ldots)$ on M that restricts to h on N. If we let V_N be the (Z-invariant) open nbhd of p in M given by the union of all the Z-translates of N, h and h' agree on V_N. By Corollary, $T(h)$, $T(h')$ agree on some open nbhd $\tilde{V}_N$ that is independent of h, h'.

Local twist $\Rightarrow$ $T(F_h(h_a)) = F_h(T(h_a))$. So if a point $(p', \tilde{p}') \in V_N \times \tilde{V}_N$ satisfies $h_a(p') = T(h_a)(\tilde{p}')$ for all the h_a, then it satisfies $h'(p') = T(h')$. Thus for $(p', \tilde{p}') \in V_N \times \tilde{V}_N$, we have $h_a(p') = T(h_a)(\tilde{p}')$ iff $h(p') = T(h)(\tilde{p}')$ for all h.

Details: $P \subset M \times \tilde{M}$ is a submanifold

Upshot: Given $(p', \tilde{p}') \in V_N \times \tilde{V}_N$, we have $h_a(p') = T(h_a)(\tilde{p}')$ iff $(p', \tilde{p}') \in P$.

To check P is a submanifold of $M \times \tilde{M}$, it is enough to check that its intersection with an open nbhd of an arbitrary $(p, \tilde{p}) \in P$ in $M \times \tilde{M}$ is a submanifold.

Define a map $V_N \times \tilde{V}_N \to \mathbb{R}^{n-1}$ sending $(p', \tilde{p}')$ to $(h_a(p') - T(h_a)(\tilde{p}'))$. Its differential at $(p, \tilde{p})$ is surjective as h_a are coordinate functions on N. By the implicit function theorem, its vanishing set i.e. $P \cap (V_N \times \tilde{V}_N)$ is a submanifold of codimension $n - 1$.

Details: $\hat{\eta}$ on $P \subset M \times \tilde{M}$

Tangent vector fields on P are $C^\infty(P)$-linear combinations of vector fields of the form $u \oplus (T(u) + a\tilde{Z})$ where a is a constant. Let the evaluation of $\hat{\eta}$ on these be a. The $\hat{\eta}$-horizontal lift of u on M is then $u \oplus T(u)$. So if $\mathscr{L}_u v = w$

$$\begin{aligned} \mathscr{L}_{\hat{u}} \hat{v} &= \mathscr{L}_u v \oplus \mathscr{L}_{T(u)} \circ T(v) = w \oplus (T(w) - \omega(u,v)\tilde{Z}) \\ &= \hat{w} - 0 \oplus \omega(u,v)\tilde{Z}, \\ \mathscr{L}_{0 \oplus \tilde{Z}} \hat{v} &= \mathscr{L}_{\tilde{Z}} \circ T(v) = 0. \end{aligned} \tag{36}$$

Thus, we have

$$\begin{aligned} \mathrm{d}\hat{\eta}(\hat{u}, \hat{v}) &= \hat{u}(\hat{\eta}(\hat{v})) - \hat{v}(\hat{\eta}(\hat{u})) - \hat{\eta}(\mathscr{L}_{\hat{u}}\hat{v}) = \omega(u,v), \\ \mathrm{d}\hat{\eta}(0 \oplus \tilde{Z}, \hat{v}) &= (0 \oplus \tilde{Z})(\hat{\eta}(\hat{v})) - \hat{v}(\hat{\eta}(0 \oplus \tilde{Z})) - \hat{\eta}(\mathscr{L}_{0 \oplus \tilde{Z}}\hat{v}) = 0. \end{aligned} \tag{37}$$

It follows that $\mathrm{d}\hat{\eta} = \pi^*\omega$.

Details: QK/HK via the twist

Let the tuples of twist data (U, Z, ω, f, η) and $(U, \tilde{Z}, \tilde{\omega}, \tilde{f}, \tilde{\eta})$ be dual. Then any tensor field S satisfies

$$\mathscr{L}_{\tilde{Z}} \circ \mathrm{tw}_{Z,f,\eta}(S) = -\frac{1}{f - \eta(Z)} \mathrm{tw}_{Z,f,\eta}(\mathscr{L}_Z S). \tag{38}$$

In particular, if we choose a local oriented orthonormal frame $(J_1 = \mu^Z / \|\mu^Z\|, J_2, J_3)$ of Q such that

$$\mathscr{L}_Z J_2 = J_3, \quad \mathscr{L}_Z J_3 = -J_2$$

then their twists $I_i = \mathrm{tw}_{Z,f_Q,\eta_Q}(J_i)$ satisfy

$$\mathscr{L}_{\tilde{Z}} I_1 = 0, \quad \mathscr{L}_{\tilde{Z}} I_2 = I_3, \quad \mathscr{L}_{\tilde{Z}} I_3 = -I_2.$$

