Reconstruction of tensor categories from their invariants

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Overview

Section 1

Introduction

Consider the finite dimensional representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The simple objects of this module category are irreducible representations V_m of dimension $m+1$ for each $m \in \mathbb{N}$. The tensor product is determined by the Glebsch-Gordan formula:

$$V_n \otimes V_m = V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{n-m+2} \oplus V_{n-m}, \text{ for } n \geq m.$$

This module category is an abelian, Krull-Schmidt tensor category.

Definition (Representation Ring)

Let $\mathcal{C}$ be an abelian, Krull-Schmidt tensor category. The Green ring or representation ring $r(\mathcal{C})$ of $\mathcal{C}$ is the abelian group generated by the isomorphism classes $[V]$ of $\mathcal{C}$ modulo the relations $[V \oplus W] = [M] + [V]$. The multiplication is given by the tensor product, i.e. $[V][W] = [V \otimes W]$.

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Consider again the representation category of $\mathfrak{sl}_2(\mathbb{C})$. It is easy to see that the representation ring is generated by $[V_m]$, the classes of simple objects. By the Glebsch-Gordan formula, $[V_n][V_m] = \sum_{k\in\mathbb{N}} c_{n,m,k}[V_k]$ with $c_{n,m,k} \in \mathbb{N}$.

Question

Which rings can arise as the representation ring of an abelian, Krull-Schmidt tensor category?

Example

Suppose that $\mathbb{Z}[i] = r(\mathcal{C})$ for some Krull-Schmidt tensor category $\mathcal{C}$. Then $\mathcal{C}$ has (at least) two indecomposable objects. We assume that the unit object for the tensor product is simple, hence this object corresponds to $1 \in r(\mathcal{C})$. Let X denote the other indecomposable object. Then X corresponds to $a + bi \in r(\mathcal{C})$ with $b \neq 0$. Since $(a + bi)^2 = -(a^2 + b^2) \cdot 1 + 2a \cdot (a + bi)$, $X \otimes X \cong -(a^2 + b^2)1 \oplus 2aX$, but this makes no sense. Hence $\mathbb{Z}[i]$ is not the representation ring of such a category. Notice that $\mathbb{Z}[i]$ is not a unital $\mathbb{Z}_+$-ring.

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Definition

- A $\mathbb{Z}_+$-basis of a ring free as a module over $\mathbb{Z}$ is a $\mathbb{Z}$-basis $B = \{r_i\}_{i \in I}$ such that for any $i, j \in I$, $r_i r_j = \sum_k c_{ijk} r_k$ with $c_{ijk} \in \mathbb{N}$.
- A $\mathbb{Z}_+$-ring is a $\mathbb{Z}$-algebra with unit endowed with a $\mathbb{Z}_+$-basis.
- A $\mathbb{Z}_+$-ring with a $\mathbb{Z}_+$-basis B is unital if $1 \in B$.
- A $\mathbb{Z}_+$-module over a $\mathbb{Z}_+$-ring is a free $\mathbb{Z}$-module M endowed with a fixed basis $\{m_j\}_{j \in J}$ such that $r_i m_j = \sum_k d_{ijk} m_k$ with $d_{ijk} \in \mathbb{N}$.

- Let A be an $\mathbb{F}$-algebra with a complete set of orthogonal primitive idempotents $\{e_i\}_{i\in I}$ satisfying $1 = \sum_i e_i$ and $|I| = n$.
- For $\mathbf{m} = (m_i)_{i\in I} \in \mathbb{N}^I$, let $|\mathbf{m}| = \sum_{i\in I} m_i$.
- For $\mathbf{m}, \mathbf{s} \in \mathbb{N}^I$, an $(\mathbf{m}, \mathbf{s})$-type matrix X over A is a block matrix

$$X = \begin{pmatrix} X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & X_{22} & \dots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \dots & X_{nn} \end{pmatrix}$$

with $X_{ij} \in M_{m_i \times s_j}(e_i A e_j)$. Notice that X is an $|\mathbf{m}| \times |\mathbf{s}|$-matrix over A.

