**Batchelor’s Theorem**

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**Remark** Let $X$ be an $m$-dimensional (smooth) manifold, let $E \to X$ be a vector bundle of rank $n$. Then we have that $(X, \mathcal{O}_E)$ is a supermanifold of dimension $m|n$, where $\mathcal{O}_E$ is the sheaf of smooth sections of $\Lambda^*E \to X$. Furthermore, $C(U) := C^\infty(U; \Lambda^0 E) \cong C^\infty(U)$ is a function factor, the related sheaf morphism $\beta_U$ is given by

$$\beta_U : \mathcal{O}_E(U) \to C^\infty(U) : \sum \epsilon f_* \theta^\epsilon \mapsto f_0, \ldots, 0,$$

where $\epsilon$ is a multiindex with value in $\{0, 1\}^n$ and $\theta^\epsilon = \theta^\epsilon_1 \cdots \theta^\epsilon_n$.

**Remark** The $C^\infty(U)$-algebra

$$\mathcal{O}^1(U) := \ker \beta_U = C^\infty(U, \bigoplus_{k \geq 1} \Lambda^k E)$$

is the set of all nilpotent elements of $\mathcal{O}_E(U)$. Hence, we can see $\beta_U$ as the projection onto $C^\infty(U)$.

**Lemma**

Let $X$ be a (smooth) manifold and let $(U_\alpha)_{\alpha \in I}$ be an open covering of $X$. Let $g_{\alpha \beta} \in C^\infty(U_\alpha \cap U_\beta; \text{GL}(n, \mathbb{R}))$, such that for all $\alpha, \beta, \gamma \in I$ on $U_\alpha \cap U_\beta \cap U_\gamma$:

$$g_{\alpha \beta} \circ g_{\beta \gamma} = g_{\alpha \gamma},$$

this is called the cocycle condition. Then there exist a up to isomorphism unique vector bundle $E \to X$ which can be trivialized on each $U_\alpha$ with smooth sections $e_{\alpha,1}, \ldots, e_{\alpha,n}$, such that $(e_{\alpha,1}, \ldots, e_{\alpha,n})$ is a frame field and for all $\alpha, \beta \in I$, $x \in U_\alpha \cap U_\beta$ the following holds true:

$$\sum_{j=1}^n g_{\alpha \beta}(x)_{ij} e_{\beta,j}(x) = e_{\alpha,i}(x).$$

**Proof** See lecture notes "Nichtkommutative Geometrie", C. Bär, page 29.

**Theorem (Bachelor, 1980)**

Let $(X, \mathcal{O}_X)$ be a supermanifold of dimension $m|n$ and let $C(X) \subset \mathcal{O}_X(X)$ be a function factor. Then there exists a vector bundle $E \to X$ of rank $n$, such that $(X, \mathcal{O}_X)$ is isomorphic to $(X, \mathcal{O}_E)$. The vector bundle $E$ is unique up to isomorphism and does not depend on the choice of $C(X)$. Furthermore, the isomorphism from $(X, \mathcal{O}_X)$ to $(X, \mathcal{O}_E)$ maps $C(X)$ to $C^\infty(X; \Lambda^0 E)$. 
\textbf{Proof}  \quad} {\textit{Roughly speaking, the proof works like this:}}

(i) We construct a $C^\infty(U)$-module on $\mathcal{O}^1(U)/(\mathcal{O}^1(U) \cdot \mathcal{O}^1(U))$.

(ii) we show that, under certain circumstances, it is free and find a basis.

(iii) Using this and the previous lemma we construct the vector bundle $E \to X$.

(iv) construct our (first just local) sheaf isomorphism $\Phi : \mathcal{O}_E \to \mathcal{O}_X$ and

(v) show that it is a (global) sheaf isomorphism and finally

(vi) we show that $E$ is unique up to isomorphy.