Details: QK/HK via the twist

We need to show I_1, I_2, I_3 are Kähler. By a lemma of Hitchin, it is enough to show that $\varpi_i = \tilde{g} \circ I_i$ are all closed. Let

$$g_{\mathrm{Q}} = K \left(\frac{\nu}{\|\mu^Z\|} \, g|_{\mathbb{H}_Q Z^\perp} - \frac{f_{\mathrm{Q}}}{\|\mu^Z\|^2} \, g|_{\mathbb{H}_Q Z} \right) \tag{39}$$

and $\sigma_i = g_{\mathrm{Q}} \circ J_i$. Then $\varpi_i = \mathrm{tw}_{Z, f_{\mathrm{Q}}, \eta_{\mathrm{Q}}}$ and we find after some computation

$$\varpi_i = \frac{K}{2} \sum_{j,k} \epsilon_{ijk} \mathrm{d} \left(\frac{\langle J_j, \nabla^g J_k \rangle}{\|\mu^Z\|} \right). \tag{40}$$

Moreover $\tilde{Z}$ is ϖ_1-Hamiltonian:

$$\iota_{\tilde{Z}} \varpi_1 = -\mathrm{d} \left(-\frac{K}{\|\mu^Z\|} \right). \tag{41}$$

Details: Normalisations

Note that

$$g_{\mathsf{Q}} = K\nu^2 \left(\frac{1}{(\nu\|\mu^Z\| + 0)}\, g|_{\mathbb{H}_Q Z^\perp} - \frac{f_{\mathsf{Q}}}{(\nu\|\mu^Z\| + 0)^2}\, g|_{\mathbb{H}_Q Z} \right). \quad (42)$$

With this choice of normalisation, Macia–Swann and our result are inverses when

$$\tilde{f}_1 = T\left(-\frac{K}{\|\mu^Z\|}\right), \quad \tilde{K} = -\frac{1}{\nu}, \quad \tilde{k} = \frac{1}{K\nu}. \quad (43)$$

Details: Uniqueness of one-loop deformation

The QK manifold (M, g, Q) is twist of an elementary deformation of a locally HK manifold $(\tilde{M}, \tilde{g}, H)$ by some rotating Killing field $\tilde{Z}$ given by QK/HK.

As we can compose twists, (M', g', Q') would then be the twist of some elementary deformation of $(\tilde{M}, \tilde{g}, H)$ by $\tilde{Z}$.

Macia–Swann then implies that this is necessarily a one-loop deformation of (M, g, Q) by Z.

Details: One-loop deformation

Consider the elementary deformation

$$g_{\mathrm{L}} = \frac{1}{1-4c\|\mu^Z\|}\, g|_{\mathbb{H}_Q Z^\perp} + \frac{1+\frac{4c}{\nu}\, f_{\mathrm{Q}}}{(1-4c\|\mu^Z\|)^2}\, g|_{\mathbb{H}_Q Z}. \tag{44}$$

Then the one-loop deformation is given by $g^c = \mathrm{tw}_{Z, f_{\mathrm{Q}}+\frac{\nu}{4c}, \eta_{\mathrm{Q}}}(g_{\mathrm{L}})$ for a suitable choice of η_{Q}. Moreover, the associated data is

$$\begin{aligned} Z^c &= \frac{1}{1+\frac{4c}{\nu}\, f_{\mathrm{Q}}}\, \mathrm{tw}_{Z, f_{\mathrm{Q}}+\frac{\nu}{4c}, \eta_{\mathrm{Q}}}(Z), \quad \left\|\mu^{Z^c}\right\| = \frac{\|\mu^Z\|}{1-4c\|\mu^Z\|}, \\ \omega^c_{\mathrm{Q}} &= \frac{1}{1+\frac{4c}{\nu}\, f_{\mathrm{Q}}}\, \mathrm{tw}_{Z, f_{\mathrm{Q}}+\frac{\nu}{4c}, \eta_{\mathrm{Q}}}(\omega_{\mathrm{Q}}), \quad f^c_{\mathrm{Q}} = \frac{f_{\mathrm{Q}}}{1+\frac{4c}{\nu}\, f_{\mathrm{Q}}}. \end{aligned} \tag{45}$$

The dual to $(Z, \omega_{\mathrm{Q}}, f_{\mathrm{Q}} + \frac{\nu}{4c})$ is $\left(-\frac{4c}{\nu}\, Z^c, \frac{4c}{\nu}\, \omega^c_{\mathrm{Q}}, \frac{4c}{\nu}(1-\frac{4c}{\nu}\, f^c_{\mathrm{Q}})\right)$.