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Let $X \in M_{\mathbf{m}\times\mathbf{s}}(A)$ and $Y \in M_{\mathbf{m}'\times\mathbf{s}}(A)$. Then we define the horizontal sum of X and Y as

$$X \underline{\oplus} Y := \begin{pmatrix} X_{11} & X_{12} & \dots & X_{1n} \\ Y_{11} & Y_{12} & \dots & Y_{1n} \\ X_{21} & X_{22} & \dots & X_{2n} \\ Y_{21} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \dots & X_{nn} \\ Y_{n1} & Y_{n2} & \dots & Y_{nn} \end{pmatrix}.$$

Similarly, we define the vertical sum of X and Y:

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Let $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_l \in \mathbb{N}^l$. For a matrix

$$X = \begin{pmatrix} X_{11} & X_{12} & \ldots & X_{1l} \\ X_{21} & X_{22} & \ldots & X_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ X_{r1} & X_{r2} & \ldots & X_{rl} \end{pmatrix}$$

over A with $X_{ij} \in M_{\mathbf{m}_i \times \mathbf{s}_j}(A)$, define a matrix $\prod(X)$ over A by

$$\prod(X) := (X_{11} \overline{\oplus} X_{12} \overline{\oplus} \ldots \overline{\oplus} X_{1l}) \underline{\oplus} \ldots \underline{\oplus} (X_{r1} \overline{\oplus} X_{r2} \overline{\oplus} \ldots \overline{\oplus} X_{rl}).$$

Then $\prod(X) \in M_{\mathbf{m} \times \mathbf{s}}(A)$ with $\mathbf{m} = \sum_{i=1}^r \mathbf{m_i}$ and $\mathbf{s} = \sum_{j=1}^l \mathbf{s}_j$.

Let A and B be two $\mathbb{F}$-algebras. Let

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1s} \\ x_{21} & x_{22} & \cdots & x_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{ms} \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1s'} \\ y_{21} & y_{22} & \cdots & y_{2s'} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m'1} & y_{m'2} & \cdots & y_{m's'} \end{pmatrix}$$

be an $m \times s$-matrix over A and an $m' \times s'$-matrix over B respectively. Then we define an $mm' \times ss'$-matrix $X \otimes_{\mathbb{F}} Y$ over $A \otimes_{\mathbb{F}} B$ by

$$\begin{pmatrix} x_{11} \otimes y_{11} & \cdots & x_{1s} \otimes y_{11} & \cdots & x_{11} \otimes y_{1s'} & \cdots & x_{1s} \otimes y_{1s'} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ x_{m1} \otimes y_{11} & \cdots & x_{ms} \otimes y_{11} & \cdots & x_{m1} \otimes y_{1s'} & \cdots & x_{ms} \otimes y_{1s'} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{11} \otimes y_{m'1} & \cdots & x_{1s} \otimes y_{m'1} & \cdots & x_{11} \otimes y_{m's'} & \cdots & x_{1s} \otimes y_{m's'} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ x_{m1} \otimes y_{m'1} & \cdots & x_{ms} \otimes y_{m'1} & \cdots & x_{m1} \otimes y_{m's'} & \cdots & x_{ms} \otimes y_{m's'} \end{pmatrix}$$

Lemma

Let $X \in M_{m\times s}(A), X_1 \in M_{s\times t}(A)$ *and* $Y \in M_{m'\times s'}(A), Y_1 \in M_{s'\times t'}(B)$. *Then* $(X \otimes Y)(X_1 \otimes Y_1) = (XX_1) \otimes (YY_1)$.

Definition

- An $(\mathbf{m}, \mathbf{s})$-type matrix X is called column-independent if for any $(\mathbf{s}, \mathbf{l})$-type matrix Y,$XY = 0 \Rightarrow Y = 0$.
- Similarly, we define row-independent.
- An $(\mathbf{s}, \mathbf{t})$-type matrix Y is a right universal annihilator of X if $XY = 0$ and $XM = 0 \Rightarrow M = YZ$ for a unique Z.
- Similarly, we define left universal annihilators.

Section 2

Construction of tensor categories from given data

Goal of the construction

- We will construct an abelian, Krull-Schmidt tensor category $\mathcal{C}$ from the data

$$(R, A, I, \{e_i \mid i \in I\}, \phi, \{a_{ijl} \mid i, j, l \in I\})$$

  s.t.
  - $R = r(\mathcal{C})$.
  - A is the Auslander algebra of $\mathcal{C}$.
- We will give a workable criterion when two such categories are tensor equivalent.

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- We will give a workable criterion when two such categories are tensor equivalent.

