Introduction to defects in Landau-Ginzburg models

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Overview

- Landau Ginzburg model: 2 dimensional theory with $N = (2,2)$ supersymmetry
- Basic ingredient: Superpotential $W(x_i)$, $W \in \mathbb{C}[x_i]$
- Bulk theory: Described by the ring $\mathbb{C}[x_i]/\langle \partial_i W \rangle$.
- Chiral superfields correspond to polynomials.
- Question: How do we describe defects? boundary conditions? Boundary fields? How do we fuse two defects?
- Applications?

(Based on several old papers with Daniel Roggenkamp)

Defects in $N = (2,2)$ theories, I

- Consider a theory with 4 anti-commuting supercharges with the usual anti-commutation relations

$$\{Q_\pm, \bar{Q}_\pm\} = H \pm P \ .$$

- We are interested in defects that preserve at least half of the supersymmetry. Just as for boundary conditions or orientifolds, there are two ways to do so.
- B-type defect: We demand that the combination $Q_B = Q_+ + Q_-$ is preserved everywhere. This means that along the interface the supercharges have to fulfill the "gluing conditions"

$$\begin{aligned} Q_+^{(1)} + Q_-^{(1)} &= Q_+^{(2)} + Q_-^{(2)}, \\ \bar{Q}_+^{(1)} + \bar{Q}_-^{(1)} &= \bar{Q}_+^{(2)} + \bar{Q}_-^{(2)} \ . \end{aligned}$$

where the superscripts refer to the two theories separated by the defect.

Defects in $N = (2,2)$ theories, II

- A-type defect: Same for $Q_A = Q_+ + \bar{Q}_-$.
- A-type and B-type defects are interchanged by mirror symmetry
- In situations with both defects and boundaries, B (A) defects preserve the same SUSY as B (A) D-branes and hence are compatible
- There are special defects that actually preserve the full supersymmetry.

$$Q_\pm^{(1)} = Q_\pm^{(2)}, \quad \bar{Q}_\pm^{(1)} = \bar{Q}_\pm^{(2)} \quad \text{on } \mathbf{R} ,$$

The SUSY algebra implies that those defects also fulfill

$$H^{(1)} = H^{(2)}, \quad P^{(1)} = P^{(2)} \quad \text{on } \mathbf{R} .$$

They preserve translational invariance in space and time.

Defects in $N = (2,2)$ theories, III

- We can define a second class of defects preserving the full supersymmetry. It can be obtained by twisting with the mirror automorphism.

$$\begin{aligned} Q_+^{(1)} &= Q_+^{(2)}, \quad \bar{Q}_+^{(1)} = \bar{Q}_+^{(2)} \\ Q_-^{(1)} &= \bar{Q}_-^{(2)}, \quad \bar{Q}_-^{(1)} = Q_-^{(2)} \end{aligned}$$

- In particular, mirror symmetry itself can be interpreted as a defect.

Defects in super-conformal field theory

- The $N = 2$ superconformal algebra is generated by the modes of the stress energy tensor T, the $U(1)$ current J and two supercurrents $G^{\pm}$.
- Just like before we can formulate A-type and B-type defects.
- Also, there are defects that preserve the full superconformal symmetry. Those in particular fulfill

$$T^{(1)} = T^{(2)} \text{ on the defect}$$

  which means that they are topological.
- Since we are considering $N = 2$ superconformal theories, we can topologically twist our theories. A(B)-type defects are compatible with the A(B)-twist.
- On the level of the topological theory any (compatible) defect can be moved around and act on boundary conditions, also those that are not topological on the level of the full conformal field theory.

Landau Ginzburg models with boundaries

- Landau Ginzburg action
$$S = \int d\theta^+ d\theta^- W(X),$$
where X is a chiral superfield, $\bar{D}_\pm X = 0$
- This action is invariant under $N = (2,2)$ SUSY for worldsheets without boundary. If there is a boundary, the B-supersymmetry variation will produce a boundary term $\sim \int_{\partial\Sigma} dt d\theta W$.
- Introduce a boundary F-term
$$\Delta S = \int_{\partial\Sigma} J(X)\pi \ ,$$
where π is a boundary fermion. It is not chiral, but fulfills $\bar{D}\pi = E(X)$. The variation of the boundary term thus cancels the unwanted term resulting from the variation of the bulk action if
$$J(X)E(X) = W(X)$$

Matrix factorizations of the superpotential

- The open string spectrum is given by the cohomology of the boundary BRST operator

$$Q = \pi J + \bar{\pi} E = \begin{pmatrix} 0 & E \\ J & 0 \end{pmatrix} \qquad Q^2 = W \ .$$

