Berezin Integration

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The statements (without proofs) can be found in a concise form in [Bä05]. In [CdG94] the change of variables formula is prooved in great detail.

1 Motivation

In classical differential geometry we can integrate densities. These are given by n-forms on a smooth compact n-dimensional manifold M. Given a coordinate system $\varphi : M \to \mathbb{R}^n$, $\varphi = (x_1, \dots, x_n)$ and an n-form $\omega = \omega(x)dx_1 \wedge \dots \wedge dx_n$, the integral of ω over M is defined to be

$$\int_M \omega := \int_{\varphi(M)} \omega(x)\, d(x_1, \dots, x_n). \tag{1}$$

By the change of variables formula this is then independent of the choice of coordinate system made.[1]
We want to define a similiar integration on supermanifolds. The crucial steps are then the definition of an integral on super domains and a change of variables for it. Using partitions of unity this can then in principal be lifted to supermanifolds.
The Berezin Integral is motivated by the rules

$$\int_{\mathbb{R}^{0|1}} 1 d\theta = 0, \qquad \int_{\mathbb{R}^{0|1}} \theta d\theta = 1 \tag{2}$$

on the super point $\mathbb{R}^{0|1}$. By formal use of the theorem of Fubini this motivates the definition of the integral on general super domains.

2 Berezin Integration and the change of variables formula

Definition 1. Let $(U, \mathcal{O}_{p|q}|_U)$ be a super domain and

$$f = \sum_{\varepsilon} f_\varepsilon \theta_1^{\varepsilon_1} \cdots \theta_n^{\varepsilon_n} \in \mathcal{O}_{p|q}(U) \tag{3}$$

be a super function with compact support. Then the *Berezin Integral* of f over $(U, \mathcal{O}_{p|q}|_U)$ is defined to be

$$\int_U f\, d(x, \theta) = (-1)^{pq+q(q-1)/2} \int_U f_{(1,\dots,1)}(x_1, \dots, x_p)\, dx_1, \dots, dx_m. \tag{4}$$

Theorem 1 (Change of variables formula)**.** *Let $(U, \mathcal{O}_{p|q}|_U)$, $(V, \mathcal{O}_{p|q}|_V)$ be super domains with coordinates (x_j, θ_j) on U and (y_j, η_j) on V. Let*

$$(\varphi, \Psi) : (U, \mathcal{O}_{p|q}|_U) \to (V, \mathcal{O}_{p|q}|_V) \tag{5}$$

be an isomorphism. Let $f \in \mathcal{O}_{p|q}(V)$ be a super function with compact support. Then the Berezin Integral transforms as

$$\int_V f\, d(y, \eta) = \pm \int_U \Psi(f) \cdot \operatorname{sdet}(J(\varphi, \Psi))\, d(x, \theta). \tag{6}$$

The negative sign appears iff φ is orientation reversing.

Example 1. It is imperative that the super functions have compact support for the change of variables formula to hold: Let $(U, \mathcal{O}_{p|q}|_U) = (V, \mathcal{O}_{p|q}|_V) = \big((0,1), \mathcal{O}_{1|2}|_{(0,1)}\big)$. Let $(\varphi, \Psi) : (U, \mathcal{O}_{p|q}|_U) \to (V, \mathcal{O}_{p|q}|_V)$ with $\varphi = \mathrm{id}_{(0,1)}$ and

$$\Psi : \begin{pmatrix} f_{(0,0)} \\ f_{(1,0)} \\ f_{(0,1)} \\ f_{(1,1)} \end{pmatrix} \mapsto \begin{pmatrix} f_{(0,0)} \\ f_{(1,0)} \\ f_{(0,1)} \\ f_{(1,1)} + f'_{0,0} \end{pmatrix} \tag{7}$$

[1] By giving a coordinate system in (1) I implicitely made a choice of orientation. Orientation reversing changes of coordinate systems will then change the sign of the integral. In any case a choice of orientation has to be made.

