Reconstruction of tensor categories from their invariants

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Universität Hamburg, June 8, 2017
Overview

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Section 1

Introduction
Consider the finite dimensional representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The simple objects of this module category are irreducible representations $V_m$ of dimension $m + 1$ for each $m \in \mathbb{N}$. The tensor product is determined by the Glebsch-Gordan formula:

$$V_n \otimes V_m = V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{n-m+2} \oplus V_{n-m}, \text{ for } n \geq m.$$ 

This module category is an abelian, Krull-Schmidt tensor category.

**Definition (Representation Ring)**

Let $\mathcal{C}$ be an abelian, Krull-Schmidt tensor category. The Green ring or representation ring $r(\mathcal{C})$ of $\mathcal{C}$ is the abelian group generated by the isomorphism classes $[V]$ of $\mathcal{C}$ modulo the relations $[V \oplus W] = [M] + [V]$. The multiplication is given by the tensor product, i.e. $[V][W] = [V \otimes W]$. 

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[V \oplus W] = [M] + [V].
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The multiplication is given by the tensor product, i.e. \([V][W] = [V \otimes W]\).
Consider again the representation category of $\mathfrak{sl}_2(\mathbb{C})$. It is easy to see that the representation ring is generated by $[V_m]$, the classes of simple objects. By the Glebsch-Gordan formula, $[V_n][V_m] = \sum_{k \in \mathbb{N}} c_{n,m,k}[V_k]$ with $c_{n,m,k} \in \mathbb{N}$.

**Question**

Which rings can arise as the representation ring of an abelian, Krull-Schmidt tensor category?

**Example**

Suppose that $\mathbb{Z}[i] = r(C)$ for some Krull-Schmidt tensor category $C$. Then $C$ has (at least) two indecomposable objects. We assume that the unit object for the tensor product is simple, hence this object corresponds to $1 \in r(C)$. Let $X$ denote the other indecomposable object. Then $X$ corresponds to $a + bi \in r(C)$ with $b \neq 0$. Since $(a + bi)^2 = -(a^2 + b^2) \cdot 1 + 2ab \cdot (a + bi)$, $X \otimes X \cong -(a^2 + b^2)1 \oplus 2aX$, but this makes no sense. Hence $\mathbb{Z}[i]$ is not the representation ring of such a category. Notice that $\mathbb{Z}[i]$ is not a unital $\mathbb{Z}_+$-ring.
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A \( \mathbb{Z}_+ \)-basis of a ring free as a module over \( \mathbb{Z} \) is a \( \mathbb{Z} \)-basis \( B = \{r_i\}_{i \in I} \) such that for any \( i, j \in I \), \( r_i r_j = \sum_k c_{ijk} r_k \) with \( c_{ijk} \in \mathbb{N} \).

A \( \mathbb{Z}_+ \)-ring is a \( \mathbb{Z} \)-algebra with unit endowed with a \( \mathbb{Z}_+ \)-basis.

A \( \mathbb{Z}_+ \)-ring with a \( \mathbb{Z}_+ \)-basis \( B \) is unital if \( 1 \in B \).

A \( \mathbb{Z}_+ \)-module over a \( \mathbb{Z}_+ \)-ring is a free \( \mathbb{Z} \)-module \( M \) endowed with a fixed basis \( \{m_j\}_{j \in J} \) such that \( r_i m_j = \sum_k d_{ijk} m_k \) with \( d_{ijk} \in \mathbb{N} \).
Let $A$ be an $F$-algebra with a complete set of orthogonal primitive idempotents $\{e_i\}_{i \in I}$ satisfying $1 = \sum_i e_i$ and $|I| = n$.

For $m = (m_i)_{i \in I} \in \mathbb{N}^I$, let $|m| = \sum_{i \in I} m_i$.

For $m, s \in \mathbb{N}^I$, an $(m, s)$-type matrix $X$ over $A$ is a block matrix

$$X = \begin{pmatrix}
X_{11} & X_{12} & \ldots & X_{1n} \\
X_{21} & X_{22} & \ldots & X_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
X_{n1} & X_{n2} & \ldots & X_{nn}
\end{pmatrix}$$

with $X_{ij} \in M_{m_i \times s_j}(e_i A e_j)$. Notice that $X$ is an $|m| \times |s|$-matrix over $A$. 

Yinhuo Zhang (joint with Huixiang Chen) (University of Hasselt) Representation rings Universität Hamburg, June 8, 2017 7 / 32
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Let $X \in M_{m \times s}(A)$ and $Y \in M_{m' \times s}(A)$. Then we define the horizontal sum of $X$ and $Y$ as

$$X \oplus Y := \begin{pmatrix}
X_{11} & X_{12} & \ldots & X_{1n} \\
Y_{11} & Y_{12} & \ldots & Y_{1n} \\
X_{21} & X_{22} & \ldots & X_{2n} \\
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\end{pmatrix}.$$ 

Similarly, we define the vertical sum of $X$ and $Y$:

$$X \oplus Y := \begin{pmatrix}
X_{11} & Y_{11} & X_{12} & Y_{12} & \ldots & X_{1n} & Y_{1n} \\
X_{21} & Y_{21} & X_{22} & Y_{22} & \ldots & X_{2n} & Y_{2n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
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Preliminaries and Notations

Let $X \in M_{m \times s}(A)$ and $Y \in M_{m' \times s}(A)$. Then we define the horizontal sum of $X$ and $Y$ as

