Defects, gerbes and the gauge anomaly

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The Gauge Principle for the multi-phase two-dimensional non-linear sigma model will be discussed from the vantage point of the model's lagrangean formulation. The rôle of the 2-category of equivariant gerbes over the target space of the field theory in the construction of the gauged sigma model will be emphasised, and an interpretation of the gauge anomaly in terms of an obstruction to the existence of topological defect networks implementing the gauge symmetry will be given.

Goals:

1. Establishing a **defect–duality correspondence** through transgression of cohomological (bi-brane) data of a conformal world-sheet defect.

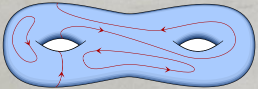
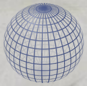
2. Deriving **the Gauge Principle** for global symmetries through generalisation of the minimal-coupling recipe and local reconstruction of the σ-model coupled to world-sheet gauge fields of arbitrary topology using **topological gauge-symmetry defect networks**, whence a symmetry-equivariant extension of σ-model data ensues.

Part I

Preliminaries

A brief recap on the multi-phase 2d non-linear σ-model

The action functional of the model for patchwise C^1-smooth maps

X :  $\hookrightarrow$ 

world-sheet (Σ, η)
with defect quiver $\Gamma \cong \bigsqcup_{d \in D} \mathbb{S}^1$

geometric string background

requires the 2-category $\mathfrak{BGrb}^\nabla(\mathscr{F})$ of bundle gerbes with connection over the composite target space $\mathscr{F} := \bigsqcup_{i \in I} M_{(i)} \sqcup \bigsqcup_{\langle i,j \rangle \in I^2} Q_{(i,j)}$.

Γ splits the spacetime into domains Σ_i separated by defect lines $\ell_{i,j}$,

$$\Sigma = \bigsqcup_{i \in I} \Sigma_i \sqcup \bigsqcup_{\langle i,j \rangle \in I^2} \ell_{i,j}$$

Domains support phases $X : \Sigma_i \longrightarrow M_{(i)}$ embedded C^1-smoothly in connected metric manifolds $(M_{(i)}, \mathrm{g}_{(i)})$ with gerbes $\mathcal{G}_{(i)}$ subject to the conformality constraints

$$\beta_{\mu\nu}\left(\mathrm{g}_{(i)}, \Gamma(\mathrm{g}_{(i)}), \mathrm{curv}(\mathcal{G}_{(i)}); \alpha'\right) = 0\,.$$

A brief recap on the multi-phase 2d non-linear σ-model – ctd.

Defect lines are loci of field discontinuity determined by C^1-smooth embeddings $X : \ell_{i,j} \longrightarrow Q_{(i,j)}$ in manifolds $Q_{(i,j)}$ of curvatures $\omega_{(i,j)} \in \Omega^2(Q_{(i,j)})$, mapping smoothly

$$\iota_1^{(i,j)} : Q_{(i,j)} \longrightarrow M_{(i)}, \qquad \iota_2^{(i,j)} : Q_{(i,j)} \longrightarrow M_{(j)},$$

and equipped with gerbe bimodules

$$\Phi_{(i,j)} : \iota_1^{(i,j)\,*}\mathcal{G}_{(i)} \xrightarrow{\cong} \iota_2^{(i,j)\,*}\mathcal{G}_{(j)} \otimes I_{\omega_{(i,j)}}.$$

In what follows, we shall consider the composite target and bi-brane

$$\mathcal{M} := \bigsqcup_{i\in I}(M_{(i)}, \mathrm{g}_{(i)}, \mathcal{G}_{(i)}) \equiv (M, \mathrm{g}, \mathcal{G}), \qquad \mathcal{B} := \bigsqcup_{\langle i,j\rangle \in I^2}(Q_{(i,j)}, \iota_\alpha^{(i,j)}, \omega_{(i,j)}, \Phi_{(i,j)}) \equiv (Q, \iota_\alpha, \mathrm{curv}(\mathcal{B}), \Phi)$$

$(\mathcal{G}, \Phi)$ are a geometric realisation of the relative integral cohomology class

$$\tfrac{1}{2\pi}\,[(\mathrm{curv}(\mathcal{G}), \mathrm{curv}(\mathcal{B}))] \in H^3(M, Q\,|\,\Delta_Q), \qquad \Delta_Q := \iota_2^* - \iota_1^*$$

and as such they define a Cheeger–Simons differential character

$$\mathrm{Hol}_{(\mathcal{G},\Phi)}(X\,|\,\Gamma) \qquad \text{aka (decorated) surface holonomy.}$$

A brief recap on the multi-phase 2d non-linear σ-model – ctd.

The string background $\mathfrak{B} := (\mathcal{M}, \mathcal{B})$ determines the action functional:

$$S_\sigma[(X\,|\,\Gamma);\eta] := -\tfrac{1}{2}\int_\Sigma \mathrm{g}(\mathrm{d}X \overset{\wedge}{,} \star_\eta \mathrm{d}X) - \mathrm{i}\log \mathrm{Hol}_{(\mathcal{G},\Phi)}(X\,|\,\Gamma)$$

that yields field equations for the phases alongside the Defect Gluing Condition (DGC) over $\Gamma \ni p$ (with tangent $\widehat{t}$ and normal $\widehat{n}$),

$$\mathrm{DGC}_{\mathrm{curv}(\mathcal{B})}(\mathrm{p}_{|1}, \mathrm{p}_{|2}, X; V)(p) := \mathrm{p}_{|1}(\iota_{1\,*}V)(p) - \mathrm{p}_{|2}(\iota_{2\,*}V)(p) - \mathrm{curv}(\mathcal{B})(X_*\widehat{t}, V)(p) \overset{!}{=} 0\,,$$

valid for $\mathrm{p}_{|\alpha} := \mathrm{g}(X_{|\alpha})(X_{|\alpha\,*}\widehat{n}, \cdot)$ and an arbitrary $V \in \mathrm{T}_{X(p)}Q$.