Details: Abstract curvature tensor fields

A $(0,4)$-tensor field C is said to be an abstract curvature tensor field if it satisfies for all vector fields s, t, u, v the following equations:

$$\begin{aligned} C(s,t,u,v) &= -C(t,s,u,v) = -C(s,t,v,u), \\ C(s,t,u,v) &+ C(t,u,s,v) + C(u,s,t,v) = 0. \end{aligned} \tag{46}$$

Given a quaternionic bundle Q, such a tensor field is additionally said to be of hyperkähler type if

$$C(s,t,u,v) = C(s,t,Ju,Jv) \tag{47}$$

for any section $J \in \Gamma(Q)$ such that $J^2 = -\mathrm{id}$.

Details: Kulkarni–Nomizu and Riemann products

The Kulkarni–Nomizu product $\owedge$ of two $(0,2)$-tensor fields g, h is given by

$$
\begin{aligned}
(g \owedge h)(s,t,u,v) = g(s,u)h(t,v) - g(s,v)h(t,u) \\
- g(t,u)h(s,v) + g(t,v)h(s,u),
\end{aligned}
\tag{48}
$$

The Riemann product $\obar$ of two 2-forms α, β is given by

$$
\alpha \obar \beta = \alpha \owedge \beta + 2\,\alpha \otimes \beta + 2\,\beta \otimes \alpha. \tag{49}
$$

For any two symmetric bilinear forms g, h and any two 2-forms α, β, the tensor fields $g \owedge h$ and $\alpha \obar \beta$ are abstract curvature tensor fields.

Details: The curvature formula

The Levi-Civita connections ∇^g and $\nabla^{\tilde{g}}$ are related via

$$\nabla^g_{\tilde{T}(u)} \circ \tilde{T}(v) = \tilde{T}\left(\nabla^{\tilde{g}}_u v + S^{\mathrm{HQ}}_u v\right),$$

where u and v are $\tilde{Z}$-invariant vector fields on $\tilde{M}$, $\tilde{T}$ is the global twist map realising HK/QK and S^{HQ} is given by

$$S^{\mathrm{HQ}}_u v = \frac{1}{2}\sum_{\alpha=0}^{3}\left(\frac{1}{\tilde{f}_{\mathrm{H}}}\tilde{\omega}_{\mathrm{H}}(I_\alpha u, v) I_\alpha \tilde{Z} - \frac{1}{\tilde{f}_1}\left(\lambda_\alpha(u) I_\alpha \circ I_1 v + \lambda_\alpha(v) I_\alpha I_1 u\right)\right),$$

with $I_0 = \mathrm{id}_{TM}$ and $\lambda_\alpha = \tilde{g}(I_\alpha \tilde{Z}, \cdot)$. The curvature formula then follows from

$$R^g = \tilde{T}\left(R^{\nabla^{\tilde{g}} + S^{\mathrm{HQ}}} - \frac{1}{\tilde{f}_{\mathrm{H}}}\tilde{\omega}_{\mathrm{H}} \otimes \left(\nabla^{\tilde{g}}\tilde{Z} + S^{\mathrm{HQ}}_{\tilde{Z}}\right)\right).$$

Details: The curvature formula

It is clear that $\tilde{g}_{\mathrm{H}}$ is symmetric while ω_{H} and $\tilde{g}_{\mathrm{H}} \circ I_i$ are antisymmetric. So $\tilde{g}_{\mathrm{H}} \owedge \tilde{g}_{\mathrm{H}}$, $\tilde{\omega}_{\mathrm{H}} \varobar \tilde{\omega}_{\mathrm{H}}$, $(\tilde{g}_{\mathrm{H}} \circ I_i) \varobar (\tilde{g}_{\mathrm{H}} \circ I_i)$ are abstract curvature tensor fields.

Consider the endomorphism field $I_{\mathrm{H}} = I_1 + 2\nabla^{\tilde{g}}\tilde{Z}$. The Killing equation for $\tilde{Z}$ along with the fact that $\tilde{Z}$ is rotating can be used to show that I_{H} is skew-symmetric and commutes with all the I_i. Then it follows that

$$\begin{aligned} \tilde{\omega}_{\mathrm{H}}(I_i u, v) &= \tilde{g}(I_{\mathrm{H}} I_i u, v) = -\tilde{g}(I_i u, I_{\mathrm{H}} v) = \tilde{g}(u, I_i \circ I_{\mathrm{H}} v) \\ &= \tilde{g}(u, I_{\mathrm{H}} \circ I_i v) = \tilde{g}(I_{\mathrm{H}} \circ I_i v, u) = \tilde{\omega}_{\mathrm{H}}(I_i v, u). \end{aligned} \tag{50}$$

So $(\tilde{\omega}_{\mathrm{H}} \circ I_i) \owedge (\tilde{\omega}_{\mathrm{H}} \circ I_i)$ is also an abstract curvature tensor field.