  Bosons are given by block-diagonal matrices, fermions by off-diagonal matrices.
- The factorization $W = W\ 1$ is trivial in the sense that its spectrum with any other factorization is empty. We can hence add such branes to any other stack of branes without changing the physics.

$$Q_{\rm bd} \sim Q_{\rm bd} \oplus Q_{\rm triv}$$

Equivalence of matrix factorizations

- Two boundary conditions specified by BRST charges $Q_{\rm bd}$ and $Q'_{\rm bd}$ are equivalent if there are homomorphisms U, V

$$Q'_{\rm bd} = U Q_{\rm bd} V \,, \qquad UV = {\rm id}' + \{Q'_{\rm bd}, O'\} \,, \quad VU = {\rm id} + \{Q_{\rm bd}, O\}$$

- The BRST cohomology does not change under these similarity transformations.
- U and V can be regarded as open string operators propagating from one brane to another in opposite directions. The condition above can be read as the existence of an identity operator in the open string spectrum.

Several boundary fermions

- Straightforward generalization

$$\{\pi^i, \bar{\pi}^j\} = \delta^{ij} \quad \{\pi^i, \pi^j\} = \{\bar{\pi}^i, \bar{\pi}^j\} = 0$$

$$Q = \sum_{i=1}^{n} \left(\pi^i J_i + \bar{\pi}^i E_i\right)$$

$$Q^2 = \sum_i E_i J_i = \sum_i W_i = W$$

- Whenever the Landau-Ginzburg superpotential W can be written in the above form, we can associate a matrix factorization and hence a D-brane. This type of factorization can be understood as a tensor product of the factorizations associated to the W_i.
- Any MF gives rise to a consistent boundary condition.

Herbst-Lazaroiu

Defects in Landau-Ginzburg theories

- In the case of defects, the supersymmetry variation gives a contribution from both the theories on the upper and lower half plane, with a relative sign because of the different relative orientations.
- Hence, defects in Landau-Ginzburg models are described by matrix factorizations of the difference of the superpotentials $W = W_1 - W_2$.
- This is in perfect agreement with the folding trick, which tells us to look for boundary conditions in the tensor product of theory 1 and theory 2, where left and right movers have been exchanged for theory 2.

$$\int d\theta^+ d\theta^- W_2(Y) \to - \int d\theta^+ d\theta^- W_2(Y)$$

The left/right exchange produces the relative minus.

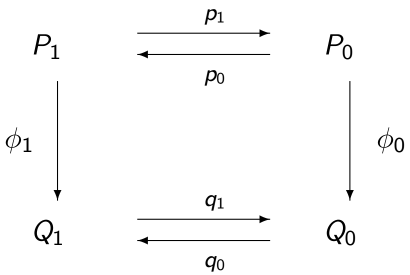
Spectrum

- Matrix factorizations are often organized in the following way:

$$Q : P_1 = \mathbb{C}[x_i]^N \underset{p_0}{\overset{p_1}{\rightleftarrows}} \mathbb{C}[x_i]^N = P_0 \, , \; p_1 p_0 = W \, .$$

(i.e. as infinite, two-periodic, twisted complexes.)

- BRST-invariant bosons are then represented by maps ϕ_0, ϕ_1 such that the following diagram commutes



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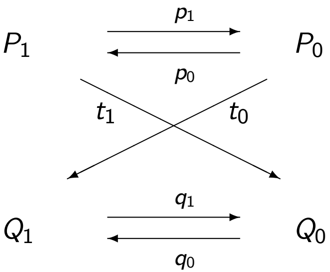


- Cohomology: Divide out exact fermions/bosons.

Tensor products of matrix factorizations

- The tensor product of two Landau-Ginzburg theories with superpotentials $W_1(x_i)$ and $W_2(y_i)$ is given by the superpotential $W = W_1(x_i) + W_2(y_i)$.
- Given MF P of W_1 and Q of W_2 one can form their tensor product:

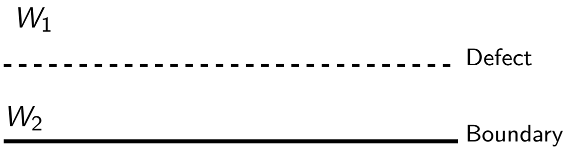
$$P \otimes Q : P_1 \otimes Q_0 \oplus P_0 \otimes Q_1 \underset{r_0}{\overset{r_1}{\rightleftarrows}} P_0 \otimes Q_0 \oplus P_1 \otimes Q_1$$

with

$$r_1 = \begin{pmatrix} p_1 \otimes \mathrm{id} & \mathrm{id} \otimes q_1 \\ -\mathrm{id} \otimes q_0 & p_0 \otimes \mathrm{id} \end{pmatrix}, \quad r_0 = \begin{pmatrix} p_0 \otimes \mathrm{id} & -\mathrm{id} \otimes q_1 \\ \mathrm{id} \otimes q_0 & p_1 \otimes \mathrm{id} \end{pmatrix}$$

- The BRST charge for the tensor product is the sum of the BRST charges of the two factors.