N.B. World-sheets with a non-empty boundary are described by boundary defects with

$$M := M_{(1)} \sqcup \{\bullet\}\,, \qquad Q := D \subset M_{(1)}\,, \qquad \iota_1 : D \hookrightarrow M_{(1)}\,,$$

$$\iota_2 : D \longrightarrow \{\bullet\}\,.$$

Part II

The defects' anatomy and physique

Conformality

For the special choice $V := X_* \widehat{t}$, the DGC yields the continuity equation

$$\lim_{\sigma \to p^-} T_{\|}(\sigma) = \lim_{\sigma \to p^+} T_{\|}(\sigma)$$

for the component $T_{\|}$ of the energy-momentum tensor

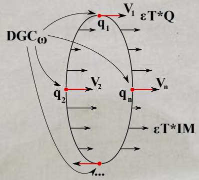
$$T^{ab} := -\frac{1}{\sqrt{|\det \eta|}} \frac{\delta}{\delta \eta_{ab}} S_\sigma[(X \,|\, \Gamma); \eta]$$

that generates diffeomorphisms of Σ preserving Γ.

Conclusion: The world-sheet defects are conformal *by construction*.

The twisted sector

In the presence of (timelike) defect lines, we may extract from S_σ – *via* first-order formalism – a canonical description of $\mathcal{B}$-twisted states



Gawędzki's transgression map

$$\tau \ : \ \left(\mathbb{H}^2\left(M, \mathcal{D}(2)^\bullet_M\right), \mathrm{g}\right) \longrightarrow \mathbb{H}^1\left(\mathsf{P}_\sigma, \mathcal{D}(1)^\bullet_{\mathsf{P}_\sigma}\right) \ : \ ([\mathcal{G}], \mathrm{g}) \longmapsto [\mathcal{L}_\sigma]$$

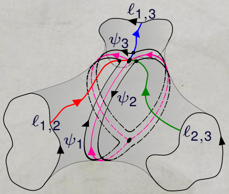
also generalises to the relative-geometric setting,

$$\tau_{\Delta_Q} \ : \ \left(\mathbb{H}^2\left(\mathscr{F}, \mathcal{D}(2)^\bullet_{\mathscr{F}} \,|\, \check{\Delta}_Q\right), \mathrm{g}\right) \longrightarrow \mathbb{H}^1\left(\mathsf{P}^{\mathrm{tw.}}_\sigma, \mathcal{D}(1)^\bullet_{\mathsf{P}^{\mathrm{tw.}}_\sigma}\right) \ : \ ([(\mathcal{G}, \Phi)], \mathrm{g}) \longmapsto [\mathcal{L}^{\mathrm{tw.}}_\sigma],$$

and thus captures geometric quantisation of the $\mathcal{B}$-twisted sector of the theory.

Fusion of twisted states and defect junctions

The splitting-joining interaction of (twisted) states



entails fusion of defects, for which we need to inject defect junctions of valence n in manifolds T_n mapping smoothly $\pi_n^{k,k+1} : T_n \longrightarrow Q$, $k \in \mathbb{Z}/n\mathbb{Z}$ and equipped with gerbe 2-isomorphisms

$$\varphi_n \ : \ \bigcirc_{k\in\mathbb{Z}/n\mathbb{Z}} \pi_n^{k,k+1\,*}\left(\Phi^{\varepsilon_n^{k,k+1}} \otimes \mathrm{Id}\right) \xRightarrow{\cong} \mathrm{Id}\,, \qquad \varepsilon_n^{k,k+1}(\text{in/out}) = +1/-1\,.$$

Through transgression, these define twisted-loop fusion

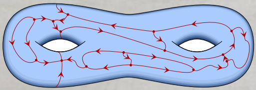
$$\left(\mathrm{pr}_1^*\mathcal{L}_\sigma^{\rm tw.} \otimes \mathrm{pr}_2^*\mathcal{L}_\sigma^{\rm tw.}\right)|_{\mathfrak{I}^{\rm tw.}_{2\to 1}} \xrightarrow{\cong} \mathrm{pr}_3^*\mathcal{L}_\sigma^{\rm tw.}|_{\mathfrak{I}^{\rm tw.}_{2\to 1}}$$

over an "interaction subspace" $\mathfrak{I}^{\rm tw.}_{2\to 1} \subset \mathsf{P}_\sigma^{\rm tw.} \times \mathsf{P}_\sigma^{\rm tw.} \times \mathsf{P}_\sigma^{\rm tw.}$.

The general multi-phase σ-model

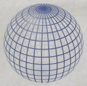
We may now conceive a theory of patchwise C^1-smooth embeddings

X :



$(\Sigma \supset \Gamma, \eta)$

$\hookrightarrow$



$\mathfrak{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J})$

in a string background $\mathfrak{B}$ composed of a target and a bi-brane as before, and of an inter-bi-brane

$$\mathcal{J} := \bigsqcup_{n \in \mathbb{N}_{\geq 3}} \left(T_n, \pi_n^{k,k+1}, \varphi_n \right)$$

The quantum Feynman amplitude

$$\mathscr{A}_{\rm F}[(X \,|\, \Gamma); \eta] := \exp\left(-\tfrac{\mathrm{i}}{2} \int_\Sigma \mathrm{g}(\mathrm{d}X \overset{\wedge}{,} \star_\eta \mathrm{d}X) \right) \cdot \mathrm{Hol}_{(\mathcal{G}, \Phi, \varphi_n)}(X \,|\, \Gamma)$$

is defined in terms of the differential character associated with the triple $(\mathcal{G}, \Phi, \varphi_n)$ that realises the integral cohomology class

$$\tfrac{1}{2\pi}[\Theta(\mathfrak{B})] \in H^3(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}), \qquad \Delta_{T_n} := \sum_{n \in \mathbb{Z}/n\mathbb{Z}} \varepsilon_n^{k,k+1} \, \pi_n^{k,k+1\,*}.$$

of the relative 3-cocycle $\Theta(\mathfrak{B}) := (\mathrm{curv}(\mathcal{G}), \mathrm{curv}(\mathcal{B}), 0)$.

The general multi-phase σ-model – ctd.