First assumptions

- R is a unital $\mathbb{Z}_+$-ring with finite unital basis $\{r_i\}_{i\in I}$ with $I = \{1, 2, \ldots, n\}$ and $r_1 = 1$ and $r_i r_j \neq 0$.
- A is a finite dimensional $\mathbb{F}$-algebra with a complete set of orthogonal primitive idempotents $\{e_i\}_{i\in I}$.
- $\mathbb{F}$ is algebraically closed.
- (KS) $e_i A e_j A e_i \subset \mathrm{rad}(e_i A e_i)$ for $i \neq j$.
- (Dec) Any $(\mathbf{m}, \mathbf{s})$-type matrix can be written as the product of CI and RI matrices.
- (RUA) Any matrix has right universal annihilator.
- (LUA) Any matrix has left universal annihilator.
- (CI) If X is CI and Y a LUA of X, then X is RUA of Y.
- (RI) If X is RI and Y a RUA of X, then Y is LUA of X.

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Construction of Category

Definition

We define a category $\mathcal{C}$ by

1. $\text{Obj}(\mathcal{C}) := \mathbb{N}^I$;
2. $\text{Hom}_{\mathcal{C}}(\mathbf{m}, \mathbf{s}) := M_{\mathbf{s}\times\mathbf{m}}(A)$;
3. Composition is usual matrix product.

Lemma

- *Let* $\mathbf{m}, \mathbf{s} \in Obj(\mathcal{C})$, *then* $\mathbf{m} + \mathbf{s} \cong \mathbf{m} \oplus \mathbf{s}$.
- *Moreover,* $\mathcal{C}$ *is an additive category over* $\mathbb{F}$.
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Krull-Schmidtness

Definition

Define $\mathbf{e_i} := (0, \ldots, 0, 1, 0 \ldots, 0)$. Then for any $\mathbf{m} \in \mathrm{Obj}(\mathcal{C})$, we have

$$\mathbf{m} \cong \bigoplus_{i \in I} m_i \mathbf{e_i}.$$

Moreover, the $\mathbf{e_i}$'s are all non-isomorphic indecomposable objects.

Proposition

$\mathcal{C}$ is an abelian Krull-Schmidt category over $\mathbb{F}$.

Corollary

The following are equivalent:

1. *$\mathcal{C}$ is a semisimple category over $\mathbb{F}$;*
2. $\dim_{\mathbb{F}}(A) = n$;
3. *$A \cong \mathbb{F}^n$ as $\mathbb{F}$-algebras.*

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- Put $\mathbf{c}_{ij} = (c_{ij1}, \dots, c_{ijn})$ with $r_i r_j = \sum_{k=1}^{n} c_{ijk} r_k$.
- We define $\mathbf{m} \otimes \mathbf{s} := \sum_{i,j}^{n} m_i s_j \mathbf{c}_{ij}$.
- Define an $\mathbb{F}$-algebra $M(R, A, I) := \bigoplus_{1 \le i, i', j, j' \le n} M_{\mathbf{c}_{i'j'} \times \mathbf{c}_{ij}}(A)$.
- If $X \in M_{\mathbf{c}_{i'j'} \times \mathbf{c}_{ij}}$, $Y \in M_{\mathbf{c}_{i''j''} \times \mathbf{c}_{i_1 j_1}}$, then XY is the usual matrix product if $(i, j) = (i'', j'')$ and zero otherwise.

Assumption

Assume that $\dim(e_1 A e_1) = 1$ and $\exists \phi : A \otimes A \to M(R, A, I)$ s.t.

1. $\phi(e_1 \otimes e_j) = E_{\mathbf{c}_{ij}} \in M_{\mathbf{c}_{ij}}(A)$;
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Let $X \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{m}_1, \mathbf{s}_1)$, $Y \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{m}_2, \mathbf{s}_2)$. We define $X \otimes Y$ over A by

- if $\mathbf{m}_1 \otimes \mathbf{m}_2 = \mathbf{0}$ and $\mathbf{s}_1 \otimes \mathbf{s}_2 = \mathbf{0}$, then $X \otimes Y := 0 \in M_{1\times 1}(A)$;
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- if $\mathbf{m}_1 \otimes \mathbf{m}_2 \neq \mathbf{0}$ and $\mathbf{s}_1 \otimes \mathbf{s}_2 \neq \mathbf{0}$, then $X \otimes Y := \prod(\phi(X \otimes_{\mathbb{F}} Y))$.