Details: $\mathbb{C}\mathbf{H}^2$

Let $\mathbb{C}^{1,2}$ denote $\mathbb{C}^3$ endowed with the standard Hermitian inner product of signature $(1,2)$ (minus signs first). Then, $\mathbb{C}\mathbf{H}^2$ is the space of complex lines of negative norm. The line spanned by (z_0, z_1, z_2) is assigned the coordinates

$$(\rho, \tau, \zeta) = \left(-\frac{\|z\|^2}{|z_0 + z_1|^2}, -\mathrm{Im}\left(\frac{z_0 - z_1}{z_0 + z_1}\right), \frac{\sqrt{2}\, z_2}{z_0 + z_1}\right). \qquad (51)$$

This is a diffeomorphism from $\mathbb{C}\mathbf{H}^2$ to $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{C}$ with inverse

$$(\rho, \tau, \zeta) \mapsto \mathbb{C}^\times \cdot \left(1, \frac{1 - (\rho + |\zeta|^2 - \mathrm{i}\tau)}{1 - (\rho - |\zeta|^2 - \mathrm{i}\tau)}, \frac{\sqrt{2}\, \zeta}{1 - (\rho - |\zeta|^2 - \mathrm{i}\tau)}\right). \qquad (52)$$

Details: $\mathbb{C}\mathbf{H}^2$

The noncompact Fubini–Study metric in the horospherical coordinates becomes

$$\frac{1}{4\rho^2}\left(\mathrm{d}\rho^2 + 2\rho|\mathrm{d}\zeta|^2 + \left(\mathrm{d}\tau + \mathrm{Im}(\zeta\,\mathrm{d}\bar{\zeta})\right)^2\right).$$

Its one-loop deformation is

$$g^c = \frac{1}{4\rho^2}\left(\frac{\rho+2c}{\rho+c}\,\mathrm{d}\rho^2 + 2(\rho+2c)|\mathrm{d}\zeta|^2 + \frac{\rho+c}{\rho+2c}\left(\mathrm{d}\tau + \mathrm{Im}(\zeta\,\mathrm{d}\bar{\zeta})\right)^2\right).$$

For $c > 0$, this is defined for all $\rho > 0$. For $c < 0$, one has to restrict the domain.

Details: $\mathbb{C}\mathbf{H}^2$

For $|c| < \rho < 2|c|$, we in fact get a metric of positive scalar curvature! Redefining $\rho \mapsto \rho + |c|$ and getting rid of an overall minus, we get

$$\frac{1}{4(\rho+|c|)^2}\left(\frac{|c|-\rho}{\rho}\,\mathrm{d}\rho^2 + 2(|c|-\rho)|\mathrm{d}\zeta|^2 + \frac{\rho}{|c|-\rho}\left(\mathrm{d}\tau + \mathrm{Im}(\zeta\,\mathrm{d}\bar{\zeta})\right)^2\right).$$

Setting $|c| = 1/b$, redefining $\tau \mapsto \tau/b$ and getting rid of the overall constant factor of b, we get

$$\frac{1}{4(b\rho+1)^2}\left(\frac{1-b\rho}{\rho}\,\mathrm{d}\rho^2 + 2(1-b\rho)|\mathrm{d}\zeta|^2 + \frac{\rho}{1-b\rho}\left(\mathrm{d}\tau + b^2\,\mathrm{Im}(\zeta\,\mathrm{d}\bar{\zeta})\right)^2\right).$$

In the $b \to 0$ limit we get the flat metric as the coordinate change $\rho = r^2$ shows.

Supplements: Przanowski–Tod Ansatz

Any QK metric of dimension 4 with $\nu \neq 0$ and at least one Killing field ∂_τ locally admits the Przanowski–Tod Ansatz:

$$g = \frac{1}{4\rho^2}\left(P\,\mathrm{d}\rho^2 + 2Pe^u|\mathrm{d}\zeta|^2 + \frac{1}{P}\left(\mathrm{d}\tau + \Theta\right)^2\right), \tag{53}$$

where P and u are ∂_τ-invariant smooth functions, and Θ is a ∂_τ-invariant 1-form satisfying

$$\begin{aligned} &\partial_\zeta\partial_{\bar\zeta}u = -\frac{1}{2}\,\partial_\rho^2(e^u), \quad P = \frac{2}{\nu}\left(\rho\,\partial_\rho u - 2\right) > 0, \\ &\mathrm{d}\Theta = \mathrm{i}\left(\left(\partial_\zeta P\,\mathrm{d}\zeta - \partial_{\bar\zeta}P\,\mathrm{d}\bar\zeta\right)\wedge \mathrm{d}\rho - \partial_\rho(Pe^u)\,\mathrm{d}\zeta\wedge\mathrm{d}\bar\zeta\right). \end{aligned} \tag{54}$$