Upshot: Classification of inequivalent mono-phase σ-models, defects between them and defect junctions joining them in terms of relative Deligne cohomology

$$\mathbb{H}^\bullet\left(\mathscr{F}, \mathcal{D}(2)^\bullet_{\mathscr{F}} \,|\, \check{\Delta}_Q, \check{\Delta}_{T_n}\right) ,$$

to wit,

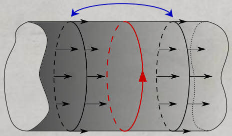
- inequivalent phases with fixed $(M, \mathrm{g}, \mathrm{curv}(\mathcal{G}))$ are classified by $H^2(M, \mathrm{U}(1))$;
- inequivalent defects between given phases with fixed $(Q, \iota_\alpha, \mathrm{curv}(\mathcal{B}))$ are classified by $H^1(Q, \mathrm{U}(1))$;
- inequivalent data for an n-valent junction of given defects with fixed $(T_n, \pi_n^{k,k+1})$ are classified by $\mathrm{U}(1)^{\pi_0(T_n)}$.

Part III

The defect-duality correspondence

Dualities from defects, ...

In conformity with the obvious world-sheet intuition,



a defect sets in correspondence states from the phases separated by it. Formally, the DGC defines an isotropic subspace

$$\mathfrak{D}_\sigma \subset \left(\mathsf{P}_\sigma \times \mathsf{P}_\sigma, \mathrm{pr}_1^*\Omega_\sigma - \mathrm{pr}_2^*\Omega_\sigma\right)$$

composed of pairs $\psi_\alpha = (X_\alpha, \mathsf{p}_\alpha) \in \mathsf{T}^*\mathsf{L}M$, $\alpha \in \{1, 2\}$ related as

$$\exists_{X \in \mathsf{L}Q} \; : \; X_\alpha = \iota_\alpha \circ X \,, \qquad \mathrm{DGC}_{\mathrm{curv}(\mathcal{B})}(\mathsf{p}_1, \mathsf{p}_2, X; \cdot) = 0 \,.$$

Bi-brane data lift this (local) symplectomorphism to a (local) bundle automorphism

$$\mathrm{pr}_2^*\mathcal{L}_\sigma|_{\mathfrak{D}_\sigma} \cong \mathrm{pr}_1^*\mathcal{L}_\sigma|_{\mathfrak{D}_\sigma} \,.$$

Dualities from defects, … – ctd.

For a proper field-theoretic (self-)duality, we must further require that

- $\mathfrak{D}_\sigma$ be a graph $\Longrightarrow$ the ι_α should be surjective submersions;
- the fundamental quantum charges of the states in correspondence match $\Longrightarrow$ $(\mathrm{pr}_2^*\mathscr{H} - \mathrm{pr}_1^*\mathscr{H})|_{\mathfrak{D}_\sigma} \stackrel{!}{=} 0$.

The last requirement, in conjunction with the DGC, translates into the statement of topologicality of duality defects,

$$\lim_{\sigma\to p^-} T(\sigma) = \lim_{\sigma\to p^+} T(\sigma)\,.$$

Amidst topological defects, there are extendible defects, with

$$\widehat{X} : \mathscr{U}_1 \cup \Gamma \cup \mathscr{U}_2 \longrightarrow Q\,, \qquad \widehat{X}|_\Gamma = X|_\Gamma \quad \wedge \quad \iota_\alpha \circ \widehat{X}|_{\mathscr{U}_\alpha} = X|_{\mathscr{U}_\alpha}\,,$$

$$\Delta_Q \mathrm{g}(\widehat{X})(\widehat{X}_*\widehat{u}^\perp, V)(\sigma) + \mathrm{curv}(\mathcal{B})(\widehat{X})(\widehat{X}_*\widehat{u}, V) = 0$$

for $\widehat{u} \in \mathrm{T}_\sigma\Sigma$ and $\sigma \in \mathscr{U}_1 \cup \Gamma \cup \mathscr{U}_2$ arbitrary.

. . . and *vice versa*

Conversely, a large class of dualities (the so-called $\partial_a X_{|\alpha}$-linear ones) can be shown to induce bi-brane data. These include T-duality-type defects and isometric defects with data

$$Q = (\mathrm{id}_M \times F)(M) \xrightarrow{\iota_\alpha = \mathrm{pr}_\alpha} M\,, \qquad \mathrm{curv}(\mathcal{B}) = 0\,, \qquad \Phi_F \ :\ \mathrm{pr}_1^* \mathcal{G} \xrightarrow{\cong} \mathrm{pr}_2^* \mathcal{G}$$

defined by arbitrary isometric diffeomorphisms $F \in \mathrm{Iso}(M, \mathrm{g})$.

N.B. The curvature of the bi-brane associated with an isometric defect vanishes identically.

Part IV

The Gauge Principle

Generalities

Point of departure: A field theory with field bundle

$$\widehat{\pi} \; : \; \widehat{\mathscr{F}} \longrightarrow \Sigma$$

of typical fibre $\widehat{\pi}^{-1}(\{\sigma\}) \cong \mathscr{F}$ over spacetime Σ. Sections $X \in \Gamma(\widehat{\mathscr{F}})$ define Feynman amplitudes

$$\mathscr{A}_{\mathrm{F}}[X] = \mathrm{e}^{\mathrm{i}\, S[X]}\,,$$

invariant under the action

$$\ell \; : \; \mathrm{G} \times \mathscr{F} \longrightarrow \mathscr{F} \; : \; (g, X) \longmapsto g.X$$

of a (global-)symmetry group $G \subset \mathrm{Diff}(\mathscr{F})$.

Goal: Render the symmetry local by passing to another bundle $\widetilde{\pi} : \widetilde{\mathscr{F}} \longrightarrow \Sigma$ of fibre isotype $\mathscr{F}$ and such that a bundle of groups $\widetilde{\mathrm{G}} \longrightarrow \Sigma$ of fibre isotype G acts fibrewise on $\widetilde{\mathscr{F}}$ through transformations that depend upon the point in the base. Use sections of $\widetilde{\pi}$ to write down the gauged field theory.

Generalities – ctd.