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$X \otimes Y \in Hom_{\mathcal{C}}(\mathbf{m}_1 \otimes \mathbf{m}_2, \mathbf{s}_1 \otimes \mathbf{s}_2)$. Moreover

1. $E_{\mathbf{m}} \otimes E_{\mathbf{s}} = E_{\mathbf{m}\otimes\mathbf{s}}$;
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The Associativity Constraint

Definition

Set $Y^{\mathbf{m}}_{i,k} = (y_1, \ldots, y_{|\mathbf{m}|}) \in M_{\mathbf{e}_i \times \mathbf{m}}(A)$, where

$$y_j = \begin{cases} e_i, & j = k + \sum_{1 \le l < i} m_l \\ 0 & \text{otherwise.} \end{cases}$$

Put $X^{\mathbf{m}}_{i,k} = (Y^{\mathbf{m}}_{i,k})^T$. Then $Y^{\mathbf{m}}_{i,k} X^{\mathbf{m}}_{i',k'} = E_{\mathbf{e}_i}$ if $(i,k) = (i',k')$ and zero otherwise, and

$$\sum_{i=1}^{n} \sum_{1 \le k \le m_i} X^{\mathbf{m}}_{i,k} Y^{\mathbf{m}}_{i,k} = E_{\mathbf{m}}.$$

Assumptions on Associatvity Constraint

Assumptions

There exists a family of matrices $a_{ijl} \in M_{\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_l}(A)$ s.t.

1. a_{ijl} is invertible;
2. $(x \otimes (y \otimes z))a_{ijl} = a_{i'j'l'}((x \otimes y) \otimes z)$ for $x \in e_{i'}Ae_i, y \in e_{j'}Ae_j, z \in e_{l'}Ae_l$;
3. $a_{i1j} = E_{\mathbf{c}_{ij}}$;
4. $$\sum_{j=1}^{n}\sum_{k=1}^{c_{i_2 i_3 j}}(e_{i_1} \otimes a_{i_2 i_3 i_4}(X_{jk}^{\mathbf{c}_{i_2 i_3}} \otimes e_{k_4}))a_{i_1 j i_4}((e_{i_1} \otimes Y^{\mathbf{c}_{i_2 i_3}})a_{i_1 i_2 i_3} \otimes e_{i_4}) =$$
$$\sum_{j,j'=1}^{n}\sum_{k=1}^{c_{i_3 i_4 j}}\sum_{k'=1}^{c_{i_1 i_2 j'}}(e_{i_1} \otimes (e_{i_2} \otimes X_{jk}^{\mathbf{c}_{i_3 i_4}}))a_{i_1 i_2 j}(X_{j'k'}^{\mathbf{c}_{i_1 i_2}} \otimes Y_{jk}^{\mathbf{c}_{i_3 i_4}})$$
$$\cdot a_{j' i_3 i_4}((Y_{j'k'}^{\mathbf{c}_{i_1 i_2}} \otimes e_{i_3}) \otimes e_{i_4}).$$

$\mathcal{C}$ is monoidal

Theorem

$(\mathcal{C}, \otimes, \mathbf{e}_1, a, l = Id, r = Id)$ is a tensor category over $\mathbb{F}$, $r(\mathcal{C}) \cong R$ and $End_{\mathcal{C}}(\bigoplus_{i=1}^{n} \mathbf{e}_i) \cong A$ as $\mathbb{F}$-algebras.

Remark

In general $\mathbf{e}_1$ is not simple, but TFAE:

- $\mathbf{e}_1$ is a simple object of $\mathcal{C}$;
- If $X \in M_{\mathbf{e}_1 \times \mathbf{m}}(A)$ is CI, then either $\mathbf{m} = 0$ and $X = 0$, or $\mathbf{m} = \mathbf{e}_1$ and $X = \alpha e_1$ for some $\alpha \in \mathbb{F}_0$.

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Other associativity constraints

Definition

$\{a_{ijl}\}$ and $\{a_{i'j'l'}\}$ are called equivalent if there exists a familiy of invertible matrices $\eta_{i,j} \in M_{\mathbf{c}_{ij}}(A)$ s.t.

