Introduction to sheaves and ringed spaces

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0 Preliminairies: Categories

A nice introduction to categories can be found in [Ric10].

Definition 1. A *category* $\mathcal{C}$ is a class of objects $Ob(\mathcal{C})$ together with for each two objects $A, B \in Ob(\mathcal{C})$ a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and for each triple $A, B, C \in Ob(\mathcal{C})$ a composition map $\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ satisfying the following:

- For $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$: $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$. (composition of morphisms)
- For each $A \in Ob(\mathcal{C})$ there is an identity morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ such that for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$: $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$.
- The composition of morphisms is associative: $f \circ (g \circ h) = (f \circ g) \circ h$.

Example 1. Sets: Sets and maps between sets, Gr: Groups with group homomorphisms, k-Vect: k-vector spaces and k-linear maps. (for a field k), k-SVect: k-super vector spaces and even k-linear maps, k-$\mathsf{Vect}^{\mathbb{Z}_2}$: $\mathbb{Z}_2$-graded vector spaces with graded k-linear maps.

Example 2. Let X be a topological space, $\mathcal{T}_X$ its topology. $\mathcal{T}_X$ is a category with inclusions of sets as morphisms.

1 Sheaves: Basic definitions

From now on I basically follow [Bä05].

Definition 2. Let X be a topological space with topology $\mathcal{T}_X$ as in example 2. Let $\mathcal{C}$ be a category. A *presheaf* with values in $\mathcal{C}$ is a contravariant functor $\mathcal{G} : \mathcal{T}_X \to \mathcal{C}$:

- $\mathcal{G}$ assigns to each open set $U \in \mathcal{T}_X$ an object $\mathcal{G}(U)$ in $\mathcal{C}$.
- $\mathcal{G}$ assigns to each pair of open sets $U \subset V$ a "restriction" morphism $\rho^{\mathcal{G}}_{U,V} \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(V), \mathcal{G}(U))$.
- $\mathcal{G}$ preserves identities and composition:

$$\rho_{U,U} = \mathrm{id}_U,$$
$$\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$$

  for all $U, V, W \in \mathcal{T}_X$.

Definition 3. A *sheaf* is a presheaf that is complete in the following sense: For each family $(U_\alpha)_{\alpha \in I}$ of open sets with $\cup_{\alpha \in I} U_\alpha = U \in \mathcal{T}_X$ we have:
Given a family $(f_\alpha)_{\alpha \in I}$, $f_\alpha \in \mathcal{G}(U_\alpha)$ such that for all $\alpha, \beta \in I$

$$\rho_{U_\alpha \cap U_\beta, U_\alpha}(f_\alpha) = \rho_{U_\alpha \cap U_\beta, U_\beta}(f_\beta), \tag{1}$$

there exists a unique $f \in \mathcal{G}(U)$ with $f_\alpha = \rho_{U_\alpha, U}(f)$ for all $\alpha \in I$.

Definition 4. Let $\mathcal{G}$ be a sheaf on X, $p \in X$. Define the *stalk* of $\mathcal{G}$ over p as:

$$\mathcal{G}_p := \left(\mathop{\dot{\bigcup}}_{\substack{U \in \mathcal{T}_X \\ p \in U}} \mathcal{G}(U) \right) \Big/ \sim \tag{2}$$

Here the equivalence relation for $f \in \mathcal{G}(U_1), g \in \mathcal{G}(U_2)$ is given as

$$f \sim g :\Leftrightarrow \exists\, U_1 \cap U_2 \supset U_3 \in \mathcal{T}_X, p \in U_3 : \rho_{U_3, U_1}(f) = \rho_{U_3, U_2}(g). \tag{3}$$

The classes $[f]_p \in \mathcal{G}_p$ are called the *germs* of f at p.

Remark 1. The stalks inherit the structure of the sheaf, i.e. $\mathcal{G}_p \in Ob(\mathcal{C})$.

Example 3.
- Let X be a topological manifold. Then C^0_X is a sheaf of rings over X: $C^0_X(U) = C^0(U)$ and for $f \in C^0(V)$, $\rho_{U,V} : C^0(V) \to C^0(U)$ is given by $\rho_{U,V} : f \mapsto f|_U$.
- Let X be a smooth manifold and $\Omega^\bullet X = \oplus_{k=0}^n \Omega^n X$ be the smooth differential forms on X. This is a sheaf of unital $\mathbb{R}$-algebras.

