Basics of linear and commutative superalgebra

1 Linear superalgebra

Definition 1.1. A $\mathbb{Z}$-graded ring R is a ring with the additional structure of a decomposition

$$R = \bigoplus_{i \in \mathbb{Z}} R_i$$

as abelian groups, such that $R_i \cdot R_j \subseteq R_{i+j}$. A $\mathbb{Z}$-graded module M over a $\mathbb{Z}$-graded ring R is a module over R equipped with a decomposition

$$M = \bigoplus_{i \in \mathbb{Z}} M_i$$

as abelian groups such that $R_i \cdot M_j \subseteq M_{i+j}$. Elements of the R_i and M_i are called homogeneous.

We note that

1. we can turn every ring R into a graded ring by setting $R_0 = R$ and $R_i = 0$ for $i \neq 0$,
2. only R_0 is a subring of R, the other R_i are modules over R_0,
3. if R is unital (which we will always assume), then $1 \in R_0$
4. this construction can be carried out with an arbitrary *monoid* M instead of $\mathbb{Z}$. E.g., we can have $\mathbb{N}$-graded rings such as the polynomials $\mathbb{K}[x]$ or differential forms $\Omega^\bullet(M)$ on a smooth manifold.

Definition 1.2. A super vector space $V = V_0 \oplus V_1$ over a field $\mathbb{K}$ is a $\mathbb{Z}_2$-graded $\mathbb{K}$-module. On the homogeneous elements we define the *parity function*

$$\begin{array}{rcl} p : (V_0 \cup V_1) \setminus \{0\} & \to & \mathbb{Z}_2 \\ p(v) & = & i \quad \text{for } v \in V_i. \end{array}$$

A *morphism* $\phi : V \to W$ of super vector spaces is a linear map that preserves the grading, i.e., $\phi(V_i) \subseteq W_i$.

Most natural constructions for vector spaces can be extended to super vector spaces:

1. direct sums: $(V \oplus W)_i := V_i \oplus W_i$
2. tensor products: $(V \otimes W)_i := \bigoplus_{i=j+k} V_j \otimes W_k$

The space $\mathrm{Hom}_{\mathsf{SVect}}(V, W)$ of morphisms of super vector spaces does *not* form a super vector space itself, but the space of all linear maps does, via

$$\mathrm{Hom}_{\mathsf{Vect}}(V, W)_i := \{\phi : V \to W \mathfrak{m} \phi(V_j) \subset W_{j+i}\}.$$

So every linear map $\phi : V \to W$ of super vector spaces splits uniquely into $\phi = \phi_0 + \phi_1$ where ϕ_0 ism.

Note: in a supercommutative algebra, all odd elements square to zero: $a^2 = \frac{1}{2}[a, a] = 0$.

2 Supermodules

Definition 2.1. A left supermodule over a supercommutative ring $R = R_0 \oplus R_1$ is simply a left $\mathbb{Z}_2$-graded module $M = M_0 \oplus M_1$ over R.

For the supermodules over a supercommutative superring R we define a *braiding*, i.e., a rule for how to interchange the factors in a tensor product of modules:

$$\begin{array}{rcl} c_{M,N} : M \otimes N & \to & N \otimes M \\ m \otimes n & \mapsto & (-1)^{p(m)p(n)} n \otimes m \end{array}$$

for all R-modules M, N and homogeneous elements $m \in M, n \in N$. This braiding distinguishes superalgebra (and supergeometry) from the plain algebra of graded rings and modules. In practice on can sum this up in the
Sign Rule: whenever in a multiplicative expression involving elements of supermodules over a supercommutative ring we exchange two neighbouring odd elements, a factor (-1) occurs.

Example: the braiding defines what "symmetric" means: a map $f : M \otimes M \to N$ between modules over a supercommutative algebra A is *supersymmetric*, if $f(m_1, m_2) = (-1)^{p(m_1)p(m_2)} f(m_2, m_1)$ for all homogeneous $m_1, m_2 \in M$. So a supercommutative algebra is commutative in this new "super" sense.

Every left supermodule M over a supercommutative algebra A can be given a right module structure by setting

$$m \cdot a := (-1)^{p(a)p(m)} a \cdot m.$$

Definition 2.2. Let M, N be left supermodules over a supercommutative $\mathbb{K}$-superalgebra A. A $\mathbb{K}$-linear map $\phi : M \to N$ is called *graded linear over* A, if for all homogeneous $m \in M, a \in A$ we have

$$\phi(a \cdot m) = (-1)^{p(a)p(\phi)} a \cdot \phi(m).$$

We write $\mathrm{Hom}_A(M, N)$ for the A-supermodule of graded linear maps $M \to N$ over A, and $\mathrm{End}_A(M)$ for $\mathrm{Hom}_A(M, M)$.

Example: left translation $L_b : M \to M$, $L_b(m) = b \cdot m$ for $b \in A$, A supercommutative, M a left A-module is graded linear of parity $p(L_b) = p(b)$.

