Supermanifolds II

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1 Compact real super manifolds $(X, \mathcal{O}_X)$ and their superalgebra $\mathcal{O}_X(X)$ of global superfunctions

The aim of this section is to prove that compact supermanifolds $(X, \mathcal{O}_X)$ are completely determined by their superalgebra $\mathcal{O}_X(X)$ of global superfunctions.

1.1 Statements and Corollaries

Let us first formulate the statement more precisely:

Theorem 1.1. *Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be compact supermanifolds such that the unital superalgebras $\mathcal{O}_X(X)$ and $\mathcal{O}_Y(Y)$ are isomorphic. Then the supermanifolds $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are isomorphic.*

More generally we prove

Theorem 1.2. *Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be supermanifolds, $\chi : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ be a unital superalgebra homomorphism and Y be compact. Then there exists exactly one morphism*

$$(\varphi, \Psi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that $\Psi_Y = \chi$.

Before we come to the technical details of the proof of Theorem 1.2 let us first see how we can use Theorem 1.2 to prove Theorem 1.1:

Let $\chi : \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ be the unital superalgebra isomorphism. From Theorem 1.2 we get morphisms of supermanifolds

$$(\varphi_1, \Psi_1) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X), \quad (\varphi_2, \Psi_2) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that $(\Psi_1)_X = \chi$ and $(\Psi_2)_Y = \chi^{-1}$. Then $(\varphi_1 \circ \varphi_2, \Psi_1 \circ \Psi_2) : (X, \mathcal{O}_X) \to (X, \mathcal{O}_X)$ and $(\mathrm{id}_X, \mathrm{id}_{\mathcal{O}_X}) : (X, \mathcal{O}_X) \to (X, \mathcal{O}_X)$ are both morphisms of the compact supermanifold $(X, \mathcal{O}_X)$ such that the corresponding unital superalgebra homorphism from $\mathcal{O}_X(X)$ to $\mathcal{O}_X(X)$ is the identity. Thus the uniqueness result in Theorem 1.1 tells us that they coincide, i.e. $\varphi_1 \circ \varphi_2 = \mathrm{id}_X$, $\Psi_1 \circ \Psi_2 = \mathrm{id}_{\mathcal{O}_X}$. Similarly, we get $\varphi_2 \circ \varphi_1 = \mathrm{id}_Y$, $\Psi_2 \circ \Psi_1 = \mathrm{id}_{\mathcal{O}_Y}$ and so $(\varphi_1, \Psi_1) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an isomorphism of supermanifolds.

Besides we like to note an interesting consequence of Theorem 1.1. We saw last time that each smooth manifold M gives rise to a supermanifold (M, C^∞_M) of dimension $n|0$. Moreover, we know from last time that if (φ, Ψ) is a morphism of supermanifolds, then φ is smooth. Hence we get

Corollary 1.3. *Let M, N be compact smooth manifolds such that the unital algebras $C^\infty(M)$ and $C^\infty(N)$ are isomorphic. Then M and N are diffeomorphic.*

Remark 1.4. Note that Corollary 1.3 is totally wrong if we are in the complex category: From Liouville's theorem we know that for all compact complex manifolds the set $Hol_X(X)$ of holomorphic functions $f : X \to \mathbb{C}$ is isomorphic as a unital algebra to $\mathbb{C}$, but there are of course many non-isomorphic compact complex manifolds.

1.2 Globalization of local superfunctions and other useful results

We remind the reader of three properties we had proven last time:

Lemma 1.5. *(a) Let M be a compact smooth manifold. The unital algebra homomorphisms $\delta : C^\infty(M) \to \mathbb{R}$ are precisely δ_p, $p \in M$ with $\delta_p(f) := f(p)$.*

(b) Let $(X, \mathcal{O}_X)$ be a supermanifold. Then there is precisely one unital sheaf homomorphism $\beta : \mathcal{O}_X \to C^\infty_X$. Moreover, for all $U \in \mathcal{T}_X$, we have the following short exact sequence of superalgebras:

$$0 \to \mathcal{O}^1(U) \to \mathcal{O}_X(U) \overset{\beta_U}{\to} C^\infty_X(U) = C^\infty(U) \to 0.$$

Thereby, $\mathcal{O}^1(U)$ is the set of all nilpotent elements in $\mathcal{O}_X(U)$.

