# From GERBES to DEFECTS
## with some side steps

Krzysztof Gawędzki
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> ”Certain defects are necessary for the existence of individuality”
>
> Johann Wolfgang von Goethe

**Plan:**

- 2D WZ actions and gerbes
- boundaries and walls
- junctions

**Warnings:**

- gerbes will be abelian
- applications will concern low dim. field theories
- descriptions will be somewhat impressionistic

**Examples** of theories under consideration:

- general $2$D **sigma models** with a **Wess-Zumino** (**WZ**) term in the action corresponding to a closed $3$-form $H$ on target $M$
- **Wess**-**Zumino**-**Witten** (**WZW**) **models** with a **Lie** group $G$ as the target (examples of **CFT**)
- **coset models** of **CFT** viewed as gauged **WZW** models
- **Chern-Simons** (**CS**) topological gauge theory (viewed as a $3$D sigma model with background **Pontryagin** closed $4$-forms) - not here

## Common features of these models:

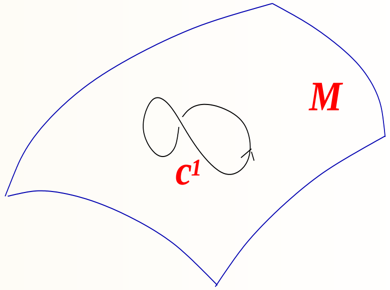
- **Feynman amplitudes** receive contributions from

  $\boxed{\textbf{higher Abelian holonomies}}$

  generalizing the case of standard **Abelian holonomy** for the electromagnetic field

- They may be treated using
  - **Deligne cohomology**
  - **Cheeger-Simons differential characters**
  - **Murray**'s **bundle gerbes**

- Standard **Abelian holonomy** for the **electromagnetic field:**
  - $A$ a $1$-form on $M$
  - $dA = F$ "field strength" - a $2$-form
  - $c^1$ a $1$-cycle in $M$



$$\exp\left[i\int_{c^1} A\right] = Hol_{\mathcal{L}}(c^1)$$

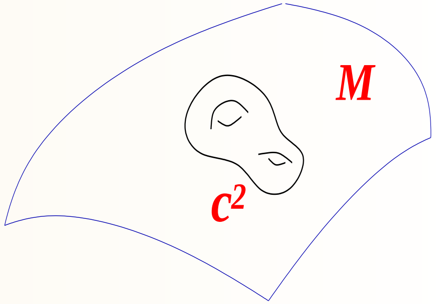
**top. Feyn. ampl.**

**line bundle** over $M$

- **RHS** makes sense for any **line bundle** $\mathcal{L}$ with connection of curvature $F$

- Such bundles exist iff $F$ is a closed $2$-form with periods in $2\pi\mathbb{Z}$ (**Dirac**'s quantization of magnetic charge)

- Degree 2 **Abelian holonomy** for the **Kalb-Ramond field:**



  - $B$ a 2-form over $M$
  - $dB = H$ "torsion" 3-form
  - $c^2$ a 2-cycle in $M$

$$\exp\left[i\int_{c^2} B\right] \;=\; Hol_{\mathcal{G}}(c^2)$$

**top. Feyn. ampl.** ↗ ↖ **gerbe** over $M$

- **RHS** makes sense for any **bundle gerbe** $\mathcal{G}$ with connection of curvature $H$ (called below a **gerbe**, for short)

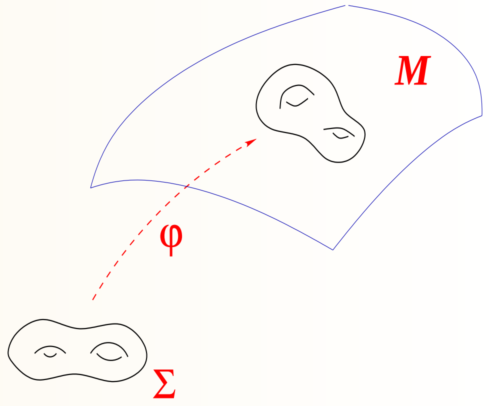
- Such gerbes exist iff $H$ is a closed 3-form with periods in $2\pi\mathbb{Z}$

