Batchelor's Theorem

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Remark Let X be an m-dimensional (smooth) manifold, let $E \to X$ be a vector bundle of rank n. Then we have that $(X, \mathcal{O}_E)$ is a supermanifold of dimension $m|n$, where $\mathcal{O}_E$ is the sheaf of smooth sections of $\Lambda^* E \to X$.

Furthermore, $C(U) := C^\infty(U; \Lambda^0 E) \cong C^\infty(U)$ is a function factor, the related sheaf morphism β_U is given by

$$\beta_U : \mathcal{O}_E(U) \to C^\infty(U) : \sum_\varepsilon f_\varepsilon \theta^\varepsilon \mapsto f_{0,\dots,0},$$

where ε is a multiindex with value in $\{0,1\}^n$ and $\theta^\varepsilon = \theta_1^{\varepsilon_1} \cdots \theta_n^{\varepsilon_n}$.

Remark The $C^\infty(U)$-algebra

$$\mathcal{O}^1(U) := \ker \beta_U = C^\infty(U, \bigoplus_{k \geq 1} \Lambda^k E)$$

is the set of all nilpotent elements of $\mathcal{O}_E(U)$. Hence, we can see β_U as the projection onto $C^\infty(U)$.

Lemma

Let X be a (smooth) manifold and let $(U_\alpha)_{\alpha \in I}$ be an open covering of X. Let $g_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta; \mathrm{GL}(n, \mathbb{R}))$, such that for all $\alpha, \beta, \gamma \in I$ on $U_\alpha \cap U_\beta \cap U_\gamma$:

$$g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma},$$

this is called the cocycle condition. Then there exist a up to isomorphism unique vector bundle $E \to X$ which can be trivialized on each U_α with smooth sections $e_{\alpha,1}, \dots, e_{\alpha,n}$, such that $(e_{\alpha,1}, \dots, e_{\alpha,n})$ is a frame field and for all $\alpha, \beta \in I$, $x \in U_\alpha \cap U_\beta$ the following holds true:

$$\sum_{j=1}^n g_{\alpha\beta}(x)_{ij} e_{\beta,j}(x) = e_{\alpha,i}(x).$$

Proof See lecture notes "Nichtkommutative Geometrie", C. Bär, page 29.

Theorem (Bachelor, 1980)

Let $(X, \mathcal{O}_X)$ be a supermanifold of dimension $m|n$ and let $C(X) \subset \mathcal{O}_X(X)$ be a function factor. Then there exists a vector bundle $E \to X$ of rank n, such that $(X, \mathcal{O}_X)$ is isomorphic to $(X, \mathcal{O}_E)$. The vector bundle E is unique up to isomorphism and does not depend on the choice of $C(X)$. Furthermore, the isomorphism from $(X, \mathcal{O}_X)$ to $(X, \mathcal{O}_E)$ maps $C(X)$ to $C^\infty(X; \Lambda^0 E)$.

Proof Roughly speaking, the proof works like this:

(i) We construct a $C^\infty(U)$-module on $\mathcal{O}^1(U)/(\mathcal{O}^1(U)\cdot\mathcal{O}^1(U))$,

(ii) we show that, under certain circumstances, it is free and find a basis.

(iii) Using this and the previous lemma we construct the vector bundle $E \to X$,

(iv) construct our (first just local) sheaf isomorphism $\Phi : \mathcal{O}_E \to \mathcal{O}_X$ and

(v) show that it is a (global) sheaf isomorphism and finally

(vi) we show that E is unique up to isomorphy.

(i) Let $C(X) \subset \mathcal{O}_X$ be a function factor. We have shown before that for any open $U \subset X$ there exist a unique function factor $C(U) \subset \mathcal{O}_X(U)$, such that

$$\rho^{\mathcal{O}_X}_{U,X}(C(X)) \subset C(U).$$

Let $\mathcal{O}^2(U)$ denote the ideal in $\mathcal{O}_X$ generated by $\mathcal{O}^1(U)\cdot\mathcal{O}^1(U)$. Since $\mathcal{O}^2(U) \subset \mathcal{O}^1(U)$, we can define $\mathcal{E}(U)$ as the quotient of $C^\infty(U)$-algebras

$$\mathcal{E}(U) := \mathcal{O}^1(U)/\mathcal{O}^2(U).$$

The $C^\infty(U)$-module structure on $\mathcal{E}(U)$ is defined as

$$f \cdot [\varphi]_{\mathcal{E}(U)} := [\sigma_U(f)\cdot\varphi]_{\mathcal{E}(U)},$$

