Topological defects in perturbed CFTs

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joint work with I. Runkel and C. Meneghelli

Workshop on Field Theories with Defects

Motivation

- Understand/generalise the Bazhanov-Lukyanov-Zamolodchikov (BLZ) construction of integrals of motion in (perturbed) CFTs
- π_λ: representation of a quantum group
- $$\mathbf{T}_\pi(\lambda) = \mathrm{tr}_{\pi_\lambda}\left(e^{2i\pi P\otimes\pi_\lambda(H)}\mathcal{P}\exp\left(\int_0^{2\pi} d\theta\left(:e^{-2\varphi^+(\theta)}:\otimes\pi_\lambda(E_0^+)\right.\right.\right.$$
  $$\left.\left.\left.+:e^{2\varphi^+(\theta)}:\otimes\pi_\lambda(E_1^+)\right)\right)\right)$$

Overview

$$\begin{tikzcd}
\text{CFT} \arrow[rr, Rightarrow, "e^{\int \Phi}"] & & \text{perturbed CFT}
\end{tikzcd}$$

$$\begin{tikzcd}
\text{topological defects} & & \\
\{\text{with YD-structure}\} \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr, Rightarrow, "e^{\int \psi^{+} + \psi^{-}}"] \arrow[d, leftrightsquigarrow] & & \text{topological defects} \arrow[d] \\
\text{quantum groups} & & \text{``Integrable structure''}
\end{tikzcd}$$

example:

free boson — sine-Gordon theory

Field theories with topological defects

Definition (For the purpose of this talk)

A field theory with topological defects is a functor

$$ft : \mathrm{Cyl}_{D,\mathcal{F}} \to \mathrm{tVect}$$

where

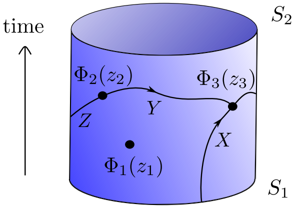
- tVect is some category of topological vector spaces
- D is an involutive monoid ("of topological defect conditions")
- $\{\mathcal{F}_{A,B}\}_{A,B\in D}$ is a family of vector spaces ("defect fields")
- $\mathrm{Cyl}_{D,\mathcal{F}}$ is the category of two dimensional Lorentzian cylinders with defects and field insertions, i.e. ...

Field theories with topological defects

...

$$\mathrm{Cyl}_{D,\mathcal{F}} := \begin{cases} \text{objects:} & \text{circles with oriented points labelled by } D \\ \mathrm{Hom}(S_1,S_2): & \text{cylinders with oriented defect network,} \\ & \text{field insertions and embeddings of } S_1,\ S_2, \\ & \text{into the boundary, modulo isotopies} \\ & \text{and fusion of defect lines} \end{cases}$$

e.g.



$$X, Y, Z \in D, \quad \Phi_1 \in \mathcal{F}_{1,1}, \\ \Phi_2 \in \mathcal{F}_{Z,Y}, \quad \Phi_3 \in \mathcal{F}_{YX,Z}$$

Conformal field theories (CFTs)

Construction [cf. Fuchs, Runkel, Schweigert et al.]

Let $\mathcal{C}$ be the category of primary fields of a given chiral theory, equip it with the braiding given by analytic continuation of correlators. Then there is a full conformal field theory with topological defects, where

- D is the monoid of iso-classes of $\mathcal{C}$,
- $\mathcal{F}_{A,B} := \{(F^+, F^-, m) \,|\, F^\pm \in \mathcal{C},\ m \in \mathrm{Hom}_{\mathcal{C}}(F^+ \otimes A \otimes F^-, B)\}$

Conformal field theories (CFTs)

Example: free boson

- Primaries: $\{V_\alpha(z) := :e^{i\alpha\varphi^+(z)}:\}_{\alpha\in\mathbb{R}}$
  Chiral correlators: $\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\rangle := (z_1 - z_2)^{\alpha_1\alpha_2}$ for $\Re z_i > \Re z_j$ $(i > j)$, for other values defined by analytic continuation
- Category $\mathcal{C} = H\text{-Mod}_{\mathbf{Vec}}$ where H is the quasitriangular Hopf algebra

$$H = \begin{cases} \overline{\mathbb{C}[h]}, \qquad \mathcal{R} = e^{\pi i h\otimes h} \ , \\ \Delta(h) = h\otimes 1 + 1\otimes h \ , \end{cases}$$