1. $(x \otimes y)\eta_{i,j} = \eta_{i',j'}(x \otimes y)$;
2. $$\sum_{t=1}^{n} \sum_{k=1}^{c_{ijt}} a_{ijl}(X_{tk}^{\mathbf{c}_{ij}} \otimes e_l)\eta(t,l)(Y_{tk}^{\mathbf{c}_{ij}}\eta(i,j) \otimes e_l)$$ $$= \sum_{t=1}^{n} \sum_{k=1}^{c_{jlt}} (e_i \otimes X_{tk}^{\mathbf{c}_{jl}})\eta(i,t)(e_i \otimes Y_{tk}^{\mathbf{c}_{jl}}\eta(j,l)a'_{ijl}).$$

Proposition

$(\mathcal{C}, \otimes, \mathbf{e}_1, a, l, r)$ and $(\mathcal{C}, \otimes, \mathbf{e}_1, a', l, r)$ are equivalent if $\{a_{ijl}\}$ and $\{a'_{ijl}\}$ are equivalent.

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Uniqueness

Theorem

$(\mathcal{C}, \otimes, \mathbf{e}_1, a, l, r)$ and $(\mathcal{C}', \otimes_{\mathcal{C}'}, \mathbf{e}'_1, a', l', r')$ are tensor equivalent if and only if $n = n'$ $(I = I')$ and there exists a $\sigma \in S(I)$ such that:

1. *$R \to R' : r_i \mapsto r'_i$ is a ring isomorphism;*
2. *there exists a $\mathbb{F}$-algebra map $\delta : A \to A'$ with $\delta(e_i) = e'_{\sigma(i)}$;*
3. *there exists an $\alpha \in \mathbb{F}_0$ and a family of invertible elements $\phi_{i,j} \in M_{\mathbf{c}'_{\sigma(i)\sigma(j)}}(A')$ s.t.*
    1. $\phi_{1,i} = \phi_{i,1} = \alpha E'_{\mathbf{e}^{\sigma}_i}$;
    2. $\phi_{i',j'}(\delta(x) \otimes_{\mathcal{C}'} \delta(y)) = P_\sigma(\mathbf{c}_{i'j'})\delta(x \otimes_{\mathcal{C}} y)P_\sigma(\mathbf{c_{ij}})^T \phi_{i,j}$;
    3. $$\sum_{t=1}^n \sum_{k=1}^{c_{ijt}} \delta(a_{i,j,l})\delta(X_{t,k}^{\mathbf{c}_{ij}} \otimes_{\mathcal{C}} e_l)P_\sigma(\mathbf{c}_{tl})^T \phi_{t,l}(Y_{\sigma(t),k}^{'\mathbf{c}'_{\sigma(i)\sigma(j)}} \phi_{i,j} \otimes_{\mathcal{C}'} e'_{\sigma(l)}) =$$
    $$\sum_{t=1}^n \sum_{k=1}^{c_{jlt}} \delta(e_i \otimes_{\mathcal{C}} X_{t,k}^{\mathbf{c}_{jl}})P_\sigma(\mathbf{c}_{it})^T \phi_{i,t}(e'_{\sigma(i)} \otimes_{\mathcal{C}'} Y_{\sigma(t),k}^{'\mathbf{c}'_{\sigma(j)\sigma(l)}} \phi_{j,l})a_{\sigma(i),\sigma(j),\sigma(l)'}$$

    for all $x \in e'_i A e_i$, $y \in e'_j A e_j$.

Section 3

Invariants of tensor categories

Let $\mathcal{C}$

- be an abelian, Krull-Schmidt tensor category over $\mathbb{F}$;
- have finitely many indecomposable objects;
- have finite-dimensional Hom-spaces;
- be strict and the unit object $\mathbf{1}$ be simple.

Let $\{V_i \mid i \in I\}$ be a set of representatives of the isomorphism classes of the indecomposable objects of $\mathcal{C}$ ($\mathbf{1} = V_1$). We also assume that $U \otimes V \neq 0$ for all nonzero objects.

Goal of the section

We will associate data $(r(\mathcal{C}), A(\mathcal{C}), I, \{e_i \mid i \in I\}, \phi_{\mathcal{C}}, \{a_{ijl} \mid i, j, l \in I\})$ to C. By the previous section we can then construct a category $\widehat{C}$. We will show that $\widehat{C}$ is tensor equivalent to $\mathcal{C}$.