- **Field theory application:**

Gerbe holonomy defines the **Wess-Zumino** contribution to the **Feynman amplitudes** of 2D sigma model fields $\varphi : \Sigma \to M$ for closed oriented worldsheets $\Sigma$:

$$\exp\left[i\, S_{WZ}(\varphi)\right] \quad := \quad Hol_{\mathcal{G}}(\varphi(\Sigma))$$

where $\mathcal{G}$ is a fixed gerbe with curvature $H$



- **Standard example:**

**WZW** model with $M = G$ - a **Lie** group - and

$$H \equiv H_k \;=\; \frac{k}{12\pi}\,\mathrm{tr}\,(g^{-1}dg)^3$$

- **Main property** of the line-bundle holonomy:

$$Hol_{\mathcal{L}}(\partial c^2) \quad = \quad \exp\Big[\, i \int_{c^2} F \,\Big]$$

i.e. $Hol_{\mathcal{L}}$ is a degree $2$ **Cheeger-Simons** **differential character**

- **Main property** of the gerbe holonomy:

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- **Remark:** **RHS**s determine **LHS**s if $H_1(M) = 0$ or $H_2(M) = 0$, respectively, and **l.-bdles** and **gerbes** are an overkill in such cases

# What are gerbes ?

- **Line bundles with connections** may be presented by **local data** $(A_\alpha, g_{\alpha\beta})$ w.r.t. to an open covering $(\mathcal{O}_\alpha)$

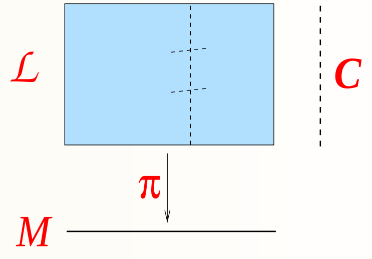
$$dA_\alpha = F\,, \qquad A_\beta - A_\alpha = i\,d\ln g_{\alpha\beta}\,, \qquad g_{\alpha\beta}\,g_{\alpha\gamma}^{-1}\,g_{\beta\gamma} = 1\,,$$

$(A_\alpha, g_{\alpha\beta})$ and $(A'_\alpha, g'_{\alpha\beta})$ representing isomorphic **l.-bdles** iff

$$A'_\alpha - A_\alpha = -i d\ln f_\alpha, \qquad g'_{\alpha\beta}\,g_{\alpha\beta}^{-1} = f_\alpha f_\beta^{-1}$$

isomorphism classes of **l.−bdles** (with connection) $\cong$ degree 2 classes of smooth **Deligne cohomology**

- But **line bundles** possess also a geometric description



## Similarly:

- **Gerbes** (with connection) may be presented by **local data** $(B_\alpha, A_{\alpha\beta}, g_{\alpha\beta\gamma})$ with

$$dB_\alpha = H\,, \quad B_\beta - B_\alpha = dA_{\alpha\beta}\,, \quad A_{\alpha\beta} - A_{\alpha\gamma} + A_{\beta\gamma} = i\,d\ln g_{\alpha\beta\gamma}\,,$$
$$g_{\alpha\beta\gamma}\, g^{-1}_{\alpha\beta\delta}\, g_{\alpha\gamma\delta}\, g^{-1}_{\beta\gamma\delta} = 1$$

and $(B_\alpha, A_{\alpha\beta}, g_{\alpha\beta\gamma})$ and $(B'_\alpha, A'_{\alpha\beta}, g'_{\alpha\beta\gamma})$ representing ($1$-)isomorphic gerbes iff

$$B'_\alpha - B_\alpha = d\Pi_\alpha, \qquad A'_{\alpha\beta} - A_{\alpha\beta} = \Pi_\beta - \Pi_\alpha - i\,d\ln\chi_{\alpha\beta},$$
$$g'_{\alpha\beta\gamma}\, g^{-1}_{\alpha\beta\gamma} = \chi^{-1}_{\alpha\beta}\chi_{\alpha\gamma}\chi^{-1}_{\beta\gamma}$$