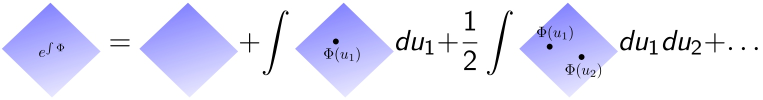
  simple objects $\mathbb{C}_\alpha$: h acting on $\mathbb{C}$ by $h.\xi := \alpha\cdot\xi$
  braiding: $\mathbb{C}_\alpha\otimes\mathbb{C}_\beta\to\mathbb{C}_\beta\otimes\mathbb{C}_\alpha$, $1\otimes 1\mapsto e^{\pi i\alpha\beta}\cdot 1\otimes 1$

Perturbed CFT

- let $\Phi \in \mathcal{F}_{1,1}$
- consider the theory determined by the perturbed Hamiltonian

$$H = H_{\mathrm{CFT}} + \int_0^{2\pi} \Phi(u)\, du.$$

- amounts to inserting a factor of $e^{\int \Phi}$ into each correlator

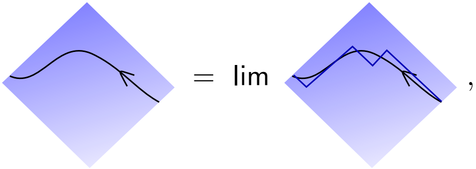


Example: sine-Gordon theory

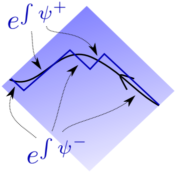
- H_{CFT} the Hamiltonian of the free boson theory,
  $\Phi(u) = :\cos(\alpha\varphi(u)): = :e^{i\alpha\varphi(u)}: + :e^{-i\alpha\varphi(u)}:$
- one can show that the above series converges for sufficiently small α [cf. Fröhlich '76].

Perturbing defects

- let $X \in D$ and $\psi_X^\pm \in \mathcal{F}_{X,X}$ be chiral/anti-chiral defect fields
- define a perturbed defect by approximating a defect line



then inserting factors "$\exp\left(\int \psi_X^\pm\right)$" into correlators

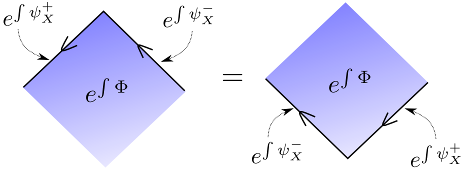


and taking the limit

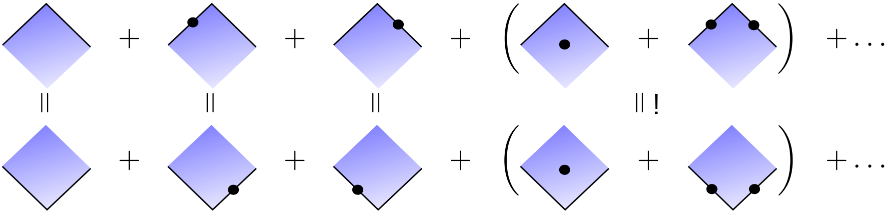
- can be shown to exist in sine-Gordon theory [BMR]

The “commutation condition”

- Q: When is a perturbed defect topological in the perturbed theory?
- Necessary condition:

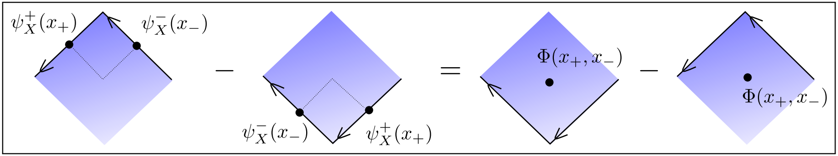


expand:

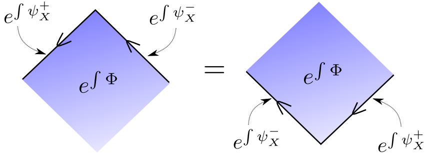


The “commutation condition”

