Seminar on Supergeometry - super domains and super manifolds

David Klein - 4 May 2011 - Reference: C. Bär - Nichtkommutative Geometrie (2005)

1 Super domains

Recall 1. U is a $\mathbb{K}$ - vector space. The **Grassmann algebra** is defined by:

$$\Lambda^* U = \bigoplus_{k \geq 0} \Lambda^k U$$

with the wedge product as multiplication. The partition of $\Lambda^* U$ into

$$(\Lambda^* U)_0 := \bigoplus_{j \geq 0} \Lambda^{2j} U \quad \text{and} \quad (\Lambda^* U)_1 := \bigoplus_{j \geq 0} \Lambda^{2j+1} U$$

gives $\Lambda^* U$ the structure of an associative super algebra.

Def. 1.1. Let $U \subset \mathbb{R}^m$ be an open subset. We consider

$$\mathcal{O}_{m|n} := C^\infty(U) \otimes_{\mathbb{R}} \Lambda^* \mathbb{R}^n$$

which can be understood more easily if we choose a basis, i.e. let $\theta_1^{\epsilon_1} \wedge \cdots \wedge \theta_n^{\epsilon_n}, \quad \epsilon_j \in \{0, 1\}$ be a basis of $\Lambda^* \mathbb{R}^n$. Hence we can express every $f \in \mathcal{O}_{m|n}(U)$ in a unique way:

$$f = \sum_{\substack{\epsilon = (\epsilon_1, \ldots, \epsilon_n) \\ \epsilon_j \in \{0,1\}}} f_\epsilon \otimes \theta_1^{\epsilon_1} \wedge \cdots \wedge \theta_n^{\epsilon_n}, \quad f_\epsilon \in C^\infty(U)$$

We call the elements f of $\mathcal{O}_{m|n}(U)$ **superfunctions** of even dimension m and odd dimension n.

Remark 1.2. Let $V \subset U \subset \mathbb{R}^n$ be open subsets. We find a morphism $\mathcal{O}_{m|n}(U) \to \mathcal{O}_{m|n}(V)$ by

$$f = \sum_\epsilon f_\epsilon \otimes \theta_1^{\epsilon_1} \wedge \cdots \wedge \theta_n^{\epsilon_n} \mapsto \sum_\epsilon f_\epsilon\big|_V \otimes \theta_1^{\epsilon_1} \wedge \cdots \wedge \theta_n^{\epsilon_n}$$

by the usual restriction $C^\infty(U) \to C^\infty(V)$. Hence, $\mathcal{O}_{m|n}$ is a **sheaf** of supercommutative super algebras. The multiplication and the neutral element in $\mathcal{O}_{m|n}(U)$ are defined by

$$(f \otimes \theta_1^{\epsilon_1} \wedge \cdots \wedge \theta_n^{\epsilon_n}) \cdot (g \otimes \theta_1^{\delta_1} \wedge \cdots \wedge \theta_n^{\delta_n}) = (fg) \otimes \theta_1^{\epsilon_1} \wedge \cdots \wedge \theta_n^{\epsilon_n} \wedge \theta_1^{\delta_1} \wedge \cdots \wedge \theta_n^{\delta_n}$$

$$1 = \sum_\epsilon f_\epsilon \otimes \theta_1^{\epsilon_1} \wedge \cdots \wedge \theta_n^{\epsilon_n}, \quad \text{with } f_\epsilon \left\{ \begin{array}{l} 1, \ \epsilon = (0, \ldots, 0) \\ 0, \ \text{otherwise} \end{array} \right\}$$

Remark 1.3. We obtain a homomorphism of algebras v_p by

$$v_p \colon \mathcal{O}_{m|n,p} \to \mathbb{R}, \quad v_p([f]_p) := f_{(0,\ldots,0)}(p)$$

We can now assemble the above concepts to obtain the triple $(\mathbb{R}^m, \mathcal{O}_{m|n}, v)$.

Theorem 1.4. $(\mathbb{R}^m, \mathcal{O}_{m|n}, v)$ is a locally ringed space.