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\end{pmatrix}.$$
Let $m_1, m_2, \ldots, m_r, s_1, s_2, \ldots, s_l \in \mathbb{N}$. For a matrix

$$X = \begin{pmatrix}
X_{11} & X_{12} & \cdots & X_{1l} \\
X_{21} & X_{22} & \cdots & X_{2l} \\
\vdots & \vdots & \ddots & \vdots \\
X_{r1} & X_{r2} & \cdots & X_{rl}
\end{pmatrix}$$

over $A$ with $X_{ij} \in M_{m_i \times s_j}(A)$, define a matrix $\prod(X)$ over $A$ by

$$\prod(X) := (X_{11} \oplus X_{12} \oplus \cdots \oplus X_{1l}) \oplus \cdots \oplus (X_{r1} \oplus X_{r2} \oplus \cdots \oplus X_{rl}).$$

Then $\prod(X) \in M_{m \times s}(A)$ with $m = \sum_{i=1}^r m_i$ and $s = \sum_{j=1}^l s_j$. 
Let $A$ and $B$ be two $\mathbb{F}$-algebras. Let

$$X = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1s} \\ x_{21} & x_{22} & \ldots & x_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \ldots & x_{ms} \end{pmatrix}$$

and

$$Y = \begin{pmatrix} y_{11} & y_{12} & \ldots & y_{1s'} \\ y_{21} & y_{22} & \ldots & y_{2s'} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m'1} & y_{m'2} & \ldots & y_{m's'} \end{pmatrix}$$

be an $m \times s$-matrix over $A$ and an $m' \times s'$-matrix over $B$ respectively. Then we define an $mm' \times ss'$-matrix $X \otimes_{\mathbb{F}} Y$ over $A \otimes_{\mathbb{F}} B$ by

$$X \otimes_{\mathbb{F}} Y = \begin{pmatrix} x_{11} \otimes y_{11} & \ldots & x_{1s} \otimes y_{11} & \ldots & x_{11} \otimes y_{1s'} & \ldots & x_{1s} \otimes y_{1s'} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{m1} \otimes y_{11} & \ldots & x_{ms} \otimes y_{11} & \ldots & x_{m1} \otimes y_{1s'} & \ldots & x_{ms} \otimes y_{1s'} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{11} \otimes y_{m'1} & \ldots & x_{1s} \otimes y_{m'1} & \ldots & x_{11} \otimes y_{m's'} & \ldots & x_{1s} \otimes y_{m's'} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{m1} \otimes y_{m'1} & \ldots & x_{ms} \otimes y_{m'1} & \ldots & x_{m1} \otimes y_{m's'} & \ldots & x_{ms} \otimes y_{m's'} \end{pmatrix}$$
Lemma

Let $X \in M_{m \times s}(A)$, $X_1 \in M_{s \times t}(A)$ and $Y \in M_{m' \times s'}(A)$, $Y_1 \in M_{s' \times t'}(B)$. Then

$$(X \otimes Y)(X_1 \otimes Y_1) = (XX_1) \otimes (YY_1).$$

Definition

- An $(m, s)$-type matrix $X$ is called column-independent if for any $(s, l)$-type matrix $Y$, $XY = 0 \Rightarrow Y = 0$.
- Similarly, we define row-independent.
- An $(s, t)$-type matrix $Y$ is a right universal annihilator of $X$ if $XY = 0$ and $XM = 0 \Rightarrow M = YZ$ for a unique $Z$.
- Similarly, we define left universal annihilators.
Section 2

Construction of tensor categories from given data
Goal of the construction

We will construct an abelian, Krull-Schmidt tensor category $C$ from the data

$$(R, A, I, \{e_i \mid i \in I\}, \phi, \{a_{ijl} \mid i, j, l \in I\})$$

s.t.

- $R = r(C)$.
- $A$ is the Auslander algebra of $C$.

We will give a workable criterion when two such categories are tensor equivalent.
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- We will give a workable criterion when two such categories are tensor equivalent.
First assumptions

- \( R \) is a unital \( \mathbb{Z}_+ \)-ring with finite unital basis \( \{r_i\}_{i \in I} \) with \( I = \{1, 2, \ldots, n\} \) and \( r_1 = 1 \) and \( r_i r_j \neq 0 \).

- \( A \) is a finite dimensional \( \mathbb{F} \)-algebra with a complete set of orthogonal primitive idempotents \( \{e_i\}_{i \in I} \).

- \( \mathbb{F} \) is algebraically closed.

- (KS) \( e_i A e_j A e_i \subset \text{rad}(e_i A e_i) \) for \( i \neq j \).

- (Dec) Any \( (m, s) \)-type matrix can be written as the product of CI and RI matrices.

- (RUA) Any matrix has right universal annihilator.

- (LUA) Any matrix has left universal annihilator.

- (CI) If \( X \) is CI and \( Y \) a LUA of \( X \), then \( X \) is RUA of \( Y \).

- (RI) If \( X \) is RI and \( Y \) a RUA of \( X \), then \( Y \) is LUA of \( X \).
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We define a category $C$ by

1. $\text{Obj}(C) := \mathbb{N}^I$;
2. $\text{Hom}_C(m, s) := M_{s \times m}(A)$;
3. Composition is usual matrix product.

**Lemma**

- Let $m, s \in \text{Obj}(C)$, then $m + s \cong m \oplus s$.
- Moreover, $C$ is an additive category over $\mathbb{F}$.
- $m \cong s$ if and only if $m = s$. 
Construction of Category

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**Definition**

Define $e_i := (0, \ldots, 0, 1, 0 \ldots, 0)$. Then for any $m \in \text{Obj}(C)$, we have

$$m \cong \bigoplus_{i \in I} m_i e_i.$$ 

Moreover, the $e_i$’s are all non-isomorphic indecomposable objects.