X a defect condition, $\psi_X^\pm$ defect fields on X; suppose



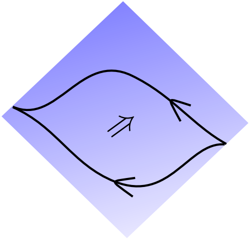
then:



The "commutation condition"

$\rightsquigarrow$ Topological (left-directed) defects in the perturbed theory:

If $(X, \psi_X^{\pm})$ satisfy the "commutation condition", the defect X perturbed by $\psi_X^{\pm}$ is topological in the perturbed theory:



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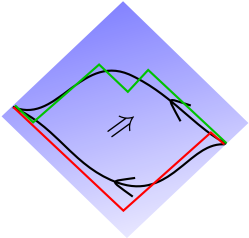

The "commutation condition"

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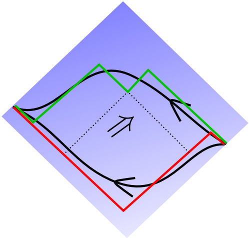

so far...

$$\begin{tikzcd}
\text{CFT} \arrow[r, Rightarrow, "e^{\int \Phi}"] & \text{perturbed CFT} \\
\text{topological defects} \arrow[d, phantom, "\cup\vert"] & \\
\{\text{with YD-structure}\} \arrow[r, Rightarrow, "e^{\int \psi^{+}+\psi^{-}}"] \arrow[d, leftrightsquigarrow] & \text{topological defects} \arrow[d] \\
\text{quantum groups} & \text{“Integrable structure”}
\end{tikzcd}$$

Now: “translate into algebra.”

“Commutation condition” $\leftrightsquigarrow$ Yetter-Drinfeld condition

In the context of the Fuchs-Runkel-Schweigert construction:

recall: $\mathcal{F}_{A,B} := \{(F^+, F^-, m) \,|\, m \in \mathrm{Hom}_{\mathcal{C}}(F^+ \otimes A \otimes F^-, B)\}$

CFT		$\mathcal{C}$
$\Phi \in \mathcal{F}_{1,1}$	$\leftrightsquigarrow$	$b : F^+ \otimes F^- \to \mathbb{1}$
$\psi_X^+ \in \mathcal{F}_{X,X}$	$\leftrightsquigarrow$	$m_+ : F^+ \otimes X \to X$
$\psi_X^- \in \mathcal{F}_{X,X}$	$\leftrightsquigarrow$	$m_- : X \otimes F^- \to X$

“Commutation condition” ⇝ Yetter-Drinfeld condition

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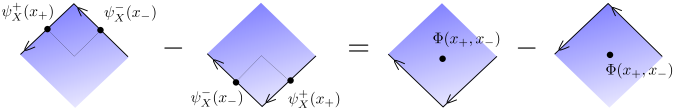
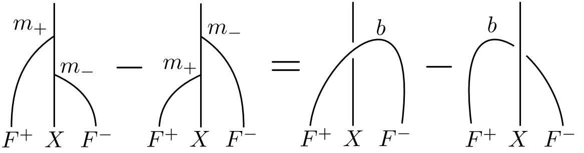
$\psi^-_X \in \mathcal{F}_{X,X}$	⇝	$m_- : X \otimes F^- \to X$



$\Longleftrightarrow$



“Commutation condition” $\leftrightsquigarrow$ Yetter-Drinfeld condition

Tensor algebras

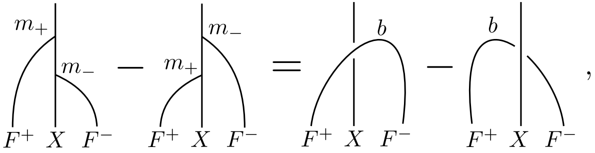
$F \in \mathcal{C} \rightsquigarrow$ if $T(F) := \bigoplus_{n\geq 0} \underbrace{F \otimes \cdots \otimes F}_{\text{n times}}$ exists, it comes equipped with