Definition 2.3. A left supermodule M over a supercommutative superalgebra A is called *free of rank* $r|s$, if there exists a homogeneous basis

$$\underbrace{e_1, \dots, e_r}_{even}, \underbrace{e_{r+1}, \dots, e_{r+s}}_{odd}$$

for M. That means that every $x \in M$ can *uniquely* be written as

$$x = \sum_{i=1}^{r+s} a^j e_j, \qquad a^j \in A.$$

Remarks:

1. One can show that the rank $r|s$ is independent of the basis chosen.

2. We can as well use a left basis like above as a right basis, i.e., we can as well write every $x \in M$ uniquely as

$$x = \sum_{i=1}^{r+s} e_j b^j, \qquad b^j \in A.$$

A graded linear morphism $\phi : M \to N$ between free A-supermodules can be written as a matrix as follows. We pick bases $e_1 \dots, e_{m+n}$ of M and $f_1, \dots, f_{r+s}$. Then we have unique expressions

$$\begin{aligned} \phi(e_j) &= \sum_{i=1}^{r+s} f_i a^i_j \\ x &= \sum_{j=1}^{m+n} e_j x^j \\ \phi(x) &= \sum_{i=1}^{r+s} f_i y^i \end{aligned}$$

for any $x \in M$. Thus

$$\phi(x) = \sum_{j=1}^{m+n} \phi(e_j) x^j = \sum_{i=1}^{r+s} \sum_{j=1}^{m+n} f_i a^i_j x^j$$

and so $y^i = \sum_j a^i_j x^j$. We can therefore think of the a^i_j as the entries of a matrix representation L of the morphism ϕ which decomposes into blocks

$$L = \left(\begin{array}{c|c} L_{00} & L_{01} \\ \hline L_{10} & L_{11} \end{array} \right) \tag{1}$$

where L_{00} is a $r \times m$-matrix, L_{01} a $r \times n$ matrix, L_{10} a $s \times m$-matrix and L_{11} a $s \times n$-matrix. When ϕ is homogeneous, then the entries of L_{ij} have parity $i + j + p(\phi)$.

Definition 2.4. We define $\mathrm{Mat}_A(m|n, r|s)$ as the A-supermodule of all matrices of block form as in (1). A matrix L is homogeneous of parity $p(L)$ if the entries of L_{ij} have parity $i + j + p(L)$. The A-supermodule structure of $\mathrm{Mat}_A(m|n, r|s)$ is given by

$$a \cdot L = \left(\begin{array}{c|c} aL_{00} & aL_{01} \\ \hline (-1)^{p(a)} aL_{10} & (-1)^{p(a)} aL_{11} \end{array} \right).$$

3 The supertrace

Definition 3.1. The supertrace is defined on the quadratic supermatrices $\mathrm{Mat}_A(m|n)$ by

$$\mathrm{str}(L) := \mathrm{tr}(L_{00}) - (-1)^{p(L)} \mathrm{tr}(L_{11}).$$

This definition is essentially (up to normalization) forced upon us by requiring that

1. $\mathrm{str} : \mathrm{Mat}_A(m|n)$ is A-linear,
2. $\mathrm{str}([X, Y]) = 0$ where $[X, Y]$ is the supercommutator of matrices (see above).

The second requirement ensures that the super trace is invariant under base changes: we can actually define the super trace to be a morphism of A-modules $\mathrm{str} : \mathrm{End}(M) \to A$ for any *free* A-module M.

One checks that str is an even A-linear map, i.e., $\mathrm{str}(a \cdot L) = a \cdot \mathrm{str}(L)$ for all square supermatrices L and all $a \in A$.

4 The superdeterminant (Berezinian)

The superdeterminant is a less obvious generalization. It can only be defined on a certain subset of the square matrices $\mathrm{Mat}_A(m|n)$.

Lemma 4.1. *Let $A = A_0 \oplus A_1$ be a supercommutative $\mathbb{K}$-superalgebra. Then*

1. *the quotient $\mathcal{A} = A/(A_1)$, where (A_1) is the ideal generated by the odd elements, is an ordinary commutative $\mathbb{K}$-algebra,*
2. *an element $a \in A$ is invertible if and only if its even part a_0 is invertible, and a_0 is invertible if and only if its image $\pi(a) \in \mathcal{A}$ is invertible. Here $\pi : A \to \mathcal{A}$ denotes the projection onto the quotient algebra.*

Theorem 4.2. *A matrix $L \in Mat_A(m|n)$ is invertible if and only if $\pi(L) \in \mathrm{Mat}_{\mathcal{A}}(m+n)$ is invertible.*

Both statements are proven in [1]. As a corollary one finds that an even matrix L is invertible if and only if L_{00} and L_{11} are invertible.

Definition 4.3. We define the general linear group of a free A-supermodule of rank $r|s$ as

$$GL_A(r|s) = \{L \in \mathrm{Mat}_A(r|s)\mathfrak{m}p(L) = 0, \quad L \text{ invertible } \}.$$

The superdeterminant (Berezinian) can only be defined on such even invertible square matrices.

Definition 4.4. The superdeteterminant is defined as

$$\begin{aligned} \mathrm{sdet} : GL_A(r|s) &\to A_0 \\ \mathrm{sdet}\left(\begin{array}{c|c} L_{00} & L_{01} \\ \hline L_{10} & L_{11} \end{array}\right) &:= \det(L_{00} - L_{01}L_{11}^{-1}L_{10})^{-1}) \det(L_{00}) \end{aligned}$$

This definition is again essentially forced upon us if we require that

1. the superdeterminant be multiplicative: $\mathrm{sdet}(A \cdot B) = \mathrm{sdet}(A) \cdot \mathrm{sdet}(B)$,
2. sdet is independent of the chosen basis for a free module, i.e., that it is actually a map from the even invertible endomorphisms to A_0 rather than from the matrices.

Theorem 4.5. *For all $r, s > 0$ the superdeterminant is a homomorphism*

$$\mathrm{sdet} : GL_A(r|s) \to A_0^\times$$

of groups. Moreover we have

$$\mathrm{sdet}(e^A) = e^{\mathrm{str}(A)}$$

for all $A \in GL_A(r|s)$.

Proof. Tough, see [2]. □

References

[1] C. Bär: *Nichtkommutative Geometrie.* Vorlesungsskript.

[2] F. Constantinescu, H.F. de Groote: *Geometrische und algebraische Methoden der Physik: Supermannigfaltigkeiten und Virasoro-Algebren.* Teubner Studienbücher, Teubner, Stuttgart 1994.