\textbf{(i)} \quad \text{Let $C(X) \subset \mathcal{O}_X$ be a function factor. We have shown before that for any open $U \subset X$ there exist a unique function factor $C(U) \subset \mathcal{O}_X(U)$, such that}

$$\rho_{U,X}(C(X)) \subset C(U).$$

Let $\mathcal{O}^2(U)$ denote the ideal in $\mathcal{O}_X$ generated by $\mathcal{O}^1(U) \cdot \mathcal{O}^1(U)$. Since $\mathcal{O}^2(U) \subset \mathcal{O}^1(U)$, we can define $\mathcal{E}(U)$ as the quotient of $C^\infty(U)$-algebras

$$\mathcal{E}(U) := \mathcal{O}^1(U)/\mathcal{O}^2(U).$$

The $C^\infty(U)$-module structure on $\mathcal{E}(U)$ is defined as

$$f \cdot [\varphi]_{\mathcal{E}(U)} := [\sigma_U(f) \cdot \varphi]_{\mathcal{E}(U)},$$

where $f \in C^\infty(U)$, $[\varphi]_{\mathcal{E}(U)} \in \mathcal{E}(U)$ and $\sigma_U := (\beta_U|_{C(U)})^{-1} : C^\infty(U) \to C(U) \subset \mathcal{O}_X(U)$. Since $\mathcal{O}^1(U)$ and $\mathcal{O}^2(U)$ are ideals in $\mathcal{O}_X(U)$, the multiplication with $f$ is well defined. Hence $\mathcal{E}(U)$ is a $C^\infty(U)$-module.

\textbf{(ii)} \quad \text{If $U \subset U'$ is in a superchart domain, then $\mathcal{E}(U)$ is a free $C^\infty(U)$-module of rank $n$. For a superchart}

$$\langle \varphi, \Psi \rangle : (U', \mathcal{O}_X|_{U'}) \to (V', \mathcal{O}_{m|n}|_{V'}),$$

we have that

$$\tilde{\theta}_i := [\Psi \cdot \theta_i]_{\mathcal{E}(U)}, \quad V = \phi(U) \subset V'$$

is a $C^\infty(U)$-basis of $\mathcal{E}(U)$, where $\theta_1, \ldots, \theta_n$ are generators of $\Lambda^* \mathbb{R}$.

\textbf{Proof} \quad \text{It suffices to do the calculations in $\mathcal{O}_{m|n}(V)$. After doing so, applying the superchart $\langle \varphi, \Psi \rangle$ shows that $\mathcal{E}(U)$ has the desired properties.}

We see that

$$\mathcal{O}_{m|n}^1(V) = \left\{ \sum_{|\epsilon| \geq 1} f_\epsilon \theta^\epsilon \right\},$$

$$\mathcal{O}_{m|n}^2(V) = \left\{ \sum_{|\epsilon| \geq 2} f_\epsilon \theta^\epsilon \right\}$$

and

$$\mathcal{E}(U) = \mathcal{O}_{m|n}^1(V)/\mathcal{O}_{m|n}^2(V) \cong \left\{ \sum_{|\epsilon| = 1} f_\epsilon \theta^\epsilon \right\} = \left\{ \sum_{i=1}^n f_i \theta_i \right\}.$$

Hence, $\mathcal{E}(U)$ is a free $C^\infty(V)$-module of rank $n$ (since $\Lambda^1 \mathbb{R}^n$ has exactly $n$ generators) and the module structure is defined as follows

$$f \cdot [\psi]_{\mathcal{E}(V)} = [f \psi]_{\mathcal{E}(V)} \quad \forall f \in C^\infty(V), \psi \in \mathcal{O}_{m|n}^1.$$

It remains to check that this module structure is consistent with (i). Since $\beta_V \sigma_v(f) = f$ we have

$$\sigma_v(f) = f + \nu_v(f).$$
where $\nu_V(f) \in \mathcal{O}_{m|n}^1(V)$. Hence

$$f \cdot [\psi]_{\mathcal{E}(V)} = \{ \sigma_V(f) \cdot [\psi]_{\mathcal{E}(V)} + \nu_V(f) \cdot [\psi]_{\mathcal{E}(V)} \} \in \mathcal{O}_{m|n}^2(V),$$

$$= [f \cdot [\psi]_{\mathcal{E}(V)}] = f \cdot [\psi]_{\mathcal{E}(V)}.$$

\(\blacksquare\)

(iii) We want to construct the vector bundle $E \to X$. Therefore, let

$$(\phi_\alpha, \Psi_\alpha) : (U_\alpha, \mathcal{O}_X|_{U_\alpha}) \to (V_\alpha, \mathcal{O}_{m|n}|_{V_\alpha})$$

be a superchart covering of $X$. For $\theta_i \in \mathcal{O}_{m|n}(V_\alpha)$ choose the standard basis of $\mathbb{R}^n$, i.e. the generators of $\Lambda^* \mathbb{R}^n$ and let

$$\bar{\theta}_{\alpha,i} = [\Psi_{\alpha,V_\alpha,\theta_i}]_{\mathcal{E}(V_\alpha)} \in \mathcal{E}(U_\alpha).$$