**Proposition**

$C$ is an abelian Krull-Schmidt category over $\mathbb{F}$.

**Corollary**

The following are equivalent:

1. $C$ is a semisimple category over $\mathbb{F}$;
2. $\dim_{\mathbb{F}}(A) = n$;
3. $A \cong \mathbb{F}^n$ as $\mathbb{F}$-algebras.
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Constructing the Tensor Product

Definition

- Put $c_{ij} = (c_{ij1}, \ldots, c_{ijn})$ with $r_ir_j = \sum_{k=1}^n c_{ijk}r_k$.
- We define $m \otimes s := \sum_{i,j} m_isjc_{ij}$.
- Define an $\mathbb{F}$-algebra $M(R, A, I) := \bigoplus_{1 \leq i, i', j, j' \leq n} M_{c_{i'i'j'j}} \times c_{ij}(A)$.
- If $X \in M_{c_{i'i'j'j}} \times c_{ij}$, $Y \in M_{c_{i'i'j'j}} \times c_{i'j1}$, then $XY$ is the usual matrix product if $(i, j) = (i'', j'')$ and zero otherwise.

Assumption

Assume that $\dim(e_1 Ae_1) = 1$ and $\exists \phi : A \otimes A \to M(R, A, I)$ s.t.

1. $\phi(e_1 \otimes e_j) = E_{c_{ij}} \in M_{c_{ij}}(A)$;
2. $\phi(e_1 \otimes a) = a = \phi(a \otimes e_1)$ for all $a \in e_1 Ae_j$. 
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- Define an $\mathbb{F}$-algebra $M(R, A, I) := \bigoplus_{1 \leq i,i',j,j' \leq n} M_{c_{i'j'}} \times c_{ij}(A)$.
- If $X \in M_{c_{i'j'}} \times c_{ij}$, $Y \in M_{c_{i''j''}} \times c_{i_1j_1}$, then $XY$ is the usual matrix product if $(i, j) = (i'', j'')$ and zero otherwise.

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- If $X \in M_{c_{i'j'}} \times_{c_{ij}} Y \in M_{c_{i''j''}} \times_{c_{i'j_1j_1}}$, then $XY$ is the usual matrix product if $(i, j) = (i'', j'')$ and zero otherwise.

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- Define an \( \mathbb{F} \)-algebra \( M(R, A, I) := \bigoplus_{1 \leq i,i',j,j' \leq n} Mc_{i',j'} \times c_{ij}(A) \).
- If \( X \in Mc_{i',j'} \times c_{ij}, Y \in Mc_{i''j''} \times c_{i'j'1} \), then \( XY \) is the usual matrix product if \( (i,j) = (i'',j'') \) and zero otherwise.

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Definition

Let $X \in \text{Hom}_C(m_1, s_1)$, $Y \in \text{Hom}_C(m_2, s_2)$. We define $X \otimes Y$ over $A$ by

- if $m_1 \otimes m_2 = 0$ and $s_1 \otimes s_2 = 0$, then $X \otimes Y := 0 \in M_{1 \times 1}(A)$;
- if $m_1 \otimes m_2 = 0$ and $s_1 \otimes s_2 \neq 0$, then $X \otimes Y := 0 \in M_{|s_1 \otimes s_2| \times 1}(A)$;
- if $m_1 \otimes m_2 \neq 0$ and $s_1 \otimes s_2 = 0$, then $X \otimes Y := 0 \in M_{1 \times |m_1 \otimes m_2|}(A)$;
- if $m_1 \otimes m_2 \neq 0$ and $s_1 \otimes s_2 \neq 0$, then $X \otimes Y := \prod(\phi(X \otimes_F Y))$.

Lemma

$X \otimes Y \in \text{Hom}_C(m_1 \otimes m_2, s_1 \otimes s_2)$. Moreover

1. $E_m \otimes E_s = E_{m \otimes s}$;
2. $E_{e_1} \otimes X = X = X \otimes E_{e_1}$.

Finally, $(X \otimes Y)(X_1 \otimes Y_1) = (XX_1 \otimes YY_1)$. 
Constructing the Tensor Product

### Definition

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- if $m_1 \otimes m_2 \neq 0$ and $s_1 \otimes s_2 = 0$, then $X \otimes Y := 0 \in M_{1 \times \|m_1 \otimes m_2\|}(A)$;
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Finally, $(X \otimes Y)(X_1 \otimes Y_1) = (XX_1 \otimes YY_1)$.
The Associativity Constraint

**Definition**

Set $Y_{i,k}^m = (y_1, \ldots, y_{|m|}) \in M_{e_i \times m(A)}$, where

$$y_j = \begin{cases} e_i, & j = k + \sum_{1 \leq l < i} m_l \\ 0, & \text{otherwise.} \end{cases}$$

Put $X_{i,k}^m = (Y_{i,k}^m)^T$. Then $Y_{i,k}^m X_{i',k'}^m = E_{e_i}$ if $(i,k) = (i',k')$ and zero otherwise, and

$$\sum_{i=1}^{n} \sum_{1 \leq k \leq m_i} X_{i,k}^m Y_{i,k}^m = E_m.$$
Assumptions on Associativity Constraint

**Assumptions**

There exists a family of matrices $a_{ijl} \in M_{e_i \otimes e_j \otimes e_l}(A)$ s.t.