Action of defects on boundary conditions



- Defect and boundary condition are specified by BRST operators $Q_{\rm def}$ and $Q_{\rm bd}$ satisfying

$$Q_{\rm def}^2 = (W_1(x_i) - W_2(y_i)), \quad Q_{\rm bd}^2 = W_2(y_i)$$

- Taking the limit where the defect coincides with the boundary all fermionic degrees of freedom are moved to the new boundary. The new BRST charge is

$$Q'_{\rm bd} = Q_{\rm def} + Q_{\rm bd}$$

- The new boundary condition is obtained from the old one by taking a tensor product.

The new BRST charge

- The sum of the BRST charges has indeed the right properties to describe a boundary condition for the theory with superpotential W_1.

$$(Q'_{\mathrm{bd}})^2 = Q^2_{\mathrm{def}} + Q^2_{\mathrm{bd}} = W_1(X_i) - W_2(Y_i) + W_2(Y_i) = W_1(X_i)$$

- Similarly, two defects can be composed by adding the BRST charges.
- This prescription is simple and natural. However, formally the variables Y_i still appear in the factorization of a superpotential $W(X_i)$ that no longer depends on those variables. So the MF is infinite dimensional over $\mathbb{C}[X_i]$.
- This matrix factorization is equivalent to one that only depends on the X_i, obtained by stripping of infinitely many trivial matrix factorizations.

Fusion

- Consider a polynomial ring $A = \mathbf{C}[X_1, \ldots, X_n]$. To a superpotential associate the ring $R = A/(W)$.
- The R-module $\mathrm{coker} p_1$ has a two-periodic R-free resolution, where the maps are given by the data entering the matrix factorization

$$\ldots \xrightarrow{p_1} R^k \xrightarrow{p_0} R^k \xrightarrow{p_1} R^k \to \mathrm{coker} p_1 \to 0 \; .$$

- In particular, we can write down such a two-periodic resolution corresponding to the tensor product Q' of two factorizations.
- It is then easy to see that the module $V = \mathrm{coker}(p_1 \otimes \mathrm{id}_{Q_0}, \mathrm{id}_{P_0} \otimes q_1)$ has an R-free resolution that becomes periodic after finitely many steps, coinciding with the resolution of the tensor product.
- V is a much simpler object than the tensor product factorization.

Example

- Consider the superpotential $W_1 = X^d$, the LG description of an $N = 2$ minimal model. A defect for this model is given by a matrix factorization of

$$W = X^d - Y^d, \quad \text{e.g.} \quad W = (X - Y)A(X, Y)$$

- A boundary condition for the model $W_2 = Y^d$ is given by factorizations $W = Y^n Y^{d-n}$ or the module $\mathbf{C}[Y]/(Y^n)$. How does it get mapped by the defect?
- Our prescription tells us that the module describing the new boundary condition is given by $V = \mathbf{C}[X, Y]/(X - Y, Y^n) \sim \mathbf{C}[X]/(X^n)$ to which we can associate the factorization $W_1 = X^n X^{d-n}$.
- The defect acts as the identity defect.

Generalization : Symmetry defects

- The type of factorization discussed above for a minimal model exists indeed for arbitrary superpotentials and one can associate to it the identity defect.

$$W(X_i) - W(Y_i) = \sum_j (X_j - Y_j) A_j(X_j, Y_j)$$

- More generally, if the theory has a symmetry g that leaves the superpotential invariant, we can also consider the defects $\mathcal{D}_g$ corresponding to the matrix factorizations

$$W(X_i) - W(Y_i) = \sum_j (X_j - g(Y_j)) A_i(X_j, Y_j)$$

- Regarded as maps of the bulk Hilbert space

$$\mathcal{D}_g : \mathcal{H} \to \mathcal{H}$$

the defect implements the symmetry g.

Properties of symmetry defects

- One can check that the defects $\mathcal{D}_g$ we associated to a symmetry g have the following properties
  - The spectrum of the theory is constant on the full plane, no extra degrees of freedom on the defect.
  - The bulk rings gets indeed mapped according to the symmetry action g, as can be verified by explicit calculations of correlators in the doubled theory.
  - The composition law for the defects is the group law

$$\mathcal{D}_g \mathcal{D}_{g'} = \mathcal{D}_{gg'} \ .$$

  - The action on branes is as it should be.
  - Twisted sector states arise as defect-changing fields, the gth twisted sector between $\mathcal{D}_g$ and $\mathcal{D}_1$. One can verify that the defect changing spectrum is in agreement with the LG orbifold prescription given by Vafa a long time ago.