2 Morphisms of sheaves

Definition 5. Let $\mathcal{F}, \mathcal{G}$ be sheaves over a space X. A *sheaf homomorphism* $\psi : \mathcal{F} \to \mathcal{G}$ is a natural transformation of functors $\mathcal{F} \Rightarrow \mathcal{G}$, i.e., a family of morphisms $(\psi_U)_{U \in \mathcal{T}_X}$, $\psi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\rho^{\mathcal{F}_{U,V}}"] \arrow[d, "\psi_V"'] & \mathcal{F}(U) \arrow[d, "\psi_U"] \\ \mathcal{G}(V) \arrow[r, "\rho^{\mathcal{G}}_{U,V}"'] & \mathcal{G}(U) \end{tikzcd} \tag{4}$$

commutes for all $V \subset U$.

Remark 2. For every $p \in X$ this induces a morphism on stalks $\psi_p : \mathcal{F}_p \to \mathcal{G}_p$, $\psi_p([f]_p) := [\psi_U(f)]_p$.

Definition 6. Let X, Y be topological spaces, $\varphi : X \to Y$ continuous, $\mathcal{F}$ a sheaf over X, $\mathcal{G}$ a sheaf over Y. φ induces a functor $\varphi^{-1} : \mathcal{T}_Y \to \mathcal{T}_X$ and therefore an image sheaf $\varphi_* \mathcal{F} = \mathcal{F} \circ \varphi^{-1}$.
A *sheaf morphism* $(\varphi, \psi) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is a continuous map $\varphi : X \to Y$ together with a sheaf homomorphism $\psi : \mathcal{G} \to \varphi_* \mathcal{F}$.

3 Ringed spaces

Definition 7. A $\mathbb{k}$-ringed space ($\mathbb{k}$ a field) $(X, \mathcal{G}, v)$ is a topological space X together with a sheaf $\mathcal{G}$ of $\mathbb{k}$-algebras on X and a family $(v_p)_{p \in X}$ of $\mathbb{k}$-algebra morphisms (evaluation maps)

$$v_p : \mathcal{G}_p \to \mathbb{k}.$$

$\mathcal{G}$ is then called the *structure sheaf* of $(X, \mathcal{G}, v)$.

Definition 8. A $\mathbb{k}$-ringed space $(X, \mathcal{G}, v)$ is called *local*, if for all $p \in X$ $\ker v_p$ is the unique maximal ideal of $\mathcal{G}_p$.

Remark 3. For local $\mathbb{k}$-ringed spaces the evaluation map v is unique.

Example 4. (X, C^0_X, ev) with $ev_p(f) = f(p)$ is a locally $\mathbb{R}$-ringed space.

Definition 9. A *morphism of* $\mathbb{k}$*-ringed spaces* $(X, \mathcal{F}, v)$, $(Y, \mathcal{G}, w)$ is a morphism $(\varphi, \psi) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ of sheaves such that

$$\begin{tikzcd} \mathcal{G}_{\varphi(p)} \arrow[rr, "\psi_p"] \arrow[dr, "w_{\varphi(p)}"'] & & \mathcal{F}_p \arrow[dl, "v_p"] \\ & \mathbb{k} & \end{tikzcd} \tag{5}$$

commutes for all $p \in X$.

Lemma 1. *Let $(\varphi, \psi) : (X, \mathcal{F}, v) \to (Y, \mathcal{G}, w)$ be a morphism of $\mathbb{k}$-ringed spaces, $p \in X$. Then*

$$\ker w_{\varphi(p)} = \psi_p^{-1}(\ker v_p). \tag{6}$$

In particular if $(X, \mathcal{F}, v)$, $(Y, \mathcal{G}, w)$ are locally ringed spaces then

$$\psi_p : \mathcal{G}_{\varphi(p)} \to \mathcal{F}_p$$

is a morphism of local rings, i.e: $\psi_p(\ker w_{\varphi(p)}) \subset \ker v_p$.

References

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[Ric10] B. Richter. Kategorientheorie mit Anwendungen in Topologie (Skript) . http://www.math.uni-hamburg.de/home/richter/cats.pdf, 2010.