Supplements: Przanowski–Tod Ansatz

Its QK/HK dual wrt Killing field ∂_τ is the Boyer–Finley Ansatz:

$$2K(\partial_\rho u)(\mathrm{d}\rho^2 + 2e^u|\mathrm{d}\zeta|^2) + \frac{8K}{\partial_\rho u}\left(\mathrm{d}\tau - \frac{1}{2}\left(\partial_y u\,\mathrm{d}x - \partial_x u\,\mathrm{d}y\right)\right)^2 \quad (55)$$

Meanwhile one-loop deformation followed by a reparametrisation amounts to replacing u by its pullback u' under the diffeomorphism $(\rho, \zeta, \tau) \mapsto (\rho + c, \zeta, \tau)$. This too satisfies the continuous Toda equation:

$$\partial_\zeta \partial_{\bar{\zeta}} u' = -\frac{1}{2}\,\partial_\rho^2(e^{u'}). \quad (56)$$

Supplements: Przanowski–Tod Ansatz

The following is a solution of the continuous Toda equation:

$$e^u = \frac{a\rho^2 + b\rho + c}{(1 + \frac{a}{2}|\zeta|^2)^2}. \tag{57}$$

Substituting this into the Przanowski–Tod Ansatz gives the metric

$$\begin{aligned} g^{a,b,c} = -\frac{1}{2\nu\rho^2} \Bigg(& \frac{b\rho + 2c}{a\rho^2 + b\rho + c}\, \mathrm{d}\rho^2 + \frac{2(b\rho + 2c)|\mathrm{d}\zeta|^2}{(1 + \frac{a}{2}|\zeta|^2)^2} \\ & + \frac{a\rho^2 + b\rho + c}{b\rho + 2c} \left(-\frac{\nu}{2}\, \mathrm{d}\tau + \frac{b\, \mathrm{Im}(\zeta\, \mathrm{d}\bar{\zeta})}{1 + \frac{a}{2}|\zeta|^2} \right)^2 \Bigg). \end{aligned} \tag{58}$$

Supplements: Przanowski–Tod Ansatz

This generically is of cohomogeneity 1. It is at most 1, because we have the transitive action of the following isometries on the constant ρ hypersurfaces:

$$(\rho, \tau, \zeta) \mapsto \left(\rho, \tau + \tau' + \frac{4b}{\nu a} \operatorname{Im}\left(\ln\left(\sqrt{\frac{a}{2}}\, v\zeta + \overline{w}\right)\right), \frac{w\zeta - \sqrt{\frac{a}{2}}\,\overline{v}}{\sqrt{\frac{a}{2}}\, v\zeta + \overline{w}}\right),$$

where $v, w \in \mathbb{C}$ and satisfy $|v|^2 + |w|^2 = 1$. Geometrically, we can interpret ζ as the stereographic coordinate on a Riemann sphere with (v, w) parametrising rotations of the sphere.

Supplements: Przanowski–Tod Ansatz

Meanwhile, its curvature norm is given by

$$\mathrm{tr}(\mathscr{R}^2) = 6\nu^2 \left(1 + b^2((b^2 - 2ac)^2 + 4a^2c^2)\left(\frac{\rho}{\rho + 2c}\right)^6\right).$$

This is an injective function of $\rho > 0$ whenever

$$b^2((b^2 - 2ac)^2 + 4a^2c^2) \neq 0, \quad c \neq 0.$$

Supplements: Przanowski–Tod Ansatz

When $a, b > 0$, we can carry out the following change of coordinates:

$$\rho = \frac{b}{2a}\left(\frac{1}{\varrho^2} - 1\right), \quad \zeta = \sqrt{\frac{2}{a}}\,\xi, \quad \tau = \frac{2b}{a}\,\theta. \tag{59}$$

Note that this is invertible when $0 < \varrho < 1$, with the inverse coordinate transformation given by

$$\varrho = \sqrt{\frac{b}{2a\rho + b}}, \quad \xi = \sqrt{\frac{a}{2}}\,\zeta, \quad \theta = \frac{a}{2b}\,\tau. \tag{60}$$

