Vector Fields on Supermanifolds

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Notation:	$(X, \mathcal{O}_X)$ supermanifold of dimension $m\|n$ with $\mathcal{O}_{m\|n}(U)$ superalgebra of superfunctions on $U \subset X$ **Sheaf structure:** $\mathcal{O}_X : U \to \mathcal{O}_X(U)$ sheaf of superalgebras on X $(U \in \mathcal{T}_X)$
Example:	**Classical manifold**, $n = 0$ M Manifold of dimension m $\mathcal{C}^\infty(M) = \mathcal{O}_{m\|0}(M) = \{f : M \to \mathbb{R} \| f \text{ smooth}\}$ $\mathcal{C}^\infty : U \to \mathcal{C}^\infty(U)$ Sheaf of $\mathbb{R}$-algebras on M $(U \in \mathcal{T}_M)$

1 Super Derivations

Definition 1. *Let $A = A_0 \oplus A_1$ be a supercommutative superalgebra over a field $\mathbb{K}$. A map $\delta \in End_{\mathbb{K}}(A)$ is called a superderivation of A if every homogeneous component η_i of δ with $i \in \{0, 1\}$ satisfies the graded Leibniz rule (p denotes the parity function):*

$$\eta_i(fg) = \eta_i(f)g + (-1)^{p(f)p(\eta_i)} f\eta_i(g) \ \forall \text{ homogeneous } f, g \in A$$

Der (A)**: $\mathbb{K}$-vector space of superderivations on** A

- Der (A) becomes an A**-left supermodule** with the following multiplication:

$$(f \cdot \delta)(g) := f \cdot \delta(g) \ \forall f \in A, \delta \in \text{Der } (A)$$

- Der (A) together with the supercommutator forms a **Lie-Superalgebra**.
  We will see later that this algebra is supercommutative.

Special case:
$A = \mathcal{O}_X(U), \text{Der } (\mathcal{O}_X(U)) := \text{Der } U$

Der $\mathcal{O}_X : U \to \text{Der } U$ becomes a sheaf of supermodules over $\mathcal{O}_X$ together with the restriction morphisms

$$\rho'_{V,U} : \text{Der } U \to \text{Der } V$$
$$\left(\rho'_{V,U}(\delta)\right)(\rho_{V,U}(f)) := \rho_{V,U}(\delta(f))$$

There are now two sheaves on the supermanifold $(X, \mathcal{O}_X)$:

Superfunctions on $(\mathcal{O}_X, X)$:	$\mathcal{O}_X : U \to \mathcal{O}_X(U)$	sheaf of superalgebras
Supervectorfields on $(\mathcal{O}_X, X)$:	Der $\mathcal{O}_X : U \to \text{Der } U$	sheaf of supermodules

Lemma 1. *[1, Lemma 1.5.2] Superderivations are local operations:*
Let $g \in \mathcal{O}_X(U)$, $\delta \in \text{Der }(U)$ and $V \subset U$ open. Then

$$\rho_{V,U}(g) = 0 \Rightarrow \rho_{V,U}(\delta(g)) = 0$$

Classical manifold

- Consider $A = \mathcal{C}^\infty(M)$
- Space of vector fields over M: Der $(\mathcal{C}^\infty(M)) = \mathfrak{X}(M)$
  Sheaf of vector fields on M $(U \subseteq M)$: Der $\mathcal{C}^\infty : U \to$ Der $\mathcal{C}^\infty(U)$
- vector field on M:

$$X \in \text{Der }(\mathcal{C}^\infty(M))$$
$$X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$
$$f \to X(f)$$

- Consider $(M, \text{Der } \mathcal{C}^\infty)$ as an $\mathbb{R}$-ringed space. The tangent vector on M in the point $p \in M$ is the evaluation map of the sheaf Der $\mathcal{C}^\infty$ in p:

$$v_p : (\text{Der } \mathcal{C}^\infty)_p \to \mathbb{R}$$

2 Super Vector Fields

Vector Fields on a Classical manifold

$(x_1, ...x_m)$ coordinates on $U \subset M$.