- algebra structure: $\underbrace{F \otimes \cdots \otimes F}_{\text{n times}} \otimes \underbrace{F \otimes \cdots \otimes F}_{\text{m times}} \xrightarrow{\text{id}} \underbrace{F \otimes \cdots \otimes F}_{\text{n+m times}}$
- coalgebra structure: induced by $\Delta|_F = \text{id}_F \otimes \text{id}_{\mathbb{1}} + \text{id}_{\mathbb{1}} \otimes \text{id}_F$
- $\rightsquigarrow$ $T(F)$ is a braided Hopf algebra
- $b : F^+ \otimes F^- \to \mathbb{1} \rightsquigarrow$ extends to $\rho : T(F^+) \otimes T(F^-) \to \mathbb{1}$ (Hopf pairing)

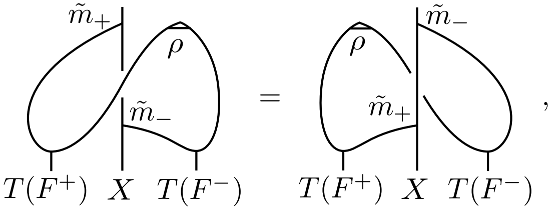
In terms of modules over tensor algebras:

CFT	$\mathcal{C}$	$T(F^\pm)$
$\Phi \in \mathcal{F}_{1,1}$	$b : F^+ \otimes F^- \to \mathbb{1}$	$\rho : T(F^+) \otimes T(F^-) \to \mathbb{1}$
$\psi_X^+ \in \mathcal{F}_{X,X}$	$m_+ : F^+ \otimes X \to X$	$X \in T(F^+)$-Mod
$\psi_X^- \in \mathcal{F}_{X,X}$	$m_- : X \otimes F^- \to X$	$X \in$ Mod-$T(F^-)$

“Commutation condition” $\leftrightsquigarrow$ Yetter-Drinfeld condition



...translating this into a condition for $T(F^+)$-left and $T(F^-)$-right modules [BR ’12]:



(the Yetter-Drinfeld/crossed module condition in the sense of Bespalov/Majid)

YD-condition $\leftrightsquigarrow$ quantum groups

“Bosonisation” [Radford]

Let $\mathcal{C} = \operatorname{Rep} H$ be the (braided) representation category of a quasitriangular Hopf algebra H and $T \in \mathcal{C}$ be a braided Hopf algebra. Then there is a Hopf algebra $B(T) := T\#H$ such that $B(T)\text{-Mod}_{\mathbf{Vec}} \cong_{\otimes} T\text{-Mod}_{\mathcal{C}}$.

Example:

Let

$$H = \begin{cases} \overline{\mathbb{C}[h]}, \quad \mathcal{R} = e^{\pi i h \otimes h} \,, \\ \Delta(h) = h \otimes 1 + 1 \otimes h \,, \end{cases} \qquad T = T(\mathbb{C}_\alpha) \,, \quad \alpha \in \mathbb{C} \,,$$

where $\mathbb{C}_\alpha \in \operatorname{Rep} H$ is $\mathbb{C}$ with h-action $h.\zeta = \alpha \cdot \zeta$.
Then $B(T) = \boxed{U_\hbar(\mathfrak{sl}_2)^+} = \langle h, e^+ \rangle \,/\, ([h, e^+] = \alpha e^+)$ with coproduct $\Delta(e^+) = e^+ \otimes 1 + e^{\pi i \alpha h} \otimes e^+$

YD-condition $\leftrightsquigarrow$ quantum groups

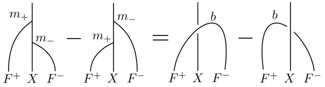
“Double-Bosonisation” [Majid]

Let $\mathcal{C} = \operatorname{Rep} H$ be the (braided) representation category of a quasitriangular Hopf algebra H, $T^+, T^- \in \mathcal{C}$ be braided Hopf algebras and $\rho : T^+ \otimes T^- \to \mathbb{1}$ be a Hopf pairing. Then there is a Hopf algebra $D(T^+, T^-, \rho)$ such that $D(T^+, T^-, \rho)\text{-}\mathrm{Mod}_{\mathbf{Vec}} \cong_{\otimes} T^+\text{-}\mathrm{YD}^{\rho}_{\mathcal{C}}\text{-}T^-$.