($1-$)isomorphism classes of **gerbes** (with connection) $\cong$ degree $3$ classes of smooth **Deligne cohomology**

- But **gerbes** possess also a geometric description due to **Murray** (1994)

$$\begin{tikzcd}
\mathcal{L} \arrow[d, "\textbf{line bundle with conn.}"'] \arrow[dr, shift left] \arrow[dr, shift right] & \\
Y \times_M Y \equiv Y^{[2]} \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & Y \arrow[d, "\pi \quad \textbf{surjective submersion}"] \\
& M
\end{tikzcd}$$

- $\mathcal{L} \rightrightarrows Y$ equipped with **groupoid multiplication** $\mu$ bilinear on fibers and preserving connection

- $Y$ is equipped with a **curving** $2$-form $B$ s.t. $F_{\mathcal{L}} = \partial_1^* B - \partial_0^* B$

- $dB = \pi^* H$

**Example** (relating local data $(B_\alpha, A_{\alpha\beta}, g_{\alpha\beta\gamma})$ to geom. definition)**:**

- $Y = \bigsqcup_\alpha \mathcal{O}_\alpha \xrightarrow{\pi} M$
- $Y^{[2]} = \bigsqcup_{(\alpha,\beta)} \mathcal{O}_\alpha \cap \mathcal{O}_\beta \rightrightarrows Y$
- **line bundle** $\mathcal{L} = Y^{[2]} \times \boldsymbol{C}$
- with **connection** form $A|_{\mathcal{O}_\alpha \cap \mathcal{O}_\beta} = A_{\alpha\beta}$
- **curving** form $B|_{\mathcal{O}_\alpha} = B_\alpha$
- **groupoid multiplication** $\mu$ in $\mathcal{L}$ given by multiplication by $g_{\alpha\beta\gamma}$

## **Facts about (bundle) gerbes (with connection)**

- **gerbes** over manifold $M$ form a $2$-**category** with $1$-morphisms between them and $2$-morphisms between $1$-morphisms

$1$-**morphism** $\alpha : \mathcal{G}_1 \to \mathcal{G}_2$ :

- $\mathcal{G}_i = (Y_i, \mathcal{L}_i, \mu_i, B_i)$ , $\alpha = (p : L \to Y = Y_1 \times_M Y_2,\ \rho)$
- $L$ is a **l.-bdle** of curvature $B_2 - B_1$
- $\rho : \mathcal{L}_1 \otimes L_1 \to L_0 \otimes \mathcal{L}_2$ is an isomorphism of **l.-bdles** over $Y^{[2]}$ associative w.r.t. $\mu_i$

$2$-**morphism** $\beta : \alpha_1 \Rightarrow \alpha_2$ for $\alpha_i : \mathcal{G}_1 \to \mathcal{G}_2$

- an isomorphism of **l.-bdles** $L$ of $\alpha_i$ intertwining $\rho$'s

## **Facts about gerbes** (cont'd)

- **gerbes** have duals (with opposite curvature), tensor products (with curvatures adding) and pullbacks (with curvatures pulling back)
  - For two gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ with same curvature $\mathcal{G}_1 \otimes \mathcal{G}_2^*$ is **flat**
- **flat gerbes** (i.e. with zero curvature) are classified up to $1$-isomorphism by cohomology classes in $H^2(M,U(1))$ - "**discrete torsion**"
- For $\varphi : \Sigma \to M$

$$Hol_{\mathcal{G}}(\varphi(\Sigma)) = \Big\langle [\Sigma], [\varphi^*\mathcal{G}] \Big\rangle$$

and such holonomy determines $\mathcal{G}$ up to $1$-isomorphism

## **Facts about gerbes** (cont'd)

- a 2-**form** $B$ defines a **gerbe** $\mathcal{I}_B$ with curvature $dB$ and holonomy

$$Hol_{\mathcal{I}_B}(c^2) = \exp\left[\, i \int_{c^2} B\right]$$

- **Transgression functor:**
  - **gerbes** over $M$ induce **line bundles** over the **loop space** $LM$

$$\mathcal{G} \quad -> \quad \mathcal{L}_{\mathcal{G}}$$

with $\quad curv(\mathcal{L}_{\mathcal{G}})(\ell) = \int_{\ell} \iota_{\dot{\ell}}\, curv(\mathcal{G}) \quad$ for $\quad \ell \in LM$