1. $a_{ijl}$ is invertible;
2. $(x \otimes (y \otimes z))a_{ijl} = a_{i'j'l'}((x \otimes y) \otimes z)$ for $x \in e_{i'} A e_i$, $y \in e_{j'} A e_j$, $z \in e_{l'} A e_l$;
3. $a_{i1j} = E_{c_{ij}}$;
4. \[
\sum_{j=1}^{n} \sum_{k=1}^{c_{i2i3,j}} (e_{i1} \otimes a_{i2i3i4}(X_{jk}^{c_{i2i3}} \otimes e_{k4}))a_{i1ji4}((e_{i1} \otimes Y^{c_{i2i3}})a_{i1i2i3} \otimes e_{i4}) = \\
\sum_{j,j'=1}^{n} \sum_{k=1}^{c_{i3i4,j}} \sum_{k'=1}^{c_{i1i2,j'}} (e_{i1} \otimes (e_{i2} \otimes X_{jk}^{c_{i3i4}}))a_{i1i2j}(X_{j'k'}^{c_{i1i2}} \otimes Y_{jk}^{c_{i3i4}})
\cdot a_{j'i3i4}((Y_{j'k'}^{c_{i1i2}} \otimes e_{i3}) \otimes e_{i4}).
\]
$C$ is monoidal

**Theorem**

$(C, \otimes, e_1, a, l = ld, r = ld)$ is a tensor category over $\mathbb{F}$, $r(C) \cong R$ and $\text{End}_C(\bigoplus_{i=1}^n e_i) \cong A$ as $\mathbb{F}$-algebras.

**Remark**

In general $e_1$ is not simple, but TFAE:

- $e_1$ is a simple object of $C$;
- If $X \in M_{e_1 \times m}(A)$ is CI, then either $m = 0$ and $X = 0$, or $m = e_1$ and $X = \alpha e_1$ for some $\alpha \in \mathbb{F}_0$. 
Theorem

\((C, \otimes, e_1, a, l = \text{Id}, r = \text{Id})\) is a tensor category over \(\mathbb{F}\), \(r(C) \cong R\) and \(\text{End}_C(\bigoplus_{i=1}^n e_i) \cong A\) as \(\mathbb{F}\)-algebras.

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Other associativity constraints

**Definition**

\( \{ a_{ijl} \} \) and \( \{ a'_{i'j'l'} \} \) are called equivalent if there exists a family of invertible matrices \( \eta_{i,j} \in M_{c_{ij}}(A) \) s.t.

1. \((x \otimes y) \eta_{i,j} = \eta_{i',j'}(x \otimes y)\);

2. 

\[
\sum_{t=1}^{n} \sum_{k=1}^{c_{ijt}} a_{ijl} (X_{tk}^{c_{ij}} \otimes e_l) \eta(t, l)(Y_{tk}^{c_{ij}} \eta(i, j) \otimes e_l) \\
= \sum_{t=1}^{n} \sum_{k=1}^{c_{jlt}} (e_i \otimes X_{tk}^{c_{jl}}) \eta(i, t)(e_i \otimes Y_{tk}^{c_{jl}} \eta(j, l) a'_{ijl}).
\]

**Proposition**

\((C, \otimes, e_1, a, l, r)\) and \((C, \otimes, e_1, a', l, r)\) are equivalent if \( \{ a_{ijl} \} \) and \( \{ a'_{ijl} \} \) are equivalent.
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**Definition**

\{ a_{ijl} \} and \{ a'_{ij'l'} \} are called equivalent if there exists a family of invertible matrices \( \eta_{ij} \in M_{c_{ij}}(A) \) s.t.

1. \((x \otimes y)\eta_{ij} = \eta_{ij'}(x \otimes y)\);

2. 
\[
\sum_{t=1}^{n} \sum_{k=1}^{c_{ijt}} a_{ijl}(X_{tk}^{c_{ij}} \otimes e_l) \eta(t, l)(Y_{tk}^{c_{ij}} \eta(i, j) \otimes e_l) = \sum_{t=1}^{n} \sum_{k=1}^{c_{jlt}} (e_i \otimes X_{tk}^{c_{jl}}) \eta(i, t)(e_i \otimes Y_{tk}^{c_{jl}} \eta(j, l)a'_{ijl}).
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**Proposition**

\((C, \otimes, e_1, a, l, r)\) and \((C, \otimes, e_1, a', l, r)\) are equivalent if \( \{ a_{ijl} \} \) and \( \{ a'_{ij'l'} \} \) are equivalent.
Uniqueness

Theorem

\((C, \otimes, e_1, a, l, r)\) and \((C', \otimes_{C'}, e'_1, a', l', r')\) are tensor equivalent if and only if \(n = n'\) \((l = l')\) and there exists a \(\sigma \in S(l)\) such that:

1. \(R \to R' : r_i \mapsto r'_i\) is a ring isomorphism;
2. there exists a \(\mathbb{F}\)-algebra map \(\delta : A \to A'\) with \(\delta(e_i) = e'_{\sigma(i)}\);
3. there exists an \(\alpha \in \mathbb{F}_0\) and a family of invertible elements \(\phi_{i,j} \in M_{c'_{\sigma(i)\sigma(j)}}(A')\) s.t.
   1. \(\phi_{1,i} = \phi_{i,1} = \alpha E'_{e_i\sigma}\);
   2. \(\phi_{i',j'}(\delta(x) \otimes C' \delta(y)) = P_{\sigma(c'_{i'j'})} \delta(x \otimes C y) P_{\sigma(c_{ij})}^T \phi_{i,j}\);
   3. 
      \[
      \sum_{t=1}^n \sum_{k=1}^{c_{ijt}} \delta(a_{i,j,l}) \delta(X_{t,k}^c \otimes C e_l) P_{\sigma(c_{itl})}^T \phi_{t,l}(Y_{\sigma(t),k}^c e_{ij}^c) \phi_{i,j} \otimes C' e'_{\sigma(l)} = \\
      \sum_{t=1}^n \sum_{k=1}^{c_{ijt}} \delta(e_i \otimes C X_{t,k}^c) P_{\sigma(c_{it})}^T \phi_{i,t}(e'_{\sigma(i)} \otimes C' Y_{\sigma(t),k}^c \phi_{j,l}) a_{\sigma(i),\sigma(j),\sigma(l')}
      \] 
      for all \(x \in e'_i Ae_i, y \in e'_j Ae_j\).
Section 3

Invariants of tensor categories
Invariants of tensor categories

Let $\mathcal{C}$

- be an abelian, Krull-Schmidt tensor category over $\mathbb{F}$;
- have finitely many indecomposable objects;
- have finite-dimensional Hom-spaces;
- be strict and the unit object $\mathbf{1}$ be simple.

Let $\{V_i \mid i \in I\}$ be a set of representatives of the isomorphism classes of the indecomposable objects of $\mathcal{C}$ ($\mathbf{1} = V_1$). We also assume that $U \otimes V \neq 0$ for all nonzero objects.

Goal of the section

We will associate data $(r(\mathcal{C}), A(\mathcal{C}), I, \{e_i \mid i \in I\}, \phi_\mathcal{C}, \{a_{ijl} \mid i, j, l \in I\})$ to $\mathcal{C}$. By the previous section we can then construct a category $\hat{\mathcal{C}}$. We will show that $\hat{\mathcal{C}}$ is tensor equivalent to $\mathcal{C}$.

Definition

Let $V = \bigoplus_{i \in I} V_i$ and $A(\mathcal{C}) = \text{End}_\mathcal{C}(V) = \text{Hom}_\mathcal{C}(V, V)$. Then $A(\mathcal{C})$ is a finite-dimensional $\mathbb{F}$-algebra. Let $\pi_i : V \rightarrow V_i$ and $\tau_i : V_i \rightarrow V$ be the canonical projections and injections. Then $\text{Id}_V = \sum_{i \in I} \tau_i \circ \pi_i$ and

\[
\pi_i \circ \tau_j = \begin{cases} 
\text{Id}_{V_i}, & \text{if } i = j, \\
0, & \text{else}.
\end{cases}
\]

Let $e_i = \tau_i \circ \pi_i \in A(\mathcal{C})$. 

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Invariants of tensor categories

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**Goal of the section**

We will associate data \((r(C), A(C), I, \{ e_i \mid i \in I \}, \phi_C, \{ a_{ijl} \mid i, j, l \in I \})\) to \( C \). By the previous section we can then construct a category \( \hat{C} \). We will show that \( \hat{C} \) is tensor equivalent to \( C \).

**Definition**

Let \( V = \bigoplus_{i \in I} V_i \) and \( A(C) = \text{End}_C(V) = \text{Hom}_C(V, V) \). Then \( A(C) \) is a finite-dimensional \( \mathbb{F} \)-algebra. Let \( \pi_i : V \to V_i \) and \( \tau_i : V_i \to V \) be the canonical projections and injections. Then

\[
\pi_i \circ \tau_j = \begin{cases} 
ld_{V_i}, & \text{if } i = j, \\
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Invariants of tensor categories

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We will associate data $(r(\mathcal{C}), A(\mathcal{C}), I, \{e_i \mid i \in I\}, \phi_C, \{a_{ijl} \mid i, j, l \in I\})$ to $\mathcal{C}$. By the previous section we can then construct a category $\hat{\mathcal{C}}$. We will show that $\hat{\mathcal{C}}$ is tensor equivalent to $\mathcal{C}$.

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Let $e_i = \tau_i \circ \pi_i \in A(\mathcal{C})$. 
The conditions are satisfied

**Proposition**

The set \( \{ e_i \mid i \in I \} \) forms a complete set of orthogonal primitive idempotents of \( A(C) \). Moreover, for all \( i \neq j \), \( e_i A(C) e_j A(C) e_i \subset \text{rad}(e_i A(C) e_i) \). Hence if \( f \in \text{Hom}(V_i, V_j) \), \( g \in \text{Hom}(V_j, V_i) \), then \( gf \in \text{End}(V_i) \).

**Proposition**

Let \( X \) be an \( (m, s) \)-matrix over \( A(C) \), then

1. \( X \) has a right universal annihilator.
2. \( X \) has a left universal annihilator.
3. There is a CI \( (m, t) \)-matrix \( X_1 \) and a RI \( (t, s) \)-matrix \( X_2 \) s.t. \( X = X_1 X_2 \).
4. If \( X \) is CI and \( Y \) is a LUA of \( X \), then \( X \) is a RUA of \( Y \).
5. If \( X \) is RI and \( Y \) is a RUA of \( X \), then \( X \) is a LUA of \( Y \).
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The tensor product

**Definition**

We write 

\[ [V_i][V_j] = \sum_{k \in I} c_{ijk} [V_k] \] in \( r(C) \). Let \( c_{ij} := (c_{ijk})_{k \in I} \in \mathbb{N}^I \). Define a vector space \( M(C) \) by

\[
M(C) := \bigoplus_{i,i',j,j' \in I} M_{c_{i'j'} c_{ij}}(A(C)).
\]

Then \( M(C) \) is an \( \mathbb{F} \)-algebra as before.