Proof 1.5 (Bä05)**.** p. 16.

Def. 1.6. Let $U \subset \mathbb{R}^m$ be a domain, i.e. open and connected. We call $(U, \mathcal{O}_{m|n}\big|_U)$ a **super domain** of dimension m | n. The cartesian coordinates $x_1, \ldots, x_m$ of $\mathbb{R}^m$ are called the **even coordinates**, the coordinates $\theta_1, \ldots, \theta_n$ are called the **odd coordinates**.

2 Super manifolds

Def. 2.1. Let $(X, \mathcal{O}_X)$ be an arbitrary $\mathbb{R}$-super ringed space. A **super chart** of dim m | n of $(X, \mathcal{O}_X)$ consists of open subsets $U \subset X,\ V \subset \mathbb{R}^m$ and an isomorphism of $\mathbb{R}$-super ringed spaces:

$$(\phi, \psi)\colon (U, \mathcal{O}_x\big|_U) \to (V, \mathcal{O}_{m|n}\big|_V)$$

Def. 2.2. A $\mathbb{R}$-super ringed space $(X, \mathcal{O}_X)$ is a **super manifold** if

1. X is Hausdorff.
2. The topology $\mathcal{T}_X$ of X possesses a countable basis.
3. Each point $p \in X$ is in the Domain of a super chart.

Def. 2.3. A **morphism of super manifolds** is a morphism of $\mathbb{R}$-ringed spaces.

Example 2.4. Let M be an m-dimensional smooth manifold. We find that (M, C^∞_M) is a super manifold of dimension m | 0, since we can construct a super chart from any chart $\phi\colon M \supset U \to V \supset \mathbb{R}^m$ of the manifold M, by chosing

$$\psi(f) = f \circ \phi = \phi^*(f)$$

and finally constructing

$$(\phi, \psi)\colon (U, C^\infty_U) \to (V, C^\infty_V)$$

which is an isomorphism of $\mathbb{R}$-super ringed spaces and

$$C^\infty_V(V') = C^\infty(V') \otimes_{\mathbb{R}} \mathbb{R} = C^\infty(V') \otimes_{\mathbb{R}} \Lambda^* \mathbb{R}^0 = \mathcal{O}_{m|0}(V')$$

Theorem 2.5. Let $(X, \mathcal{O}_X)$ be a super manifold. It holds:

1. X possesses exactly one differentiable structure such that for any super chart $(\phi, \psi)\colon (U, \mathcal{O}_x\big|_U) \to (V, \mathcal{O}_{m|n}\big|_V)$, the map $\phi\colon U \to V$ is a diffeomorphism.
2. There exists exactly one 1-preserving homomorphism

   $$\beta\colon \mathcal{O}_X \to C^\infty_X$$

   of sheaves of $\mathbb{R}$-algebras.
3. Let $U \in \mathcal{T}_X$. We define $\mathcal{O}^1(U) := \{f \in \mathcal{O}_X(U) | \text{f is nilpotent}\}$.

   Then we find that the sequence

   $$0 \to \mathcal{O}^1(U) \hookrightarrow \mathcal{O}_X(U) \xrightarrow{\beta_U} C^\infty(U) \to 0$$

   is exact, i.e. β is surjective and $\ker\beta_U = \mathcal{O}^1(U)$.
4. Let $(Y, \mathcal{O}_Y)$ be another super manifold and $(\phi, \psi)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of super manifolds. Then $\phi\colon X \to Y$ is smooth.
5. For any $V \in \mathcal{T}_Y$ the diagram

   $$\begin{array}{ccc} \mathcal{O}_X(\phi^{-1}(V)) & \xleftarrow[\psi_V]{} & \mathcal{O}_Y(V) \\ \big\downarrow{\scriptstyle \beta_{\phi^{-1}(V)}} & & \big\downarrow{\scriptstyle \beta_V} \\ C^\infty(\phi^{-1}(V)) & \xleftarrow{\phi^*} & c^\infty(V) \end{array}$$

   commutes.

Proof 2.6 (Bä05)**.** p. 21.