\(\forall x \in U_\alpha \cap U_\beta\) we have the following:

$$\bar{\theta}_{\alpha,i}(x) = \sum_j g_{\alpha\beta}(x)_{ij} \bar{\theta}_{\beta,j}(x),$$

this is in fact the basis transformation of the basis of $\mathcal{E}(U_\alpha \cap U_\beta) \subset \mathcal{E}(U_\beta)$ to the basis in $\mathcal{E}(U_\beta)$, the coefficients $g_{\alpha\beta}(\cdot)_{ij}$ are in $C^\infty(U_\alpha \cap U_\beta)$. We can see the matrices $g_{\alpha\beta}$ as smooth maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{GL}(n, \mathbb{R})$$

since $g_{\alpha\beta} \in \text{GL}(n, C^\infty(U_\alpha \cap U_\beta))$. To use the previous lemma we have to check the cocycle property $g_{\alpha\gamma} = g_{\alpha\beta} \cdot g_{\beta\gamma}. \quad \text{We have}$$

$$\bar{\theta}_{\beta,i} = \sum_k g_{\beta\gamma}(x)_{ik} \bar{\theta}_{\gamma,k}(x)$$

and

$$\bar{\theta}_{\alpha,i} = \sum_j g_{\alpha\beta}(x)_{ij} \bar{\theta}_{\beta,j}(x) = \sum_{j,k} g_{\alpha\beta}(x)_{ij} g_{\beta\gamma}(x)_{jk} \bar{\theta}_{\gamma,k}(x).$$

This is effectively a change of a basis in every $x \in U_\alpha \cap U_\beta \cap U_\gamma$. Thus, linear algebra yields that

$$\sum_{j,k} g_{\alpha\beta}(x)_{ij} g_{\beta\gamma}(x)_{jk} \bar{\theta}_{\gamma,k}(x) = \sum_k g_{\alpha\gamma}(x)_{ik} \bar{\theta}_{\gamma,k}(x).$$

So we choose $E \to X$ to be the vector bundle defined by $g_{\alpha\beta}$ as in the previous lemma.

(iv) We construct the sheaf isomorphism $\Phi : \mathcal{O}_E \to \mathcal{O}_X$ first just for open $U \subset X$, such that $U$ is contained in a superchart domain. For $V \subset \mathbb{R}^m$ we know that

$$\mathcal{O}_{m|n}(V) \cong \bigoplus_{k=0}^n \left( \mathcal{O}_{m|n}^k(V)/\mathcal{O}_{m|n}^{k+1}(V) \right),$$

where

$$\mathcal{O}_{m|n}^k(V) = \left\{ \sum_{|\xi| \leq k} f_\xi \theta^\xi \right\}.$$ 

To see this isomorphism, we see that for any $k \in \{0, \ldots, n\}$:

$$\mathcal{O}_{m|n}^k(V)/\mathcal{O}_{m|n}^{k+1}(V) \cong \left\{ \sum_{|\xi| = k} f_\xi \theta^\xi \mid f_\xi \in C^\infty(V) \right\}.$$
Let \( U \subset U_\alpha \) for some superchart \((\phi_\alpha, \Psi_\alpha) : (U_\alpha, \mathcal{O}_X|U_\alpha) \to (V_\alpha, \mathcal{O}_{m|n}|V_\alpha)\) of \((X, \mathcal{O}_X)\) and let \( V := \phi_\alpha(U) \subset V_\alpha \). We constructed \( E \to X \) with the previous lemma and hence \( E|_{U_\alpha} \to U_\alpha \) is trivial and admits the local frame field \((e_\alpha, \ldots, e_{\alpha,n})\). We define
\[
\Phi_{k,U,\alpha} : C^\infty(U, \Lambda^k E) \to \mathcal{O}_X^k(U)/\mathcal{O}_X^{k+1}(U)
\]
\[
f = \sum_{|\varepsilon| = k} f_{\varepsilon} e_{\alpha,1}^{\varepsilon_1} \cdots e_{\alpha,n}^{\varepsilon_n} \mapsto \left[ \sum_{|\varepsilon| = k} (f_{\varepsilon} \circ \varphi^{-1}) \theta_{\varepsilon_1}^1 \cdots \theta_{\varepsilon_n}^n \right] \in \mathcal{O}_X^k(U)/\mathcal{O}_X^{k+1}(U).
\]
One can show that for \( U \subset U_\alpha \cap U_\beta \)
\[
\Phi_{k,U,\alpha} = \Phi_{k,U,\beta}.
\]
Hence we get for each \( U \) that is contained in a superchart domain an isomorphism
\[
\Phi_{1,U} := \sum_{k=0}^n \Phi_{k,U} : \mathcal{O}_E(U) = C^\infty(U, \Lambda^* E) \to \bigoplus_{k=0}^n (\mathcal{O}_X^k(U)/\mathcal{O}_X^{k+1}(U)), \quad \Phi_{1,U} = \sum_{k=0}^n \Phi_{k,U}.
\]
Each superchart \((\phi, \Psi) : (U, \mathcal{O}_X|U) \to (V, \mathcal{O}_{m|n}|V)\) yields an isomorphism
\[
\bigoplus_{k=0}^n (\mathcal{O}_X^k(U)/\mathcal{O}_X^{k+1}(U)) \cong \bigoplus_{k=0}^n (\mathcal{O}_{m|n}^k(V)/\mathcal{O}_{m|n}^{k+1}(V)) \cong \mathcal{O}_{m|n}(V) \cong \mathcal{O}_X(U).
\]
But this isomorphism depends on the choice of the superchart. With the help of a covering of \( X \) with supercharts and an associated smooth partition of unity we can construct for each \( U \) that is contained in a superchart an isomorphism
\[
\Phi_{2,U} : \bigoplus_{k=0}^n (\mathcal{O}_X^k(U)/\mathcal{O}_X^{k+1}(U)) \to \mathcal{O}_X(U)
\]
that is compatible with the restriction map \( \rho \mathcal{O}_X \) of \( \mathcal{O}_X \). Hence we have at least for each \( U \) that is contained in a superchart domain the desired sheaf homomorphism
\[
\Phi : \mathcal{O}_E \to \mathcal{O}_X, \quad \Phi_U := \Phi_{2,U} \circ \Phi_{1,U}.
\]
\((\text{v})\) Now we want to have a sheaf isomorphism \( \Phi : \mathcal{O}_E \to \mathcal{O}_X \) not just for special \( U \)'s. To get such an isomorphism, we use the gluing axiom of sheaves.

Let \( U \subset X \) be an arbitrary open set, let \((U_\alpha)\) be a basis of the topology of \( X \) that consist of all superchart domains (this actually is a basis of the topology of \( X \)) and write \( U = \bigcup_\alpha U_\alpha \). For \( f \in \mathcal{O}_E(U) \) let
\[
g_\alpha := \Phi_{U_\alpha}(\rho_{U_\alpha,U}(f)).
\]
With \( f_\alpha = \rho_{U_\alpha,U}(f) \) and \( f_\beta = \rho_{U_\beta,U} \), we have for all \( \alpha, \beta \)
\[
\rho_{U_\alpha \cap U_\beta, U_\alpha}(g_\alpha) = \Phi_{U_\alpha \cap U_\beta}(\rho_{U_\alpha \cap U_\beta, U_\alpha}(f_\alpha))
\]
\[
= \Phi_{U_\alpha \cap U_\beta}(\rho_{U_\alpha \cap U_\beta, U_\beta}(f))
\]
\[
= \Phi_{U_\alpha \cap U_\beta}(\rho_{U_\alpha \cap U_\beta, U_\beta}(f_\beta))
\]
\[
= \rho_{U_\alpha \cap U_\beta, U_\alpha}(g_\beta).
\]
Hence, there is a unique \( g \in \mathcal{O}_X(U) \), such that \( \rho_{U, U}(g) = g_\alpha \). So we just have to set \( \Phi_U(f) := g \) and thus have our desired isomorphism of sheaves.

\((\text{vi})\) To show the uniqueness of \( E \to X \) we remark that \( \mathcal{O}_E \) defines \( \Lambda^* E \) up to algebra bundle isomorphism. Hence, \( E \) is unique up to vector bundle isomorphisms, since
\[
E = \left( \bigoplus_{k \geq 1} \Lambda^k E \right) \big/ \left( \bigoplus_{k \geq 2} \Lambda^k E \right).
\]
\( \bigoplus_{k \geq 1} \Lambda^k E \) is the ideal of nilpotent elements, \( \bigoplus_{k \geq 2} \Lambda^k E = \bigoplus_{k \geq 1} \Lambda^k E \cdot \Lambda^* E \bigoplus \Lambda^k E \).

\( \square \)