- coordinate vector fields: $\dfrac{\partial}{\partial x_i}$ with the property $\dfrac{\partial}{\partial x_j} x_i = \delta_{ij}$
- $\dfrac{\partial}{\partial x_i}$ is a basis of Der $(\mathcal{C}^\infty(M))|_U$ as a module over $\mathcal{C}^\infty(U)$ (local frame of Der $(\mathcal{C}^\infty(M))|_U$)
- All deritvations $X \in$ Der $(\mathcal{C}^\infty(M))$ can locally be written uniquely as

$$X|_U = \sum_{i=1}^{m} X(x_i) \frac{\partial}{\partial x_i}$$

Super vector fields on $(\mathbb{R}^m, \mathcal{O}_{m|n})$:

Coordinates on $(\mathbb{R}^m, \mathcal{O}_{m|n}) : (\xi_1, ...\xi_{m+n}) = (x_1, ...x_m, \theta_1, ...\theta_n)$

- For even coordinates x_i we define even supervectorfields $\frac{\partial}{\partial x_i}$ by

$$\frac{\partial}{\partial x_i}(\sum_\epsilon f_\epsilon \theta_1^{\epsilon_1} \cdots \theta_n^{\epsilon_n}) := \sum_\epsilon \frac{\partial f_\epsilon}{\partial x_i} \theta_1^{\epsilon_1} \cdots \theta_n^{\epsilon_n}$$

  They are called **even coordinate fields**.
- For odd coordinates θ_j we define odd supervectorfields $\frac{\partial}{\partial \theta_j}$ by

$$\frac{\partial}{\partial \theta_j}(\sum_\epsilon f_\epsilon \theta_1^{\epsilon_1} \cdots \theta_n^{\epsilon_n}) := \sum_\epsilon \cdot \epsilon_j \cdot (-1)^{\epsilon_1 + ... + \epsilon_{j-1}} \theta_1^{\epsilon_1} \cdots \widehat{\theta_j^{\epsilon_j}} \cdots \theta_n^{\epsilon_n}$$

  They are called **odd coordinate fields**.

- The coordinate fields satisfy $\dfrac{\partial}{\partial \xi_j}\xi_i = \delta_{ij}$
  In detail: $\dfrac{\partial}{\partial x_j}x_i = \delta_{ij}, \dfrac{\partial}{\partial \theta_j}x_i = 0, \dfrac{\partial}{\partial x_j}\theta_i = 0, \dfrac{\partial}{\partial \theta_j}\theta_i = \delta_{ij}$
- The supercommutator of any two coordinate vector fields vanishes, i.e. the Lie algebra of supervectorfields is supercommutative. In particular:

$$\left(\frac{\partial}{\partial \theta_j}\right)^2 = 0 \ \forall j = 1, ...n$$

Lemma 2. *[1, Lemma 1.5.6]*
Let V be an open subset of $\mathbb{R}^m$.
Der $\mathcal{O}_{m|n}(V)$ *is a free $\mathcal{O}_{m|n}(V)$-supermodule with adapted basis*

$$\left\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}, \frac{\partial}{\partial \theta_1}, ... \frac{\partial}{\partial \theta_n}\right\}$$

3 Differential Calculus

Differential of a classical manifold

Let M and N be classical manifolds with local coordinates $(x_1, ...x_m)$ and $(y_1, ...y_n)$, respectively. Let $\phi : M \to N$ be a smooth map. The differential of ϕ in the point $p \in M$ acts on the coordinate vector fields as follows:

$$\begin{aligned} d\phi_p\left(\frac{\partial}{\partial x_i}\right) &= \sum_{j=1}^{n} \frac{\partial \phi_j}{\partial x_i}(p)\frac{\partial}{\partial y_j}\bigg|_{\phi(p)} \\ &= \underbrace{J(\phi)_p}_{\text{Jacobi matrix in } p\in M} \frac{\partial}{\partial y_j}\bigg|_{\phi(p)} \end{aligned} \tag{1}$$

In Matrix notation:

$$\begin{pmatrix} d\phi_p\left(\frac{\partial}{\partial x_1}\right) \\ \vdots \\ d\phi_p\left(\frac{\partial}{\partial x_m}\right) \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial \phi_1}{\partial x_1}(p) & & \\ & \ddots & \\ & & \frac{\partial \phi_n}{\partial x_m}(p) \end{pmatrix}}_{\text{transpose of Jacobi matrix } J(\phi,\Psi)^{st}} \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix}$$

