Supersymmetric Mechanics of Point Particles

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1 Classical Mechanics

Problem:
Given two points (spacetime or in any general manifold representing some states of the system), find the "trajectory" of the system in the configuration space.
solution: **Functional Integrals**

Definition 1 (ActionFunctional) *Given a manifold M, two points in it $p_0, p_1 \in M$, two real parameters $t_0 < t_1$ and a smooth function : $L : TM \to M$. We define the **action functional** by:*

$$S : \Omega = \{c\epsilon C^{\infty}([t_0, t_1], M) | c(t_0) = p_0, c(t_1) = p_1\} \to \mathbb{R}$$
$$c(t) \mapsto \int_{t_0}^{t_1} L(\dot{c}(t))\, dt$$

We want to find the minimum of this action. We parametrize the possible trajectories by a parameter $s \in \mathbb{R}$. For some curve $c(t) = (x_1(t), \ldots, x_m(t))$ we have $\dot{c}(t) = (x_1(t), \ldots, x_m(t), \dot{(x)}_1(t), \ldots, \dot{(x)}_m(t))$. We define:

$$v_j(t) := \frac{\partial}{\partial s} x_j(s,t)\Big|_{s=0}$$

It is obvious we look for curves with the property $\frac{d}{ds} S(c_s)\Big|_{s=0} = 0$. We calculate:

$$\begin{aligned}
\frac{d}{ds} S(c_s)\Big|_{s=0} &= \int_{t_0}^{t_1} \frac{\partial}{\partial s} L(x_1(s,t), \ldots, x_m(s,t), \dot{x}_1(s,t), \ldots, \dot{x}_m(s,t))\, dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L(x(t), \dot{x}(t))}{\partial x_i} \frac{\partial x_i}{\partial s} + \frac{\partial L(x(t), \dot{x}(t))}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial s} \right) dt \Big|_{s=0} \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L(x_1, \ldots, x_m, \dot{x}_1, \ldots, \dot{x}_m)}{\partial x_i} \frac{\partial x_i}{\partial s} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial s} \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i} \right) \frac{\partial x_i}{\partial s} \right) dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} \right) v_i(t)\, dt
\end{aligned}$$

By the fundamental lemma of calculus of variations we arrive at the well known Euler-Lagrange equations:

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i}(x(t), \dot{x}(t)) = \frac{\partial L}{\partial x_i}(x(t), \dot{x}(t)),\ i = 1, \ldots, m}$$

Example: *Geodesics*

In a n-dim manifold we take:

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$$

The Euler-Lagrange equations coincide with the geodesic equations:

$$\frac{\nabla}{dt} \dot{x}^i = 0 \,, \; i = 1, \ldots, n$$

Theorem 1 (Noether's Theorem) *If a system has a continuous symmetry, then there exists a conserved quantity which is called* ***Noether's Charge****.*

In our simple case:

$$Q_i = \frac{\partial L}{\partial \dot{x}^j} X_i^j - \left(\frac{\partial L}{\partial \dot{x}^j} \dot{x}^j - L \right) T_i \tag{1}$$

With $\Delta x^j = X_i^j \epsilon^i$ and $\Delta t = T_i \epsilon^i$. And ϵ^i are some real small parameters.
In a realistic case we take some transformation check if it leaves invariant the action, and then try to find the correspodint charge. The next example shows the way we can do it:

<u>**Example**</u>: *Time Translations*

So we take $t' \mapsto t + \alpha$ and $\delta x = \dot{x}\alpha$, where $\alpha > 0$ is some small parameter. For simplicity we take $L = \frac{m}{2}\dot{x}^2 - V(x)$. The E-L equation read $m\ddot{x} = -\dfrac{\partial V}{\partial x}$. We try so see if the action remains unchanged:

$$\delta S = \int_{t_0}^{t_1} \left(m\dot{x}\delta\dot{x} - \frac{\partial V}{\partial x}\delta x \right) dt = \int_{t_0}^{t_1} \left(m\alpha\dot{x}\ddot{x} - \frac{\partial v}{\partial x}\dot{x} \right) dt = \int_{t_0}^{t_1} \frac{d}{dt} \left(\frac{m}{2}\alpha\dot{x}^2 - \alpha V \right) dt$$

And since we ask for all physical quantities to vanish at the boundaries we get $\delta S = 0$, so the above transformation is a symmetry. To find the conserved charge we consider the parameter as a function of time and try to bring the variation to the form $\delta S = \int_{t_0}^{t_1} \dot{\alpha} Q \, dt$ where Q will be our conserved charge.