(c) Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be supermanifolds and $(\varphi, \Psi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of supermanifolds. Let $\varphi^* : C^\infty(Y) \to C^\infty(X)$ be the pullback of C^∞-functions with φ, i.e. $\varphi^*(f) = f \circ \varphi$. Then the following diagramm commutes:

$$\begin{array}{ccc} \mathcal{O}_X(X) & \overset{\chi}{\leftarrow} & \mathcal{O}_Y(Y) \\ \beta_X \downarrow & & \downarrow \beta_Y \\ C^\infty(X) & \overset{\varphi^*}{\leftarrow} & C^\infty(Y) \end{array}$$

For a local C^∞-function $f : U \to \mathbb{R}$ on a smooth manifold M we know that for each $y \in U$ we can find a global function $f_y : M \to \mathbb{R}$ such that $f_y|_{W_Y} = f|_{W_y}$ for some (in general smaller) open neighborhood $W_y \subseteq U$ of $y \in M$. We like to prove the same result for superfunctions. This result will play the crucial role in the proof of Theorem 1.2 and it is the reason why a similar result in the complex holomorphic setting fails.

Lemma 1.6. *Let $(X, \mathcal{O}_X)$ be a supermanifold, $U \in \mathcal{T}_X$, $f \in \mathcal{O}_X(U)$, $y \in U$. Then there exists $U \supseteq W_y \in \mathcal{T}_X$, $y \in W_y$ and $f_y \in \mathcal{O}_X(X)$ such that $\rho_{W_y,X}(f_y) = \rho_{W_y,U}(f)$.*

Proof. We may assume, without loss of generality, that U is a superchart neighborhood, i.e. $\mathcal{O}_X(U) \cong \mathcal{O}_{m|n}(U)$ and

$$f = \sum_\epsilon f_\epsilon \otimes \theta_1^{\epsilon_1} \cdot \ldots \cdot \theta_n^{\epsilon_n}$$

with $f_\epsilon \in C^\infty(U)$. Then there exists $\rho_y \in C^\infty(U)$ such that $\rho_y|_{W_y} \equiv 1$ and $\mathrm{supp}(\rho_y) \subsetneq U$. Thus

$$g := \sum_\epsilon f_\epsilon \cdot \rho_y \otimes \theta_1^{\epsilon_1} \cdot \ldots \cdot \theta_n^{\epsilon_n}$$

fulfills $\rho_{W_y,U}(f) = \rho_{W_y,U}(g)$ and $\rho_{U \setminus \mathrm{supp}(\rho_y),U}(g) = 0$. Let now $f_y \in \mathcal{O}_X(X)$ be the unique function such that $\rho_{U,X}(f_y) = g$ and $\rho_{X \setminus \mathrm{supp}(\rho_y),X}(f_y) = 0$. Note that such a superfunction exists since

$$\rho_{U \cap (X \setminus \mathrm{supp}(\rho_y)),U}(g) = \rho_{U \setminus \mathrm{supp}(\rho_y),U}(g) = 0 = \rho_{U \setminus \mathrm{supp}(\rho_y), X \setminus \mathrm{supp}(\rho_y)}(0).$$

Then $\rho_{W_y,X}(f_y) = \rho_{W_y,U}(g) = \rho_{W_y,U}(f)$ as claimed. $\square$

1.3 Proof of Theorem 1.2

Construction of $\varphi : X \to Y$:

To construct φ, the first step is to show that there exists a unique unital algebra homomorphism $\tilde{\chi} : C^\infty(Y) \to C^\infty(X)$such that the following diagram commutes (think of Lemma 1.5 (c)):

$$\begin{array}{ccc} \mathcal{O}_X(X) & \overset{\chi}{\leftarrow} & \mathcal{O}_Y(Y) \\ \beta_X \downarrow & & \downarrow \beta_Y \\ C^\infty(X) & \overset{\tilde{\chi}}{\leftarrow} & C^\infty(Y) \end{array}$$