  - 1-**isomorphisms** $\alpha : \mathcal{G}_1 \to \mathcal{G}_2$ isduces **1.-bdle isomorphisms**

$$\alpha \quad -> \quad \psi_\alpha$$

## Application to (non-diagonal) **WZW** field theory:

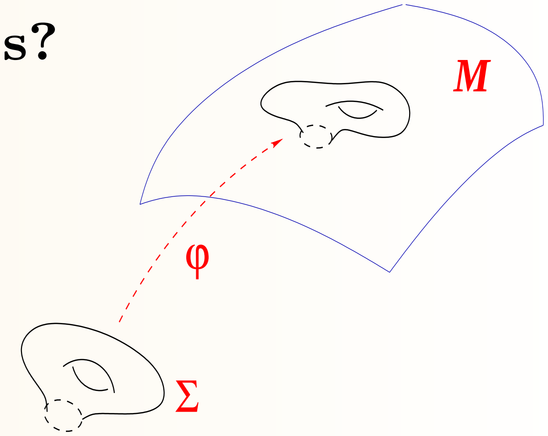
- For $G$ compact, simple, $\pi_1(G)$ arbitrary, and $H_k = \frac{k}{12\pi}\,\mathrm{tr}\,(g^{-1}dg)^3$
  - $H_k$ has periods in $2\pi\boldsymbol{Z}$ for discrete values of "**level**" $k$
  - explicit constructions of gerbes $\mathcal{G}_k$ with curvature $H_k$ known

- **WZW** theory for such $G$ may be quantized by gerbe transgression and **Borel-Weil-Segal-Presley** construction of affine algebra representations

  $\Rightarrow$ modular-invariant **partition fcts** (**Felder-G.-Kupiainen** 1988)

- **WZW** **correlation functions** may be found using geometric arguments via the scalar product of conformal blocks (**G.** 1989)

# What about **WZ** actions on open worldsheets?



- For $\Sigma$ with $\partial\Sigma \cong S^1$ and $\varphi : \Sigma \to M$

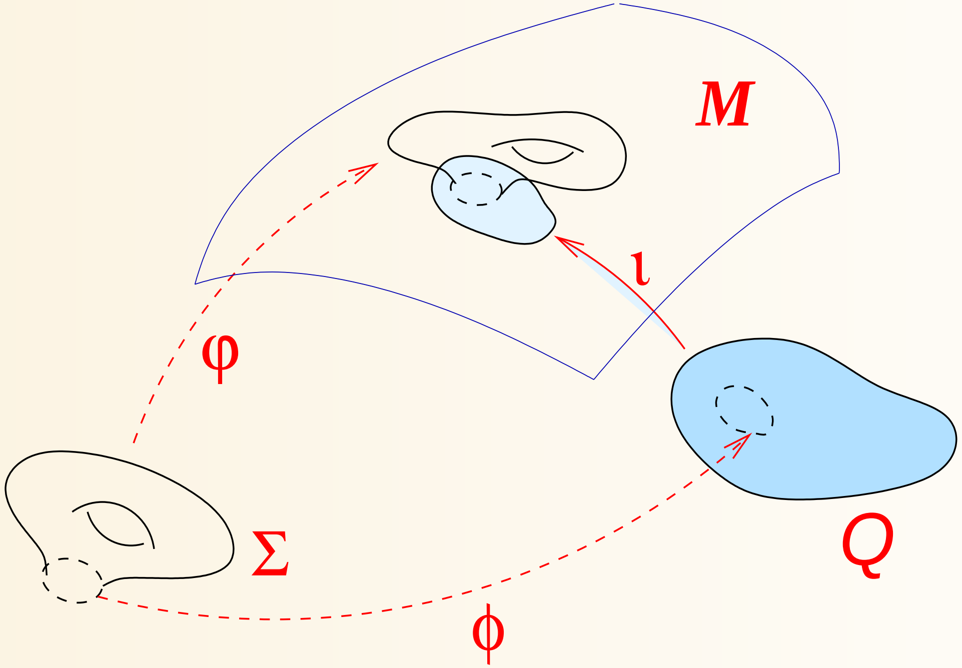
$$Hol_{\mathcal{G}}(\varphi) \in \left(\mathcal{L}_{\mathcal{G}}\right)_{\varphi|\partial\Sigma}$$