Idea: Extend $\widehat{\mathscr{F}} \longmapsto \mathsf{P} \times_\Sigma \widehat{\mathscr{F}}$ by an arbitrary principal G-bundle $\pi_{\mathsf{P}} : \mathsf{P} \longrightarrow \Sigma$ with principal G-connection $\mathcal{A} \in \Omega^1(\mathsf{P}) \otimes \mathfrak{g}$, $\mathfrak{g} := \mathrm{Lie}\,\mathrm{G}$, and subsequently descend to the associated bundle,

$$\mathsf{P} \times_\Sigma \widehat{\mathscr{F}} \longmapsto (\mathsf{P} \times_\Sigma \widehat{\mathscr{F}})/\mathrm{G}\,,$$

admitting a fibrewise action of the adjoint bundle

$$(\mathsf{P} \times \mathrm{G})/\mathrm{G} \equiv \mathsf{P} \times_{\mathrm{Ad}} \mathrm{G}\,.$$

Challenges:

1. Extension to $\mathsf{P} \times_\Sigma \widehat{\mathscr{F}}$ of the metric and cohomological structure from over $\widehat{\mathscr{F}}$ entering the definition of $\mathscr{A}_{\mathrm{F}}$, in a manner structurally compatible with the amplitudes.
2. Descent of the extended structure to the smooth quotient $(\mathsf{P} \times_\Sigma \widehat{\mathscr{F}})/\mathrm{G}$, giving rise to a gauged field theory with this field bundle and invariant with respect to $\mathsf{P} \times_{\mathrm{Ad}} \mathrm{G}$.

Generalities – ctd.

Strategy:

1. Identification of the algebraic structure on the set of global symmetries.
2. Coupling of a topologically trivial gauge field and derivation of constraints for a consistent gauging (the gauge anomaly).
3. Motivation for and reconstruction (through local trivialisation) of topologically nontrivial gauge bundles.
4. Extraction of a G-equivariant structure on the string background and classification thereof.
5. The gauged σ-model and the coset theory.

Ad 1. The algebra of global symmetries

Symmetry transformations

$$\mathrm{G} \times \mathscr{F} \longrightarrow \mathscr{F} \; : \; (g, X) \longmapsto g.X\,, \qquad \mathscr{A}_{\mathrm{F}}[g.X] = \mathscr{A}_{\mathrm{F}}[X]$$

are described, on the infinitesimal level, by vector fields $\mathscr{K} \in \Gamma(\mathrm{T}\mathscr{F})$ with local flows $\psi_{\cdot} :]-\varepsilon, \varepsilon[\times \mathscr{F} \longrightarrow \mathscr{F}$ such that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathscr{A}_{\mathrm{F}}[\psi_t \circ X] = 0\,.$$

These describe symmetries iff

$$\widetilde{\mathscr{L}_{\mathscr{K}}}\mathrm{g} = 0 \qquad \wedge \qquad \widetilde{\imath}_{\mathscr{K}}(\mathrm{curv}(\mathcal{G}), \mathrm{curv}(\mathcal{B}), 0) = -\widetilde{\mathrm{d}}(\kappa, k, 0)$$

for some $(\kappa, k, 0) \in \Omega^1(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n})$, the latter combining with $\mathscr{K}$ to a σ-symmetric section $\mathscr{K} \oplus (\kappa, k, 0)$ of the generalised tangent bundles

$$\mathrm{E}\mathscr{F} := (\mathrm{T}M \oplus \mathrm{T}^*M) \sqcup (\mathrm{T}Q \oplus (Q \times \mathbb{R})) \sqcup \bigsqcup_{n \geq 3} (\mathrm{T}T_n \oplus (T_n \times \mathbb{R})_0) \to \mathscr{F}\,.$$

Ad 1. The algebra of global symmetries – ctd.

On $\Gamma(\mathsf{E}\mathscr{F})$, there is an essentially unique bracket

$$[\mathscr{V}\oplus\upsilon,\mathscr{W}\oplus\varpi]_{\mathrm{C}}^{\Theta(\mathfrak{B})} := [\mathscr{V},\mathscr{W}]\oplus\Big(\widetilde{\mathscr{L}_{\mathscr{V}}}\varpi - \widetilde{\mathscr{L}_{\mathscr{W}}}\upsilon - \tfrac{1}{2}\,\widetilde{\mathsf{d}}(\widetilde{\imath}_{\mathscr{V}}\varpi - \widetilde{\imath}_{\mathscr{W}}\upsilon) + \widetilde{\imath}_{\mathscr{V}}\widetilde{\imath}_{\mathscr{W}}\Theta(\mathfrak{B})\Big)$$

that closes on σ-symmetric sections $\Gamma_\sigma(\mathsf{E}\mathscr{F})$ and gives rise to the relative $\Theta(\mathfrak{B})$-twisted Courant algebroid

$$\mathfrak{C}_\sigma(\mathfrak{B}) := \Big(\mathsf{E}\mathscr{F},[\cdot,\cdot]_{\mathrm{C}}^{\Theta(\mathfrak{B})},(\cdot,\cdot)_{\lrcorner},\alpha_{\mathsf{T}\mathscr{F}}\Big)\,.$$

$\mathfrak{C}_\sigma(\mathfrak{B})$ furnishes a target-space model of the Poisson algebra of Noether charges of the symmetry.

N.B. $\mathfrak{C}_\sigma(\mathfrak{B})$ is not a Lie algebroid (Jacobi and Leibniz fail). In the fundamental basis

$$\mathfrak{K}_A := \mathscr{K}_A\oplus(\kappa_A,k_A,0)\equiv\mathscr{K}_A\oplus\mathrm{K}_A\,,\qquad [\mathscr{K}_A,\mathscr{K}_B]=f_{ABC}\,\mathscr{K}_C\,,$$

we find

$$[\mathfrak{K}_A,\mathfrak{K}_B]_{\mathrm{C}}^{\Theta(\mathfrak{B})} = f_{ABC}\,\mathfrak{K}_C + 0\oplus\alpha_{AB}\,,\qquad \text{with}$$

$$\alpha_{AB} := \widetilde{\mathscr{L}_{\mathscr{K}_A}}\mathrm{K}_B - f_{ABC}\,\mathrm{K}_C - \widetilde{\mathsf{d}}(\mathfrak{K}_A,\mathfrak{K}_B)_{\lrcorner}\,.$$

Ad 2. The coupling of the trivial ...