Application: Flow defects and bulk perturbations

with Daniel Roggenkamp

- Consider CFT on a disk. Turn on a relevant (or marginal) bulk perturbation. The perturbation will induce a flow in both the bulk and boundary sector; UV boundary conditions will flow to boundary conditions of the IR theory. Given a boundary condition for the UV theory, which boundary condition in the IR will it flow to?
- String theory language: Behavior of branes under closed string tachyon condensation.
- "Standard" procedures: Coupled bulk-boundary RG flow. [Not always possible to employ.]
- Interfaces can provide an alternative method to handle the regularization and renormalization.

Flow defects

- Basic idea: Two step procedure
  - Restrict the perturbation to a subset of the Disk. Obtain an interface separating IR and UV of the theory.
  - Alternative construction of the flow defect: Folded theory.
  - Bring the interface to the boundary. This is a singular procedure, the interface will not be topological.

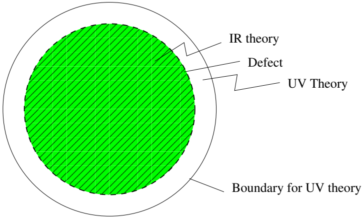


Figure: Perturbation restricted to a domain U (shaded). UV and IR theory are separated by an interface line.

Discussion of the method

- The interface connects directly UV and IR of the theory. The interface approach is hence non perturbative (in the bulk couplings).
- The fusion process is complicated. However, one can consider $N = (2, 2)$ supersymmetric models, which can be topologically twisted. On the level of the topological theory, the interfaces compatible with the twist are topological, and one can work out the fusion.

Supersymmetry preserving perturbations in N=(2,2)

- Two types of perturbations that preserve SUSY in the bulk
  - (c,c) perturbations $\Delta S = \int d^2x d\theta^+ d\theta^- \Phi|_{\bar{\theta}^\pm=0}$
  - (a,c) perturbations $\Delta S = \int d^2x d\theta^+ d\bar{\theta}^- \Psi|_{\bar{\theta}^+=\theta^-=0}$
- In theories with boundaries, supersymmetry can be preserved if the perturbation is (c,c) [(a,c)] and the boundary is A-type [B-type]. Hori-Iqbal-Vafa
- Expect that (c,c) [(a,c)] perturbations are described by A [B] type defects, and the behavior of D-branes under such perturbations can be described by fusion.
- This can be made explicit for A-branes in $N = 2$ minimal models $W = X^{k+2}$. Under relevant chiral bulk perturbations, these models flow to minimal models with $k' < k$.

Flow defects in $N = 2$ minimal models

- MF describe B-type defects. However, we can use mirror symmetry to use them to construct A-type defects.
- The mirror of the LG model with $W = X^d$ is its $\mathbb{Z}_d$ orbifold. The generator of $\mathbb{Z}_d$ acts on X by phase multiplication.

$$g : X \to e^{\frac{2\pi i}{d}} X$$

- Defects in orbifolds can then be constructed by associating a $\mathbb{Z}_d$ representation to $\mathbb{Z}_d$ suitable matrix factorizations.

$$\gamma^{-1} Q(g(X_i))\gamma = Q(X_i)$$

- Alternatively:

$$Q : P_1 = \mathbb{C}[x_i]^N \underset{p_0}{\overset{p_1}{\rightleftarrows}} \mathbb{C}[x_i]^N = P_0\,.$$

Regard the P_i become graded modules and p_i respect the grading.

- The fusion process in the orbifold theory then also involves a $\mathbb{Z}_d$-projection.

Flow defects in $N = 2$ minimal models

- For the case of $N = 2$ minimal models, the flow defects can be constructed by lifting to the linear sigma model.
- They can be fused explicitly.
- Applying the flow defects to boundary conditions reproduces and extends previous results on bulk induced boundary flows. Here, a perturbative CFT analysis was not possible, but indirect methods had to be employed. Gaberdiel-Lawrence
- Recently, such flow defects have also been constructed for non-supersymmetric minimal models. Gaiotto

Summary

- Defects gluing two LG theories with superpotentials W_1 and W_2 are described by matrix factorizations of $W = W_1 - W_2$.
- Defects can be merged and defects act on boundary conditions. This operation is described by taking the tensor product.
- The spectrum of defect changing operators arises as a BRST cohomology.
- Application: Flow defects.

Further developments

- A formula for correlation functions was derived by Kapustin-Li using path-integral localization
- Carqueville/Runkel have put all of this in a nice mathematical framework
- Carqueville/Murfet computed knot invariants following Khovanov-Rozansky.
- ... (e.g. comparisons LG-CFT, applications to CY-compactifications)