- To compensate, use $\mathcal{G}$-**brane** $\mathcal{Q} = (Q, B, \alpha)$ s. t.
  - $\iota : Q \to M$
  - $B$ is a $2$-form on $Q$ s. t. $\iota^* H = dB$
  - $\alpha : \iota^*\mathcal{G} \to \mathcal{I}_B$ is a gerbe $1$-isomorphism
  - For loops $\phi : S^1 \to Q$ the connection of l.-bdle $L$ of $\alpha$ permits to define $Hol_\alpha(\phi(S^1)) \in \left(\mathcal{L}^*_{\iota^*\mathcal{G}}\right)_{\phi|_{S^1}} = \left(\mathcal{L}^*_{\mathcal{G}}\right)_{\iota\circ\phi}$

- Upon imposing the **boundary condition** $\varphi|_{\partial\Sigma} = \iota \circ \phi$ the amplitude

$$Hol_{\mathcal{G}}(\varphi(\Sigma))\, Hol_{\alpha}(\phi(S^1))$$

becomes a number (**Kapustin** 2000, **Carey-Johnson-Murray** 2002, **G.-Reis** 2002)



## **WZW example** (**Alekseev-Schomerus** 1998, **G.** 2004 for $\pi_1(G) \neq \{1\}$)

$M = G$, $\mathcal{G}_k$ a gerbe over $G$ with curvature $H_k = \frac{k}{12\pi}\,\mathrm{tr}\,(g^{-1}dg)^3$

- On conjugacy classes $\iota : \mathcal{C} \hookrightarrow G$

$$\iota^* H_k|_{\mathcal{C}} = dB_k \qquad \text{for} \qquad \boxed{B_k = \frac{k}{8\pi}\,\mathrm{tr}\,(g^{-1}dg)\frac{1+Ad_g}{1-Ad_g}(g^{-1}dg)}$$

- $\mathcal{G}_k$**-branes** $(\mathcal{C}, B_k, \alpha_k)$ exist for a discret series of $C \subset G$
- are called **symmetric branes** as they preserve the diagonal affine-algebra symmetries of the **WZW** model: $J^L = J^R$ on $\partial\Sigma$
- The open-sector gerbe **transgression** allows unambiguous quantization of the boundary **WZW** theory

$\Rightarrow$ explicit boundary **partition functions** and boundary **OPE**!

## **Coset** $G/H$ **models example:**

- For $H \subset G$ one gauges the $g \mapsto hgh^{-1}$ symmetry of the group $G$ **WZW** model
- In general an $H$-**equivariant structure** on gerbe $\mathcal{G}_k$ is needed for that (**G.-Suszek-Waldorf** 2012)
- There exists a family of branes with

$$Q \;=\; (C^G \times C^H) \xrightarrow{\iota} G\,, \qquad \iota(g,h) = gh$$

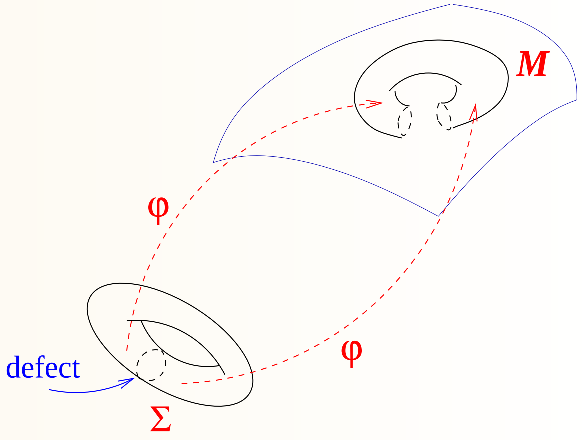
$$B(g,h) \;=\; B_k(g) + B_k(h) + \tfrac{k}{4\pi}\,\mathrm{tr}(g^{-1}dg)(hdh^{-1})$$