where $f \in C^\infty(U)$, $[\varphi]_{\mathcal{E}(U)} \in \mathcal{E}(U)$ and $\sigma_U := \left(\beta_U|_{C(U)}\right)^{-1} : C^\infty(U) \to C(U) \subset \mathcal{O}_X(U)$. Since $\mathcal{O}^1(U)$ and $\mathcal{O}^2(U)$ are ideals in $\mathcal{O}_X(U)$, the multiplication with f is well defined. Hence $\mathcal{E}(U)$ is a $C^\infty(U)$-module.

(ii) If $U \subset U'$ is in a superchart domain, then $\mathcal{E}(U)$ is a free $C^\infty(U)$-module of rank n. For a superchart

$$(\varphi, \Psi) : (U', \mathcal{O}_X|_{U'}) \to (V', \mathcal{O}_{m|n}|_{V'})$$

we have that

$$\overline{\theta}_i := [\Psi_V \theta_i]_{\mathcal{E}(U)}, \quad V = \phi(U) \subset V'$$

is a $C^\infty(U)$-basis of $\mathcal{E}(U)$, where $\theta_1, ..., \theta_n$ are generators of $\Lambda^*\mathbb{R}$.

Proof It suffices to do the calculations in $\mathcal{O}_{m|n}(V)$. After doing so, applying the superchart (φ, Ψ) shows that $\mathcal{E}(U)$ has the desired properties.

We see that

$$\mathcal{O}^1_{m|n}(V) = \left\{ \sum_{|\varepsilon|\geq 1} f_\varepsilon \theta^\varepsilon \right\},$$

$$\mathcal{O}^2_{m|n}(V) = \left\{ \sum_{|\varepsilon|\geq 2} f_\varepsilon \theta^\varepsilon \right\}$$

and

$$\mathcal{E}(U) = \mathcal{O}^1_{m|n}(V)/\mathcal{O}^2_{m|n}(V) \cong \left\{ \sum_{|\varepsilon|=1} f_\varepsilon \theta^\varepsilon \right\} = \left\{ \sum_{i=1}^n f_i \theta_i \right\}.$$

Hence, $\mathcal{E}(U)$ is a free $C^\infty(V)$-module of rank n (since $\Lambda^1\mathbb{R}^n$ has exactly n generators) and the module structure is defined as follows

$$f \bullet [\psi]_{\mathcal{E}(V)} = [f\psi]_{\mathcal{E}(V)} \quad \forall f \in C^\infty(V), \psi \in \mathcal{O}^1_{m|n}.$$

It remains to check that this module structure is consistent with (i). Since $\beta_V \sigma_v(f) = f$ we have

$$\sigma_V(f) = f + \nu_V(f),$$

where $\nu_V(f) \in \mathcal{O}^1_{m|n}(V)$. Hence

$$\begin{aligned}
&f \cdot [\psi]_{\mathcal{E}(V)} \\
&\overset{(i)}{=} [\sigma_V(f) \cdot \psi]_{\mathcal{E}(V)} \\
&= [f \cdot \psi]_{\mathcal{E}(V)} + [\underbrace{\nu_V(f) \cdot \psi}_{\in \mathcal{O}^2_{m|n}(V)}]_{\mathcal{E}(V)} \\
&= [f \cdot \psi]_{\mathcal{E}(V)} \\
&= f \bullet [\psi]_{\mathcal{E}(V)}.
\end{aligned}$$

□

(iii) We want to construct the vector bundle $E \to X$. Therefore, let

$$(\phi_\alpha, \Psi_\alpha) : (U_\alpha, \mathcal{O}_X|_{U_\alpha}) \to (V_\alpha, \mathcal{O}_{m|n}|_{V_\alpha})$$

be a superchart covering of X. For $\theta_i \in \mathcal{O}_{m|n}(V_\alpha)$ choose the standard basis of $\mathbb{R}^n$, i.e. the generators of $\Lambda^*\mathbb{R}^n$ and let

$$\overline{\theta}_{\alpha,i} = [\Psi_{\alpha,V_\alpha}\theta_i]_{\mathcal{E}(U_\alpha)} \in \mathcal{E}(U_\alpha).$$