**Proposition**

There exists an algebra map \( \phi_C : A(C) \otimes_{\mathbb{F}} A(C) \to M(C) \) s.t.

1. \( \phi_C(e_i \otimes e_j) = E_{c_{ij}} \in M_{c_{ij}}(A(C)) \);
2. \( \phi_C(e_1 \otimes a) = \phi_C(a \otimes e_1) = a \) for all \( a \in e_i A(C) e_j \).
The tensor product

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M(C) := \bigoplus_{i,i',j,j' \in I} M_{c_{i,j};c_{i',j'}}(A(C)).
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2. \(\phi_C(e_1 \otimes a) = \phi_C(a \otimes e_1) = a\) for all \(a \in e_i A(C) e_j\).
Tensor product of morphisms

- For \( \mathbf{m} = (m_i)_{i \in I}, \mathbf{s} = (s_i)_{i \in I} \), we identify \( \text{Hom}_C(\bigoplus_{i \in I} m_i V_i, \bigoplus_{i \in I} s_i V_i) = M_{s \times m}(A(C)) \).

- For \( f \in \text{Hom}_C(\bigoplus_{i \in I} m_i V_i, \bigoplus_{i \in I} s_i V_i) \), the corresponding matrix is

\[
\begin{pmatrix}
  f_{11} & f_{12} & \cdots & f_{1m} \\
  f_{21} & f_{22} & \cdots & f_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{s1} & f_{s2} & \cdots & f_{sm}
\end{pmatrix}
\]

where \( f_{kl} = Y_{i,k_1}^s f X_{j,k_2}^m \) if \( k = \sum_{t=1}^{i-1} s_t + k_1 \) and \( l = \sum_{t=1}^{j-1} m_t + k_2 \).

- \( Y_{i,k_1}^s \in \text{Hom}_C(\bigoplus_{j \in I} s_j V_j, V_i) \) is the projection from \( \bigoplus_{j \in I} s_j V_j \) to the \( k_1 \)-th \( V_i \) of the direct summand \( s_i V_i \) of \( \bigoplus_{j \in I} s_j V_j \).

- \( X_{j,k_2}^m \in \text{Hom}_C(V_j, \bigoplus_{i \in I} m_i V_i) \) is the embedding of \( V_j \) into the \( k_2 \)-th \( V_j \) of the direct summand \( m_j V_j \) of \( \bigoplus_{i \in I} m_i V_i \).

- As before, let \( \mathbf{m} \otimes \mathbf{s} = \sum_{i,j \in I} m_i s_j c_{ij} \).

- Let \( X \in M_{s_1 \times m_1}(A(C)), Y \in M_{s_2 \times m_2}(A(C)) \). Then

\[
X \in \text{Hom}_C(\bigoplus_{i \in I} m_i V_i, \bigoplus_{i \in I} s_i V_i), \ Y \in \text{Hom}_C(\bigoplus_{i \in I} m_i V_i, \bigoplus_{i \in I} s_i V_i).
\]
For $m = (m_i)_{i \in I}, s = (s_i)_{i \in I}$, we identify $\text{Hom}_C(\bigoplus_{i \in I} m_i V_i, \bigoplus_{i \in I} s_i V_i) = M_{s \times m}(A(C))$.

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where $f_{kl} = Y_{i,k_1}^s f X_{j,k_2}^m$ if $k = \sum_{t=1}^{i-1} s_t + k_1$ and $l = \sum_{t=1}^{j-1} m_t + k_2$.

$Y_{i,k_1}^s \in \text{Hom}_C(\bigoplus_{j \in I} s_j V_j, V_i)$ is the projection from $\bigoplus_{j \in I} s_j V_j$ to the $k_1$-th $V_i$ of the direct summand $s_i V_i$ of $\bigoplus_{j \in I} s_j V_j$.

$X_{j,k_2}^m \in \text{Hom}_C(V_j, \bigoplus_{i \in I} m_i V_i)$ is the embedding of $V_j$ into the $k_2$-th $V_j$ of the direct summand $m_j V_j$ of $\bigoplus_{i \in I} m_i V_i$.

As before, let $m \otimes s = \sum_{i,j \in I} m_i s_j c_{ij}$.

Let $X \in M_{s_1 \times m_1}(A(C)), Y \in M_{s_2 \times m_2}(A(C))$. then

$$
X \in \text{Hom}_C(\bigoplus_{i \in I} m_1_i V_i, \bigoplus_{i \in I} s_1_i V_i), Y \in \text{Hom}_C(\bigoplus_{i \in I} m_2_i V_i, \bigoplus_{i \in I} s_2_i V_i).
$$
Tensor product of morphisms