The only possibility to define $\tilde{\chi}$ is by setting $\tilde{\chi}(f) := \beta_X(\chi(F))$ for some $F \in \mathcal{O}_Y(Y)$ with $\beta_Y(F) = f$. Note that due to the surjectivity of β_Y (Lemma 1.5 (b)) such an F exists. $\tilde{\chi}$ is a unital algebra homomorphism if it is well-defined. Therefore, let $F_1, F_2 \in \mathcal{O}_Y(Y)$ with $f = \beta_Y(F_1) = \beta_Y(F_2)$. In particular, then $\beta_Y(F_1 - F_2) = 0$ so that $F_1 - F_2$ is by Lemma 1.5 (b) nilpotent. Thus also $\chi(F_1 - F_2) \in \mathcal{O}_X(X)$ is nilpotent and so $\chi(F_1 - F_2) \in \ker \beta_X$. Thus $\beta_X(\chi(F_1)) - \beta_X(\chi(F_2)) = \beta_X(\chi(F_1 - F_2)) = 0$ and $\tilde{\chi}$ is well-defined.

Next, let $x \in X$. Then $\delta_x \circ \tilde{\chi} : C^\infty(Y) \to \mathbb{R}$ is a unital algebra homomorphism. Thus, by Lemma 1.5 (a) there exists exactly one $y \in Y$ such that $\delta_x \circ \tilde{\chi} = \delta_y$. Set now $\varphi(x) := y$.

Continuity of $\varphi : X \to Y$:

Notice that for all $f \in C^\infty(Y)$ and all convergent sequences $(x_i)_i$ with limit x we have

$$f(\varphi(x_i)) = \delta_{\varphi(x_i)}(f) = (\delta_{x_i} \circ \tilde{\chi})(f) = \tilde{\chi}(f)(x_i) \overset{i\to\infty}{\to} \tilde{\chi}(f)(x) = f(\varphi(x)).$$

Suppose now that nevertheless $(\varphi(x_i))_i$ does not converge to $\varphi(x)$. Then there exists an open neighborhood U of $\varphi(x)$ and a subsequence $(\varphi(x_{i_j}))_j$ such that for all $j \in \mathbb{N}$ we have $\varphi(x_{i_j}) \notin U$. There exists a smooth function $f \in C^\infty(Y)$ such that $\mathrm{supp}(f) \subsetneq U$ and $f|V \equiv 1$ for some open neighborhood $V \subsetneq U$ of $\varphi(x)$. But then

$$0 = \lim_{j\to\infty} f(\varphi(x_{i_j})) \neq 1 = f(\varphi(x)),$$

a contradiction. Thus $(\varphi(x_i)_i$ converges to $\varphi(x)$ and φ is continuous.

Uniqueness of $\varphi : X \to Y$:

Suppose that $(\varphi_0, \Psi_0) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of supermanifolds with $(\Psi_0)_Y = \chi$. Then Lemma 1.5 tells us that $\beta_X \circ (\Psi_0)_Y = \beta_X \circ \chi = \varphi_0^* \circ \beta_Y$ and the uniqueness result for $\tilde{\chi}$ at the beginning tells us $\tilde{\chi} = \varphi_0^*$, i.e. $f(\varphi(x)) = \tilde{\chi}(f)(x) = \varphi_0^*(f)(x) = f(\varphi_0(x))$ for all $f \in C^\infty(M)$. But then $\varphi(x) = \varphi_0(x)$ for all $x \in X$, since for $X \ni y \neq z \in X$ there is always $f \in C^\infty(M)$ such that $f(y) = 1$ and $f(z) = 0$.

Construction of Ψ:

For the rest of the proof we will frequently use that $\chi : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is *local*, i.e. if $U \in \mathcal{T}_Y$ and $\rho_{U,Y}(f) = 0$ for some $f \in \mathcal{O}_Y(Y)$, then $\rho_{\varphi^{-1}(U),X}(\chi(f)) = 0$. We do not prove this here.