Supplements: Przanowski–Tod Ansatz

The metric $g^{a,b,c}$ in these coordinates becomes

$$\frac{-1}{\nu(1-\varrho^2)^2}\left(\frac{1+k\varrho^2}{1+k\varrho^4}\,\mathrm{d}\varrho^2+\varrho^2(1+k\varrho^2)(\varsigma_1^2+\varsigma_2^2)+\frac{\varrho^2(1+k\varrho^4)}{1+k\varrho^2}\varsigma_3^2\right),$$

where k is given by

$$k=\frac{4ac}{b^2}-1,$$

and $\varsigma_1,\varsigma_2,\varsigma_3$ are SU(2)-invariant 1-forms on $\mathbf{S}^3$ wrt the Hopf parametrisation:

$$\varsigma_1=\frac{\mathrm{Re}\left(e^{\mathrm{i}\theta}\mathrm{d}\xi\right)}{1+|\xi|^2},\quad \varsigma_2=\frac{\mathrm{Im}\left(e^{\mathrm{i}\theta}\mathrm{d}\xi\right)}{1+|\xi|^2},\quad \varsigma_3=\frac{1}{2}\,\mathrm{d}\theta+\frac{\mathrm{Im}(\xi\,\mathrm{d}\bar{\xi})}{1+|\xi|^2}.$$

Supplements: Przanowski–Tod Ansatz

When $b < 0 < a$ and $2a\rho + b < 0$, the same change of coordinates works but we get $\varrho > 1$ as can be seen from

$$\frac{b}{2a\rho + b} = \frac{|b|}{|b| - 2a\rho} > 1.$$

In particular, for $k = 0$, we get the real hyperbolic 4-space metric:

$$\frac{-1}{\nu(1 - \varrho^2)^2} \left(\mathrm{d}\varrho^2 + \varsigma_1^2 + \varsigma_2^2 + \varsigma_3^2 \right).$$

Supplements: Przanowski–Tod Ansatz

When $b < 0 < a$ and $2a\rho + b > 0$, we instead set

$$\rho = -\frac{b}{2a}\left(\frac{1}{\varrho^2} + 1\right), \quad \zeta = \sqrt{\frac{2}{a}}\,\xi, \quad \tau = \frac{2b}{a}\,\theta. \tag{61}$$

The inverse of this change of coordinates is

$$\varrho = \sqrt{\frac{-b}{2a\rho + b}}, \quad \xi = \sqrt{\frac{a}{2}}\,\zeta, \quad \theta = \frac{a}{2b}\,\tau. \tag{62}$$

The metric now becomes

$$\frac{1}{\nu(1+\varrho^2)^2}\left(\frac{1-k\varrho^2}{1+k\varrho^4}\,\mathrm{d}\varrho^2 + \varrho^2(1-k\varrho^2)(\varsigma_1^2+\varsigma_2^2) + \frac{\varrho^2(1+k\varrho^4)}{1-k\varrho^2}\varsigma_3^2\right),$$

Note that $k = 0$ gives the standard metric on the 4-sphere.

Supplements: Przanowski–Tod Ansatz

The vanishing of k amounts to the vanishing of the discriminant $b^2 - 4ac$ of the quadratic polynomial $a\rho^2 + b\rho + c$. This means that it can be written as $a(\rho + C)^2$.

Under the (reparametrised) one-loop deformation by ∂_τ, ρ is replaced by $\rho + c'$ for some constant c'. Since the discriminant (and hence k) is still zero, it follows that the 4-sphere and hyperbolic 4-plane are fixed points of the one-loop deformation.

Supplements: Przanowski–Tod Ansatz

Meanwhile setting $k = -1$ (which corresponds to $c = 0$) gives

$$\frac{1}{\nu(1+\varrho^2)^2}\left(\frac{\mathrm{d}\varrho^2}{1-\varrho^2} + \varrho^2(1+\varrho^2)(\varsigma_1^2+\varsigma_2^2) + \varrho^2(1-\varrho^2)\varsigma_3^2\right).$$

This is the Fubini–Study metric on $\mathbb{C}\mathbf{P}^2$ (with $\varrho^{-2} = 2\|z\|^{-2} + 1$).

The vanishing of c means that $a\rho^2 + b\rho + c$ takes the form $a\rho(\rho + C)$. Carrying out a one-loop deformation amounts to replacing this by $a(\rho + c')(\rho + c' + C)$ so that

$$k = \frac{4c'(C+c')}{(C+2c')^2} - 1.$$

In the $c' \to \infty$ limit, this becomes zero, and we get the $\mathbf{S}^4$ metric.

Supplements: Przanowski–Tod Ansatz

To study the $k \to \infty$ limit, set $k = 1/\ell$ and carry out a coordinate redefinition $(\xi, \bar{\xi}) \mapsto (\sqrt{\ell}\,\xi, \sqrt{\ell}\,\bar{\xi})$ to get

$$\frac{1}{\nu(1+\varrho^2)^2}\left(\frac{\ell+\varrho^2}{\ell+\varrho^4}\,\mathrm{d}\varrho^2 + \varrho^2(\ell+\varrho^2)(\varsigma_1'^2+\varsigma_2'^2) + \frac{\varrho^2(\ell+\varrho^4)}{\ell+\varrho^2}\varsigma_3'^2\right),$$

where $\varsigma_1', \varsigma_2', \varsigma_3'$ are given by

$$\varsigma_1' = \frac{\mathrm{Re}\left(e^{\mathrm{i}\theta}\mathrm{d}\xi\right)}{1+\ell|\xi|^2}, \quad \varsigma_2' = \frac{\mathrm{Im}\left(e^{\mathrm{i}\theta}\mathrm{d}\xi\right)}{1+\ell|\xi|^2}, \quad \varsigma_3' = \frac{1}{2}\,\mathrm{d}\theta + \ell\,\frac{\mathrm{Im}(\xi\,\mathrm{d}\bar{\xi})}{1+\ell|\xi|^2}.$$