- For $m = (m_i)_{i \in I}, s = (s_i)_{i \in I}$, we identify $\text{Hom}_C(\bigoplus_{i \in I} m_i V_i, \bigoplus_{i \in I} s_i V_i) = M_{s \times m}(A(C))$.
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- $Y_{i,k_1}^s \in \text{Hom}_C(\bigoplus_{j \in I} s_j V_j, V_i)$ is the projection from $\bigoplus_{j \in I} s_j V_j$ to the $k_1$-th $V_i$ of the direct summand $s_i V_i$ of $\bigoplus_{j \in I} s_j V_j$.
- $X_{j,k_2}^m \in \text{Hom}_C(V_j, \bigoplus_{i \in I} m_i V_i)$ is the embedding of $V_j$ into the $k_2$-th $V_j$ of the direct summand $m_j V_j$ of $\bigoplus_{i \in I} m_i V_i$.
- As before, let $m \otimes s = \sum_{i,j \in I} m_i s_j c_{ij}$.
- Let $X \in M_{s_1 \times m_1}(A(C)), Y \in M_{s_2 \times m_2}(A(C))$. Then
  \[
  X \in \text{Hom}_C(\bigoplus_{i \in I} m_{1i} V_i, \bigoplus_{i \in I} s_{1i} V_i), Y \in \text{Hom}_C(\bigoplus_{i \in I} m_{2i} V_i, \bigoplus_{i \in I} s_{2i} V_i).
  \]
For $m = (m_i)_{i \in I}$, $s = (s_i)_{i \in I}$, we identify $\text{Hom}_C(\bigoplus_{i \in I} m_i V_i, \bigoplus_{i \in I} s_i V_i) = M_s \times m(A(C))$.

For $f \in \text{Hom}_C(\bigoplus_{i \in I} m_i V_i, \bigoplus_{i \in I} s_i V_i)$, the corresponding matrix is

$$
\begin{pmatrix}
  f_{11} & f_{12} & \ldots & f_{1m} \\
  f_{21} & f_{22} & \ldots & f_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{s1} & f_{s2} & \ldots & f_{sm}
\end{pmatrix}
$$

where $f_{kl} = Y_{s_i,k_1}^s f X_{j,k_2}^m$ if $k = \sum_{t=1}^{i-1} s_t + k_1$ and $l = \sum_{t=1}^{j-1} m_t + k_2$.

$Y_{s_i,k_1}^s \in \text{Hom}_C(\bigoplus_{j \in I} s_j V_j, V_i)$ is the projection from $\bigoplus_{j \in I} s_j V_j$ to the $k_1$-th $V_i$ of the direct summand $s_i V_i$ of $\bigoplus_{j \in I} s_j V_j$.

$X_{j,k_2}^m \in \text{Hom}_C(V_j, \bigoplus_{i \in I} m_i V_i)$ is the embedding of $V_j$ into the $k_2$-th $V_j$ of the direct summand $m_j V_j$ of $\bigoplus_{i \in I} m_i V_i$.

As before, let $m \otimes s = \sum_{i,j \in I} m_i s_j c_{ij}$.

Let $X \in M_{s_1 \times m_1}(A(C)), Y \in M_{s_2 \times m_2}(A(C))$. then

$$
X \in \text{Hom}_C(\bigoplus_{i \in I} m_i V_i, \bigoplus_{i \in I} s_i V_i), Y \in \text{Hom}_C(\bigoplus_{i \in I} m_i V_i, \bigoplus_{i \in I} s_i V_i).
$$
Tensor product of morphisms

Let $X \otimes Y$ denote the tensor product of $X$ and $Y$ in $\mathcal{C}$. Then

$$X \otimes Y \in \text{Hom}_\mathcal{C}((\bigoplus_{i \in I} m_1 V_i) \otimes (\bigoplus_{i \in I} m_2 V_i), (\bigoplus_{i \in I} s_1 V_i) \otimes (\bigoplus_{i \in I} s_2 V_i)).$$

Let $X\hat{\otimes} Y := \prod (\phi_C(X \otimes_{\mathbb{F}} Y)) \in M_{(s_1 \otimes s_2) \times (m_1 \otimes m_2)}(A(\mathcal{C})).$ We now have two different tensor products!

Lemma

Let $x \in e_i A(\mathcal{C}) e_i, y \in e_j A(\mathcal{C}) e_j$. Then the following diagram commutes:

\[
\begin{array}{ccc}
V_i \otimes V_j & \xrightarrow{\theta_{ij}} & \bigoplus_{k \in I} c_{ijk} V_k \\
\downarrow x \otimes y & & \downarrow x \hat{\otimes} y \\
V_i' \otimes V_j' & \xrightarrow{\theta_{i'j'}} & \bigoplus_{k \in I} c_{i'j'k} V_k
\end{array}
\]
Tensor product of morphisms

1. Let $X \otimes Y$ denote the tensor product of $X$ and $Y$ in $\mathcal{C}$. Then

$$X \otimes Y \in \text{Hom}_\mathcal{C}((\bigoplus_{i \in I} m_i V_i) \otimes (\bigoplus_{i \in I} m_i V_i), (\bigoplus_{i \in I} s_i V_i) \otimes (\bigoplus_{i \in I} s_i V_i)).$$

2. Let $X \widetilde{\otimes} Y := \prod (\phi \mathcal{C}(X \otimes \mathbb{F} Y)) \in M_{(s_1 \otimes s_2) \times (m_1 \otimes m_2)}(A(\mathcal{C}))$. We now have two different tensor products!

**Lemma**

Let $x \in e_i' A(\mathcal{C}) e_i$, $y \in e_j' A(\mathcal{C}) e_j$. Then the following diagram commutes:

$$
\begin{align*}
V_i \otimes V_j & \xrightarrow{\theta_{ij}} \bigoplus_{k \in I} c_{ijk} V_k \\
x \otimes y & \downarrow \quad \downarrow x \otimes y \\
V_{i'} \otimes V_{j'} & \xrightarrow{\theta_{i'j'}} \bigoplus_{k \in I} c_{i'j'k} V_k
\end{align*}
$$
Tensor product of morphisms

- Let $X \otimes Y$ denote the tensor product of $X$ and $Y$ in $\mathcal{C}$. Then

$$X \otimes Y \in \text{Hom}_\mathcal{C}((\bigoplus_{i \in I} m_1 V_i) \otimes (\bigoplus_{i \in I} m_2 V_i), (\bigoplus_{i \in I} s_1 V_i) \otimes (\bigoplus_{i \in I} s_2 V_i)).$$

- Let $X \tilde{\otimes} Y := \prod(\phi_C(X \otimes \mathbb{F} Y)) \in M_{(s_1 \otimes s_2) \times (m_1 \otimes m_2)}(A(C))$. We now have two different tensor products!