$\forall x \in U_\alpha \cap U_\beta$ we have the following:

$$\overline{\theta}_{\alpha,i}(x) = \sum_j g_{\alpha\beta}(x)_{ij}\overline{\theta}_{\beta,j}(x),$$

this is in fact the basis transformation of the basis of $\mathcal{E}(U_\alpha \cap U_\beta) \subset \mathcal{E}(U_\beta)$ to the basis in $\mathcal{E}(U_\beta)$, the coefficients $g_{\alpha\beta}(\cdot)_{ij}$ are in $C^\infty(U_\alpha \cap U\beta)$. We can see the matrices $g_{\alpha\beta}$ as smooth maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$$

since $g_{\alpha\beta} \in \mathrm{GL}(n, C^\infty(U_\alpha \cap U\beta))$. To use the previous lemma we have to check the cozycle property $g_{\alpha\gamma} = g_{\alpha\beta} \cdot g_{\beta\gamma}$. We have

$$\overline{\theta}_{\beta,i} = \sum_k g_{\beta\gamma}(x)_{ik}\overline{\theta}_{\gamma,k}(x)$$

and

$$\overline{\theta}_{\alpha,i} = \sum_j g_{\alpha\beta}(x)_{ij}\overline{\theta}_{\beta,j}(x) = \sum_{j,k} g_{\alpha\beta}(x)_{ij}g_{\beta\gamma}(x)_{jk}\overline{\theta}_{\gamma,k}(x).$$

This is effectively a change of a basis in every $x \in U_\alpha \cap U_\beta \cap U_\gamma$. Thus, linear algebra yields that

$$\sum_{j,k} g_{\alpha\beta}(x)_{ij}g_{\beta\gamma}(x)_{jk}\overline{\theta}_{\gamma,k}(x) = \sum_k g_{\alpha\gamma}(x)_{ik}\overline{\theta}_{\gamma,k}(x).$$

So we choose $E \to X$ to be the vector bundle defined by $g_{\alpha\beta}$ as in the previous lemma.

(iv) We construct the sheaf isomorphism $\Phi : \mathcal{O}_E \to \mathcal{O}_X$ first just for open $U \subset X$, such that U is contained in a superchart domain. For $V \subset \mathbb{R}^m$ we know that

$$\mathcal{O}_{m|n}(V) \cong \bigoplus_{k=0}^{n} \left(\mathcal{O}^k_{m|n}(V)/\mathcal{O}^{k+1}_{m|n}(V)\right),$$

where

$$\mathcal{O}^k_{m|n}(V) = \left\{ \sum_{|\varepsilon| \geq k} f_\varepsilon \theta^\varepsilon \right\}.$$

To see this isomorphism, we see that for any $k \in \{0, ..., n\}$:

$$\mathcal{O}^k_{m|n}(V)/\mathcal{O}^{k+1}_{m|n}(V) \cong \left\{ \sum_{|\varepsilon| = k} f_\varepsilon \theta^\varepsilon \,\middle|\, f_\varepsilon \in C^\infty(V) \right\}.$$

Let $U \subset U_\alpha$ for some superchart $(\phi_\alpha, \Psi_\alpha) : (U_\alpha, \mathcal{O}_X|_{U_\alpha}) \to (V_\alpha, \mathcal{O}_{m|n}|_{V_\alpha})$ of $(X, \mathcal{O}_X)$ and let $V := \phi_\alpha(U) \subset V_\alpha$. We constructed $E \to X$ with the previous lemma and hence $E|_{U_\alpha} \to U_\alpha$ is trivial and admits the local frame field $(e_{\alpha,1}, ..., e_{\alpha,n})$. We define

$$\begin{aligned} \Phi_{k,U,\alpha} &: C^\infty(U, \Lambda^k E) \to \mathcal{O}_X^k(U)/\mathcal{O}_X^{k+1}(U) \\ f = \sum_{|\varepsilon|=k} f_\varepsilon e_{\alpha,1}^{\varepsilon_1} \cdots e_{\alpha,n}^{\varepsilon_n} &\mapsto \left[\Psi_{\alpha,V} \left(\sum_{|\varepsilon|=k} (f_\varepsilon \circ \varphi_\alpha^{-1}) \theta_1^{\varepsilon_1} \cdots \theta_n^{\varepsilon_n} \right) \right]_{\mathcal{O}_X^k(U)/\mathcal{O}_X^{k+1}(U)} . \end{aligned}$$