(**G.** 2002, **Elitzur-Sarkissian** 2002)

- Such branes exist also in ungauged **WZW** model breaking the diagonal affine symmetry to the one corresponding to $H$

## Wall-defects

(**Oshikawa-Affleck** 1997, ...
**Petkova-Zuber** 2001,
**Bachas-de Boer-Dijkgraaf-Ooguri** 2002, ... )



- One may compensate the holonomy of a surface $\widetilde{\Sigma}$ (connected or not) obtained by cutting surface $\Sigma$ along a circular defect with a jump of the field $\varphi$ using a $\mathcal{G}$-**bibrane** $\mathcal{Q} = (Q, \omega, \alpha)$ s. t.
  - $\iota_{1,2} : Q \to M$
  - $B$ is a $2$-form on $Q$ s. t. $i_1^* H - \iota_2^* H = dB$
  - $\alpha : \iota_1^* \mathcal{G} \to \iota_2^* \mathcal{G} \otimes \mathcal{I}_B$ is a gerbe $1$-isomorphism $\iota_{1,2} : Q \to M$
  - For loops $\phi : S^1 \to Q$ the connection of l.-bdle $L$ of $\alpha$ permits to define $Hol_\alpha(\phi(S^1)) \in \big(\mathcal{L}^*_{\iota_1^* \mathcal{G}}\big)_{\phi|_{S^1}} \otimes \big(\mathcal{L}_{\iota_2^* \mathcal{G}}\big)_{\phi|_{S^1}}$

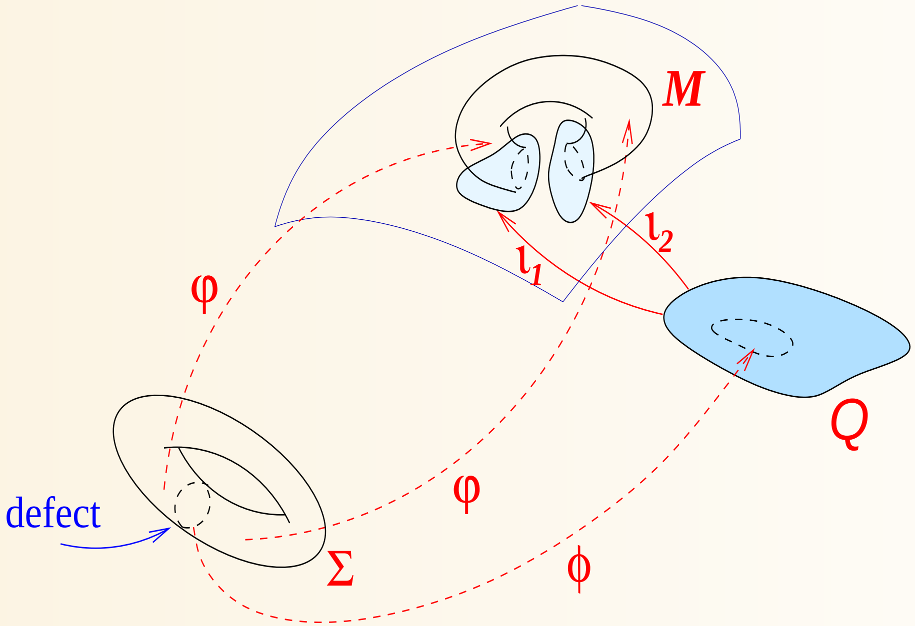
- Upon imposing the **boundary conditions**

$$\varphi|_{\partial_1\widetilde{\Sigma}} = \iota_1 \circ \phi\,, \qquad \varphi|_{-\partial_2\widetilde{\Sigma}} = \iota_2 \circ \phi$$