The $\ell \to 0$ limit is now well-defined. Carrying out a change of coordinates $\varrho^2 = 2r$ gives us

$$\frac{1}{\nu(1+\varrho^2)^2}\left(\frac{\mathrm{d}\varrho^2}{\varrho^2} + \varrho^4|\mathrm{d}\xi|^2 + \frac{\varrho^4}{4}\,\mathrm{d}\theta^2\right) = \frac{4r^2}{\nu(2r+1)^2}\left(\mathrm{d}r^2 + |\mathrm{d}\xi|^2 + \frac{1}{4}\,\mathrm{d}\theta^2\right).$$

Supplements: One-loop flows

Naïve one-loop flow:

$$\frac{\mathrm{d}g^c}{\mathrm{d}c} = -\frac{8}{\nu}\,\eta^c_{\mathrm{Q}}\,\iota_{Z^c} g^c + 4\left\|\mu^{Z^c}\right\|\,g^c - \frac{4}{\nu}\frac{g^c(Z^c, Z^c)}{\|\mu^{Z^c}\|}\,g^c|_{\mathbb{H}_{Q^c}Z^c},$$
$$\frac{\mathrm{d}Z^c}{\mathrm{d}c} = -\frac{4}{\nu}\,(f^c_{\mathrm{Q}} - \eta^c_{\mathrm{Q}}(Z^c))Z^c,$$
$$\frac{\mathrm{d}Q^c}{\mathrm{d}c} = -\frac{4}{\nu}\,[Q^c, \eta^c_{\mathrm{Q}} \otimes Z^c], \quad \frac{\mathrm{d}\eta^c_{\mathrm{Q}}}{\mathrm{d}c} = -\frac{4}{\nu}\,f^c_{\mathrm{Q}}\,\eta^c_{\mathrm{Q}}.$$

Undeformed metric: g^0.

Supplements: One-loop flows

Reparametrised one-loop flow:

$$\frac{\mathrm{d}g^{[c]}}{\mathrm{d}c} = \mathscr{L}_{W^{[c]}} g^{[c]} - \frac{8}{\nu}\, \eta_{\mathrm{Q}}^{[c]}\, \iota_{Z^{[c]}} g^{[c]} + 4 \left\| \mu^{Z^{[c]}} \right\| g^{[c]} - \frac{4}{\nu} \frac{g^{[c]}(Z^{[c]}, Z^{[c]})}{\left\| \mu^{Z^{[c]}} \right\|}\, g^{[c]}|_{\mathbb{H}_{Q^{[c]}} Z^{[c]}},$$

$$\frac{\mathrm{d}Z^{[c]}}{\mathrm{d}c} = \mathscr{L}_{W^{[c]}} Z^{[c]} - \frac{4}{\nu} \left(f_{\mathrm{Q}}^{[c]} - \eta_{\mathrm{Q}}^{[c]}(Z^{[c]}) \right) Z^{[c]},$$

$$\frac{\mathrm{d}Q^{[c]}}{\mathrm{d}c} = \mathscr{L}_{W^{[c]}} Q^{[c]} - \frac{4}{\nu} \left[Q^{[c]}, \eta_{\mathrm{Q}}^{[c]} \otimes Z^{[c]} \right],$$

$$\frac{\mathrm{d}\eta_{\mathrm{Q}}^{[c]}}{\mathrm{d}c} = \mathscr{L}_{W^{[c]}} \eta_{\mathrm{Q}}^{[c]} - \frac{4}{\nu}\, f_{\mathrm{Q}}^{[c]}\, \eta_{\mathrm{Q}}^{[c]}.$$

where $W^{[c]} = 4 \left\| \mu^{Z^{[c]}} \right\| \mu^{Z^{[c]}} Z^{[c]} / g^{[c]}(Z^{[c]}, Z^{[c]})$.

