Coordinate systems on supermanifolds

1 Coordinates on ordinary manifolds

Coordinates on a (smooth) manifold M are collections of functions playing a particular role for the local description of the manifold and for morphisms to other manifolds.

The model spaces $\mathbb{R}^m$ (and therefore all domains $U \subset \mathbb{R}^m$) have the *standard coordinates*, say $x_1, \ldots, x_m$. They are defined to be the dual basis of $\mathbb{R}^m$, i.e., the linear maps

$$x_i : \mathbb{R}^m \to \mathbb{R}$$

for which $x_i(e_j) = \delta_{ij}$ holds for all standard basis vectors e_j of $\mathbb{R}^m$.

The coordinates x_i generate the algebra $\mathbb{R}[x_1, \ldots, x_m]$ of polynomial functions on $\mathbb{R}^m$. In a sense that we do not want to make precise here, the coordinates actually determine the entire algebra of *smooth* functions on $\mathbb{R}^m$. The statement that will be interesting to us is the following.

Theorem 1.1. *Let M be a smooth manifold and $U \subset \mathbb{R}^m$ an open domain. Then there is a bijection between*

1. $\mathrm{Hom}_{\mathsf{Man}}(M, U)$*, i.e., the set of smooth maps $M \to U$, and*
2. *collections of m smooth functions $y_1, \ldots, y_m \in C^\infty(M)$ such that*

   $$(y_1(x), \ldots, y_m(x)) \in U$$

   for all $x \in M$.

In light of the theorem from last talk that stated that

$$\mathrm{Hom}_{\mathsf{Man}}(M, N) \cong \mathrm{Hom}_{\mathsf{Alg}}(C^\infty(N), C^\infty(M))$$

the above theorem tells us that a morphism $\phi : M \to U$, corresponding to $\psi : C^\infty(U) \to C^\infty(M)$ is completely specified as soon as we know the images

$$\psi(x_1), \ldots, \psi(x_m).$$

That coincides with our usual intuition that a smooth map $\phi : M \to U$ can be specified by prescribing where every point $x \in M$ gets mapped to. That amounts to prescribing an image

$$x \mapsto (x_1(\phi(x)), \ldots, x_m(\phi(x)))$$

for every $x \in M$. So in the world of ordinary manifolds, the map $\psi : C^\infty(U) \to C^\infty(M)$ is simply given by *pullback*, i.e., by assigning

$$\psi : f \mapsto f \circ \phi.$$

2 Coordinates on supermanifolds

The above intuition cannot be carried over in a one-to-one fashion to arbitrary ringed spaces. Indeed, on algebraic varieties and schemes the concept of coordinates often does not make sense. But the above classical theorem does extend to supermanifolds:

Theorem 2.1. *Let $\mathcal{M}$ be a supermanifold and $\mathcal{U} = (U, \mathcal{O}_{m|n}|_U)$ be a superdomain. Then there are bijections between*

1. *the set* $\mathrm{Hom}_{\mathsf{SMan}}(\mathcal{M}, \mathcal{U})$ *of morphisms of supermanifolds,*
2. *the set* $\mathrm{Hom}_{\mathsf{SAlg}}(\mathcal{O}_{\mathcal{U}}(U), \mathcal{O}_{\mathcal{M}}(M))$ *of morphisms of superalgebras and*
3. *the set of collections of m even functions $\phi_1, \ldots, \phi_m \in \mathcal{O}_{\mathcal{M}}(M)_0$ and n odd functions $\xi_1, \ldots, \xi_n \mathcal{O}_{\mathcal{M}}(M)_1$ such that*
$$((\beta_M(\phi_1))(x), \ldots, (\beta_M(\phi_m))(x)) \in U$$
for all $x \in M$.

That plainly means that a supermanifold has coordinates, namely the pullbacks of the standard coordinates from $\mathbb{R}^{m|n}$ and that a morphism between supermanifolds is completely determined by what it does to the coordinates.

To make this explicit we have to give a prescription for what $\psi : \mathcal{O}_{\mathcal{U}}(U) \to \mathcal{O}_{\mathcal{M}}(M)$ assigns to an arbitrary
$$\mathcal{O}_{\mathcal{U}}(U) \ni f = \sum_{\epsilon} f_\epsilon(x) \theta_1^{\epsilon_1} \cdots \theta_n^{\epsilon_n}$$
if we know that
$$\psi(x_i) = \phi_i, \quad \psi(\theta_j) = \xi_j$$
where x_i, θ_j, $1 \leq i \leq m$, $1 \leq j \leq n$ are the standard coordinates on $\mathbb{R}^{m|n}$.

From being a homomorphism, it is clear that
$$\psi(f) = \sum_{\epsilon} \psi(f_\epsilon) \xi_1^{\epsilon_1} \cdots \xi_n^{\epsilon_n}.$$
So we have to make precise what
$$\psi(f_\epsilon) ='' f_\epsilon(\phi_1, \ldots, \phi_m)''$$
is supposed to mean. Since the ϕ_i are not just maps $\mathcal{M} \to \mathbb{R}$, but abstract elements of a supercommutative algebra, we cannot straightforwardly interpret ψ as a pullback map. After picking a function factor on $\mathcal{M}$ each $\phi_i \in \mathcal{O}_{\mathcal{M}}(M)_0$ can be written as
$$\phi_i^0 + \phi_i^{nil}$$
where ϕ_i^{nil} is the nilpotent part. Then we define $\psi(f_\epsilon)$ by Taylor expansion:
$$\psi(f_\epsilon) = \sum_{u \in \mathbb{Z}^m_{\geq 0}} \frac{1}{\mu!} \frac{\partial^\mu f_\epsilon(\phi_1^0, \ldots, \phi_m^0)}{\partial^{\mu_1} \phi_1^0 \cdots \partial^{\mu_m} \phi_m^0} \left(\phi_1^{nil}\right)^{\mu_1} \cdots \left(\phi_m^{nil}\right)^{\mu_m}$$

References

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[2] F. Constantinescu, H.F. de Groote: *Geometrische und algebraische Methoden der Physik: Supermannigfaltigkeiten und Virasoro-Algebren.* Teubner Studienbücher, Teubner, Stuttgart 1994.