Let now $U \in \mathcal{T}_Y$ and $f \in \mathcal{O}_Y(U)$. Set $V := \varphi^{-1}(U)$. We have to define $\Psi_V(f) \in \mathcal{O}_X(V)$. Therefore, we like to use $\chi : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Hence we have to globalize f as in Lemma 1.6. So, for each $y \in U$ we have an open neighborhood $U_y \subseteq U$ of y and $f_y \in \mathcal{O}_Y(Y)$ such that $\rho_{U_y,Y}(f_y) = \rho_{U_y,U}(f)$. In particular,

$$\rho_{U_y\cap U_z,Y}(f_y - f_z) = \rho_{U_y\cap U_z,U_y} \circ \rho_{U_y,Y}(f_y) - \rho_{U_y\cap U_z,U_z} \circ \rho_{U_z,Y}(f_z) = \rho_{U_y\cap U_z,U}(f) - \rho_{U_y\cap U_z,U}(f) = 0. \tag{1.1}$$

for all $y, z \in U$.

Set $V_y := \varphi^{-1}(U_y) \subseteq V$ and $h_y := \rho_{V_y,X}(\chi(f_y))$ for all $y \in U$. We like to build out of the superfunctions $h_y \in \mathcal{O}_X(V_y)$ a superfunction on V. Therefore we have to check that the restrictions are the same on the overlaps. We get

$$\rho_{V_y\cap V_z,V_y}(h_y) = \rho_{V_y\cap V_z,X}(\chi(f_y)) = \rho_{V_y\cap V_z,X}(\chi(f_z)) + \rho_{V_y\cap V_z,X}(\chi(f_y - f_z)) \overset{(1.1),\,\chi\text{ local}}{=} \rho_{V_y\cap V_z,V_z}(h_z).$$

Hence there exists a unique $h \in \mathcal{O}_X(V)$ such that $\rho_{V_y,V}(h) = h_y$. Now set $\Psi_U(f) := h$. If Ψ_U is well-defined, i.e. independent of f_y and U_y, then we see from the construction that Ψ_U is a unital superalgebra homomorphism. Moreover, $\Psi_Y = \chi$, since in this case we can choose $f_y = f$ and $U_y = Y$ for all $y \in Y$. Furthermore, if everything is well-defined we also see easily that $\rho_{V_0,V} \circ \Psi_U = \Psi_{U_0} \circ \rho_{U_0,U}$ for all $U_0 \subseteq U$, $V_0 := \varphi^{-1}(U_0)$.

Well-definedness of Ψ:

Let $\tilde{f}_y \in \mathcal{O}_Y(Y)$ with $\rho_{\tilde{U}_y,Y}(\tilde{f}_y) = \rho_{\tilde{U}_y,U}(f)$, $\tilde{h}_y := \rho_{\tilde{V}_y,X}(\chi(\tilde{f}_y)) \in \mathcal{O}_X(\tilde{V}_y)$, $\tilde{V}_y := \varphi^{-1}(\tilde{U}_y)$ and $\tilde{h} \in \mathcal{O}_X(V)$ be the unique superfunction with $\rho_{\tilde{V}_y,V}(\tilde{h}) = \tilde{h}_y$. Then, for all $y \in U$, we have

$$\begin{aligned}\rho_{U_y\cap\tilde{U}_y,Y}(f_y - \tilde{f}_y) &= (\rho_{U_y\cap\tilde{U}_y,U_y} \circ \rho_{U_y,Y})(f_y) - (\rho_{U_y\cap\tilde{U}_y,\tilde{U}_y} \circ \rho_{\tilde{U}_y,Y})(f_y)\\ &= (\rho_{U_y\cap\tilde{U}_y,U_y} \circ \rho_{U_y,U})(f) - (\rho_{U_y\cap\tilde{U}_y,\tilde{U}_y} \circ \rho_{\tilde{U}_y,U})(f) = 0.\end{aligned}$$

Hence, by the locality of χ:

$$0 = \rho_{V_y \cap \tilde{V}_y, X}(\chi(f_y - \tilde{f}_y)) = \rho_{V_y \cap \tilde{V}_y, V_y}(h_y) - \rho_{V_y \cap \tilde{V}_y, \tilde{V}_y}(\tilde{h}_y) = \rho_{V_y \cap \tilde{V}_y, V}(h - \tilde{h}).$$

But then $h - \tilde{h} = 0$ since $\bigcup_{y \in U}(V_y \cap \tilde{V}_y) = V$. Thus Ψ_U is well-defined.