Begin with

$$\mathrm{P} = \Sigma \times \mathrm{G}\,, \qquad\qquad \mathrm{A} = \mathrm{A}^A \otimes t_A \in \Omega^1(\Sigma) \otimes \mathfrak{g}\,.$$

Analysis of a G-invariant tensorial string background leads to the introduction of an extended string background $\mathfrak{B}_{\mathrm{A}}$ with components:

(T) the extended target $\Sigma \times M$ with

$$\mathrm{g}_{\mathrm{A}} := \mathrm{g}_2 - \mathrm{g}(\mathscr{K}_A, \cdot)_2 \otimes \mathrm{A}_1^A - \mathrm{A}_1^A \otimes \mathrm{g}(\mathscr{K}_A, \cdot)_2 + \mathrm{g}(\mathscr{K}_A, \mathscr{K}_B)_2\, \mathrm{A}_1^A \otimes \mathrm{A}_1^B \quad \text{and}$$

$$\mathcal{G}_{\mathrm{A}} := \mathcal{G}_2 \otimes I_{\rho_{\mathrm{A}}}\,, \quad \text{where} \quad \rho_{\mathrm{A}} := \kappa_{A\,2} \wedge \mathrm{A}_1^A - \frac{1}{2}(\iota_{\mathscr{K}_A}\kappa_B)_2\, \mathrm{A}_1^A \wedge \mathrm{A}_2^B\,;$$

(BB) the extended bi-brane $\mathfrak{E}_\Gamma \times Q$ with

$$\omega_{\mathrm{A}} := \omega_{2*} - \underline{\Delta}_Q \rho_{\mathrm{A}} + \mathrm{d}\lambda_{\mathrm{A}} \quad \text{and} \quad \Phi_{\mathrm{A}} := \Phi_2 \otimes J_{\lambda_{\mathrm{A}}}\,, \quad \text{where} \quad \lambda_{\mathrm{A}} := -k_{A\,2}\, \mathrm{A}_1^A\,;$$

(IBB) the extended inter-bi-brane $\mathfrak{V}_\Gamma^{(n)} \times T_n$ with

$$\varphi_{n\,\mathrm{A}} := \varphi_{n\,2}\,.$$

Ad 2. …and the obstruction

The gauged Feynman amplitude for extended maps $\phi = (\mathrm{id}_\Sigma, X)$,

$$\mathscr{A}_{\mathrm{F}}[(X \mid \Gamma); \mathrm{A}, \eta] := \exp\left(-\tfrac{\mathrm{i}}{2} \int_\Sigma \mathrm{g}_{\mathrm{A}}(\mathrm{d}\phi \overset{\wedge}{,} \star_\eta \mathrm{d}\phi)\right) \mathrm{Hol}_{(\mathcal{G}_{\mathrm{A}}, \Phi_{\mathrm{A}}, \varphi_{n\,\mathrm{A}})}(\phi \mid \Gamma)\,,$$

is invariant under small (id-homotopic) gauge transformations iff

$$\mathfrak{S}_{\mathfrak{B}} := \left(\bigoplus_{A=1}^{\dim \mathfrak{g}} C^\infty(\mathscr{F}, \mathbb{R})\, \mathfrak{K}_A, [\cdot, \cdot]_{\mathrm{C}}^{\Theta(\mathfrak{B})}, \alpha_{\mathrm{T}\mathscr{F}}\right) \cong \mathfrak{g} \ltimes_\ell \mathscr{F} \equiv \mathrm{Lie}\,(\mathrm{G} \ltimes_\ell \mathscr{F})\,.$$

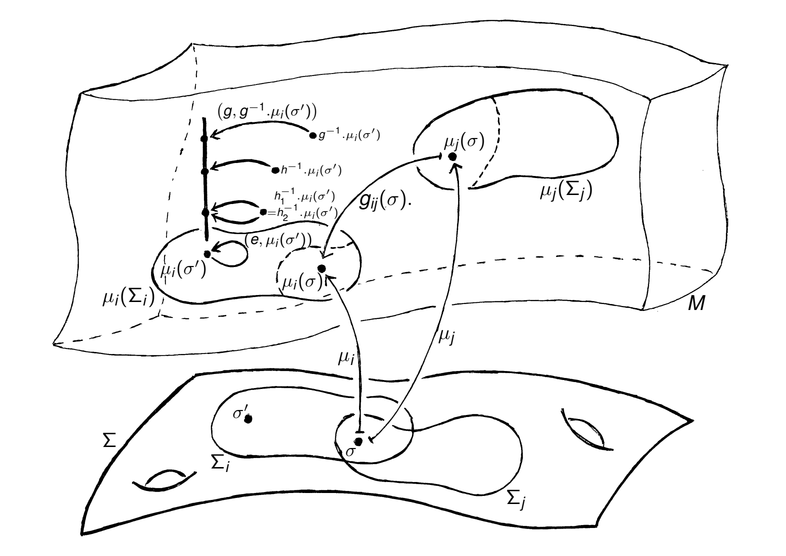
Conclusion: $\mathfrak{C}_\sigma(\mathfrak{B})$ encodes the Small Gauge Anomaly.

Question: Whence $\mathrm{G} \ltimes_\ell \mathscr{F}$ as the symmetry of the gauged σ-model?

Answer: Recall that data of the gauged σ-model are: a principal G-bundle $\mathsf{P} \longrightarrow \Sigma$ with the property that $\mathsf{P} \times_{\mathrm{G}} \mathscr{F} \longrightarrow \Sigma$ admits a global section. But we have equivalence of groupoids

$$\mathrm{G}\text{-}\mathfrak{Bun}(\Sigma \,\|\, \mathscr{F}) \cong \mathrm{G} \ltimes_\ell \mathscr{F}\text{-}\mathfrak{Bun}(\Sigma)$$ with an intuitive interpretation . . .

The world-sheet of the gauged σ-model



Ad 2. …and the obstruction – ctd.

Conclusions:

- The action groupoid $\mathrm{G}\ltimes_\ell\mathscr{F}$ captures the symmetries of the gauged σ-model.
- We need topologically non-trivial gauge fields to account for the existence of the G-twisted sector.

Goal: Get physical, *i.e.*, account for connections on P and for the metric and cohomological structure on $\mathscr{F}$.

Idea: Local trivialisation in conjunction with results for the topologically trivial case (TBC …).