$$\begin{aligned} \delta S = \int_{t_0}^{t_1} \left(m\dot{x} \left(\dot{\alpha}\dot{x} + \alpha\ddot{x} \right) - \frac{\partial V}{\partial x}\alpha\dot{x} \right) dt &= \int_{t_0}^{t_1} \left(m\dot{x}^2\dot{\alpha} - \alpha \frac{d}{dt} \left(\frac{m}{2}\dot{x}^2 - V \right) \right) dt \\ &= \int_{t_0}^{t_1} \dot{\alpha} \left(m\dot{x}^2 - \frac{m}{2}\dot{x}^2 + V \right) dt \end{aligned}$$

And so the conserved charge is $Q = \frac{m}{2}\dot{x}^2 + V = H$. So the energy is conserved

2 Classical Mechanics on Supermanifolds

We would like now to do the same procedure but in the case of a Supermanifold. Every new structure will come from the odd part. So lets say we do not have any even variables and $L \in \mathcal{O}_{0|2n}(0)$, so it is a polynomial in the variables $\theta_1, \ldots, \theta_n, \eta_1, \ldots, \eta_n$. So the Lagrangian here is defined as:

$$L : \underbrace{\left(\Lambda^* \mathbb{R}^n\right)_1 \times \cdots \times \left(\Lambda^* \mathbb{R}^n\right)_1}_{2n} \to \Lambda \mathbb{R}^n \tag{2}$$

We define by $\Omega_1 := \left\{ c = (c_1, \ldots, c_n) \in C^\infty \left([t_0, t_1], \left(\Lambda^* \mathbb{R}^n\right)_1 \times \cdots \times \left(\Lambda^* \mathbb{R}^n\right)_1 \right) \middle| c(t_0) = p_0, c(t_1) = p_1 \right\}$.
The action is defined as:

$$S : \Omega_1 \to \Lambda^* \mathbb{R}^n S(c) = \int_{t_0}^{t_1} L(c_1(t), \ldots, c_n(t), \dot{c}_1(t), \ldots, \dot{c}_n(t)) \, dt \tag{3}$$

By following the previous arguments we get:

$$\begin{aligned}\frac{d}{ds}S(c_s)\Big|_{s=0} &= \int_{t_0}^{t_1}\frac{\partial}{\partial s}L\left(\theta_1(s,t),\ldots,\theta_n(s,t),\eta_1(s,t),\ldots,\eta_n(s,t)\right)\Big|_{s=0} dt\\ &= \int_{t_0}^{t_1}\left(\frac{\partial L\left(\theta(t),\eta(t)\right)}{\partial\theta_j}\frac{d\theta_j}{ds}\Big|_{s=0}+\frac{\partial L\left(\theta(t),\eta(t)\right)}{\partial\eta_j}\frac{d\eta_j}{ds}\Big|_{s=0}\right)dt\end{aligned}$$

but $\dot{\theta}^j=\eta^j$. So:

$$\begin{aligned}\frac{d}{ds}S(c_s)\Big|_{s=0} &= \int_{t_0}^{t_1}\left(\frac{\partial L\left(\theta(t),\eta(t)\right)}{\partial\theta_j}\frac{d\theta_j}{ds}\Big|_{s=0}+\frac{d}{dt}\left(\frac{\partial L(\theta(t),\eta(t))}{\partial\eta_i}\frac{d\theta_j}{ds}\Big|_{s=0}\right)-\frac{d}{dt}\left(\frac{\partial L(\theta(t),\eta(t))}{\partial\eta_i}\right)\frac{d\theta_j}{ds}\Big|_{s=0}\right)dt\\ &= \int_{t_0}^{t_1}\left(\frac{\partial L(\theta(t),\eta(t))}{\partial\theta_i}-\frac{d}{dt}\left(\frac{\partial L(\theta(t),\eta(t))}{\partial\eta_i(t)}\right)\right)\frac{d\theta_i}{ds}\Big|_{s=0} dt\end{aligned}$$

So once again we get :

$$\boxed{\frac{\partial L(\theta(t),\eta(t))}{\partial\theta_i}=\frac{d}{dt}\left(\frac{\partial L(\theta(t),\eta(t))}{\partial\eta_i(t)}\right)}$$

Remark: *Noether's Charge*

Of course the Noether's theorem apply also to the case of Supermanifolds. Let us see an example:

Example: *Supercharge*

We take the most trivial supersymmetric Lagrangian $L=\frac{1}{2}\dot{x}^2+\frac{\imath}{2}\theta\dot{\theta}$. So we have only one even and one odd variable in a flat space. We consider the transformations $\delta(\epsilon)x=-i\epsilon\theta$ and $\delta(\epsilon)\psi=\epsilon\dot{x}$ where ϵ is some small odd parameter. To have $\delta S=0$ We get (following the same procedure as for the time translations) that there is a conserved quantity:

$$Q=i\dot{x}\theta$$

3 Pseudoclassical Mechanincs

We consider the Lagrangian: $L:=\frac{1}{2}\sum_{i=1}^{3}\theta_i\eta_i+b_1\theta_2\theta_3+b_2\theta_3\theta_1+b_3\theta_1\theta_2$. Where the quantities b_i are just real parameters.

We calculate:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\eta_j}\right)=\frac{1}{2}\sum_{i=1}^{3}\frac{d}{dt}\left(-\frac{\partial\eta_i}{\partial\eta_j}\theta_i\right)=-\frac{1}{2}\sum_{i=1}^{3}\frac{d}{dt}\left(\delta_{ij}\theta_i\right)=-\frac{1}{2}\dot{\theta}_j$$

The same way we find:

$$\frac{\partial L}{\partial\theta_i}=\frac{1}{2}\dot{\theta}_i-\sum_{ij}\epsilon_{ijk}b_j\theta_k$$

So combining the previous:

$$\dot{\theta}_i=\sum_{jk}\epsilon_{ijk}b_j\theta_k$$

Something weird happened $\Rightarrow$ ***spin***. Classicaly there is no analogue of spin. The appearence here is credited to the use of the (nonstandard) grassman variables.We consider a general Lagrangian in a supermanifold $(TM, \mathcal{O}_{TM})$ where $\mathcal{O}_{TM}(U) = C^\infty(U) \otimes \Lambda^* \mathbb{R}^{2n}$. The equations of motion are a combined system of the ones we found so far:

$$\frac{d}{dt}\frac{\partial L}{\partial y_i} = \frac{\partial L}{\partial x_i}\ ,\ i = 1, \dots, m \tag{4}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \eta_i} = \frac{\partial L}{\partial \theta_i}\ ,\ i = 1, \dots, n \tag{5}$$

Example We take $M = \mathbb{R}^3$. $\vec{B} = (b_1(x), b_2(x), b_3(x))$. Consider:

$$L(x, y, \theta, \eta) = \frac{m}{2}\langle y, y\rangle - V(x) + \frac{i}{2}\sum_{k=1}^{3}\theta_k\eta_k + i\,(b_1(x)\theta_2\theta_3 + b_2(x)\theta_3\theta_1 + b_3(x)\theta_1\theta_2) \tag{6}$$

So we get:

$$m\ddot{x} = -\nabla V + \imath\,(\nabla b_1(x)\theta_2\theta_3 + \nabla b_2(x)\theta_3\theta_1 + \nabla b_3(x)\theta_1\theta_2)$$
$$\dot{\theta} = \vec{B}(x) \times \theta$$

Grassman variables are not meassurable by experiments. We introduce the expectation value of a function of the odd variables:

$$\langle f\rangle = i\int f(\theta)\rho(\theta, t)d^3\theta \tag{7}$$

demanding also that $\langle 1\rangle = 1$, $\langle\vec{S}\rangle = \vec{C}$ we get:

$$\rho(\theta) = -\frac{i}{6}\epsilon_{abc}\theta^a\theta^b\theta^c + C^a\theta^a \tag{8}$$

But for $\langle f^* f\rangle \geq 0$ we have to get $\vec{C} = 0$. Which is very trivial. This is solved with quantization. The simplest way to do that is in the "canonical" way:

$$[,]_P \to \frac{\imath}{\hbar}[,] \tag{9}$$

The grassman variables up to some rescaling go to the Pauli matrices, $\sigma^i = \sqrt{\frac{2}{\hbar}}\theta^i$. The spin now takes the (familiar) form $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$. These turn the Grassmann algebra $[\theta^i, \theta^j]_P = i\delta^{ij}$ to the Clifford algebra $[\sigma_i, \sigma_j] = 2\delta_{ij}$. We must also redefine the expectation value by $i\int f\rho\, d\theta \ \to\ Tr\,(\rho f)$. Where :

$$\rho(\theta) = 2\left(\frac{1}{2} + \vec{C}\frac{\vec{\sigma}}{\hbar}\right) \tag{10}$$

is defined as the density matrix. So the new condition for positive difiniteness reads:$|\vec{C}| \leq \frac{1}{2}\hbar$

References

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