Definition

Let $V = \bigoplus_{i \in I} V_i$ and $A(\mathcal{C}) = \mathrm{End}_{\mathcal{C}}(V) = \mathrm{Hom}_{\mathcal{C}}(V, V)$. Then $A(\mathcal{C})$ is a finite-dimensional $\mathbb{F}$-algebra. Let $\pi_i : V \to V_i$ and $\tau_i : V_i \to V$ be the canonical projections and injections. Then $Id_V = \sum_{i \in I} \tau_i \circ \pi_i$ and

$$\pi_i \circ \tau_j = \begin{cases} Id_{V_i}, & \text{if } i = j, \\ 0, & \text{else.} \end{cases}$$

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The condtions are satisfied

Proposition

The set $\{e_i \mid i \in I\}$ forms a complete set of orthogonal primitive idempotents of $A(\mathcal{C})$. Moreover, for all $i \neq j$, $e_iA(\mathcal{C})e_jA(\mathcal{C})e_i \subset rad(e_iA(\mathcal{C})e_i)$. Hence if $f \in Hom(V_i, V_j), g \in Hom(V_j, V_i)$, then $gf \in End(V_i)$.

Proposition

Let X be an $(\mathbf{m}, \mathbf{s})$-matrix over $A(\mathcal{C})$, then

1. *X has a right universal annihilator.*
2. *X has a left universal annihilator.*
3. *There is a CI $(\mathbf{m}, \mathbf{t})$-matrix X_1 and a RI $(\mathbf{t}, \mathbf{s})$-matrix X_2 s.t. $X = X_1X_2$.*
4. *If X is CI and Y is a LUA of X, then X is a RUA of Y.*
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Definition

We write $[V_i][V_j] = \sum_{k\in I} c_{ijk}[V_k]$ in $r(\mathcal{C})$. Let $\mathbf{c}_{ij} := (c_{ijk})_{k\in I} \in \mathbb{N}^I$. Define a vector space $M(\mathcal{C})$ by

$$M(\mathcal{C}) := \bigoplus_{i,i',j,j'\in I} M_{\mathbf{c}_{i'j'}\times \mathbf{c}_{ij}}(A(\mathcal{C})).$$

Then $M(\mathcal{C})$ is an $\mathbb{F}$-algebra as before.

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There exists an algebra map $\phi_{\mathcal{C}} : A(\mathcal{C}) \otimes_{\mathbb{F}} A(\mathcal{C}) \to M(\mathcal{C})$ s.t.

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Tensor product of morphisms

- For $\mathbf{m} = (m_i)_{i\in I}, \mathbf{s} = (s_i)_{i\in I}$, we identify $\mathrm{Hom}_{\mathcal{C}}(\bigoplus_{i\in I} m_i V_i, \bigoplus_{i\in I} s_i V_i) = M_{\mathbf{s}\times\mathbf{m}}(A(\mathcal{C}))$.
- For $f \in \mathrm{Hom}_{\mathcal{C}}(\bigoplus_{i\in I} m_i V_i, \bigoplus_{i\in I} s_i V_i)$, the corresponding matrix is

$$\begin{pmatrix} f_{11} & f_{12} & \dots & f_{1m} \\ f_{21} & f_{22} & \dots & f_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s1} & f_{s2} & \dots & f_{sm} \end{pmatrix}$$

where $f_{kl} = Y^{\mathbf{s}}_{i,k_1} f X^{\mathbf{m}}_{j,k_2}$ if $k = \sum_{t=1}^{i-1} s_t + k_1$ and $l = \sum_{t=1}^{j-1} m_t + k_2$.

- $Y^{\mathbf{s}}_{i,k_1} \in \mathrm{Hom}_{\mathcal{C}}(\bigoplus_{j\in I} s_j V_j, V_i)$ is the projection from $\bigoplus_{j\in I} s_j V_j$ to the k_1-th V_i of the direct summand $s_i V_i$ of $\bigoplus_{j\in I} s_j V_j$.
- $X^{\mathbf{m}}_{j,k_2} \in \mathrm{Hom}_{\mathcal{C}}(V_j, \bigoplus_{i\in I} m_i v_i)$ is the embedding of V_j into the k_2-th V_j of the direct summand $m_j V_j$ of $\bigoplus_{i\in I} m_i V_i$.
- As before, let $\mathbf{m} \otimes \mathbf{s} = \sum_{i,j\in I} m_i s_j \mathbf{c}_{ij}$.
- Let $X \in M_{\mathbf{s}_1\times\mathbf{m}_1}(A(\mathcal{C}))$, $Y \in M_{\mathbf{s}_2\times\mathbf{m}_2}(A(\mathcal{C}))$. then

$$X \in \mathrm{Hom}_{\mathcal{C}}(\bigoplus_{i\in I} m_{1i} V_i, \bigoplus_{i\in I} s_{1i} V_i), Y \in \mathrm{Hom}_{\mathcal{C}}(\bigoplus_{i\in I} m_{2i} V_i, \bigoplus_{i\in I} s_{2i} V_i).$$