the amplitude

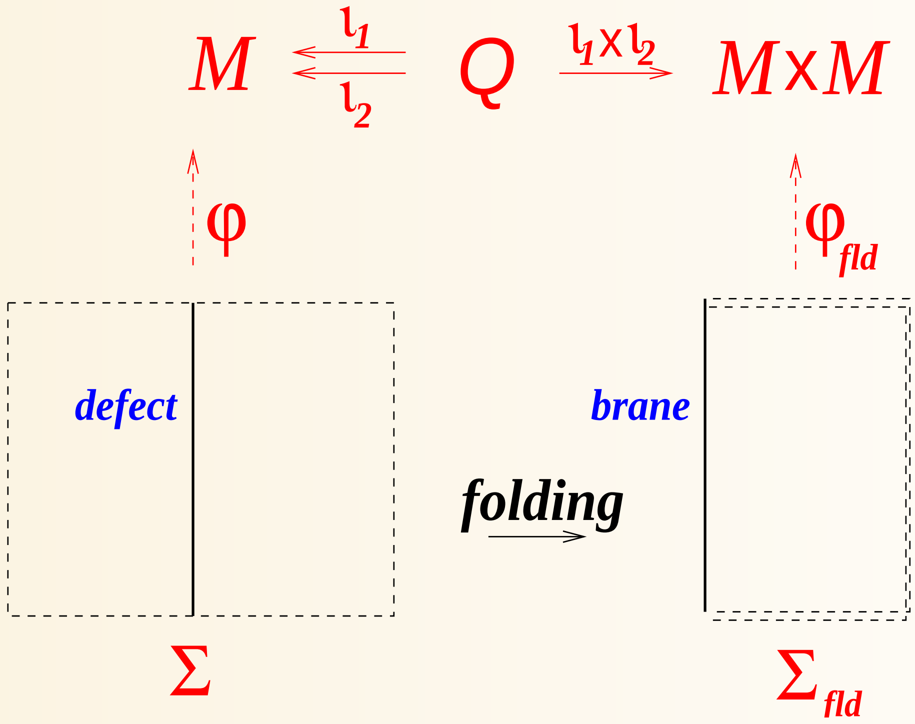
$$Hol_{\mathcal{G}}(\varphi(\Sigma))\, Hol_{\alpha}(\phi(S^1))$$

becomes a number (**Fuchs**-**Schweigert**-**Waldorf** 2008)



## **Folding trick** (**Wong-Affleck** 1994)

- **Bi-branes** correspond to **branes** on $M \times M$ with **gerbe** $\mathcal{G}_1 \otimes \mathcal{G}_2^*$



## **WZW** examples:

- For a conjugacy class $C \subset G$ from the same discrete class take

$$Q \;=\; \{\,(g_1,g_2)\in G\times G \mid g_1 g_2^{-1}\in C\,\} \;\underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}}\; G$$

$$B(g_1,g_2) \;=\; B_k(g_1 g_2^{-1}) - \tfrac{k}{4\pi}\,\mathrm{tr}(g_1^{-1}g_1)(g_2^{-1}dg_2)$$

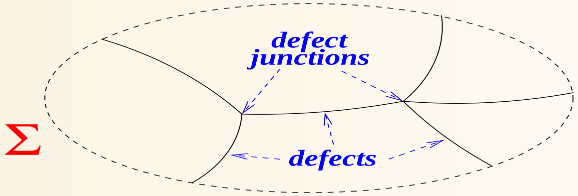
- These gives rise to topological defects with continuity of $J^L,\, J^R$ and $T^L,\, T^R$ across it