for a super function $f = \sum_{(\varepsilon_1,\varepsilon_2)} f_{(\varepsilon_1,\varepsilon_2)}\eta_1^{\varepsilon_1}\eta_2^{\varepsilon_2}$. Then $\operatorname{sdet}(J(\varphi,\Psi)) = 1$. Now let $f \in \mathcal{O}_{1|2}((0,1))$ be given by $f(y) = y$. Then

$$\int_{(0,1)} f\, d(y,\eta_1,\eta_2) = 0 \tag{8}$$

but

$$\int_{(0,1)} \Psi(f)\operatorname{sdet}(J(\varphi,\Psi))\, d(x,\theta_1,\theta_2) = \int_{(0,1)} (x+\theta_1\theta_2)\cdot 1\; d(x,\theta_1,\theta_2) = (-1)^{2\cdot 1+\frac{2\cdot 1}{2}} \int_{(0,1)} 1dx = -1. \tag{9}$$

3 Proof of the change of variables formula

Just a sketch of the proof is provided here. For the missing details the reader is referred to the literature. Write $f = f_0 + f_1$ with $f_1 := f_{(1,\dots,1)}\eta_1\cdots\eta_q$, $f_0 := f - f_1$. f_0 can then be written as

$$f_0 = \sum_{i=1}^{q} \frac{\partial}{\partial\eta_i}\tilde{f}_i. \tag{10}$$

It is obvious that $\int_V \frac{\partial}{\partial\eta_i}\tilde{f}_i\, d(y,\eta) = 0$. It can be shown (see [CdG94]) that

$$\Psi\left(\frac{\partial}{\partial\eta_i}\tilde{f}_i\right)\cdot \operatorname{sdet}\left(J(\varphi,\Psi)\right) \tag{11}$$

can still be written as a sum of terms of the form $\frac{\partial}{\partial(x,\theta)_i}h$ $(i = 1,\dots,p+q)$ for some super functions $h \in \mathcal{O}_{p|q}|_U$ with compact support. (But now even derivatives may appear). It follows with Stokes' theorem and compact support of the functions h that

$$\int_U \frac{\partial}{\partial(x,\theta)_i}h = 0 \tag{12}$$

for all $i = 1,\dots,p+q$. We can therefore assume w.r.o.g. that $f = f_{(1,\dots,1)}\eta_1\cdots\eta_q$.

Denote with $\mathcal{I}_U$ the ideal generated by $\theta_1,\dots,\theta_q$ and with $\mathcal{I}_V$ the ideal generated by $\eta_1,\dots,\eta_q$. Since Ψ is an isomorphism $\Psi(\mathcal{I}_V) = \mathcal{I}_U$ and by abuse of notation denote both ideals with $\mathcal{I}$.
We have that $f \in \mathcal{I}^q$. Then $I^q \ni \Psi(f) = h\theta_1\cdots\theta_q$ for some $h \in C_0^\infty(U)$. By the definition of the super determinant we have

$$\operatorname{sdet}\left(\mathrm{J}(\varphi,\Psi)\right) = \det\left(J(\varphi,\Psi)\right)_{00}\cdot\det\left(J(\varphi,\Psi)\right)_{11}^{-1} \mod \mathcal{I}. \tag{13}$$

We have

$$\Psi(\eta_l) = \sum_j \theta_j J(\varphi,\Psi)_{jl} \mod \mathcal{I}^2 \tag{14}$$

and a calculation shows that

$$\Psi(\eta_1)\Psi(\eta_2)\cdots\Psi(\eta_q) = \det\left(J(\varphi,\Psi)\right)_{11}\theta_1\cdots\theta_q \mod I^{q+1} = \det\left(J(\varphi,\Psi)\right)_{11}\theta_1\cdots\theta_q. \tag{15}$$

Therefore we can identify h with $\Psi(f_{(0,\dots,0)})\det\left(J(\varphi,\Psi)\right)_{11}$. Putting things together we get

$$\Psi(f)\cdot\operatorname{sdet}\left(J(\varphi,\Psi)\right) = \Psi(f_{(1,\dots,1)})\cdot\det\left(J(\varphi,\Psi)\right)_{00} \mod I. \tag{16}$$

But $(\det\left(J(\varphi,\Psi)\right)_{00} \mod I)$ is just the usual determinant of the underlying diffeomorphism and the theorem follows from the classical change of variables formula.

References

[Bä05] C. Bär. Nichtkommutative Geometrie (Skript). `http://geometrie.math.uni-potsdam.de/documents/baer/skripte/skript-NKommGeo.pdf`, 2005.

[CdG94] F. Constantinescu and H.F. de Groote. *Geometrische und algebraische Methoden der Physik: Supermannigfaltigkeiten und Virasoro-Algebren.* BG Teubner, 1994.