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- $Y^{\mathbf{s}}_{i,k_1} \in \mathrm{Hom}_{\mathcal{C}}(\bigoplus_{j\in I} s_j V_j, V_i)$ is the projection from $\bigoplus_{j\in I} s_j V_j$ to the k_1-th V_i of the direct summand $s_i V_i$ of $\bigoplus_{j\in I} s_j V_j$.
- $X^{\mathbf{m}}_{j,k_2} \in \mathrm{Hom}_{\mathcal{C}}(V_j, \bigoplus_{i\in I} m_i v_i)$ is the embedding of V_j into the k_2-th V_j of the direct summand $m_j V_j$ of $\bigoplus_{i\in I} m_i V_i$.
- As before, let $\mathbf{m} \otimes \mathbf{s} = \sum_{i,j\in I} m_i s_j \mathbf{c}_{ij}$.
- Let $X \in M_{\mathbf{s}_1\times\mathbf{m}_1}(A(\mathcal{C}))$, $Y \in M_{\mathbf{s}_2\times\mathbf{m}_2}(A(\mathcal{C}))$. then

$$X \in \mathrm{Hom}_{\mathcal{C}}(\bigoplus_{i\in I} m_{1i} V_i, \bigoplus_{i\in I} s_{1i} V_i), Y \in \mathrm{Hom}_{\mathcal{C}}(\bigoplus_{i\in I} m_{2i} V_i, \bigoplus_{i\in I} s_{2i} V_i).$$

Tensor product of morphisms

- For $\mathbf{m} = (m_i)_{i \in I}, \mathbf{s} = (s_i)_{i \in I}$, we identify $\mathrm{Hom}_{\mathcal{C}}(\bigoplus_{i \in I} m_i V_i, \bigoplus_{i \in I} s_i V_i) = M_{\mathbf{s} \times \mathbf{m}}(A(\mathcal{C}))$.
- For $f \in \mathrm{Hom}_{\mathcal{C}}(\bigoplus_{i \in I} m_i V_i, \bigoplus_{i \in I} s_i V_i)$, the corresponding matrix is

$$\begin{pmatrix} f_{11} & f_{12} & \dots & f_{1m} \\ f_{21} & f_{22} & \dots & f_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s1} & f_{s2} & \dots & f_{sm} \end{pmatrix}$$

where $f_{kl} = Y^{\mathbf{s}}_{i,k_1} f X^{\mathbf{m}}_{j,k_2}$ if $k = \sum_{t=1}^{i-1} s_t + k_1$ and $l = \sum_{t=1}^{j-1} m_t + k_2$.

- $Y^{\mathbf{s}}_{i,k_1} \in \mathrm{Hom}_{\mathcal{C}}(\bigoplus_{j \in I} s_j V_j, V_i)$ is the projection from $\bigoplus_{j \in I} s_j V_j$ to the k_1-th V_i of the direct summand $s_i V_i$ of $\bigoplus_{j \in I} s_j V_j$.
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Tensor product of morphisms

- Let $X \otimes Y$ denote the tensor product of X and Y in $\mathcal{C}$. Then

$$X \otimes Y \in \mathrm{Hom}_{\mathcal{C}}((\bigoplus_{i\in I} m_{1i} V_i) \otimes (\bigoplus_{i\in I} m_{2i} V_i), (\bigoplus_{i\in I} s_{1i} V_i) \otimes (\bigoplus_{i\in I} s_{2i} V_i)).$$

- Let $X \widetilde{\otimes} Y := \prod(\phi_{\mathcal{C}}(X \otimes_{\mathbb{F}} Y)) \in M_{(\mathbf{s}_1 \otimes \mathbf{s}_2)\times(\mathbf{m}_1 \otimes \mathbf{m}_2)}(A(\mathcal{C}))$. We now have two different tensor products!