- Such defects give rise to **symmetries** of the **boundary CFT**

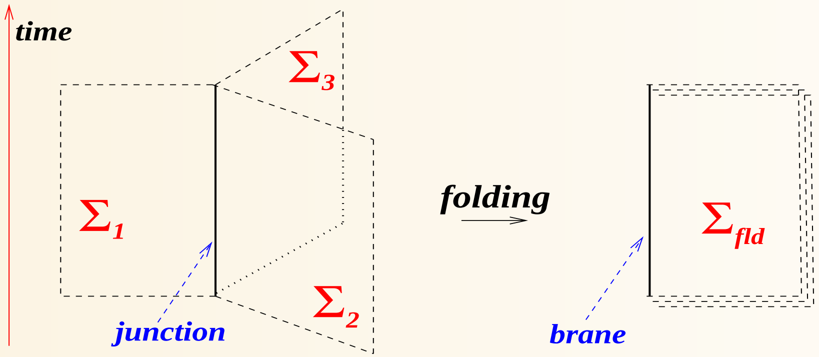
- Upon folding $\big($and replacement $g_2 \mapsto g_2^{-1}\big)$ they correspond to symmetric **permutation branes** (**Figueroa-O'Farrill-Stanciu** 2000) in $G\times G$ **WZW** model with $Q = \big\{(g_1,g_2)\mid g_1 g_2 \in C\big\}$

## Junctions

- One may consider **nets** of wall-defects with **defect junctions** (**Fröhlich-Fuchs-Runkel-Schweigert** 2007, **Runkel-Suszek** 2009)



- But one may also study **junctions** where several boundaries meet



(**Schwarz** 1996, $\cdots$, **Bachas-de Boer-Dijkgraaf-Ooguri** 2002, **Oshikawa-Chamon-Affleck** 2003)

- The latter junctions were studied in strings and in integrable $(1+1)$D **QFT** and, recently, in **CFT**, as models of contacts of **quantum wires**
  - The wires are modeled by bosonic free fields at $x \geq 0$ s. t.

$$\boxed{J_i^L(t,0) = \sum_j S_i^{\,j} J_j^R(t,0)} \qquad S \text{ orthogonal}$$

One looks for charge transport in response to change of potentials or temperatures in the wires or for **non-equilibrium steady states**

  - The **Green-Kubo** formula of linear response gives for the zero-temperature conductivity $G_i^{\,j} = \frac{\partial I_i}{\partial V_j}$

$$G_i^{\,j} = \frac{4\pi^2 e^2}{h}(x_1+x_2)^2 \Big\langle J_i^L(t,x_1)\, J_j^R(t,x_2)\Big\rangle$$

(**Rahmani-Hou-Feiguin-Chamon-Affleck** 2010)

  - A steady state for wires in different temperatures has been constructed recently (**Mintchev-Sorba** 2012, see also **Bernard-Doyon** 2012)

## **WZW** examples

- One has to consider appropriate **branes** in group $G^n$ **WZW** model.
  - Symmetric permutation branes with $Q=\{(g_1,..,g_n)\in G^n\,|\,g_1\cdots g_n\in C\}$ give

$$\boxed{J_i^L(t,0)\;=\;J_{i+1}^R(t,0)}$$

  - More interesting coset-type $G^n/diag(G)$ branes with $Q=\{(g_1,..,g_n)\in G^n\,|\,g_i=h_i\gamma,\;h_i\in C_i,\;\gamma\in C\}$ lead to

$$\boxed{J_i^L(t,0)\;=\;Ad_{\gamma(t)}J_i^R(t,0)+\frac{1}{n}\sum_{j=1}^{n}(1-Ad_{\gamma(t)})J_j^R(t,0)}$$

in classical theory, with overall conservation of charge and energy:

$$\sum_i J_i^L(t,0)\;=\;\sum_i J_i^R(t,0)\qquad\qquad\sum_i T_i^L(t,0)\;=\;\sum_i T_i^R(t,0)$$

  - Quantization: work in progress with **Clément Tauber**

# Conclusions and Ramifications

- Bundel gerbes are useful tools to handle topological ambiguities in low dimensional field theories, e.g. **WZW** models with $\pi_1(G) \neq \{1\}$
- With some additional refinements they work as well in the presence of boundaries and defects
- They promise to be helpful in nascent non-equilibrium **CFT** where new type of (**Minkowskian**) defects appears
- Gerbes help handle global anomalies in $2$D sigma models (**Clément Tauber**'s talk)
- They play an important role, omitted here, in twisted $K$-theory and its applications to strings (brane charges) and classification of topological insulators in condensed matter