Ad 2. …and the obstruction – completed

The necessary and sufficient condition for the invariance of the gauged Feynman amplitude under large (non-id-homotopic) gauge transformations,

$$(X, \mathrm{A}) \longmapsto \left(\chi.X, \mathrm{Ad}_\chi \mathrm{A} - \mathrm{d}\chi\,\chi^{-1}\right), \qquad \chi \not\sim \mathrm{id},$$

is the existence of the gerbe 1-isomorphism

$$\Upsilon \;:\; \ell^*\mathcal{G} \xrightarrow{\cong} \mathrm{pr}_2^*\mathcal{G} \otimes I_{\rho_{\theta_L}}$$

over $\mathrm{G} \times M$, and of the gerbe 2-isomorphism

$$\Xi \;:\; \ell^*\Phi \overset{\cong}{\Longrightarrow} \left[\left((\mathrm{id}_\mathrm{G} \times \iota_2)^*\Upsilon^{-1} \otimes \mathrm{Id}\right) \circ (\mathrm{pr}_2^*\Phi \otimes \mathrm{Id}) \circ (\mathrm{id}_\mathrm{G} \times \iota_1)^*\Upsilon\right] \otimes J_{\lambda_{\theta_L}}$$

over $\mathrm{G} \times Q$, subject to additional coherence constraints

$$\ell^*\varphi_n = \alpha_n \bullet \mathrm{pr}_2^*\varphi \bullet \beta_n \bullet \bigcirc_{k\in\mathbb{Z}/n\mathbb{Z}} (\mathrm{id}_\mathrm{G} \times \pi_n^{k,k+1})^* \left(\Xi^{\varepsilon_n^{k,k+1}} \otimes \mathrm{Id}\right)$$

over the $\mathrm{G} \times T_n$.

Ad 3. The reconstruction …

Having understood the necessity of incorporating topologically nontrivial gauge fields into the description, we may piece the corresponding gauged σ-model together from its local trivialisations,

$$\tau_i \,:\, \pi_{\mathrm{P}}^{-1}(\mathcal{O}_i) \longrightarrow \mathcal{O}_i \times \mathrm{G}\,, \qquad \mathcal{O}_\Sigma := \{\mathcal{O}_i\} \subset \mathscr{T}_\Sigma\,,$$

$$X_i \,:\, \mathcal{O}_i \longrightarrow \mathscr{F}\,, \qquad \mathrm{A}_i \in \Omega^1(\mathcal{O}_i) \otimes \mathfrak{g}\,,$$

related on double intersections $\mathcal{O}_i \cap \mathcal{O}_j =: \mathcal{O}_{ij} \ni \sigma$ as

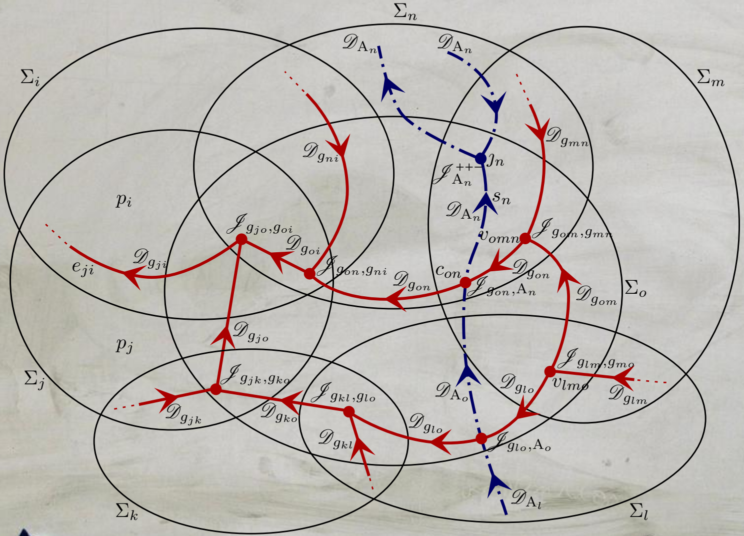
$$X_i(\sigma) = g_{ij}(\sigma).X_j(\sigma)\,, \qquad \mathrm{A}_i(\sigma) = \mathrm{Ad}_{g_{ij}(\sigma)}\mathrm{A}_j(\sigma) - \mathrm{d}g_{ij}(\sigma)\, g_{ij}(\sigma)^{-1} \equiv {}^{g_{ij}}\mathrm{A}_j(\sigma)$$

in terms of transition maps

$$g_{ij} \in C^\infty(\mathcal{O}_{ij}, \mathrm{G})\,, \qquad (\check{\delta} g)_{ijk} = e\,.$$

To this end, we associate to $\mathcal{O}_\Sigma$ a Γ-transversal and Γ-simple oriented trivalent graph $\Gamma_{\mathcal{O}_\Sigma} \subset \Sigma$ …

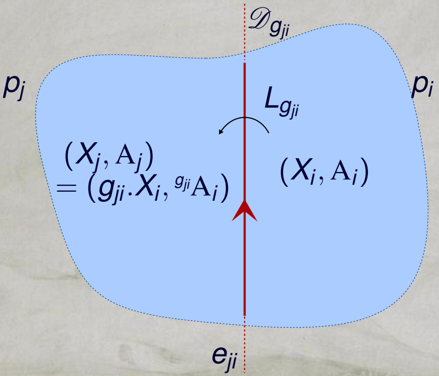
Ad 3. …through Trivialisation



Ad 3. RtT: Data assignment

To a free edge $e_{ji} \in \Gamma_{\mathcal{O}_\Sigma}$, we pull back the (flat) transition bi-brane

$$\mathcal{B}_{g_{ji}} := \Big(\{g_{ji}\} \times \mathcal{O}_{ij} \times M, L_{g_{ji}}, \mathrm{id}_{\mathcal{O}_{ij} \times M}, 0, (g_{ji} \times \mathrm{id}_M)^* \Upsilon =: \Upsilon_{g_{ji}} \Big)$$

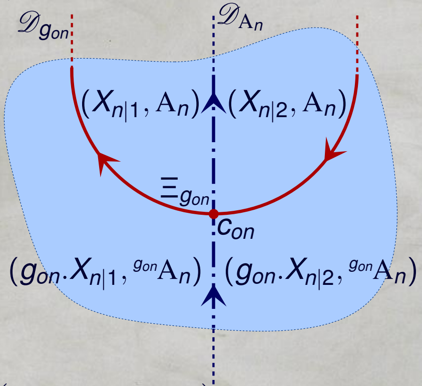


The ensuing transition defect $\mathscr{D}_{g_{ji}}$ is manifestly extendible within $\mathcal{O}_{ij}$, and hence topological.