Lemma

Let $x \in e_{i'} A(\mathcal{C}) e_i, y \in e_{j'} A(\mathcal{C}) e_j$. Then the following diagram commutes:

$$\begin{array}{ccc} V_i \otimes V_j & \xrightarrow{\theta_{ij}} & \bigoplus_{k\in I} c_{ijk} V_k \\ \downarrow{\scriptstyle x\otimes y} & & \downarrow{\scriptstyle x\widetilde{\otimes} y} \\ V_{i'} \otimes V_{j'} & \xrightarrow{\theta_{i'j'}} & \bigoplus_{k\in I} c_{i'j'k} V_k \end{array}$$

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Towards another associativity constraint

Definition

Let $\mathbf{m} \in \mathbb{N}^I$, then we define $V^{(\mathbf{m})} := \bigoplus_{i \in I} m_i V_i$. For $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{N}^I$, define a morphism $\theta(\mathbf{m}_1, \mathbf{m}_2) : V^{(\mathbf{m}_1)} \otimes V^{(\mathbf{m}_2)} \to V^{(\mathbf{m}_1 \otimes \mathbf{m}_2)}$ in $\mathcal{C}$ by

$$\theta(\mathbf{m}_1, \mathbf{m}_2) = \sum_{i,j=1}^{n} \sum_{k_1=1}^{m_{1i}} \sum_{k_2=1}^{m_{2j}} (X_{i,k_1}^{\mathbf{m}_1} \widetilde{\otimes} X_{j,k_2}^{\mathbf{m}_2}) \theta_{ij} (Y_{i,k_1}^{\mathbf{m}_1} \otimes Y_{j,k_2}^{\mathbf{m}_2}).$$

Then $\theta(\mathbf{e}_i, \mathbf{e}_j) = \theta_{ij}$.

Lemma

Each $\theta(\mathbf{m}_1, \mathbf{m}_2)$ is an isomorphism. Moreover, the following diagram commutes:

$$\begin{array}{ccc} V^{(\mathbf{m}_1)} \otimes V^{(\mathbf{m}_2)} & \xrightarrow{\theta(\mathbf{m}_1, \mathbf{m}_2)} & V^{(\mathbf{m}_1 \otimes \mathbf{m}_2)} \\ \downarrow {\scriptstyle X \otimes Y} & & \downarrow {\scriptstyle X \widetilde{\otimes} Y} \\ V^{(\mathbf{s}_1)} \otimes V^{(\mathbf{s}_2)} & \xrightarrow{\theta(\mathbf{s}_1, \mathbf{s}_2)} & V^{(\mathbf{s}_1 \otimes \mathbf{s}_2)} \end{array}$$

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The associativity constraint

Definition

Define $a_{ijl} \in M_{\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_l}(A(\mathcal{C})) = \mathrm{End}_{\mathcal{C}}(V^{(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_l)})$ by

$$a_{ijl} := \theta(\mathbf{e}_i, \mathbf{e}_j \otimes \mathbf{e}_l)(E_{\mathbf{e}_i} \otimes \theta(\mathbf{e}_j, \mathbf{e}_l))(\theta(\mathbf{e}_i, \mathbf{e}_j)^{-1} \otimes E_{\mathbf{e}_l})\theta(\mathbf{e}_i \otimes \mathbf{e}_j, \mathbf{e}_l)^{-1}.$$

Proposition

The family $\{a_{ijl}\}$ satisfies the four conditions encountered earlier w.r.t. the tensor product $\widetilde{\otimes}$.

Denote by $\widehat{\mathcal{C}}$ the category associated to the data $(r(\mathcal{C}), A(\mathcal{C}), I, \{e_i \mid i \in I\}, \phi_{\mathcal{C}}, \{a_{ijl} \mid i, j, l \in I\})$. The tensor products of $\mathcal{C}$ and $\widehat{\mathcal{C}}$ are denoted by $\otimes$ and $\widetilde{\otimes}$ respectively.

Theorem

$\widehat{\mathcal{C}}$ and $\mathcal{C}$ are tensor equivalent.

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Define $a_{ijl} \in M_{\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_l}(A(\mathcal{C})) = \mathrm{End}_{\mathcal{C}}(V^{(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_l)})$ by

$$a_{ijl} := \theta(\mathbf{e}_i, \mathbf{e}_j \otimes \mathbf{e}_l)(E_{\mathbf{e}_i} \otimes \theta(\mathbf{e}_j, \mathbf{e}_l))(\theta(\mathbf{e}_i, \mathbf{e}_j)^{-1} \otimes E_{\mathbf{e}_l})\theta(\mathbf{e}_i \otimes \mathbf{e}_j, \mathbf{e}_l)^{-1}.$$

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Thank You!