**Lemma**

Let $x \in e_i A(C) e_i$, $y \in e_j A(C) e_j$. Then the following diagram commutes:

\[
\begin{array}{ccc}
V_i \otimes V_j & \xrightarrow{\theta_{ij}} & \bigoplus_{k \in I} c_{ijk} V_k \\
\uparrow x \otimes y & & \downarrow x \tilde{\otimes} y \\
V_{i'} \otimes V_{j'} & \xrightarrow{\theta_{i'j'}} & \bigoplus_{k \in I} c_{i'j'k} V_k
\end{array}
\]
Towards another associativity constraint

**Definition**

Let \( m \in \mathbb{N}^I \), then we define \( V(m) := \bigoplus_{i \in I} m_i V_i \). For \( m_1, m_2 \in \mathbb{N}^I \), define a morphism \( \theta(m_1, m_2) : V(m_1) \otimes V(m_2) \to V(m_1 \otimes m_2) \) in \( C \) by

\[
\theta(m_1, m_2) = \sum_{i,j=1}^{m_1 i} \sum_{k_1=1}^{m_2 j} (X_{i,k_1}^{m_1} \otimes X_{j,k_2}^{m_2}) \theta_{ij}(Y_{i,k_1}^{m_1} \otimes Y_{j,k_2}^{m_2}).
\]

Then \( \theta(e_i, e_j) = \theta_{ij} \).

**Lemma**

Each \( \theta(m_1, m_2) \) is an isomorphism. Moreover, the following diagram commutes:

\[
\begin{array}{ccc}
V(m_1) \otimes V(m_2) & \xrightarrow{\theta(m_1, m_2)} & V(m_1 \otimes m_2) \\
X \otimes Y & \downarrow & X \otimes Y \\
V(s_1) \otimes V(s_2) & \xrightarrow{\theta(s_1, s_2)} & V(s_1 \otimes s_2)
\end{array}
\]
Towards another associativity constraint

**Definition**

Let \( m \in \mathbb{N}^I \), then we define \( V(m) := \bigoplus_{i \in I} m_i V_i \). For \( m_1, m_2 \in \mathbb{N}^I \), define a morphism \( \theta(m_1, m_2) : V(m_1) \otimes V(m_2) \rightarrow V(m_1 \otimes m_2) \) in \( C \) by

\[
\theta(m_1, m_2) = \sum_{i,j=1}^{n} \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} (X_{i,k_1}^{m_1} \tilde{\otimes} X_{j,k_2}^{m_2}) \theta_{ij}(Y_{i,k_1}^{m_1} \otimes Y_{j,k_2}^{m_2}).
\]

Then \( \theta(e_i, e_j) = \theta_{ij} \).

**Lemma**

*Each \( \theta(m_1, m_2) \) is an isomorphism. Moreover, the following diagram commutes:*

\[
\begin{array}{ccc}
V(m_1) \otimes V(m_2) & \xrightarrow{\theta(m_1, m_2)} & V(m_1 \otimes m_2) \\
Y \otimes X & \xrightarrow{\theta(s_1, s_2)} & Y \otimes X \\
V(s_1) \otimes V(s_2) & \xrightarrow{\theta(s_1, s_2)} & V(s_1 \otimes s_2)
\end{array}
\]
The associativity constraint

Definition

Define $a_{ijl} \in M_{e_i \otimes e_j \otimes e_l}(A(C)) = \text{End}_C(V^{(e_i \otimes e_j \otimes e_l)})$ by

$$a_{ijl} := \theta(e_i, e_j \otimes e_l)(E_{e_i} \otimes \theta(e_j, e_l))(\theta(e_i, e_j)^{-1} \otimes E_{e_l})\theta(e_i \otimes e_j, e_l)^{-1}.$$ 

Proposition

The family $\{a_{ijl}\}$ satisfies the four conditions encountered earlier w.r.t. the tensor product $\tilde{\otimes}$.

Denote by $\tilde{C}$ the category associated to the data $(r(C), A(C), I, \{e_i | i \in I\}, \phi_C, \{a_{ijl} | i, j, l \in I\})$. The tensor products of $C$ and $\tilde{C}$ are denoted by $\otimes$ and $\tilde{\otimes}$ respectively.

Theorem

$\tilde{C}$ and $C$ are tensor equivalent.
The associativity constraint

Definition

Define \(a_{ijl} \in M_{e_i \otimes e_j \otimes e_l}(A(C)) = \text{End}_C(V^{(e_i \otimes e_j \otimes e_l)})\) by

\[
a_{ijl} := \theta(e_i, e_j \otimes e_l)(E_{e_i} \otimes \theta(e_j, e_l))(\theta(e_i, e_j)^{-1} \otimes E_{e_l})\theta(e_i \otimes e_j, e_l)^{-1}.
\]

Proposition

The family \(\{a_{ijl}\}\) satisfies the four conditions encountered earlier w.r.t. the tensor product \(\tilde{\otimes}\).

Theorem

\(\hat{C}\) and \(C\) are tensor equivalent.