Ad 3. RtT: Data assignment – ctd.

To a Γ-crossing $c_{on} \in \Gamma_{\mathcal{O}_\Sigma} \cap \Gamma$, we pull back the trans-defect transition inter-bi-brane

$$\mathcal{J}_{g_{on};\mathrm{A}_n} := \Big(\{g_{on}\} \times \mathcal{O}_{on} \times Q, \widetilde{\pi}_4^{k,k+1}, (g_{on} \times \mathrm{id}_Q)^* \Xi \Big)$$



$$\Xi_{g_{on}} := (g_{on} \times \mathrm{id}_Q)^* \Xi \; : \; L^*_{g_{on}} \Phi_{g_{on}\mathrm{A}_n} \xrightarrow{\cong} \Big((\mathrm{id}_G \times \iota_2)^* \Upsilon^{-1}_{g_{on}} \otimes \mathrm{Id} \Big) \circ (\Phi_{\mathrm{A}_n} \otimes \mathrm{Id}) \circ (\mathrm{id}_G \times \iota_1)^* \Upsilon_{g_{on}} ,$$

$$\widetilde{\pi}_4^{1,2}(g_{on}, \sigma, q) = (\sigma, g_{on}(\sigma).q) , \qquad \widetilde{\pi}_4^{2,3}(g_{on}, \sigma, q) = (g_{on}, \sigma, \iota_2(q)) ,$$

$$\widetilde{\pi}_4^{3,4}(g_{on}, \sigma, q) = (\sigma, q) , \qquad \widetilde{\pi}_4^{4,1}(g_{on}, \sigma, q) = (g_{on}, \sigma, \iota_1(q)) .$$

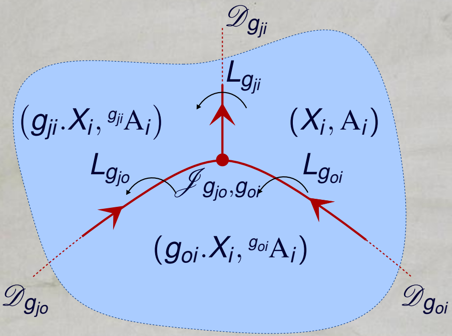
Ad 3. RtT: Coherence

Independence of the gauged Feynman amplitudes of the arbitrary choices made along the way is tantamount to the topologicality of the gauge defect network and the existence of extra structure …

Ad 3. RtT: Coherence – ctd.

We need the (elementary) transition inter-bi-brane

$$\mathcal{J}^{++-}_{(g_{jo},g_{oi})} := \Big(\{(g_{jo},g_{oi})\}\times\mathcal{O}_{joi}\times M,\widetilde{d}^{(2)}_{\bullet},((g_{jo},g_{oi})\times\mathrm{id}_M)^*\gamma =: \gamma_{(g_{jo},g_{oi})}\Big)$$

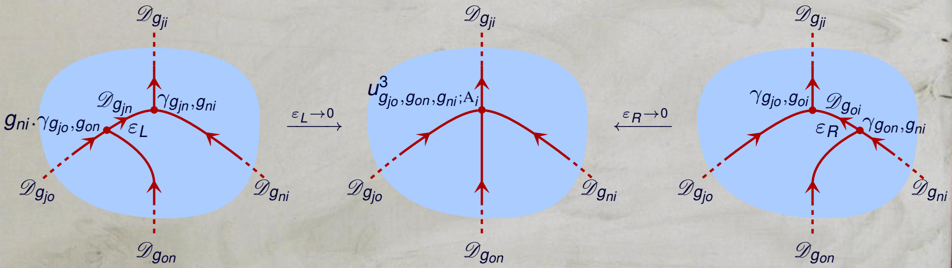


carrying the data of the gerbe 2-isomorphism (over $\mathrm{G}^2\times M$)

$$\gamma\ :\ \big(d_0^{(2)\,*}\Upsilon\otimes\mathrm{Id}\big)\circ d_2^{(2)\,*}\Upsilon\xrightarrow{\cong} d_1^{(2)\,*}\Upsilon\ \ldots$$

Ad 3. RtT: Coherence – ctd.

…and satisfying the coherence (associativity) condition



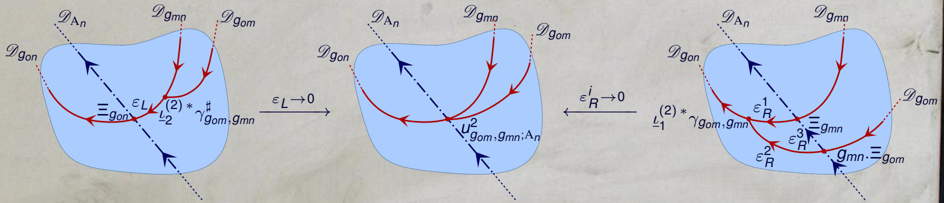
with $$u^3_{g_{jo},g_{on},g_{ni};\mathrm{A}_i} = \delta^{(2)}_{C^\infty(\mathcal{O}_{joni},\mathrm{G})} \gamma_{g_{jo},g_{on},g_{ni}} \stackrel{!}{=} 1 \,,$$

which enforces the cocycle condition (over $\mathrm{G}^3 \times M$)

$$d_1^{(3)\,*}\gamma \bullet \left(\mathrm{Id} \circ d_3^{(3)\,*}\gamma\right) = d_2^{(3)\,*}\gamma \bullet \left(\left(d_0^{(3)\,*}\gamma \otimes \mathrm{Id}\right) \circ \mathrm{Id}\right) .$$

Ad 3. RtT: Coherence – ctd.