Supplements: One-loop flows

Rescaled one-loop flow:

$$\frac{\mathrm{d}}{\mathrm{d}t} g^{(t)} = -\frac{2\ell}{L\nu^{(t)}} \eta_{\mathrm{Q}}^{(t)} \iota_{Z^{(t)}} g^{(t)} + \left(\frac{\ell}{L} \|\mu^{Z^{(t)}}\| + \ell \right) g^{(t)}$$
$$- \frac{\ell}{L\nu^{(t)}} \frac{g^{(t)}(Z^{(t)}, Z^{(t)})}{\|\mu^{Z^{(t)}}\|} \; g^{(t)}\Big|_{\mathbb{H}_{Q^{(t)}} Z^{(t)}},$$
$$\frac{\mathrm{d}}{\mathrm{d}t} Z^{(t)} = -\frac{\ell}{L\nu^{(t)}} \left(f_{\mathrm{Q}}^{(t)} - \eta_{\mathrm{Q}}^{(t)}(Z^{(t)}) \right) Z^{(t)},$$
$$\frac{\mathrm{d}}{\mathrm{d}t} Q^{(t)} = -\frac{\ell}{L\nu^{(t)}} \left[Q^{(t)}, \eta_{\mathrm{Q}}^{(t)} \otimes Z^{(t)} \right],$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \eta_{\mathrm{Q}}^{(t)} = -\frac{\ell}{L\nu^{(t)}} f_{\mathrm{Q}}^{(t)} \, \eta_{\mathrm{Q}}^{(t)},$$

Undeformed metric: $\lim_{t\to-\infty} e^{-\ell t} g^{(t)}$.

Supplements: Completeness under one-loop deformation?

Consider the elementary deformation

$$g_{\mathsf{L}} = \frac{1}{1 - 4c\|\mu^Z\|}\, g|_{\mathbb{H}_Q Z^{\perp}} + \frac{1 + \frac{4c}{\nu} f_{\mathsf{Q}}}{(1 - 4c\|\mu^Z\|)^2}\, g|_{\mathbb{H}_Q Z}.$$

Then the one-loop deformation is given by $g^c = \mathrm{tw}_{Z, f_{\mathsf{Q}} + \frac{\nu}{4c}, \eta_{\mathsf{Q}}}(g_{\mathsf{L}})$ for a suitable choice of η_{Q}. Consider also the endomorphism fields

$$M_{\mathsf{L}} = \sqrt{\frac{1}{1 - 4c\|\mu^Z\|}}\, \mathrm{pr}_{\mathbb{H}_Q Z^{\perp}} + \sqrt{\frac{1 + \frac{4c}{\nu} f_{\mathsf{Q}}}{(1 - 4c\|\mu^Z\|)^2}}\, \mathrm{pr}_{\mathbb{H}_Q Z}, \quad M_{\mathrm{tw}} = \mathrm{tw}^{-1}_{Z, f_{\mathsf{Q}} + \frac{\nu}{4c}, \eta_{\mathsf{Q}}}$$

and let $M = M_{\mathsf{L}} M_{\mathrm{tw}}$. Then

$$g^c(u, v) = g^c(Mu, Mv) = g^c(u, M^{\dagger g} M v).$$

Supplements: Completeness under one-loop deformation?

Let the pointwise minimum eigenvalue of the self-adjoint endomorphism field $M^{\dagger_g}M$ be bounded below by a constant $C \geq 0$. Then for all vectors u

$$\frac{g^c(u,u)}{g(u,u)} \geq C.$$

For C positive, we have $g^c(u,u) \geq Cg(u,u)$ for some $C > 0$. Then, completeness of $g(u,u) \Rightarrow$ completeness of $g^c(u,u)$.

For reparametrised one-loop deformation, a similar argument would work with an extra factor:

$$\frac{g^{[c]}(u,u)}{g(u,u)} = \phi_c^*\left(\frac{g^c(u,u)}{g(u,u)}\right)\frac{\phi_c^*(g(u,u))}{g(u,u)}$$

Supplements: Completeness under one-loop deformation?

In the case of the positive c one-loop deformation of $\mathbb{C}\mathbf{H}^2$

$$\frac{1}{4\rho^2}\left(\frac{\rho+2c}{\rho+c}\,\mathrm{d}\rho^2+2(\rho+2c)|\mathrm{d}\zeta|^2+\frac{\rho+c}{\rho+2c}\left(\mathrm{d}\tau+\mathrm{Im}(\zeta\,\mathrm{d}\bar{\zeta})\right)^2\right),$$

the minimum eigenvalue is realised by $Z=\partial_\tau$ and we have the following inequality:

$$\frac{g^{[c]}(u,u)}{g(u,u)}\geq\frac{g^{[c]}(Z,Z)}{g(Z,Z)}=\frac{\rho+c}{\rho+2c}>\frac{1}{2}.$$

Guess: Z works in general. So we try to show

$$g^{[c]}(u,u)\geq\frac{g^{[c]}(Z,Z)}{g(Z,Z)}\,g(u,u).$$

This is work in progress.