Example:

H as before, $T^+ := T(\mathbb{C}_\alpha)$, $T^- := T(\mathbb{C}_{-\alpha})$ and ρ the Hopf pairing induced $\mathbb{C}_\alpha \otimes \mathbb{C}_{-\alpha} \to \mathbb{1} = \mathbb{C}_0$, $1 \otimes 1 \mapsto 1$:

$$D = \boxed{U_\hbar(\mathfrak{sl}_2)} = \frac{\langle h, e^+, e^- \rangle}{([h, e^\pm] = \pm\alpha e^\pm,\ \underbrace{[e^+, e^-] = e^{\pi i\alpha h} - e^{-\pi i\alpha h}})}$$



YD-condition $\leftrightsquigarrow$ quantum groups

Example: sine-Gordon theory

Similarly, for sine-Gordon theory:

$$\Phi(u) = :e^{i\alpha\varphi(u)}: + :e^{-i\alpha\varphi(u)}: \quad \leftrightsquigarrow \quad T^{+} = T^{-} := T(\mathbb{C}_{\alpha} \oplus \mathbb{C}_{-\alpha})$$

$$\rightsquigarrow \boxed{U_{\hbar}(L\mathfrak{sl}_2)}$$

what's left...

$$\begin{tikzcd}
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\end{tikzcd}$$

$$\begin{tikzcd}
\text{topological defects} & & \\
\rotatebox{90}{$\subseteq$} & & \\
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Integrability

...means, in particular, to have

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- commuting
- integrals of motion

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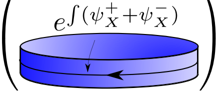
Integrals of motion as defect operators from topological defects:

$$\mathcal{O}(X) := \text{cft}\Bigg($$



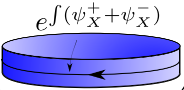
$$\Bigg) : \mathcal{H} \to \mathcal{H}$$

Integrability

...means, in particular, to have

- infinitely many ⇐ ∃ infinitely many non-isomorphic,
- commuting ⇐ commuting (in tensor product up to iso; or in Grothendieck ring)
- integrals of motion ⇐ Yetter-Drinfeld modules

Integrals of motion as defect operators from topological defects:

$\mathcal{O}(X) := \mathrm{cft}\Big($ 

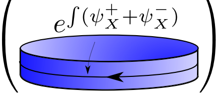
$\Big) : \mathcal{H} \to \mathcal{H}$

Integrability

...means, in particular, to have ($U_\hbar(L\mathfrak{sl}_2)$ provides an example)

- infinitely many $\Leftarrow$ $\exists$ infinitely many non-isomorphic,
  $U_\hbar(L\mathfrak{sl}_2)$-irreps are evaluation representations
- commuting $\Leftarrow$ commuting (in tensor product up to iso; or in Grothendieck ring)
  $K_0(U_\hbar(L\mathfrak{sl}_2))$-Mod is commutative
- integrals of motion $\Leftarrow$ Yetter-Drinfeld modules
  $U_\hbar(L\mathfrak{sl}_2)$-modules are "double bosonisations"

Integrals of motion as defect operators from topological defects:

$\mathcal{O}(X) := \mathrm{cft}\Bigg($  $\Bigg) : \mathcal{H} \to \mathcal{H}$

Integrability: the BLZ construction

Get commuting integrals of motion for a free boson by

$$\mathbf{T}_{\pi}(\lambda) = \mathrm{tr}_{\pi_\lambda}\left(e^{2i\pi P\otimes\pi_\lambda(H)}\mathcal{P}\exp\left(\int_0^{2\pi} d\theta\left(:e^{-2\varphi^+(\theta)}:\otimes\pi_\lambda(E_0^+)\right.\right.\right.$$
$$\left.\left.\left.+:e^{2\varphi^+(\theta)}:\otimes\pi_\lambda(E_1^+)\right)\right)\right),$$

where $\lambda \in \mathbb{C}^\times$ and π_λ is an evaluation representation of $U_\hbar(L\mathfrak{sl}_2)$.

This formula is fully reproduced by our prescription.

Thank you!

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