One can show that for $U \subset U_\alpha \cap U_\beta$

$$\Phi_{k,U,\alpha} = \Phi_{k,U,\beta}.$$

Hence we get for each U that is contained in a superchart domain an isomorphism

$$\Phi_{1,U} := \sum_{k=0}^n \Phi_{k,U} : \mathcal{O}_E(U) = C^\infty(U, \Lambda^* E) \to \bigoplus_{k=0}^n (\mathcal{O}_X^k(U)/\mathcal{O}_X^{k+1}), \quad \Phi_{1,U} = \sum_{k=0}^n \Phi_{k,U}.$$

Each superchart $(\phi, \Psi) : (U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_{m|n}|_V)$ yields an isomorphism

$$\bigoplus_{k=0}^n (\mathcal{O}_X^k(U)/\mathcal{O}_X^{k+1}) \cong \bigoplus_{k=0}^n (\mathcal{O}_{m|n}^k(V)/\mathcal{O}_{m|n}^{k+1}(V)) \cong \mathcal{O}_{m|n}(V) \cong \mathcal{O}_X(U).$$

But this isomorphism depends on the choice of the superchart. With the help of a covering of X with supercharts and an associated smooth partition of unity we can construct for each U that is contained in a superchart an isomorphism

$$\Phi_{2,U} : \bigoplus_{k=0}^n (\mathcal{O}_X^k(U)/\mathcal{O}_X^{k+1}) \to \mathcal{O}_X(U)$$

that is compatible with the restriction map $\rho^{\mathcal{O}_X}$ of $\mathcal{O}_X$. Hence we have at least for each U that is contained in a superchart domain the desired sheaf homomorphism

$$\Phi : \mathcal{O}_E \to \mathcal{O}_X, \quad \Phi_U := \Phi_{2,U} \circ \Phi_{1,U}.$$

(v) Now we want to have a sheaf isomorphism $\Phi : \mathcal{O}_E \to \mathcal{O}_X$ not just for special U's. To get such an isomorphism, we use the gluing axiom of sheaves.

Let $U \subset X$ be an arbitrary open set, let (U_α) be a basis of the topology of X that consist of all superchart domains (this actually is a basis of the topology of X) and write $U = \cup_\alpha U_\alpha$. For $f \in \mathcal{O}_E(U)$ let

$$g_\alpha := \Phi_{U_\alpha}(\rho^{\mathcal{O}_E}_{U_\alpha,U}(f)).$$

With $f_\alpha = \rho^{\mathcal{O}_E}_{U_\alpha,U}(f)$ and $f_\beta = \rho^{\mathcal{O}_E}_{U_\beta,U}$ we have for all α, β

$$\begin{aligned} \rho^{\mathcal{O}_X}_{U_\alpha \cap U_\beta, U_\alpha}(g_\alpha) &= \Phi_{U_\alpha \cap U_\beta}(\rho^{\mathcal{O}_E}_{U_\alpha \cap U_\beta, U_\alpha}(f_\alpha)) \\ &= \Phi_{U_\alpha \cap U_\beta}(\rho^{\mathcal{O}_E}_{U_\alpha \cap U_\beta, U}(f)) \\ &= \Phi_{U_\alpha \cap U_\beta}(\rho^{\mathcal{O}_E}_{U_\alpha \cap U_\beta, U_\beta}(f_\beta)) \\ &= \rho^{\mathcal{O}_X}_{U_\alpha \cap U_\beta, U_\beta}(g_\beta). \end{aligned}$$

Hence, there is a unique $g \in \mathcal{O}_X(U)$, such that $\rho^{\mathcal{O}_X}_{U_\alpha,U}(g) = g_\alpha$. So we just have to set $\Phi_U(f) := g$ and thus have our desired isomorphism of sheaves.

(vi) To show the uniqueness of $E \to X$ we remark that $\mathcal{O}_E$ defines $\Lambda^* E$ up to algebra bundle isomorphism. Hence, E is unique up to vector bundle isomorphisms, since

$$E = \left(\bigoplus_{k \geq 1} \Lambda^k E \right) \bigg/ \left(\bigoplus_{k \geq 2} \Lambda^k E \right).$$

$\bigoplus_{k \geq 1} \Lambda^k E$ is the ideal of nilpotent elements, $\bigoplus_{k \geq 2} \Lambda^k E = \bigoplus_{k \geq 1} \Lambda^k E \cdot_{\Lambda^* E} \bigoplus_{k \geq 1} \Lambda^k E$.

□