Uniqueness of Ψ:

Let $\tilde{\Psi}$ be a different sheaf homomorphism $\tilde{\Psi} : (Y, \mathcal{O}_Y) \to (Y, \varphi_* \mathcal{O}_X)$. Let $U \in \mathcal{T}_Y$ and $f \in \mathcal{O}_Y(U)$. Then, if U_y, f_y, V_y, V are defined as before, we must have

$$\rho_{V_y,V}(\tilde{\Psi}_U(f)) = \tilde{\Psi}_{U_y}(\rho_{U_y,U}(f)) = \tilde{\Psi}_{U_y}(\rho_{U_y,Y}(f_y)) = \rho_{V_y,X}(\tilde{\Psi}_Y(f_y)) = \rho_{V_y,X}(\chi(f_y)) = \rho_{V_y,V}(\Psi_U(f)).$$

Thus $\tilde{\Psi}_U(f) = \Psi_U(f)$ due to $\bigcup_{y \in U} V_Y = V$.

2 Function factors

Definition 2.1. Let $(X, \mathcal{O}_X)$ be a supermanifold and $U \in \mathcal{T}_X$. A subalgebra $C(U)$ of the even superfunctions $\mathcal{O}_X(U)_0$ is called *function factor on U* if $\beta_U|_{C(U)} : C(U) \to C^\infty(U)$ is a unital algebra isomorphism.

Remarks 2.2.

- There is a natural embedding of $C^\infty(U)$ into $\mathcal{O}_{m|n}(U) = C^\infty(U) \otimes \Lambda^* \mathbb{R}^n$ as the even subalgebra $C^\infty(U) \otimes \Lambda^0 \mathbb{R}^n$. Hence, locally function factors always exist.
- Function factors are in general not unique, even locally. Therefore, consider e.g. $(\mathbb{R}, \mathcal{O}_{1|2}(\mathbb{R}))$ and note that $\mathcal{O}_{1|2}(U)_0 = C^\infty(U) \oplus C^\infty(U)\theta_1\theta_2$ for all open $U \subseteq \mathbb{R}$. Then, for any $h \in C^\infty(U)$ the set
$$C_h(U) := \{f + h \cdot f' \cdot \theta_1 \cdot \theta_2 | f \in C^\infty(U)\}$$
is an even subalgebra of $\mathcal{O}_{1|2}(U)$, which is isomorphic to $C^\infty(U)$ via β_U.
- Each supermanifold $(X, \mathcal{O}_X)$ possesses a global function factor $C(X)$. This can be proven with similar techniques as Theorem 1.2.
- After a choice of a global function factor $C(X)$ one can prove that for each $U \in \mathcal{T}_X$ there exists a unique function factor $C(U)$ with $\rho_{U,X}(C(X)) \subseteq C(U)$. The map $U \mapsto C(U)$ is a sheaf. In general, $\rho_{U,X}(C(X)) \neq C(U)$ (consider e.g. $(X, \mathcal{O}_X) = (X, C^\infty_X)$ with the natural global function factor).
- So $(\beta|_{C(U)})^{-1}$ gives us a splitting of the following exact sequence of sheaves:
$$0 \to \mathcal{O}^1 \to \mathcal{O}_X \xrightarrow{\beta} C^\infty_X \to 0.$$

References

[1] C. Bär, *Nichtkommutative Geometrie (Skript)*, http://geometrie.math.uni-potsdam.de/documents/baer/skripte/skript-NKommGeo.pdf, (2005).