Supermanifold

Let $(V, \mathcal{O}_{m|n}|_V)$ and $(W, \mathcal{O}_{r|s}|_W)$ be superdomains with coordinates $\{\xi_1, ...\xi_{m+n}\} = \{x_1, ...x_m, \theta_1, ...\theta_n\}$ and $\{\eta_1, ...\eta_{r+s}\} = \{y_1, ...y_r, \tau_1, ...\tau_s\}$, respectively. We define the morphism

$$(\phi, \Psi) : (V, \mathcal{O}_{m|n}|_V) \to (W, \mathcal{O}_{r|s}|_W)$$

with

$$\begin{aligned} \phi : V &\to W \\ \Psi_W : \mathcal{O}_{r|s}(W) &\to \mathcal{O}_{m|n}(V) \\ \Psi_k &:= \Psi_W(\eta_k) \in \mathcal{O}_{m|n}(V) \ \forall k = 1, ...r+s \end{aligned}$$

Lemma 3. *[1, Lemma 1.6.1]*
The following equation holds:

$$\frac{\partial}{\partial \xi_i} \circ \Psi_W = \sum_{k=1}^{r+s} \frac{\partial \Psi_k}{\partial \xi_i} \cdot \left(\Psi_W \circ \frac{\partial}{\partial \eta_k} \right) : \mathcal{O}_{r|s}(W) \to \mathcal{O}_{m|n}(V)$$

In Matrix notation:

$$\begin{pmatrix} \frac{\partial}{\partial \xi_1} \Psi_W(f) \\ \vdots \\ \frac{\partial}{\partial \xi_{m+n}} \Psi_W(f) \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial \Psi_1}{\partial \xi_1} & & \\ & \ddots & \\ & & \frac{\partial \Psi_{r+s}}{\partial \xi_{m+n}} \end{pmatrix}}_{\text{supertranspose of the Jacobi matrix } J(\phi,\Psi)^{st}} \begin{pmatrix} \Psi_W \left(\frac{\partial f}{\partial \eta_1} \right) \\ \vdots \\ \Psi_W \left(\frac{\partial f}{\partial \eta_{r+s}} \right) \end{pmatrix}$$

Definition 2. *Jacobi matrix of the morphism* (ϕ, Ψ)*:*

$$J(\phi, \Psi) = \begin{pmatrix} \frac{\partial \Psi_1}{\partial \xi_1} & & \\ & \ddots & \\ & & \frac{\partial \Psi_{r+s}}{\partial \xi_{m+n}} \end{pmatrix}^{st}$$

4 Chain rule

Classical manifold

Consider the manifolds M,N and P and the following maps:

$$\phi : M \to N$$
$$\Psi : N \to P$$

Chain rule:

$$d(\Psi \circ \phi)_p = d\Psi_{\phi(p)} \circ d\phi_p$$

With equation 1, the chain rule reads

$$d(\Psi \circ \phi)_p \left(\frac{\partial}{\partial x_i} \right) = J(\Psi \circ \phi)_p \frac{\partial}{\partial y_j} = J(\Psi)_{\phi(p)} \circ J(\phi)_p \frac{\partial}{\partial y_j}$$
$$\Leftrightarrow J(\Psi \circ \phi)_p = J(\Psi)_{\phi(p)} \circ J(\phi)_p$$

Supermanifold

Proposition 1. *[1, Proposition 1.6.4]*
Let $(V, \mathcal{O}_{m|n}(V))$*,* $(W, \mathcal{O}_{r|s}(W))$ *and* $(Q, \mathcal{O}_{p|q}(Q))$ *be superdomains and*

$$(\phi_1, \Psi_1) : (V, \mathcal{O}_{m|n}(V)) \to (W, \mathcal{O}_{r|s}(W))$$
$$(\phi_2, \Psi_2) : (W, \mathcal{O}_{r|s}(W)) \to (Q, \mathcal{O}_{p|q}(Q))$$

morphisms. Then the following equation holds:

$$J((\phi_2, \Psi_2) \circ (\phi_1, \Psi_1)) = \Psi_{1,W}(J(\phi_2, \Psi_2)) \cdot J(\phi_1, \Psi_1)$$

References

[1] Christian Bär. *Nichtkommutative Geometrie.* Universität Hamburg, 2005.

[2] Vincente Cortés. *Differentialgeometrie.* Universität Hamburg, 2009.