Similar consistency considerations in the vicinity of Γ,



yield

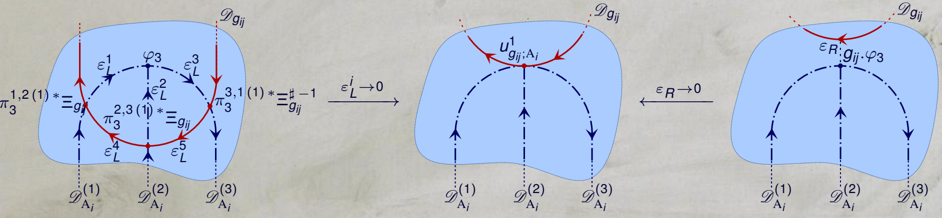
$$u^2_{gom,gmn;\mathrm{A}_n} := \left(\mathrm{Id}\circ \underline{\iota}_1^{*}\gamma_{gom,gmn}\right)\bullet\left(\mathrm{Id}\circ(\Xi_{gmn}\otimes \mathrm{Id})\circ\mathrm{Id}\right)\bullet L^*_{gmn}\Xi_{gom}\bullet\Xi^{-1}_{gon}\bullet\left(\left(\underline{\iota}_1^{*}\gamma^\sharp_{gom,gmn}\otimes\mathrm{Id}\right)\circ\mathrm{Id}\right)^{-1}\overset{!}{=}1\,,$$

and thus enforce (over $\mathrm{G}^2\times Q$)

$$\left(\left(\iota_2^{(1)\,*}\gamma^\sharp\otimes\mathrm{Id}\right)\circ\mathrm{Id}\right)\bullet d_1^{(2)\,*}\Xi=\left(\mathrm{Id}\circ\iota_1^{(1)\,*}\gamma\right)\bullet\left(\mathrm{Id}\circ\left(d_0^{(2)\,*}\Xi\otimes\mathrm{Id}\right)\circ\mathrm{Id}\right)\bullet d_2^{(2)\,*}\Xi\,.$$

Ad 3. RtT: Coherence – ctd.

Finally, coherence constraints imposed in the topologically trivial setting remove the obstruction $u^1_{g_{ij};\mathrm{A}_i}$ against



and thus bring the reconstruction procedure to a completion.

Upshot: The data

$$(\mathcal{G}, \Upsilon, \gamma)\,, \qquad (\Phi, \Xi)\,, \qquad (\varphi_n)_{n \in \mathbb{N}_{\geq 3}}\,,$$

subject to the coherence conditions listed compose a G-equivariant string background.

Ad 4. Extraction and classification

Formally, a G-equivariant structure on $\mathfrak{B}$ is a geometric realisation of the class in the G-cohomology extension of the relative Čech–Deligne cohomology extending $[(\mathcal{G}, \Phi, \varphi_n)]$. Thus, answers to questions of its existence and uniqueness are neatly captured by the cohomology of the 4-complex

$$\check{C}^\bullet\left(\mathcal{O}, \underline{\Omega^\bullet(\mathrm{G}^\bullet \times \mathscr{M}^\bullet)}\right).$$

As has been demonstrated and shall be argued further in what follows, these answers translate into classificatory statements about gauged σ-models (more on this in the WZNW context in C. Tauber's talk).

Ad 5. The gauged σ-model ...

We formulate the theory in the presence of an *arbitrary* principal G-bundle $\mathsf{P} \longrightarrow \Sigma$ as in the topologically trivial setting through replacements $\quad \Sigma \times \mathscr{F} \longmapsto \mathsf{P} \times \mathscr{F}\,, \qquad \mathrm{A} \longmapsto \mathcal{A}\,.$

Theorem: **[Gawędzki,Waldorf,rrS]** The P-extended string background $\widetilde{\mathfrak{B}}_{\mathcal{A}}$ descends to $\mathsf{P} \times_{\mathrm{G}} \mathscr{F}$ (through Stevenson's Categorial Descent)

if

$\mathfrak{B}$ carries an arbitrary G-equivariant structure. In this case,

$$S_\sigma^{\rm gauge}[(\zeta\,|\,\Gamma);\eta] := -\tfrac{1}{2}\int_\Sigma \mathrm{g}_{\mathcal{A}}(\mathsf{d}\zeta \overset{\wedge}{,} \star_\eta \mathsf{d}\zeta) - \mathsf{i}\,\log \mathrm{Hol}_{\mathfrak{B}_{\mathcal{A}}}(\zeta\,|\,\Gamma)$$

where $\zeta \in \Gamma(\mathsf{P} \times_{\mathrm{G}} \mathscr{F})$ (global) and $\widehat{\varpi}^*_{\mathsf{P}\times\mathscr{F}}\mathfrak{B}_{\mathcal{A}} := \widetilde{\mathfrak{B}}_{\mathcal{A}}$ (unique).
The model is invariant wrt. an action of the gauge group $\mathsf{P} \times_{\mathrm{Ad}} \mathrm{G}$.
The obstruction to the existence of the G-equivariant structure on $\mathfrak{B}$ is termed the Large Gauge Anomaly.

Ad 5. …and the coset theory

In topologically favourable circumstances (in particular, for $\mathscr{F} \longrightarrow \mathscr{F}/\mathrm{G}$ a principal G-bundle), the multi-phase gauged σ-model induces a multi-phase σ-model on the quotient $\mathscr{F}/\mathrm{G}$ through integration of the Lagrange multipliers A.

Part V

Conclusions & Outlook

Conclusions

- We have established an algebroidal interpretation of the Small Gauge Anomaly, consistent with the groupoidal geometry underlying the (dual) data of the gauged σ-model.
- We have reinterpreted the Large Gauge Anomaly as the obstruction to the existence of a topological gauge defect network implementing a local trivialisation of the gauged σ-model coupled to a topologically nontrivial gauge field.
- We have demonstrated the necessity of the existence of a full-fledged symmetry-equivariant structure on the string background of the σ-model.

Outlook

- Construction of intrinsically non-Abelian bi-branes of the type originally considered by Gawędzki in the context of the orbifold boundary WZNW σ-model, and of the associated inter-bi-branes.
- Comparison with the relevant predictions of the categorial quantisation scheme.
- Application of the gauge principle in the study of T-duality in the context of the gerbe theory of the σ-model.
- A world-sheet construction of the σ-model on an orbispace of a target space with respect to the action of *bona fide* dualities, based on the defect-duality correspondence (*e.g.*, T-folds).